Calculation of Bead Space Cohomology

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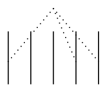
Abstract

This paper describes one approach for calculating the cohomology of a particular complex which arises in knot theory and which is used to prove the existence of 'associators', a key ingredient in the construction of a universal knot invariant. The approach taken is to show that the relevant complex can be interpreted, essentially, as the cohomology of the functor $Hom^{CC}(K, C)$, where C is the coalgebra of commutative power series in a finite number of variables, and the *Homs* are bi-comodule morphisms from the ground field K into a cofree bi-comodule resolution of C. Techniques from homological algebra and Koszul duality theory are then used to compute this cohomology. The original content of this article consists only of the interpretation of the knot theory complex as the complex described above. Otherwise, the paper serves primarily to provide a description of the relevant homological and Koszul theoretic tools used.

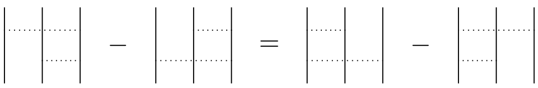
Contents

1 Knot Theoretic Background

This section will provide some very rudimentary knot theoretic background to motivate the main content of this paper. The overall goal of the paper is to exhibit a means to calculate the cohomology of a particular complex which arises in knot theory and which is used in proving the existence of an associator. To describe this complex we begin by defining the following space, for each integer $n > 0$:

$AP_n := \Big\{$  - n vertical strands
- trivalent graphs attached to strands $\Big\}$ *mod* $4T$

Ie, the vector space of diagrams with n vertical strands (usually indicated by solid lines), with a (possibly empty or disconnected) trivalent graph (indicated usually by dotted lines), all of whose endpoints are attached to strands. When more than on endpoint is attached to a particular strand, different orderings of the endpoints on the strand give different diagrams. Finally, this space is subject to the relation $4T$, which stands for '4-term relation', ie the subspace generated by the following relation:



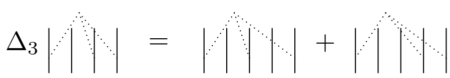
plus similar relations obtained by permuting the strands in each diagram the same way, or by adding one or more blank vertical strands to either side or in between the shown strands.

One can then form a sequence:

$$\ldots \to AP_{n-1} \to AP_n \to AP_{n+1} \to \ldots \qquad (1)$$

which will become a complex once we define a differential d (and show that $d^2 = 0$). To define the differential, we first need to define two operations which exist on the spaces AP_n, which are used in defining the differential. They are:

- Blank strand addition: Given a particular diagram in AP_n, we can add a blank strand to the left, or to the right, of the diagram (we call the resulting operations L and R).
- Strand doubling: Given a particular diagram in AP_n, we can double the ith strand, and sum over the ways of lifting graph endpoints that ended on strand i to the two newly-formed strands. We get i operations Δ_i, for $i = 1 \ldots n$. To help clarify this definition, operation Δ_3 operates as follows on the indicated diagram:

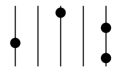


We can now define the differential d as follows:

$$d := L + \Sigma_{i=1}^{n}(-1)^i \Delta_i + (-1)^{n+1} R \qquad (2)$$

One then verifies that the sequence (1) becomes a complex using this differential. A key motivator for this paper is the fact that knowing certain parts of the cohomology of this complex allows one to prove the existence of associators (see [2]).

In fact, it turns out that knowing certain parts of the cohomology of a much simpler but related complex is sufficient. Specifically, one simplifies the spaces AP_n by retaining in each diagram only the vertical strands and the endpoints of trivalent graphs (marked as 'beads' which sit at different heights on the strands), but excising the graphs. One is left with vector spaces B_n, where:

$B_n := \Big\{$  - n vertical strands
- beads placed on strands at different heights $\Big\}$

where we note that since the trivalent graphs are gone, the $4T$ relation is no longer relevant. We get a sequence:

$$\ldots \to B_{n-1} \to B_n \to B_{n+1} \to \ldots \tag{3}$$

in which the horizontal arrows are the same differentials as in the complex (1). This makes sense since the definition of d used only the endpoints of graphs, not their interiors, so d carries over to the bead spaces B_n.

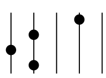
There is a clever trick one can use to recover the cohomology of (1) from the cohomology of (3), see [2]. The purpose of this paper is to describe a procedure whereby one can compute the cohomology of the complex (3).

2 Algebraic Model

2.1 An Algebraic Model for B_n

Our plan of attack is to find a way to translate the pictorial complex (3) into an algebraic complex amenable to the tools of homological algebra. We first note though that the differential d does not affect the number or height of each bead, so we can partition the complex (3) into subcomplexes each containing a fixed number k of beads (at different heights), where k is any positive integer. The cohomology of the total bead complex will then be the direct sum (over k) of the cohomologies of the subcomplexes with k beads. Henceforth, the terms in the bead complex (3) should be understood as spaces B_n^k, of bead diagrams on n vertical strands and with some fixed by unspecified number k of beads (at different heights). We will now exhibit an algebraic model for these B_n^k.

We illustrate the procedure with an example. We will model the following bead diagram by means of the indicated monomial:

 $\longleftrightarrow \quad x_2 \otimes x_1x_3 \otimes 1 \otimes x_4 \otimes 1$

We now define the space $C_4 := K[[x_1, \ldots, x_4]]$, the commutative power series in 4 variables over the ground field K (which here and throughout is assumed to have characteristic 0), and note that the monomial on the right sits inside $C_4^{\otimes 5}$. Thus the procedure can be described as follows:

- The ith strand is represented by the ith component in the tensor monomial.
- The ith highest bead is represented by the variable x_i, located in the component corresponding to the strand on which the bead rests.
- The presence of two or more beads on a particular strand is represented by the product of the corresponding variables, situated in the relevant component of the monomial. These variables must commute, since the order in which the variables are written does not have any meaning within the bead diagram (all the relevant information, relating to the height of the beads, is encoded in the indices).
- A blank strand is indicated by a 1 in the relevant component.

More generally, we will consider the spaces $C_k^{\otimes n}$, where k corresponds to the number of beads and n to the number of strands. The bead spaces B_n^k will in fact be the subspaces of $C_k^{\otimes n}$ in which each variable $x_i,\ i = 1 \ldots k$ appears exactly once in each monomial. However it will become clear that the differential that we will define between the spaces $C_k^{\otimes n}$ does not change the number or type of the variables, so that in fact we can proceed to deal with complexes involving the whole spaces $C_k^{\otimes n}$, and restrict ourselves as needed to the subcomplexes with exactly one variable of each kind.

We will thus consider sequences:

$$0 \to C_k \to C_k^{\otimes 2} \to C_k^{\otimes 3} \to \ldots \tag{4}$$

We still need to explain how to define a differential which models the differential in our bead complexes.

2.2 An Algebraic Model for the Differential

2.2.1 Preliminaries

We will need a familiarity with the concept of coalgebra, whose definition we now recall.

Definition 1. *Coalgebra*

A coalgebra C is a vector space over the field K with maps

- $\Delta :\ C \to C \otimes C$ and
- $\epsilon :\ C \to K$

Here Δ satisfies the following 'coassociativity' property:

$$(\Delta \otimes id)\ \Delta = (id \otimes \Delta)\ \Delta$$

and Δ and ϵ together must satisfy the following compatibility requirement:

$$(\epsilon \otimes id)\ \Delta = (id \otimes \epsilon)\ \Delta = id$$

Example 1. *Coalgebra of Power Series*

We make C_k into a coalgebra by defining

$$\begin{aligned}\Delta(1) &:= 1 \otimes 1 \\ \Delta(x_i) &:= x_i \otimes 1 + 1 \otimes x_i, \quad \forall i = 1 \dots k\end{aligned}$$

and

$$\begin{aligned}\epsilon(1) &:= 1 \\ \epsilon(x_i) &:= 0, \quad \forall i = 1 \dots k\end{aligned}$$

and extending Δ and ϵ to products of the x_i multiplicatively. It is readily verified that C_k with these operations forms a coalgebra.

We can now define operations $\Delta_i,\ i = 1 \dots n$ on tensor powers $C_k^{\otimes n}$ which 'replicate' the strand doubling operation, by defining Δ_i to be the coproduct Δ defined above, acting on the ith tensor component in $C_k^{\otimes n}$. It is again readily verified that such Δ_i do indeed replicate the effect of strand doubling.

In order to find an algebraic description of the differential we still need to algebraically describe left and right (blank) strand addition. The most obvious approach would be simply to set

$$\begin{aligned}L(c) &:= 1 \otimes c \\ R(c) &:= c \otimes 1\end{aligned}$$

This 'works' in the sense that it does indeed replicate the effect of the pictorial strand addition operations. However, it leaves us in no better position to compute cohomology than when we simply faced the complex (3) (we have simply changed the terminology). We will therefore take a more roundabout approach that will pay off in the end by permitting the use of some powerful homological algebra tools to compute cohomology.

To this end we need to recall a number of definitions and basic observations concerning comodules and their resolutions. The reader who is familiar with these notions can quickly skim section 2.2.2.

2.2.2 Comodules, Their Resolutions, and Hochschild Cohomology

We first recall the notion of comodule over a coalgebra C.

Definition 2. *Comodule*

A vector space M is a (left-) comodule over a coalgebra C if there is a linear 'structure' map

$$\lambda : M \to\ C \otimes M$$

such that $(C \otimes \lambda) \circ \lambda = (\Delta \otimes M) \circ \lambda$, and

$$(\epsilon \otimes M) \circ \lambda = M.$$

Note that, here and from now on, we will often use the convention whereby the identity map id_V of a vector space V is denoted V (when no confusion is likely - e.g., when id_V appears as part of a tensor product). Thus, for instance, in the above $(C \otimes \lambda)$ means $(id_C \otimes \lambda)$.

There is a corresponding definition of right-comodule (with structure map ρ), and of bi-comodule, which is a left- and right-comodule with compatibility condition

$$(C \otimes \rho) \circ \lambda = (\lambda \otimes C) \circ \rho$$

Example 2. *Comodule structure on C.*

C has a structure of left-, right- or bi-comodule over itself, where the left and right structure maps are simply the coproduct in C. Compatibility comes from coassociativity of the coproduct.

Example 3. *Trivial Comodule Structure on K*

The ground field K has a trivial (left-) comodule structure given by $\Delta k := 1 \otimes k,$ for all $k \in K$ (and similar right- and bi-comodule structures).

Definition 3. *Comodule Homomorphisms*

If M and N are left-comodules over a coalgebra C, then a comodule homomorphism $\phi : M \rightarrow \; N$ is a linear map of the underlying vector spaces which satisfies

$$(C \otimes \phi) \circ \lambda_M = \lambda_N \circ \phi$$

Similar concepts exist for right- and bi-comodules.

Definition 4. *Cofree Comodules*

Let V be a vector space, C a coalgebra and M a comodule (left-, right- or bi-) over C. We define a functor *Forget* : **Comod** $\rightarrow$ **Vect** which sends any comodule M to its underlying vector space (here **Comod** refers to the category of C comodules (left-, right- or bi-), and **Vect** refers to the category of vector spaces over the underlying ground field K).

A cofree comodule generated by V over C is a comodule $CoFree(V)$ such that for any comodule M (left-, right- or bi-) over C,

$$Hom^C(M, CoFree(V)) \simeq Hom^K(\mathit{Forget}(M), V) \tag{5}$$

where, on the left, Hom^C refers to homomorphisms of left-, right- or bi-comodules as the case may be, and as usual Hom^K means vector space maps.

A different way of saying this is that $CoFree(V)$ is a vector space over K, together with a vector space surjection $\pi : \; CoFree(V) \rightarrow V$, such that for any comodule M (left-, right- or bi-) over C, and any (vector space) morphism $f: \mathit{Forget}(M) \to V$, there is a unique lifting f' of f, ie a comodule morphism (of the appropriate type) $f': M \to CoFree(V)$ such that $\pi \circ f' = f$.

Exercise 1. *Show that for any vector space V, the cofree left-, right- or bi-comodule generated by V over C is $C \otimes V$, $V \otimes C$ or $C \otimes V \otimes C$, respectively. In the case of left cofree comodules, the projection π is just $\epsilon_C \otimes V$, and f' is given by the composition $(C \otimes f) \circ \lambda_M$ (and similarly for right cofree comodules). In the case of cofree bi-comodules, the projection π is just $\epsilon_C \otimes V \otimes \epsilon_C$, and f' is given by the composition $(C \otimes f \otimes C) \circ (\lambda_M \otimes C) \circ \rho_C$.*

The following specialization of the constructions in Example 1 will be essential in the sequel.

Example 4. *We take the case $M = K$ with the bi-comodule structure set out in Example 3 and let $f: K \to V$ be any vector space map. Then the lifting of f to a bi-comodule map $f': K \to C \otimes V \otimes C$ is given by the composition*

$$(C \otimes f \otimes C) \circ (\lambda \otimes C) \circ \rho$$

and thus sends an element $k \in K$ to $1 \otimes f(k) \otimes 1$.

Note that we can identify $Hom^K(K, V)$ with V simply by associating any map f with the value $f(1)$. Taking this observation together with the previous example, we see that we have the isomorphisms:

$$Hom^{CC}(K, C \otimes V \otimes C) \to Hom^K(K, V) \to V \tag{6}$$

We refer to the composition of these isomorphisms as α (itself an isomorphism). We note that, under α, a map $f' \in Hom^{CC}(K, C \otimes V \otimes C)$ gets sent to $(\epsilon \otimes V \otimes \epsilon) \circ f'(1)$. Conversely, to any $v \in V$ the isomorphism α^{-1} will associate the bi-comodule morphism which sends $1 \in K$ to $1 \otimes v \otimes 1 \in C \otimes V \otimes C$. The reader may find it useful to verify these assertions, particularly as they will be important in what follows.

Before we proceed further, we need to introduce one more concept, that of cofree resolution.

Definition 5. *Cofree Comodule Resolution*

A cofree resolution of a comodule M (left-, right- or bi-) over C is an exact sequence of cofree comodules M_i (left-, right- or bi-) over C and differentials d_i:
$0 \to M \xrightarrow{d_{-1}=\epsilon} M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} \ldots$

There is a standard resolution of C by bi-comodules, known as the cobar resolution:

Definition 6. *Cobar Resolution*

This is the sequence:

$$0 \to C \to C \otimes C^0 \otimes C \to C \otimes C^1 \otimes C \to C \otimes C^2 \otimes C \to \ldots \tag{7}$$

Here the zeroth term of the resolution, $C \otimes (K = C^0) \otimes C \simeq C \otimes C$ in a canonical way. The differentials are given by

$$d_n = \sum_{i=0\dots n} (-1)^i C \otimes \cdots \otimes C \otimes \Delta \otimes C \otimes \cdots \otimes C$$

where the Δ appears in the ith position in the ith term. From exercise (1) we know that the $C \otimes C^n \otimes C$ are cofree generated by C^n. To show exactness, we construct a homotopy $h : C \otimes C^{n+1} \otimes C \to\ C \otimes C^n \otimes C$ such that $dh + hd = id$, for $n \geq -1$.

Specifically, we let

$$h(c_0 \otimes c_1 \otimes \cdots \otimes c_{n+1}) = \epsilon(c_0)\ c_1 \otimes \cdots \otimes c_{n+1}$$

Then

$$dh(c_0 \otimes c_1 \otimes \cdots \otimes c_{n+1}) = \epsilon(c_0)\ d(c_1 \otimes \cdots \otimes c_{n+1})$$

while

$$\begin{aligned} hd(c_0 \otimes c_1 \otimes \cdots \otimes c_{n+1}) &= h(\Delta(c_0)\ c_1 \otimes \cdots \otimes c_{n+1}) - h(c_0 \otimes d(c_1 \otimes \cdots \otimes c_{n+1})) \\ &= ((\epsilon \otimes C) \circ \Delta)(c_0) \otimes c_1 \otimes \cdots \otimes c_{n+1} - h(c_0 \otimes d(c_1 \otimes \cdots \otimes c_{n+1})) \\ &= c_0 \otimes c_1 \otimes \cdots \otimes c_{n+1} - \epsilon(c_0) d(c_1 \otimes \cdots \otimes c_{n+1}) \end{aligned}$$

In going from the second to third line we have used the identity $(\epsilon \otimes id) \circ \Delta = id$, valid in any coalgebra. It follows that adding $dh + hd$ gives the identity, as required.

Note that there are analogous cobar resolutions by left- or right- comodules, in which the term $C \otimes C^n \otimes C$ is replaced by $C \otimes C^n$ or $C^n \otimes C$, as the case may be (and the differentials keep the same form).

We note in passing that, if we took $C = C_k$, the cobar resolution would involve the same spaces as the sequence (4), but nonetheless the cobar resolution cannot be the complex we are looking for as the differentials have the wrong form (in particular they do not involve left and right 'strand addition').

We have almost completed our review of algebraic preliminaries. All we need now is the concept of Hochschild cohomology of a coalgebra C.

Definition 7. *Hochschild Cohomology of a Coalgebra C*

The Hochschild cohomology of a coalgebra C is the cohomology that results from applying the functor $Hom^{CC}(K,\ \ )$ to the cobar resolution of C by cofree bi-comodules. In the case of the coalgebras C_k, we get the sequence:

$$0 \to Hom^{CC}(K, C_k) \to Hom^{CC}(K, C_k^{\otimes 2}) \to Hom^{CC}(K, C_k^{\otimes 3}) \to \dots \tag{8}$$

The differential in this sequence is induced from the differential on the cobar complex, via the formula:

$$df := f \circ d, \quad \forall f \in Hom^{CC}(K, C_k^{\otimes n})$$

2.2.3 Derivation of the Algebraic Complex Differential

We now apply this algebra to show how the differential on the algebraic complex (4) arises in a natural way. In the following diagram we have displayed two parallel complexes, the bottom one of which is the algebraic complex (4). The top row is what we get after applying the isomorphism α^{-1} (see (6)) with $V = C_k^{\otimes n},\ C_k^{\otimes n+1}, \ldots$.

$$\begin{tikzcd}
{} \arrow[r] & Hom^{CC}(K, C \otimes C_k^{\otimes n} \otimes C) \arrow[r] & Hom^{CC}(K, C \otimes C_k^{\otimes n+1} \otimes C) \arrow[r] \arrow[d, "\alpha"] & {} \\
{} \arrow[r] & C_k^{\otimes n} \arrow[u, "\alpha^{-1}"] \arrow[r] & C_k^{\otimes n+1} \arrow[r] & {}
\end{tikzcd}$$

We have also shown the isomorphism α^{-1} going up and α going down.

We take the top row to be the Hochschild cohomology of C_k (see (8)) and we will compute the effect of the composition $\alpha \circ d \circ \alpha^{-1}$ on an element $c \in C_k^{\otimes n}$. We will see that through this composition we recover the desired differential for (4). We have, for $c \in C_k^{\otimes n}$:

$$\begin{aligned}
\alpha \circ d \circ \alpha^{-1}(c) &= \alpha \circ d(1 \otimes c \otimes 1) \\
&= \alpha(1 \otimes 1 \otimes c \otimes 1 + \sum_{i=1}^{n} (-1)^i 1 \otimes \Delta_i(c) \otimes 1 + (-1)^{n+1} 1 \otimes c \otimes 1 \otimes 1) \\
&= 1 \otimes c + \sum_{i=1}^{n} (-1)^i \Delta_i(c) + (-1)^{n+1} c \otimes 1
\end{aligned}$$

where in going from the first to second line we used the fact that $\Delta(1) = 1 \otimes 1$ (see Example 1 re the coalgebra of power series and Example 7 re the cobar resolution). Moreover, in going from the second to third line, we used the fact that $\alpha = \epsilon \circ C_k^{\otimes n} \otimes \epsilon$ and that $\epsilon(1) = 1$ (again see Example 1 re the latter point).

Thus as promised we see that we recover the differential for the algebraic complex (4) as to be the composition $\alpha^{-1} \circ d \circ \alpha$. To say this a little differently, we have shown that the algebraic complex (4) with the desired differential is the cohomology of the functor $Hom^{CC}(K, \ )$ applied to the cobar resolution of $C_k^{\otimes n}$ by cofree bi-comodules. This is the real benefit to our rather circuitous method of defining the differential: we can now use the basic fact from homological algebra that the cohomology of a functor applied to a resolution is independent of the particular resolution chosen. In other words, we are free to pick any other, simpler resolution of C_k, for which the calculation of cohomology will be easer (if we can find one). And here we are again fortunate: it turns out that there is indeed another well-known resolution of the power series coalgebra C_k, which is much simpler and indeed whose cohomology under the functor $Hom^{CC}(K, \ )$ has been well studied: it is essentially the De Rham complex.

The fact that all resolutions (of the same type - e.g., by cofree bi-comodules) give the same cohomology will not be reproduced here: see, for instance, [3], Theorem 17.1(6), where this is established in the (dual) case of projective (including free) resolutions of modules over a ring.

2.3 A Simpler Resolution for C_k

To explain the simpler resolution that we will use, we begin by introducing some notation for the exterior algebras on finite sets $\{x_1 \dots , x_k\}$ of variables.

Definition 8. *Exterior Algebra on* $\{x_1 \dots , x_k\}$

For k a positive integer, we define

$$\begin{aligned} A_k &:= \Lambda(x_1 \dots , x_k) \\ &= K \oplus \Lambda^1 \oplus \Lambda^2 \oplus \dots \oplus \Lambda^k \end{aligned}$$

where the Λ refers to the exterior algebra over the ground field K generated by the indicated symbols, and the superscripts in each Λ^i indicate the subspaces of degree i.

We can now define the Koszul resolution for C_k by bi-comodules. To do this we first define the Koszul resolution of K by left (and right) C_k comodules.

Definition 9. *Left (Right) Koszul Complex for Power Series Algebras*

The Koszul resolution of K as left C_k comodule is the following:

$$0 \to K \to C_k \to C_k \otimes \Lambda^1 \to C_k \otimes \Lambda^2 \to \dots$$

Here the second arrow is the trivial left structure map for K as C_k comodule (ie, $k \mapsto 1 \otimes k$). Subsequent arrows are the differential:

$$x_{i_1} \dots x_{i_l} \otimes a \mapsto \sum_{s=1}^{l} x_{i_1} \dots \hat{x}_{i_s} \dots x_{i_l} \otimes (x_{i_s} \wedge a)$$

where $x_{i_1} \dots x_{i_l} \in C_k$, $a \in A_k$ and the hat in $\hat{x}_{i_s}$ means that that symbol is omitted. It is readily verified that this differential squares to 0. It is finally noted that the above complex is really a direct sum of subcomplexes of fixed degree $n \geq 0$, with K concentrated in degree 0. In other words, we really have the complexes:

$$0 \to K \to K \to 0$$
$$0 \to 0 \to C_k^n \to C_k^{n-1} \otimes \Lambda^1 \to C_k^{n-2} \otimes \Lambda^2 \to \dots$$

where the first row is in degree 0 and the second is in degree $n > 0$.

To show that the complex is exact, we exhibit a contracting homotopy. Specifically, we define:

$$h: \; C_k \otimes \Lambda^{n+1} \to C_k \otimes \Lambda^n$$
$$x \otimes (x_{i_1} \wedge \dots \wedge x_{i_{n+1}}) \mapsto \sum_{s=1}^{n+1} (-1)^s (x \; x_{i_s}) \otimes (x_{i_1} \wedge \dots \wedge \hat{x}_{i_s} \wedge \dots \wedge x_{i_{n+1}})$$

where $x \in C_k$ and $x_{i_1} \wedge \dots \wedge x_{i_{n+1}} \in \Lambda^{n+1}$.

It is left to the reader to check that $hd + dh = (degx + n + 1) \; id$.

The Koszul resolution of K by cofree right comodules is analogous.

Definition 10. *Two-sided Koszul Complex for Power Series Algebras*

The Koszul resolution of C_k by cofree bi-comodules is based on the sequence:

$$0 \to C_k \to C_k \otimes C_k \to C_k \otimes \Lambda^1 \otimes C_k \to C_k \otimes \Lambda^2 \otimes C_k \to \dots$$

The differential is given by the co-product Δ in degree -1, and by:

$$d := d^l \otimes C_K + (-1)^{n+1} C_k \otimes d^r$$

where d^l and d^r refer to the left and right Koszul differentials, respectively. Again, one can verify that this differential satisfies $d^2 = 0$.

The fact that the above complex is exact is more involved, and we will skip it. The reader may refer to [1], Proposition 3.12, from which it follows that the exactness of the two-sided Koszul complex is equivalent to the exactness of either one-sided complex. A dual version of this Proposition appears in [4] as Proposition 19.

2.4 The Cohomology of the Bead Space Complex

Finally, we cite a theorem which gives the cohomology of the functor $Hom^{CC}(K, \;)$ (also known as Hochschild cohomology) as applied to the two-sided Koszul resolution:

Theorem 1. *Hochschild Cohomology of C_k*

$$H^n(Hom^{CC}(K, C_k^{bi})) \;\simeq\; A_k^n \tag{9}$$

where C_k^{bi} refers to the two-sided Koszul resolution of C_k, and A_k^n is the degree n subspace of A_k.

We will again skip the proof, a more general version of which can be found in [1], Corollary 3.25. A related result is given in [5], Theorem 14(ii), and [4], Corollary 4, which state that $H_n(Hom_{A_k}(K,K)) \simeq C_k^n$ (where Hom_{A_k} refers to one-sided A_k algebra morphisms, and the first copy of K refers to the one-sided Koszul resolution of K). This is a dual result, but only a partial dual as it deals only with one-sided cohomology, not the two-sided version that we use.

With this theorem, we are done, since we know that the degree n cohomology of the bead complex (3) (with k beads) is the same as that of the subcomplex of the algebraic complex (4) with k variables (in which each variable appears exactly once per monomial). This in turn is the subspace of A_k^n in which each variable appears exactly once per monomial - hence is 0 if $n \neq k$, and is isomorphic to K if $n = k$.

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