

# CLOSED-FORM ASSOCIATOR in an ALMOST COMMUTATIVE QUOTIENT

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THE SETTING: Finite-type invariants

[1]

Includes: Alexander, Jones, HOMFLYPT, Reshetikhin-Turaev, most other.

**CENTRAL FACT:** There is a bialgebra  $A$  and an essentially 1-1 correspondence:

$$\left\{ \begin{array}{l} \text{Finite-type} \\ \text{Q-valued} \\ \text{invariants} \end{array} \right\} \xrightarrow[\text{hard}]{} \xleftarrow[\text{easy}]{} A^* := \left\{ \begin{array}{l} \text{Linear maps} \\ A \rightarrow Q \end{array} \right\}$$

[2]

What is  $A$ ?

[3]

**OUR GOAL**

[4]

Construct  $Z$  so that  
given any  $W$  we get a  $V$ :

$$\left\{ \begin{array}{l} \text{Knots} \\ \text{and} \\ \text{Tangles} \end{array} \right\} \xrightarrow{Z} A$$

$$V \xrightarrow{?} W \xleftarrow{?} Q$$

$$A = \left\langle \begin{array}{l} \text{Chord} \\ \text{diagrams} \\ \text{on 3} \\ \text{strands} \end{array} \right\rangle \text{eg: } \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \cdots \quad | \quad | \\ \uparrow \quad \uparrow \end{array} \right\rangle / \begin{array}{l} 4TR \\ + \text{other} \end{array}$$

$$4TR: \boxed{H} + \boxed{H} = \boxed{H} + \boxed{H} \quad \text{and cyclic permutations of strands}$$

We require multiplicativity

$$Z(\text{tangle}) = Z\left(\begin{array}{c} 1 \\ \times \\ 1 \end{array}\right) = Z(\text{X}) \cdot Z(1)$$

[5]

**3-POINT PLAN:** ① Define  $Z$  on

[6]

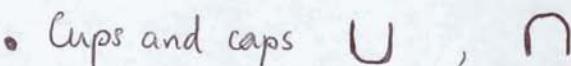
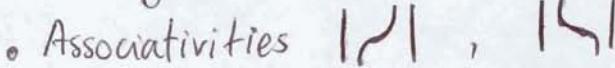
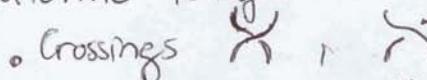
② Extend to knots and

[6']

"atomic" tangles  
• Crossings  
• Associativities  
• Cups and caps

general tangles by  
composition of  
tangles

•



③ Check RELATIONS, eg Reidemeister.

[7]

R3 is particularly hard

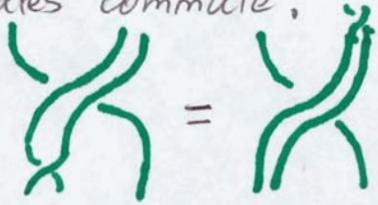
[8]

so we introduce **CABLING + SCALE**

**CABLING:** Double, triple or cable strands together:

$$\text{eg. } (\Delta_{12} \otimes I_3)(\text{X}) := \text{X}_2 \text{X}_3$$

**SCALE** Events at different scales commute:



We get R3!

[9] To "unpack"

[10] need "distributivity":

$$Z(\text{X}) = Z(\text{YX}) = Z(Y) \cdot Z(X) \quad \Delta$$

Z will have to satisfy a relation something like this

**SAD BUT TRUE:** It is not possible to find such Z.

Let us see this in  $A^{\text{hor}}$ : horizontal chords only  
a special case: Put  $a = \text{H}\text{H}$ ,  $b = \text{H}\text{H}$ ,  $c = \text{H}\text{H}$

$$\text{case: } -^{\text{1h}} = Q\langle\langle a, b, c \rangle\rangle / 4\text{TR}, \text{ 4TR: } [a+b, c] = [b+c, a] = [c+a, b] = c$$

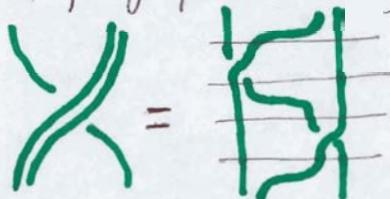
[11] Fact: Can take

$$Z(X) = \exp(H)$$

$$\Delta: \exp(b+c) = \exp(b) \cdot \exp(c)$$

FALSE, for non-commutative  $a, b, c$ ..

So insert fudge factors



The "hexagon"

**MORE BAD NEWS** About

the only knot for which we can calculate Z:



About

[15] LETS FREEZE OUR FEET:

Consider  $\tilde{A}^{\text{hor}} = A^{\text{hor}} / \text{1 FROZEN}$

**1FF:**  $u \cdot x \cdot y = u \cdot y \cdot x$  for  $u \in \{a, b, c\}$ ,  $x, y \in A^{\text{hor}}$

WLOG:  $Z(1/\text{l}) = \exp \varphi$

with  $\varphi$  a Lie series

$$\text{over } \mathcal{L} = Q\langle\langle a, b, c \rangle\rangle / 4\text{TR}$$

$$\text{and } [x, y] = xy - yx$$

$$\text{or } \tilde{\mathcal{L}} = \mathcal{L} / \text{1FF}$$

[16] "Lie words"  $[x_n, \dots, [x_1, [a, b]], \dots]$

$$= \pm [a, b] \underbrace{x_1, \dots, x_n}_{\text{commute}}$$

via 1FF,  
JACOBI

$$\text{Hence } \tilde{\varphi}(a, b) = [a, b] \lambda(a, b)$$

with  $\lambda$  ordinary power series in commutative variables.

[17] **LINEARIZATION**

$$\exp(\tilde{\varphi}(a, b))$$

$$= 1 + [a, b] \lambda(a, b)$$

(1FF)

[22] Fully solved by V. Kurlin

arXiv 0408398. Eg:

[23] **THE FUTURE:**

- freeze both feet?

- relate to

Alexander kernel?

$$2\lambda(x, y) = \left\{ \frac{\sinh(xy)}{(xy)}, \left( \frac{1}{\sinh x} + \frac{1}{\sinh y} - 1 \right) - 1 \right\} / xy$$

Kurlin was looking for Z s.t.  $Z(1/\text{l}) = \exp \tilde{\varphi}$   
 $\tilde{\varphi} \in \tilde{\mathcal{L}} = \mathcal{L} / [[\mathcal{L}, \mathcal{L}], [\mathcal{L}, \mathcal{L}]]$ , and got the same hexagon.

$$\lambda(ab)e^{btc} + \lambda(b,c)e^c + \lambda(ac)$$

=

$$-\left\{ \frac{e^{btc}}{ab} + \frac{e^c}{bc} + \frac{1}{ac} \right\}$$