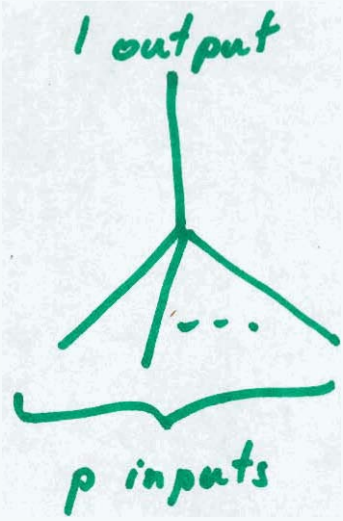
HOCHSCHILD COHOMOLOGY

- A associative algebra

- Define $C^p(A) := \mathrm{Hom}(A^{\otimes p}, A)$

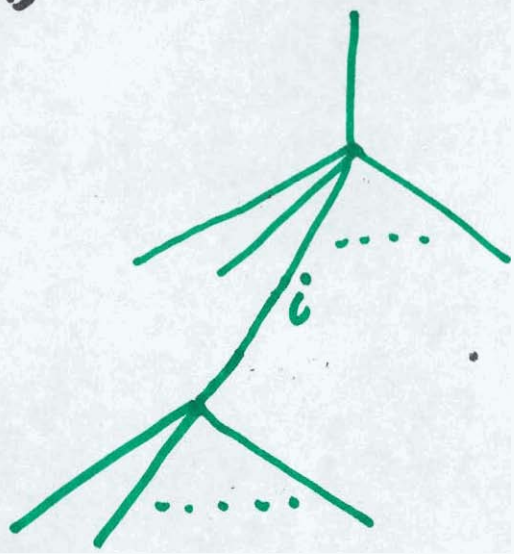
- $C \in C^p(A) \longleftrightarrow$



- Vertical compositions "$\circ_i$" ie $C \circ_i D$

$C \in C^p$

$D \in C^q$



Gerstenhaber product $C \diamond D$

"Signed sum over insertions of $\underset{q}{D}$ into $\underset{p}{C}$"

$$C \diamond D := \sum_{i=1}^{p} (-1)^{(i-1)(q-1)} \, C \circ_i D$$

Gerstenhaber bracket

$$[C, D] := C \diamond D - (-1)^{(p-1)(q-1)} \, D \diamond C$$

FACT $C(A)$ is a graded Lie algebra with $[-,-]$

ASSOCIATIVITY CRITERION

Any $m \in C^2(A_\lambda)$ defines a product $*_m$

FACT $[m, m] = 0 \iff *_m$ associative

PROOF: $[m, m](x, y, z) =$

$$m(m(x, y), z) - m(x, m(y, z))$$

$$= (x * y) * z - x * (y * z)$$

$$= 0$$

$\iff *$ associative.

DIFFERENTIAL ON $C(A)$

Given $\mu \in C^2(A) = \mathrm{Hom}(A \otimes A, A)$

with $[\mu, \mu] = 0$

ie $*_\mu$ associative

<u>Define</u> : $d_\mu : C^p \longrightarrow C^{p+1}$

$$d_\mu(c) := [c, \mu]$$

<u>FACT</u> $d^2 = 0$

<u>Proof</u> : graded Jacobi, As

Proof of $d^2 = 0$

$C \in C^p(A)$

$$d_\mu \circ d_\mu (C) := d_\mu [C, \mu]$$

$$:= [[C, \mu], \mu]$$

$$= [[\mu, \mu], C] + (-1)^{p-1} [[\mu, C], \mu]$$

(from graded Jacobi)

$$= 0 - (-1)^{p-1}(-1)^{p-1} [[C, \mu], \mu]$$

(from graded AS)

$$= -[[C, \mu], \mu]$$

Hence $d^2 C = [[C, \mu], \mu] = 0$ as required.