Hochschild Cohomology

- A associative algebra
- Define $C^p(A) := \text{Hom}(A^\otimes p, A)$
- $c \in C^p(A)$

Vertical compositions "\(D_i\)" i.e. $C \circ i \circ D$
Gerstenhaber product $COD$

"Signed sum over insertions of $D$ into $C$"

\[ C \odot D := \sum_{i=1}^{p} (-1)^{(i-1)(q-1)} C_{o_i} D \]

Gerstenhaber bracket

\[ [C,D] := COD - (-1)^{(p-1)(q-1)} DOC \]

FACT $C(A)$ is a graded Lie algebra with $[-,-]$
**Associativity Criterion**

Any \( m \in C^\infty(A) \) defines a product \( \cdot_m \)

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**Fact**  \[ [m, m] = 0 \iff \cdot_m \text{ associative} \]

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**Proof:**  \[ [m, m](x, y, z) = \]

\[ m \left( m \left( m(x, y), z \right) - m(x, m(y, z)) \right) \]

\[ = (x \cdot y) \cdot z - x \cdot (y \cdot z) \]

\[ = 0 \]

\[ \iff \cdot \text{ associative}. \]
Differential on $\mathcal{C}(A)$

Given $\mu \in \mathcal{C}^2(A) = \text{Hom}(A \otimes A, A)$ with $[\mu, \mu] = 0$

ie $\ast \mu$ associative

Define: $d_\mu : \mathcal{C}^p \to \mathcal{C}^{p+1}$

$d_\mu (\mathcal{C}) := [\mathcal{C}, \mu]$

Fact: $d^2 = 0$

Proof: graded Jacobi, as
Proof of $d^2 = 0$

\[ C \in C^p(A) \]

\[ d_\mu \circ d_\nu (c) := d_\mu \left[ c, \mu \right] \]

\[ := \left[ \left[ c, \mu \right], \mu \right] \]

\[ = \left[ \left[ \mu, \mu \right], c \right] + (-1)^{p-1} \left[ \left[ \mu, c \right], \mu \right] \]

(from graded Jacobi)

\[ = 0 - (-1)^{p-1} (-1)^{p-1} \left[ \left[ c \mu \right], \mu \right] \]

(from graded AS)

\[ = -\left[ \left[ c \mu \right], \mu \right] \]

Hence \( d^2 C = \left[ \left[ c \mu \right], \mu \right] = 0 \) as required.