- Hochschild cohomology

$$H_\mu^\bullet(C(A), d_\mu)$$

- <u>FACT</u>: $C(A)$ is a "differential graded Lie algebra" DGLA

ie $d$ is a graded derivation with respect to $[-,-]$

ie: $d[a,b] = [da,b] \pm [a,db]$

# ASSOCIATIVITY — MAURER-CARTAN CRITERION

. Let $(A, \mu_A)$ be associative.

. Let $\mu := \mu_A + \mu_{def}$

$$\mu_{def} := \sum_{r \geq 1} \mu_r t^r$$

$$[\mu, \mu] = \underbrace{[\mu_A, \mu_A]}_{=0} + \underbrace{[\mu_A, \mu_{def}]}_{= [\mu_{def}, \mu_A]}$$

$$+ [\mu_{def}, \mu_A] + [\mu_{def}, \mu_{def}]$$

Hence:

$$[\mu, \mu] = 2[\mu_{def}, \mu_A] + [\mu_{def}, \mu_{def}]$$

So: $\mu$ associative

$$\iff [\mu, \mu] = 0$$

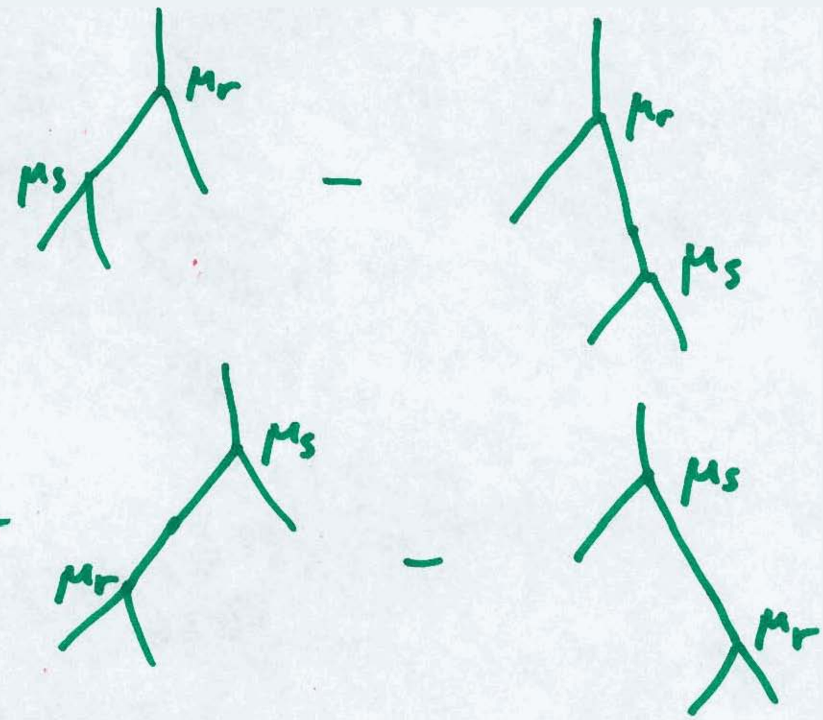
$$\iff d_\mu(\mu_{def}) = -\frac{1}{2}[\mu_{def}, \mu_{def}]$$

"MAURER- CARTAN"

MORE SPECIFICALLY... for any $a,b,c \in \mathcal{A}$,

$$d\mu_n(a,b,c) = -\frac{1}{2}\sum_{\substack{r+s=n \\ r,s\geq 1}} [\mu_r, \mu_s](a,b,c)$$

In pictures: $[\mu_r, \mu_s] =$




So: $$d\mu_n(a,b,c) = -\sum_{\substack{r+s=n \\ r,s\geq 1}} (\mu_r(\mu_s(ab),c) - (\mu_r(a,\mu_s(bc))$$

# LET'S LOOK AT LOW DEGREES

Write $\mu_0(a,b) = a \cdot b = ab$

(original product in $\mathcal{A}$)

## DEGREE 1

$$d\mu_1 = -\frac{1}{2} \sum_{\substack{r+s=1 \\ r,s \geq 1}} [\mu_r, \mu_s] = 0$$

BUT $d\mu_1(a,b,c) := [\mu_1, \mu_0](a,b,c)$

$$= \mu_1(ab,c) - \mu_1(a,bc) + \mu_1(a,b) \cdot c - a \cdot \mu_1(b,c) = 0$$

"Eqn 1"

Now apply $\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$ to (Eqn 1)

$$\left.\begin{aligned} &\mu_1(ba,c) - \mu_1(b,ac) \\ &+ \mu_1(ba)\cdot c - b\cdot\mu_1(a,c) = 0 \end{aligned}\right\} \text{(Eqn 2)}$$

Also apply $\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$ to (Eqn 1)

$$\begin{aligned} &\mu_1(ac,b) - \mu_1(a,cb) \\ &+ \mu_1(a,c)\cdot b - a\cdot\mu_1(c,b) \end{aligned} \quad \text{(Eqn 3)}$$

Now take Eqn 1 − Eqn 2 − Eqn 3 :

(and define $\bar{\mu}_1(a,b) := \frac{1}{2}(\mu_1(ab) - \mu_1(ba))$

Get: $\bar{\mu}_1(b,ac) = \bar{\mu}_1(b,a)c + a\cdot\bar{\mu}_1(bc)$

ie Leibniz

In similar fashion, the degree 2 equation (with suitable permutations) <u>gives</u> JACOBI for $\bar{\mu}_1$.

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Thus any quantization $\mu = \sum_{i \geq 0} \mu_i t^i$ of $A$ gives ~~us~~ a Poisson structure $\{\ ,\ \} := \bar{\mu}_1$ on $\mathcal{A}$.

QUESTION: Can we reverse this?

If $A$ has a Poisson bracket $\{\,,\,\}$,

we could impose $\bar{\mu}_1 = \{\,,\,\}$, find $\mu_1$,

and ask if $\mu_0 + \mu_1 \cdot t$ can be completed

to an associative product

$$\mu = \sum \mu_i t^i$$

$\rightsquigarrow$ This would give a quantization

of $A$ "in the direction of $\{\,,\,\}$"

# COHOMOLOGICAL INTERPRETATION OF ASSOCIATIVITY

**GOAL**: Find $\mu = \sum \mu_i t^i = \mu_A + \mu_{def}$

st: $d\mu_{def} = -\frac{1}{2}[\mu_{def}, \mu_{def}]$ (MC)

**IN DEGREE $n$**:

$$d\mu_n = -\frac{1}{2} \sum_{\substack{r+s=n \\ r,s \geqslant 1}} [\mu_r, \mu_s]$$

WE SOLVE FOR $\mu_n$ degree by degree

DEGREE 1 in $t$

$$d(\mu_1) = 0$$

ie $\mu_1$ is a 2-cocycle

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DEGREE $n$ in $t$

Suppose that for $k \leq n$,

$$d\mu_k = -\frac{1}{2} \sum_{\substack{r+s=k \\ r,s \geq 1}} [\mu_r, \mu_s]$$

We want to find $\mu_{n+1}$

CONSIDER $E_{n+1} = -\frac{1}{2} \sum_{\substack{r+s=n+1 \\ r,s \geq 1}} [\mu_r, \mu_s]$

"degree $n+1$ error"

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$d_\mu(E_{n+1}) = -\sum [\underbrace{d\mu_r}_{\text{expand}}, \mu_s]$ (d is graded derivation)

$= \sum_{\substack{r+s+v=n+1 \\ r,s,v \geq 1}} [\mu_r, [\mu_s, \mu_v]]$

$= \frac{1}{3} \sum [\mu_r, [\mu_s, \mu_v]] + \text{cyclic}$

$= 0$ (Jacobi)

---

SO, if $H^3(A, d_{\mu_n}) = 0$

we can solve for $\mu_n$, $\forall n$

ie we can find $\mu_n$ st $d\mu_{n+1} = E_{n+1}$

# RECAP

To construct $\mu = \mu_A + \sum_{r \geq 1} \mu_r t^r$ :

1. Take $\mu_1$ to be any co-cycle in $C^2(A)$

2. Calculate $H^3(A)$ (!!)

3. If $H^3(A) = 0$, $\mu_1$ can be extended to $\mu$.

# EQUIVALENCE OF DEFORMATIONS

A "change of coordinates"

is an invertible map $g : \mathcal{A}_\lambda \to \mathcal{A}_\lambda$

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First define $g : \mathcal{A} \longrightarrow \mathcal{A}_\lambda$

and extend linearly in $\mathbb{C}[[\lambda]]$.

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As with deformations of "$\cdot$", we take $g$

of the form:

$$\boxed{g(a) := \sum_{r=0}^{\infty} g_r(a)\,\lambda^r}$$

where $g_r : \mathcal{A} \longrightarrow \mathcal{A}$

and $g_0(a) \equiv 1$

Such $g$ is indeed invertible, and if $h = g^{-1}$

$$h_0 = 1$$

$$h_n = -\sum_{r=0}^{n-1} h_r \, g_{n-r} \qquad n > 0$$

$$(h_1 = -g_1)$$

**PROP** Such $g$'s form a group, $G$.

**PROP** $G$ acts on deformations $\mu$ of $A$.

via:

$$\begin{tikzcd}
A_\lambda \times A_\lambda \arrow[rr, "\mu"] \arrow[d, shift right=3, "g \times g"'] & & A_\lambda \arrow[d, "g"] \\
A_\lambda \times A_\lambda \arrow[u, shift right=3, "h \times h"'] \arrow[rr, dashed, "\mu'"'] \arrow[urr, phantom, "\circlearrowright"] & & A_\lambda
\end{tikzcd}$$

$\mu'$ is a new deformation of $\mathcal{A}$

$$\mu' = \sum_{r \geq 0} \mu'_r \lambda^r$$

where : $\mu'_r(a,b) = \sum_{m+k+l+j=r} g_m\left(\mu_k(h_l(a), h_j(b))\right)$

DEGREE 1 : $\quad h_1 = -g_1$

and $\mu'_1(a,b) = g_1(a.b) + \mu_1(a,b) - g_1(a)\cdot b - a\cdot g_1(b)$

Hence $\mu'_1(a,b) - \mu_1(a,b) = g_1(ab) - g_1(a)\cdot b - a\cdot g_1(b)$

$$= dg_1(a,b)$$

$\boxed{\text{ie } \mu'_1 - \mu_1 \in B^1(\mathcal{A})}$