# Closed-Form Associators and Braidors in a Partly Commutative Quotient

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**Abstract**

Construction of a universal finite-type invariant can be reduced, under suitable assumptions, to the solution of certain equations (the hexagon and pentagon equations) in a particular associative algebra of chord diagrams. An explicit, closed-form solution to these equations may, indirectly, give information about various interesting properties of knots, such as which knots are ribbon. However, while closed-form solutions (as opposed to solutions which can only be approximated to successively higher degrees) are needed for this purpose, such solutions have proven elusive, partly as a result of the non-commutative nature of the algebra. To make the problem more tractable, we restrict our attention to solutions of the equations in a certain partly commutative quotient of the subalgebra of horizontal chord diagrams. We show that this restriction leads in a straightforward and fairly short way to a reduction of the hexagon and pentagon equations to a simpler equation taken over the algebra of power series in two commuting variables. This equation had been found and solved explicitly by Kurlin under a superficially different set of reduction assumptions, which we show here are in fact equivalent to ours. This paper thus provides a simpler and more concise derivation of Kurlin's equation. The paper also considers an alternative equation which can be used to obtain finite-type invariants, namely the braidor equation, and shows that in the quotient considered, solutions to this equation are in a 1–1 relation with solutions to the hexagons and pentagon.

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# 1 Introduction

The fundamental theorem of finite-type invariants assures us that a universal finite type invariant exists which takes values in the associative algebra of chord diagrams on the circle (modulo the 4-term relation). Construction of such an invariant can be reduced, under suitable assumptions, to solving certain known equations (the hexagon and pentagon equations) in the related algebra $\mathcal{A}_n$ of chord diagrams on $n$ vertical strands (modulo similar relations). Solving these equations explicitly has proved to be extremely difficult, partly because multiplication in $\mathcal{A}_n$ is associative but not commutative.

A universal formula for solutions to the equations has been given by Drinfel'd [8] using integral methods. However, for applications, closed-form solutions (as opposed to solutions which can only be approximated term-by-term) are often needed. For instance, it is expected that finite-type invariants can be constructed which may give information about the genus of knots, or which knots are ribbon knots [6]. But finite degree approximations are not generally expected to suffice in this regard — in particular, Ng [11] has shown that no finite degree approximation to a universal finite-type invariant can give information about ribbon knots.

A closed–form solution has been found by Lieberum [10] which is applicable where the algebra is the universal enveloping algebra of gl(1|1). However, the difficulty of solving the equations explicitly for universal finite-type invariants has prompted researchers to consider various simplified versions of the problem. One approach, proposed by Drinfel'd in [8], is to assume solutions to the equations are group-like elements of the relevant associative algebra, and to consider the image of the equations under the logarithm function. The result is certain equations over a Lie algebra $\mathcal{L}$ which has the same generators as the associative algebra. Drinfel'd discussed the solution of these equations in the quotient $\mathcal{L}/[[\mathcal{L},\mathcal{L}],[\mathcal{L},\mathcal{L}]]$, without however giving explicit formulae.

Kurlin [9] subsequently took up Drinfel'd's approach and achieved a significant simplification. Specifically he showed that the logarithmic hexagon and pentagon equations, in the given quotient of the Lie algebra $\mathcal{L}$ (which he called 'compressed' equations), were equivalent to a single equation, this time taken over the algebra of ordinary power series in two formal commuting variables $x$ and $y$:

$$\lambda(x,y) + e^{-y}\ \lambda(y,z)) + e^{x}\ \lambda(x,z)) \ = \ \frac{1}{xy} + \frac{e^{-y}}{yz} + \frac{e^{x}}{xz} \tag{1}$$

Here $\lambda$ is a power series in $x$ and $y$ (for which we are solving) and $z$ stands for $-(x+y)$.

Translating equations over a Lie algebra into an equation over an ordinary commuting power series algebra constituted an important development. However Kurlin was also able to solve this equation and specify explicitly all of the solutions.

In this paper we take an alternative approach to simplifying the hexagon and pentagon equations, which turns out to be equivalent but is simpler and easier to extend. Like Kurlin we restrict ourselves to the algebra of chord diagrams in which all chords are horizontal. In the relevant case where our diagrams lie on three vertical strands, we essentially get a power series algebra in three non-commuting variables subject to certain additional relations. Unlike Kurlin, we consider those equations at the associative algebra level, but over the quotient algebra in which:

$$u[x,y] = 0$$

where $x$ and $y$ are any elements of the algebra, and $u$ represents a product of one or more chords (we call this the 'one frozen foot', or 1FF, quotient). In a later part of the paper, we generalize somewhat to the case where $u$ is a product of any two or more chords (the two frozen feet, or 2FF, quotient).

We show that the hexagon and pentagon equations, modulo 1FF, again reduce to Kurlin's equation defined over the algebra of commuting power series in two variables. Moreover, we show that solving Kurlin's compressed equations at the Lie algebra level is equivalent to solving the hexagons and pentagon at the associative algebra level modulo 1FF. Thus in particular we get an alternative, but simpler and shorter, derivation of Kurlin's simplified equation (1).

We also show how to solve the hexagon and pentagon equations in the 2FF quotient and show that all such solutions are derived, in an explicitly given way, from solutions over 1FF. It is expected that such solutions over 2FF may have important connections with the Alexander polynomial.

Part 2 of the paper is organized as follows: in Section 2.1, we review various relevant algebras of chord diagrams and their related Lie algebras. In Section 2.2, we set forth the equations we wish to solve and the assumptions we place *ab initio* on the solutions. In Section 2.3 we derive some basic properties of the chord diagram algebras and of solutions to the hexagon and pentagon equations, based on the given assumptions. In Section 2.4 we show how the hexagon and pentagon equations, in the 1FF quotient, reduce to Kurlin's equation (1). We also show that the hexagon and pentagon equations hold (with group-like solutions) modulo 1FF if and only if their logarithmic images hold (with Lie series solutions) modulo $[[\mathcal{L}, \mathcal{L}], [\mathcal{L}, \mathcal{L}]]$, thus establishing the equivalence of our results with the results of Kurlin. Finally, in Section 2.5 we extend the main result of Section 2.4 to two frozen feet. Specifically, we show how the hexagon and pentagon equations can be solved in the 2FF quotient by a specific extension of solutions in the 1FF quotient.

Part 3 of the paper applies the technology developed in Part 2 for studying the 1FF quotient to study an equation related to the hexagons, specifically the braidor equation. The braidor equation can be viewed as the equation one gets if one adds a 'shield' strand on the left of the Reidemeister III move. The shield strand is 'inert' in that it is never flipped with any of the other strands. However, one thinks of the shield as potentially 'interacting' with the other strands, in that the chord diagrams associated to crossings of the ordinary strands by the sought-for knot invariant can include chords resting on the shield as well as on the strands being flipped.

One then asks whether one can associate to any crossing of two strands a chord diagram (or series) $B \in \hat{\mathcal{A}}^3$ in such a way that the Reidemeister III equation (and the Reidemeister II equation) is satisfied. It is known that, given an associator $\Phi \in \hat{\mathcal{A}}_3$, one can construct a braidor as $B = \Phi \cdot R^{23} \cdot (\Phi^{-1})^{132}$, where $R^{ij}$ is the power series of chord diagrams representing the flipping of strands $i$ and $j$. However, at this time it is an open question as to whether all braidors come from associators.

We investigate the question whether one can find a symmetric power series of chord diagrams $\Psi$ on 3 strands (not necessarily an associator) such that if we define $B = \Psi \cdot R^{23} \cdot (\Psi^{-1})^{132}$, the braidor equation is satisfied modulo 1FF. We show that, indeed, one can construct all such $\Psi$ in an explicit way. In addition, we show that any $\Psi$ satisfying the braidor equation must also satisfy the equality:

$$Hex^{+} + \ Hex^{-} = 0$$

where $Hex^{+} = 0$ is the positive hexagon and $Hex^{-} = 0$ is the negative hexagon, both modulo 1FF. Indeed, we show that, modulo 1FF, there is a 1-1 relation between solutions to the braidor and solutions to the hexagons and pentagon, and explicitly show how the two are related.

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# 2 Closed-Form Associator

## 2.1 Algebras of Chord Diagrams

### 2.1.1 The Algebras $\hat{\mathcal{A}}$ and $\mathcal{A}^{1FF}$

In this paper, we consider the algebra

$$\hat{\mathcal{A}} := \widehat{\mathcal{A}^{hor}} \tag{2}$$

where $\mathcal{A}^{hor}$ refers to the algebra of chord diagrams on $n$ vertical strands (modulo the 4T relation - see below), but in which we allow only horizontal chords. The $n$ is usually implicit, but sometimes it is indicated, as in $\mathcal{A}_n$. The hat on the RHS above means we take the formal completion. We can also define

$$\mathcal{A}^{1FF} := \hat{\mathcal{A}}/ \ 1FF, \tag{3}$$

where, again, 1FF stands for '1 Frozen Foot', i.e. allowing all chords after the first (counting from the bottom) to commute. In a later portion of the paper we consider the quotient $\mathcal{A}^{2FF} = \hat{\mathcal{A}}/ \ 2FF$, in which all chords after the second (counting from the bottom) commute.

All algebras in this paper are considered over the ground field $\mathbb{Q}$.

### 2.1.2 Notation

In this paper we will mostly be concerned with the case $n = 3$. The resulting algebra can be expressed in purely algebraic terms as an (almost) commutative polynomial algebra on three letters $a, b, c$:

$$\hat{\mathcal{A}} \ := \ \mathbb{Q} \ll a, b, c \gg \ / \ 4T, 1FF \tag{4}$$

where for convenience we use the convention (followed through much of the paper):

$a$ =  $b$ =  $c$ = 

The 4-term relation 4T here translates to

$$[a+b, c] \ = \ [b+c, a] \ = \ [c+a, b] = \ 0$$

or equivalently

$$[a+b+c, w] = 0, \ \forall w \in \mathcal{A}$$

and 1FF translates to:

$$uxy = uyx$$

where

$$u, x, y \in \mathcal{A} \text{ and } deg\ u \geq 1$$

### 2.1.3 Operations on $\hat{\mathcal{A}}$

There are $n$ 'degeneracy' operations $\eta_i$, $i \in \{1, \ldots, n\}$, defined on $\mathcal{A}_n$ (and therefore on $\hat{\mathcal{A}_n}$) by the fact that $\eta_i$ acts on individual chord diagrams by deleting the $i$-th strand (and relabeling subsequent strands) if no chord ends on the $i$-th strand, and sending the diagram to zero if any chord ends on the $i$-th strand.

There are also strand doubling operations $\Delta_i$, where $i$ may range over the number of strands in a diagram. $\Delta_i$ acts by doubling the $i$-th strand, and summing over the number of ways of 'lifting' all strands previously ending on the $i$-th strand to the two new strands. For instance, viewing $a$ as an element of $\mathcal{A}_2$ in the obvious way, $\Delta_1(a) = b + c$. In Part 3 of the paper, we will only allow doubling of the first (shield) strand, that is we only allow the doubling operation $\Delta_1$.

When $\Delta_i$ acts on a diagram with just one strand, we write $\Delta$. This gives rise to the notation $\Delta_i = (1 \otimes \cdots \otimes \Delta \otimes \cdots \otimes 1)$ where the $\Delta$ is in the $i$-th position. We sometimes write this $(1 \ldots 1\Delta 1 \ldots 1)$, omitting the tensor symbols.

The symmetric group $\mathbb{S}_n$ acts on diagrams with $n$ strands by permuting the strands. If $D$ is a diagram with $n$ strands, we indicate the action of the permutation which sends $(12...n)$ to $(xy...z)$ on $D$ as $D^{xy \cdots z}$. For instance, we have

$$a^{132} = c$$

### 2.1.4 Induced Lie Algebras

The algebras $\mathcal{A}_n$ (and their completions) induce Lie algebras $\mathcal{L}_n$ (and their completions $\hat{\mathcal{L}_n}$) with generators the single-chord diagrams in $\mathcal{A}_n$ and with the commutator bracket: $[x, y] := xy - yx$.

We can also form the quotient

$$\mathcal{L}^{CC} := \hat{\mathcal{L}} / \ [[\mathcal{L}, \mathcal{L}], [\mathcal{L}, \mathcal{L}]]$$

where the superscript '$CC$' stands for 'commutators commute'.

## 2.2 Criteria for $\Phi$

We are looking for invertible $R \in \hat{\mathcal{A}}_2$ and $\Phi \in \hat{\mathcal{A}}_3$ satisfying the following primary conditions:

- The Hexagons

  Positive Hexagon:

$$(\Delta 1)R \ = \ \Phi \cdot R^{23} \cdot (\Phi^{-1})^{132} \cdot R^{13} \cdot \Phi^{312} \tag{5}$$

  Negative Hexagon:

$$(\Delta 1)(R^{-1}) \ = \ \Phi \cdot (R^{-1})^{23} \cdot (\Phi^{-1})^{132} \cdot (R^{-1})^{13} \cdot \Phi^{312} \tag{6}$$

- The Pentagon

$$\Phi^{123} \cdot (1\Delta 1)\Phi \cdot \Phi^{234} = (\Delta 11)\Phi \cdot (11\Delta)\Phi \tag{7}$$

We also impose the following ancillary conditions:

- Symmetry

$$\Phi \cdot \Phi^{321} = 1 \tag{8}$$

- Non-Degeneracy

$$\eta_1 \Phi = \eta_2 \Phi = \eta_3 \Phi = 0 \tag{9}$$

- Group-like

$$\Phi = \exp(\phi), \text{ for some } \phi \in \hat{\mathcal{L}}_3 \tag{10}$$

It can be shown that (under the assumptions in this paper, and particularly the restriction to horizontal chords, which forces $R$ to be symmetric) any $\Phi$ which satisfies the hexagons must satisfy the symmetry (or 'unitarity') condition (see [8] at equation (2.10) *et seq.*, or [2] Prop. 3.7).

We also take $R$ to have the form

$$R = \exp(a) \tag{11}$$

where $a$ is viewed as an element of $\mathcal{A}_2$.

### 2.3 Some Basic Manipulations in $\mathcal{A}^{1FF}$

We are looking for a non-commutative power series $\Phi(a,b)$ which is group-like in the sense that $\Phi(a,b) = \exp(\phi(a,b))$, where $\phi(a,b)$ is a Lie series in $a$, $b$, i.e. $\phi(a,b) \in \hat{\mathcal{L}}_3$. In fact, we take the terms of $\phi$ to be of degree at least two, so that the leading term of $\phi(a,b)$ is $[a,b]$ (up to some multiplicative factor). We will now derive a number of simple consequences of the assumptions we have made for $\mathcal{A}^{1FF}$, $\phi$ and $\Phi$.

**Remark 1.** *Lie Word Notation*

The following notation will be useful: for $w$ a word in the alphabet $\{a, b\}$, that is $w = w_1 w_2 \dots w_n$ for $w_i \in \{a, b\}$, we write $[w]$ for

$$[w_1, [w_2, [\dots [w_{n-1}, w_n] \dots]]]$$

However we do sometimes still write the commutator of two terms as $[x, y]$.

**Lemma 1.** *Using the definition of $a, b, c$ above,*
*(i)* $[ab] = [bc] = [ca]$
*(ii) Modulo 1FF, we have*

$$[ab]c = [ab](-a - b)$$

*and more generally*

$$[ab]c^n = [ab](-a - b)^n$$

*Equivalently, in any expression which is pre-multiplied by the commutator $[ab]$, we can take $c = (-a - b)$ modulo 1FF.*
*(iii)* $[c^n ab] = [(-a - b)^n ab]$ *in* $\hat{\mathcal{A}}$.

*Proof.* (i) This follows from the centrality of $a+b+c$, i.e. $[a, c] = [a, c-a-b-c] = [a, -b] = [b, a]$ and similarly for the other equality.
(ii) This is just a simple calculation.
(iii) This is another immediate consequence of the centrality of $a+b+c$. □

**Proposition 1.** *Modulo $[[\mathcal{L}, \mathcal{L}], [\mathcal{L}, \mathcal{L}]]$ (and hence also modulo 1FF and 2FF, as these subalgebras contain $[[\mathcal{L}, \mathcal{L}], [\mathcal{L}, \mathcal{L}]]$),*
*(i) for $w$ a word on $\{a, b\}$ of length at least two (so that $[w]$ is a commutator), we have $[abw] = [baw]$;*
*(ii) for $w$ a word on $\{a, b\}$ with $n$ a's and $m$ b's, we have $[wab] = [a^n b^m ab]$;*
*(iii) any Lie series $l(a, b, c)$ beginning in degree two (or higher) can be written*

$$l(a, b, c) = [w(a, b)ab] \tag{12}$$

*where $w(a, b)$ is a unique commutative power series of words on $\{a, b\}$ and the notation $[wab]$ has been extended in the obvious way; and*
*(iv) for Lie series $l(a, b)$ and $l'(a, b)$, the following are equivalent:*

$$l = l' mod\ 1FF$$
$$l = l' mod\ [[\mathcal{L}, \mathcal{L}], [\mathcal{L}, \mathcal{L}]]$$

*Proof.* The Jacobi relation gives us

$$[abw] = -[b[w]a] - [[w]ab] = -[b[w]a] = [ba[w]]$$

The remaining assertions follow immediately. □

The following is clear from the previous proposition.

**Proposition 2.** *Modulo 1FF, if $w$ is a word on $\{a, b\}$ with $n$ a's and $m$ b's, we have* $[wab] = (-1)^{n+m}[ab]a^n b^m$.

**Remark 2.** *Lie Algebra Decomposition and Vector Space Bases*

*It follows from Lemma 1 and Propositions 1 and 2 that:*

*(i) As Lie algebras,* $\mathcal{L} \cong \mathbb{Q}c \oplus \mathcal{L}(a, b)$, *where* $\mathcal{L}(a, b)$ *is the Lie algebra generated over* $\mathbb{Q}$ *by* $a, b$ *modulo 4T.*

*(ii)* $\mathcal{L}^{CC} = \hat{\mathcal{L}}/[[\mathcal{L}, \mathcal{L}], [\mathcal{L}, \mathcal{L}]]$ *has vector space basis* $\{a, b, c\} \cup \{[a^n b^m ab] : n, m \geq 0\}$.

This remark recaptures some results from [9], see in particular Proposition 3.4 thereof.

We now explore the implication of the symmetry requirement (8) for the form $\Phi$ must take.

**Proposition 3.** *Taking* $\Phi(a, b) = \exp \phi(a, b)$ *where* $\phi(a, b) = [w(a, b)ab]$ *and* $w(a, b)$ *is a power series in* $a$ *and* $b$*, the symmetry requirement (8) is equivalent to* $w(a, b)$ *being symmetric.*

*Proof.* A key simplification valid in both 1FF and 2FF is the following 'linearization' effect:

$$\begin{aligned}\exp([w(a,b)ab]) &= 1 + [w(a,b)ab] + 1/2![w(a,b)ab]^2 + \dots \\ &= 1 + [w(a,b)ab]\end{aligned}$$

As a result we will look for $\Phi$ of the form:

$$\Phi(a, b) = 1 + [w(a, b)ab] \tag{13}$$

Now $\Phi$ must also satisfy the 'symmetry' (or 'unitarity') property, $\Phi^{-1} = \Phi^{321}$. Again using the 'linearization' effect valid in 1FF or 2FF we have:

$$\begin{aligned}\Phi(a,b)^{-1} &= 1 - [w(a,b)ab] + [w(a,b)ab]^2 - \dots \\ &= 1 - [w(a,b)ab] \qquad (14)\end{aligned}$$

Note that if the strands in a chord diagram are permuted according to the 'unitary' transformation $(123) \rightarrow (321)$ the chords $a, b, c$ transform as follows:

$$\begin{aligned} a &\rightarrow b \\ b &\rightarrow a \\ c &\rightarrow c \end{aligned}$$

So we get:

$$\begin{aligned}\Phi(a,b)^{321} =& 1 + [w(b,a)ba] \\ =& 1 - [w(b,a)ab]\end{aligned}$$

Comparing the expressions for $\Phi^{-1}$ and $\Phi^{321}$, we see that equation (8) is equivalent to $w$'s being symmetric. □

The following lemma further refines the form that a Lie series may take over 1FF or 2FF.

**Lemma 2.** *Modulo 2FF, a Lie series $[w(a,b)ab]$ where $w(a,b)$ is an ordinary power series in commutative variables $a, b$ can be written:*

$$[w(a,b)ab] = [ab]\lambda(a,b) - a[ab]\partial_a\lambda(a,b) - b[ab]\partial_b\lambda(a,b) \tag{15}$$

*where $\lambda(a,b) := w(-a,-b)$. Modulo 1FF this reduces to:*

$$[w(a,b)ab] = [ab]\lambda(a,b) \tag{16}$$

*Proof.* It is enough to check the case where $\lambda$ is a monomial. We have

$$(-1)^n[a^n ab] = [ab]a^n - a[ab]na^{n-1}$$

Indeed, the result is clear for $n = 1$, and for $n > 1$ we have:

$$\begin{aligned}(-1)^n[a^n ab] &= (-1)^n[a \cdot a^{n-1}ab] \\ &= (-1)\big\{a[ab]a^{n-1} - a^2[ab](n-1)a^{n-2} - [ab]a^{n-1} \cdot a \\ &\qquad + a[ab](n-1)a^{n-2} \cdot a\big\} \\ &= [ab]a^n - a[ab]na^{n-1}\end{aligned}$$

This generalizes readily to

$$(-1)^{n+m}[a^n b^m ab] = [ab]a^n b^m - a[ab]na^{n-1}b^m - b[ab]a^n mb^{m-1}$$

and the result follows over 2FF. The reduction to 1FF is immediate. □

Based on the preceding lemmas we have:

**Proposition 4.** *Modulo 2FF, $\Phi$ must take the form:*

$$\Phi(a,b) = 1 + [ab]\lambda(a,b) - a[ab]\partial_a\lambda(a,b) - b[ab]\partial_b\lambda(a,b) \tag{17}$$

*with $\lambda(a,b)$ a commutative, symmetric power series, and modulo 1FF, the form:*

$$\Phi(a,b) = 1 + [ab]\lambda(a,b) \tag{18}$$

*with $\lambda(a,b)$ a commutative, symmetric power series.*

## 2.4 Solving the Hexagons Modulo 1FF

### 2.4.1 The Positive Hexagon

We will gradually work the positive hexagon into a comparatively simpler form, which can be solved. Specifically, we will prove:

**Theorem 1.** *Modulo 1FF, and under the assumptions (8), (9) and (10) on* $\Phi$*, and the assumption (11) on* $R$*, the positive hexagon (5) becomes:*

$$\lambda(a,b) + e^{-b}\ \lambda(b,c) + e^{a}\ \lambda(a,c) \ = \ \{\frac{1}{ab} + \frac{e^{-b}}{bc} + \frac{e^{a}}{ac}\} \tag{19}$$

**Remark 3.** *Comparison with Kurlin's Compressed Hexagon*

This equation is equivalent to the 'compressed' equation obtained by Kurlin [9]. However, if the equation (19) is summarized as $L(\lambda) = R$, the equation obtained by Kurlin is $L(\lambda) = -R$. The sign is due to the fact that Kurlin writes the hexagons (5) and (6) with the associators on the RHS appearing in the opposite order to ours. We have followed the convention used in [7] and [2].

The first step in proving Theorem 1 is:

**Proposition 5.** *Modulo 1FF, and under the assumptions (8), (9) and (10) on* $\Phi$*, and the assumption (11) on* $R$*, the positive hexagon (5) becomes:*

$$e^{b+c} \ - \ e^{b}\cdot e^{c} \ = \ [a,b]\{\ \lambda(a,b)\ e^{b+c} \ + \ \lambda(b,c)\ e^{c} \ + \ \lambda(a,c)\ \} \tag{20}$$

*Proof.* Plugging the expression (11) for $R$ and (18) for $\Phi$ into the positive hexagon (5) we get:

$$\begin{aligned} e^{b+c} \ =& \Big[1+[a,b]\lambda(a,b)\Big] \ \cdot \ e^{b} \ \cdot \ \Big[(1+[a,b]\ \lambda(a,b))^{-1}\Big]^{132} \\ & \cdot \ e^{c} \ \cdot \ \Big[1+[a,b]\ \lambda(a,b)\Big]^{312} \end{aligned} \tag{21}$$

Now note that if the strands in a chord diagram are permuted according to the transformation $(123) \ \rightarrow \ (132)$ the chords $a, b, c$ transform as follows:

$$\begin{aligned} a &\rightarrow c \\ b &\rightarrow b \\ c &\rightarrow a \end{aligned}$$

Applying this to (14) and looking modulo 1FF, we get:

$$(\Phi^{-1})^{132} = 1 + [a,b]\ \lambda(b,c)$$

Similarly, under the strand permutation (123) $\rightarrow$ (312) the chords transform as:

$$\begin{array}{rcl} a & \rightarrow & c \\ b & \rightarrow & a \\ c & \rightarrow & b \end{array}$$

Therefore,

$$\Phi^{312} = 1 + [a,b]\ \lambda(c,a)$$

Accordingly, the hexagon becomes:

$$e^{b+c}\ =\ (1 + [a,b]\lambda(a,b)) \cdot e^b \cdot (1 + [a,b]\lambda(b,c)) \cdot e^c\ (1 + [a,b]\lambda(c,a))$$

We now multiply out on the RHS. By the frozen feet property, we need keep only the terms at most linear in [a,b]. Using also that $\lambda$ is symmetric, we get

$$e^{b+c}\ -\ e^b \cdot e^c\ =\ [a,b]\{\ \lambda(a,b)\ e^b e^c\ +\ \lambda(b,c)\ e^c\ +\ \lambda(a,c)\ \}$$

The term $e^b e^c$ in the RHS becomes $e^{b+c}$, because it is pre-multiplied by $[ab]$ and hence the $b$ and $c$ commute. □

We now want to find a better form for the expression $e^{b+c} - e^b e^c$. First, though, we make a comment concerning power series. In light of the importance of the first 'frozen foot' in any product, we will find it useful to split power series in one or more variables (such as the exponential function) into summands, each with a different 'foot'. These feet do not commute with subsequent variables, but the variables in the series which follow do commute among each other. Thus for instance $\exp(x + y)$ will become

$$e^{(x+y)} = 1 + x \cdot \frac{e^{(x+y)} - 1}{x + y} + y \cdot \frac{e^{(x+y)} - 1}{x + y}$$

This notation will be very convenient, but it must be remembered that, in general, the $x+y$ (or other variables) in the denominator should only be treated as dividing (in this example) $(e^{(x+y)} - 1)$ and its summands, but not the feet $x$ and $y$. More generally, in a term premultiplied by $a, b$ or $c$, a subsequent denominator should only be treated as dividing factors that follow the foot, not the foot itself. Put more technically, the feet act as a set of generators for a module on which the commutative ring $\mathbb{Q}[[a, a^{-1}, b, b^{-1}, c, c^{-1}]]$ acts by right multiplication. By abuse of notation, we use the same letters to denote corrresponding module and ring generators.

Further to this point, we note that the second equality in the following would be incorrect:

$$[xy] = xy - y\frac{xy}{y} = xy - \frac{y}{y}xy = xy - xy = 0$$

Instead, we have the rather trivial but still very useful lemma:

**Lemma 3.** *For $x$, $y$ any two different elements of $\{a,\ b,\ c\}$,*

$$x\frac{1}{x} - y\frac{1}{y} = [xy]\frac{1}{xy} \tag{22}$$

*Proof.* Proof is a simple calculation. □

Using this lemma, we can derive the following proposition.

**Proposition 6.** *Modulo 1FF, we have*

$$e^{b+c} - e^b \cdot e^c = [a,b] \cdot \big(\frac{e^{-a}}{ab} + \frac{e^c}{bc} + \frac{1}{ac}\big) \tag{23}$$

*Proof.*

$$\begin{aligned} e^{b+c} - e^b e^c &= \big(1 + (b+c)\frac{e^{b+c}-1}{b+c}\big) - \big(1 + b\frac{e^b-1}{b}\big)\big(1 + c\frac{e^c-1}{c}\big) \\ &= (b+c)\frac{e^{b+c}-1}{b+c} - c\frac{e^c-1}{c} - b\frac{e^b-1}{b}e^c \\ &= \big(c\frac{1}{c} - (b+c)\frac{1}{b+c}\big) + \big((b+c)\frac{1}{b+c} - b\frac{1}{b}\big)e^{b+c} - \big(c\frac{1}{c} - b\frac{1}{b}\big)e^c \\ &= [c, b+c]\frac{1}{c(b+c)} + [b+c, b]\frac{1}{(b+c)b}e^{b+c} - [c,b]\frac{1}{cb}e^c \\ &= [ab]\big(\frac{e^{-a}}{ab} + \frac{e^c}{bc} + \frac{1}{ac}\big) \end{aligned}$$

as required (note that in the last line, we have in particular used Lemma 1(i) and (ii)). □

Putting together the last two propositions, dropping the $[a,b]$ on both sides and multiplying by $e^a$, we get Theorem 1.

**Remark 4.** *Singular Solution*

The simplified equation in Theorem 1 has the singular solution $\lambda(x,y) = \frac{1}{xy}$. We note in passing that this is the factor which appears pre-multiplied by $[x,y]$ in the difference $x\frac{1}{x} - y\frac{1}{y}$ (see Lemma 3). Non-singular solutions are discussed below.

### 2.4.2 The Negative Hexagon

It will be recalled that the associator is in fact required to solve two hexagon equations, the positive and negative hexagons. The positive hexagon was set out above and simplified. As mentioned earlier, it can be shown (see [7] at equation (2.10) and [2] Prop. 3.7) that any solution of the positive hexagon automatically satisfies the negative hexagon, provided it is symmetric - in our notation, this means $\Phi \cdot \Phi^{321} = 1$, or $\lambda(a,b) = \lambda(b,a)$ (which we have assumed of our $\Phi$ and $\lambda$).

In principle therefore, we need not concern ourselves further with the negative hexagon. However, it will be useful at various points in the remainder of this paper to have a statement of the negative hexagon as simplified modulo 1FF, and so we set it out here. The original version is:

$$(\Delta 1)R^{-1} \;=\; \Phi\ (R^{-1})^{23}\ (\Phi^{-1})^{132}\ (R^{-1})^{13}\ \Phi^{312} \tag{24}$$

In terms of $a, b, c$ this means:

$$\begin{aligned} e^{-(b+c)} \;=\; & \Big[1+[a,b]\lambda(a,b)\Big] \;\cdot\; e^{-b} \cdot \Big[(1+[a,b]\ \lambda(a,b))^{-1}\Big]^{132} \;\cdot\; e^{-c} \\ & \cdot\; \Big[1+[a,b]\ \lambda(a,b)\Big]^{312} \end{aligned} \tag{25}$$

This in turn simplifies to:

$$\lambda(a,b) + e^{b}\ \lambda(b,c) + e^{-a}\ \lambda(a,c) \;=\; \frac{1}{ab} + \frac{e^{b}}{bc} + \frac{e^{-a}}{ac} \tag{26}$$

by the same method as the positive hexagon.

### 2.4.3 Solution to the Hexagons

Subject to the same constraints (symmetry, non-degeneracy and group-like form for $\Phi$, and exponential form for $R$), Kurlin [9] considered the image of the hexagon and pentagon equations (in a slightly different form) under the map $Log : \mathcal{GA} \longrightarrow \mathcal{L}$, where $\mathcal{GA}$ refers to the group-like elements of $\mathcal{A}$ and $Log$ is given by the usual power series. Kurlin considered the resulting equations modulo the Lie ideal $[[\mathcal{L},\mathcal{L}],[\mathcal{L},\mathcal{L}]]$ (calling these equations the 'compressed' hexagons and pentagon). Kurlin found that the compressed hexagons were equivalent to (19) and (26) (up to a change of sign due to different conventions – see Remark 3), and proceeded to derive a full set of solutions.

Theorem 2 below shows that solving the compressed equations (i.e. $Log$ of the hexagons and pentagon, mod $[[\mathcal{L},\mathcal{L}],[\mathcal{L},\mathcal{L}]]$) is equivalent to solving the hexagons and pentagon modulo $1FF$. Thus Theorem 1 and the corresponding derivation of (26) provide an alternative proof that the compressed hexagons are equivalent to (19) and (26).

I note, out of interest, one of the solutions found by Kurlin:

$$\lambda(a,b) \;=\; \Big(\frac{\sinh(a+b)}{(a+b)} \;\cdot\; \frac{\omega}{\sinh\omega} \;-1\Big)\ /a\cdot b \tag{27}$$

where $\omega \;=\; (a^2+ab+b^2)^{1/2}$.

Before stating and proving Theorem 2, we need a few lemmas.

**Lemma 4.** *Let $\alpha + A$ be a Lie series in $\hat{\mathcal{L}}$, with $\alpha$ the linear part (i.e. of degree one). Then, modulo 1FF:*

$$e^{\alpha+A} = e^{\alpha} + A\frac{e^{\alpha}-1}{\alpha}$$

*Proof.* We consider the degree $n$ part of $e^{\alpha+A}$:

$$\frac{1}{n!}(\alpha + A)^n = \frac{1}{n!}\alpha^n + A \cdot \alpha^{n-1} \quad mod\ 1FF$$

Hence

$$\begin{aligned} e^{\alpha+A} &= 1 + (\alpha + A) + \cdots + 1/n!(\alpha^n + A \cdot \alpha^{n-1}) + \ldots \\ &= e^{\alpha} + A\{1 + 1/2\alpha + \cdots + 1/n!a^{n-1} + \ldots \\ &= e^{\alpha} + A\ \frac{e^{\alpha} - 1}{\alpha} \end{aligned}$$

as required. □

**Proposition 7.** *Let $x + X,\ y + Y$ be Lie series in $\hat{\mathcal{L}}$ with $x$ and $y$ the linear parts. Then the equality of group-like elements*

$$e^{x+X} = e^{y+Y}$$

*holds modulo 1FF if and only if the equality of their logarithmic images*

$$x + X = y + Y$$

*holds modulo* $[[\mathcal{L}, \mathcal{L}], [\mathcal{L}, \mathcal{L}]]$.

*Proof.* We have

$$e^{x+X} = e^{y+Y}$$

if and only if

$$1 + x + X + \text{h.o.} = 1 + y + Y + \text{h.o.}$$

(where 'h.o.' means 'higher order terms') and hence, by looking at degree one terms, we must have $x = y$.

Then

$$e^{x+X} = e^{x+Y} \quad mod\ 1FF$$

$$\Longleftrightarrow$$

$$e^x + X\frac{e^x - 1}{x} = e^x + Y\frac{e^x - 1}{x} \quad mod\ 1FF$$

$$\Longleftrightarrow$$

$$X = Y \quad mod\ 1FF$$

$$\Longleftrightarrow$$

$$X = Y \quad mod\ [[\mathcal{L}, \mathcal{L}], [\mathcal{L}, \mathcal{L}]]$$

as required. Note that in going from the second to the third line, we used the summation

$$\frac{e^x - 1}{x} = 1 + \frac{1}{2}x + \frac{1}{3!}x^2 + \ldots$$

which shows that $\frac{e^x-1}{x}$ is invertible, and in going from the third to the fourth lines, we used Proposition 1. □

**Theorem 2.** *The hexagons and the pentagon hold modulo 1FF if and only if their logarithmic images hold modulo* $[[\mathcal{L},\mathcal{L}],[\mathcal{L},\mathcal{L}]]$.

*Proof.* The LHS and RHS of the hexagons and pentagon are (products of) group-like elements, hence are themselves group-like elements. We can now apply Proposition 7. □

**Remark 5.** *Campbell-Hausdorff-Baker Formula, Modulo 1FF or* $[[\mathcal{L},\mathcal{L}],[\mathcal{L},\mathcal{L}]]$

Lemma 4 can be used to give a short, simple derivation of Kurlin's formula for the Campbell-Hausdorff-Baker formula modulo $[[\mathcal{L},\mathcal{L}],[\mathcal{L},\mathcal{L}]]$ (see [9] at Prop. 2.8 and Prop. 2.12), which also holds modulo *1FF*. Indeed, we seek a power series $C(x,y)$ in commutative variables $x,y$ such that

$$\exp(b+c+[bc]C(b,c)) = \exp(c)\exp(b)$$

But from Lemma 4, we have

$$e^{(b+c+[bc]C(b,c))} = e^{(b+c)} + [bc]C(b,c)\ \frac{e^{(b+c)}-1}{(b+c)}$$

(where technically we should perhaps have converted $[bc]C(b,c)$ to the Lie series $[C(-b,-c)bc]$ using Proposition 2 before applying Lemma 4, and then converted back to $[bc]C(b,c)$). But then

$$\begin{aligned}[bc]C(b,c)\ \frac{e^{(b+c)}-1}{(b+c)} &= e^c e^b - e^{(b+c)} \\ &= -[ac]\big(\frac{e^{-a}}{ac} + \frac{e^b}{bc} + \frac{1}{ab}\big)\end{aligned}$$

from Proposition 6 (after the exchange $b \leftrightarrow c$). From this we readily derive:

$$C(b,c) = \frac{e^b-1}{bc}\ \big(\frac{b+c}{e^{b+c}-1} - \frac{b}{e^b-1}\big)$$

This result is valid modulo *1FF* and, pursuant to Proposition 7, modulo $[[\mathcal{L},\mathcal{L}],[\mathcal{L},\mathcal{L}]]$.

### 2.4.4 Solving the Pentagon Modulo 1FF

It so happens that the pentagon is automatically satisfied modulo 1FF, for any function $\Psi$ of the form $\Psi = [ab]\ \mu(a,b)$, where $\mu$ is a commutative power series of two variables which is symmetric in its arguments. In particular, the 1FF associator $\Phi$ satisfies the pentagon since it has the required symmetry property (and this, independent of the particular form $\Phi$ must take to solve the hexagons). Since the proof already appears in [9] Proposition 5.10, in this section we merely present notation, and state and prove results in the form that will be used in the balance of this paper. This material largely reproduces results by Kurlin but the results and proofs are given here in a somewhat more general and concise form.

We will find expressions for $(\Delta 11)\Psi$, $(1\Delta 1)\Psi$ and $(11\Delta)\Psi$ as sums of terms of the form $[x,y]\lambda(u,v)$, where $x,\ y,\ u,\ v$ are all single-chord diagrams on four strands. In fact, each of $(\Delta 11)\Psi$, $(1\Delta 1)\Psi$ and $(11\Delta)\Psi$ will be a sum of 4 terms of that form. Then, as shown by Kurlin [9] in his Proposition 5.10, it can be seen that cancellations occur in the (linearized) pentagon, leading to the desired equality.

Since the pentagon lives in the space of chord diagrams on four strands, we need to make precise what this space is. Thus we will will be concerned with the space $\mathcal{A}_4$ (or rather its completion $\hat{\mathcal{A}}_4$) generated by single horizontal chord diagrams on 4 vertical strands, ie

$$\begin{aligned} a &:= t^{12}, \quad b := t^{23}, \quad c := t^{13} \\ d &:= t^{24}, \quad e := t^{34}, \quad f := t^{14} \end{aligned} \tag{28}$$

where $t^{ij}$ represents the chord diagram with a single horizontal chord resting on strands $i$ and $j$. Obviously, $t^{ij} = t^{ji}$.

For each $l = 1, \ldots, 4$, we get the expected $4T$ relations among the chord diagrams whose endpoints rest on the three strands other than strand $l$, ie

$$[t^{ij}, t^{jk}] = [t^{jk}, t^{ki}] = [t^{ki}, t^{ij}] \tag{29}$$

where $l \notin \{i,j,k\}$.

These are just the $4T$ relations we would get if we dropped all chord diagrams with chords resting on strand $l$, and viewed the remaining chord diagrams as forming a copy of $\mathcal{A}_3$. In addition, however, we have a new kind of relation, known as 'locality in space' relations, which provide that we can commute any two chords whose four endpoints rest on four different strands. In other words,

$$[t^{ij}, t^{kl}] = 0, \quad \textit{whenever } \#\{i,j,k,l\} = 4 \tag{30}$$

We will use the notation $\mathcal{A}^{1FF}$ to refer to $\hat{\mathcal{A}}_4$ modulo 1FF. Of course, 1FF in this context refers to the relations

$$u[xy] = 0 \quad for\ u, x, y \in \mathcal{A}_4, deg(u) \geq 1$$

One can easily check that the element $a+b+c+d+e+f = \sum_{1 \leq i < j \leq 4} t^{ij}$ is central in $\hat{\mathcal{A}}_4$. Since $a+b+c$ is central in the algebra $\hat{\mathcal{A}}_3$ generated by $\{a,b,c\}$, it follows also that

$$[d+e+f, [ab]] = 0 \tag{31}$$

and hence

$$[ab]f = [ab](-e-d)$$

modulo 1FF.

We will now also take $\mathcal{L}$ to be the Lie algebra generated by the symbols $\{t^{ij}\}_{1\leq i<j\leq 4}$, with the commutation relations given by $4T$ and the locality relations. In fact we really deal with the completion $\hat{\mathcal{L}}$, but will still write $\mathcal{L}$. Moreover, we take $\mathcal{L}^{CC}$ to refer to (the completed) $\mathcal{L}$ modulo $[[\mathcal{L},\mathcal{L}],[\mathcal{L},\mathcal{L}]]$.

Finally we introduce the notation:

- $\overline{t^{ij}} := t^{4k} \quad \textit{if } \{i,j,k\} = \{1,2,3\}$
- $\overline{t^{4i}} := t^{4i} \quad \textit{if } i \in \{1,2,3\}$

Thus, 'barring' a chord amounts to replacing it with its 'complementary' chord (ie, resting on complementary strands), with which it commutes, if the chord does not sit on strand 4 (ie, $\bar{a} = e,\ \bar{b} = f,\ \bar{c} = d$). Barring a chord that sits on strand 4 has no effect (ie $\bar{d} = d,\ \bar{e} = e,\ \bar{f} = f$).

We will also use a short-hand notation for commutators: if $l \in \{1,2,3,4\}$, and $(i,j,k) = (1,...,\hat{l},...,4)$, we write:

$$[\hat{l}] := [t^{ij}, t^{jk}]$$

ie we identify a (degree two) commutator by the strand it does not touch, with sign given by the stated assumption on the order of $i,j,k$.

We now derive some basic results about the Lie algebra $\mathcal{L}^{CC}$. We first give a set of vector space generators of $\mathcal{L}^{CC}$ (compare [9] Lemma 5.5).

**Proposition 8.** *$\mathcal{L}^{CC}$ is generated as a vector space by the elements:*

1. *$[(t^{ij})^r(t^{jk})^s[\hat{l}]]$, where $(i,j,k) = (1,...,\hat{l},...,4)$, and $r,s \in \mathbb{N}$. These just generate the Lie subalgebras obtained by dropping strand $l$, and viewing diagrams on the remaining strands as constituting a copy of $\hat{\mathcal{L}}_3$; and*
2. *$[(t^{i4})^r(t^{j4})^s[\hat{4}]]$, where $\{i,j\} \subseteq \{1,...,\hat{l},...,4\}$, and $r,s \in \mathbb{N}$.*

To prove Proposition 8, we need three lemmas, starting with this lemma which applies in $\mathcal{L}_4$ (compare [9] Claim 5.2(c)):

**Lemma 5.** *In $\mathcal{L}_4$ we have*

$$[t^{rs}[\hat{s}]] = (-1)^s[\overline{t^{rs}}[\hat{4}]], \quad r,s \textit{ distinct elements of } \{1,2,3,4\}$$

*Proof.* We let $i \in \{1,2,3\}$, and then choose $j,k$ so that $(ijk)$ is a cyclic permutation of $(123)$. Thus $[t^{ij}t^{jk}] = [\hat{4}]$.

We note that, by 4T, $[t^{ij}t^{jk}] = [t^{jk}t^{ki}]$. Hence, $[t^{4i}t^{ij}t^{jk}]$ can also be written $[t^{4i}t^{jk}t^{ki}]$, and:

$$[t^{4i}t^{ij}t^{jk}] = -[t^{ij}t^{jk}t^{4i}] - [t^{jk}t^{4i}t^{ij}] = -[t^{jk}t^{4i}t^{ij}]$$
$$[t^{4i}t^{jk}t^{ki}] = -[t^{jk}t^{ki}t^{4i}] - [t^{ki}t^{4i}t^{jk}] = -[t^{jk}t^{ki}t^{4i}]$$

where the first equality in each line is the Jacobi relation, and the second equality comes from the locality in space relations.

From the assumptions on $i, j, k$, we see that $(4, i, j)$ is some permutation of $(1, ..., \hat{k}, ...4)$, and one can readily confirm that the permutation is in fact *cyclic* iff k is odd. Hence we get:

$$[t^{jk}[\hat{k}]] = (-1)^k[t^{4i}t^{ij}t^{jk}] = (-1)^k[\overline{t^{jk}}[\hat{4}]] \tag{32}$$

By similar reasoning one can see that:

$$[t^{jk}[\hat{j}]] = (-1)^j[t^{4i}t^{jk}t^{ki}] = (-1)^j[\overline{t^{jk}}[\hat{4}]] \tag{33}$$

Next we can repeat the process to get:

$$[t^{4i}[t^{ij}t^{jk}]] = -[t^{jk}t^{4i}t^{ij}] = -[t^{jk}t^{j4}t^{4i}] = [t^{j4}t^{4i}t^{jk}] + [t^{4i}t^{jk}t^{j4}] = [t^{4i}t^{jk}t^{j4}]$$

Here, for the first equality we used the $4T$ relations to get $[t^{4i}t^{ij}] = [t^{j4}t^{4i}]$, for the second equality we used Jacobi, and for the third we used the locality in space relations.

We now note that $[t^{jk}t^{4j}] = (-1)^{\eta}[\hat{\imath}]$, where $\eta = +1$ (or $-1$) when $(k, j, 4)$ is (or is not) a cyclic permutation of $(1, ..., \hat{\imath}, ...4)$. Moreover, $(k, j, 4)$ is a cyclic permutation of $(1, ..., \hat{\imath}, ..., 4)$ if and only if $i$ is even. Hence:

$$[t^{4i}[\hat{\imath}]] = (-1)^i[t^{4i}t^{ij}t^{jk}] = (-1)^i[t^{4i}[\hat{4}]]$$

When $s \neq 4$, Equation (32) gives us the desired result in the case $r = j < k = s$, Equation (33) in the case $s = j < k = r$, and the last equation in the case $r = 4 \neq s$. The case $s = 4$ is trivial.

□

We now move to our various quotient spaces. We first note the remark (see also [9] Lemma 5.4(a)):

**Remark 6.** *By an obvious generalization of the proof of Proposition 1, if w is a word on $\{a, b, c, d, e, f\}$ with at least two letters, and $u, v \in \{a, b, c, d, e, f\}$, we have*

$$[uvw] = [vuw] \quad mod\ [[\mathcal{L}, \mathcal{L}], [\mathcal{L}, \mathcal{L}]]$$
$$[w]uv = [w]vu \quad mod\ 1FF\ or\ 2FF$$

In $\mathcal{L}^{CC}$ we get the result (compare [9] Lemma 5.4(b)):

**Lemma 6.** *In $\mathcal{L}^{CC}$, if $\omega$ is a commutative power series in the $t^{ij}$, $1 \leq i < j \leq 4$, and $k = 1, 2, 3$, we have*

$$[\omega\ t^{4k}[\hat{4}]] = [\overline{\omega}\ t^{4k}[\hat{4}]]$$

*ie the presence of the factor $t^{4k}$ multiplying the commutator $[\hat{4}]$ allows us to replace all $t^{ij}$s by $\overline{t^{ij}}$s.*

*Proof.* It suffices to show this for monomials of the given form, and also we may assume $\{i, j, k\} = \{1, 2, 3\}$ since the result is trivial when $i$ or $j$ is equal to 4. We first consider the case $[t^{ij}t^{4j}[\hat{4}]]$:

$$[t^{ij}t^{4j}[\hat{4}]] = (-1)^j[t^{ij}t^{4j}[\hat{j}]] = (-1)^j[t^{4j}t^{ij}[\hat{j}]] = [t^{4j}t^{4k}[\hat{4}]] = [t^{4k}t^{4j}[\hat{4}]]$$

(where the third equality uses Lemma convertkto4) as needed, for this case.

Next we take the case $[t^{ij}t^{4k}[\hat{4}]]$. We note that $[t^{ij}[\hat{4}]] = [(-t^{jk} - t^{ki})[\hat{4}]]$ by 4T, so:

$$\begin{aligned}[t^{ij}t^{4k}[\hat{4}]] &= [t^{4k}t^{ij}[\hat{4}]] = [t^{4k}(-t^{jk} - t^{ki})[\hat{4}]] \\ &= [t^{4k}(-t^{4i} - t^{4j})[\hat{4}]] = [t^{4k}t^{4k}[\hat{4}]]\end{aligned}$$

where in the second last equality we applied the first case twice, and in the last equality we applied Equation (31). This completes the proof. □

We now give a lemma, valid in $\mathcal{L}^{CC}$, which in particular will allow us to derive explicit expressions for the action of strand doubling in $\mathcal{L}^{CC}$ (see [9] Claim 5.7):

**Lemma 7.** *We take $l \in \{1, 2, 3, 4\}$, and $(i, j, k)$ a cyclic permutation of $(1, ..., \hat{l}, ..., 4)$ (so that $[t^{ij}, t^{jk}] = [\hat{l}]$). We also take $u \in \{i, j, k\}$ and $m$ a non-negative integer. Then, in $\mathcal{L}^{CC}$,*

$$[(t^{ij} + t^{ul})^m[\hat{l}]] = [(t^{ij})^m[\hat{l}]] - (-1)^l[(\overline{t^{ij}})^m[\hat{4}]] + (-1)^l[(\overline{t^{ij}} + \overline{t^{ul}})^m[\hat{4}]]$$

*Proof.*

$$\begin{aligned}[(t^{ij} + t^{ul})^m[\hat{l}]] &= [(t^{ij})^m[\hat{l}]] + \sum_{r=1}^{m} \binom{m}{r} [(t^{ij})^{m-r}(t^{ul})^r[\hat{l}]] \\ &= [(t^{ij})^m[\hat{l}]] + (-1)^l \sum_{r=1}^{m} \binom{m}{r} [(t^{ij})^{m-r}(\overline{t^{ul}})^r[\hat{4}]] \\ &= [(t^{ij})^m[\hat{l}]] + (-1)^l \sum_{r=1}^{m} \binom{m}{r} [(\overline{t^{ij}})^{m-r}(\overline{t^{ul}})^r[\hat{4}]] \\ &= [(t^{ij})^m[\hat{l}]] - (-1)^l[(\overline{t^{ij}})^m[\hat{4}]] + (-1)^l[(\overline{t^{ij}} + \overline{t^{ul}})^m[\hat{4}]]\end{aligned}$$

as required (where in going from the first to second lines we used Lemma 5, and in going from the second to third lines Lemma 6 applies to $t^{ij}$ since, in the summation, $\overline{t^{ul}}$ appears with the power $r \geq 1$).

□

*Proof of Proposition 8.* The proof of Proposition 8 is now a straightforward combination of the past three lemmas. □

We can now derive explicit expressions for the action of $\Delta 11$, $11\Delta$ *and* $1\Delta 1$ on $\Psi(t^{12}, t^{23}) = [t^{12}t^{23}]\lambda(t^{12}, t^{23})$ ([9] Lemma 5.8).

**Proposition 9.** *Mod* $[[\mathcal{L}, \mathcal{L}], [\mathcal{L}, \mathcal{L}]]$*, we have:*

$$
\begin{aligned}
(\Delta 11)\Psi &= [\lambda(t^{13}, t^{34})[\hat{2}]] + [\lambda(t^{23}, t^{34})[\hat{1}]] - [\lambda(t^{24}, t^{34})[\hat{4}]] + [\lambda(t^{14}, t^{34})[\hat{4}]] \\
(11\Delta)\Psi &= [\lambda(t^{12}, t^{24})[\hat{3}]] + [\lambda(t^{12}, t^{23})[\hat{4}]] - [\lambda(t^{34}, t^{14})[\hat{4}]] + [\lambda(t^{34}, t^{24})[\hat{4}]] \\
(1\Delta 1)\Psi &= [\lambda(t^{12}, t^{24})[\hat{3}]] + [\lambda(t^{13}, t^{34})[\hat{2}]] + [\lambda(t^{34}, t^{24})[\hat{4}]] - [\lambda(t^{24}, t^{34})[\hat{4}]]
\end{aligned}
$$

*Proof.* It suffices to prove the statements for monomials. We begin with:

$$
\begin{aligned}
(\Delta 11)[(t^{12})^k (t^{23})^l [\hat{4}]] &= [(t^{13} + t^{23})^k (t^{34})^l [t^{13} + t^{23}, t^{34}]] \\
&= [(t^{13} + t^{23})^k (t^{34})^l [\hat{2}]] + [(t^{13} + t^{23})^k (t^{34})^l [\hat{1}]] \\
&= [(t^{13})^k (t^{34})^l [\hat{2}]] - [(t^{24})^k (t^{34})^l [\hat{4}]] + [(t^{24} + t^{14})^k (t^{34})^l [\hat{4}]] \\
&\quad + [(t^{23})^k (t^{34})^l [\hat{1}]] + [(t^{14})^k (t^{34})^l [\hat{4}]] - [(t^{24} + t^{14})^k (t^{34})^l [\hat{4}]] \\
&= [(t^{13})^k (t^{34})^l [\hat{2}]] + [(t^{23})^k (t^{34})^l [\hat{1}]] - [(t^{24})^k (t^{34})^l [\hat{4}]] \\
&\quad + [(t^{14})^k (t^{34})^l [\hat{4}]]
\end{aligned}
$$

where we used Lemma 7 in going from the second to the third lines.

The proof of the relation for $(11\Delta)$ and $(1\Delta 1)$ is similar (though involving iterated use of Lemma 7 in the case of $(1\Delta 1)$). □

We note finally that, while the results of this section have been stated in terms of the Lie algebras $\mathcal{L}$ and $\mathcal{L}^{CC}$, by means of Proposition 2 these results have immediate analogues in the language of $\mathcal{A}$ and $\mathcal{A}^{1FF}$.

## 2.5 Solving the Positive Hexagon Modulo 2FF

### 2.5.1 Overview

We follow the same general strategy for reworking the positive hexagon into a usable form modulo 2FF as we used modulo 1FF. However, the details are more involved, and so an overview of the specifics may be useful.

We are taking $\Phi$ to be of the form

$$\Phi(a, b) = 1 + [ab]\lambda(a, b) - a[ab]\partial_a \lambda(a, b) - b[ab]\partial_b \lambda(a, b) \tag{34}$$

where $\lambda(a, b)$ is a symmetric power series in the commutative variables $a$ and $b$ (see equation (17)).

We plug this expression into the positive hexagon, and rework it into the form

$$e^{b+c} - e^b e^c = \{\text{expression in } \lambda \text{ and its partials}\}$$

where we refer to the LHS as the 'triangle'. We find that the RHS is a sum of terms premultiplied by $[ab]$, $b[ab]$ and $c[ab]$.

We then find an expression for the triangle which, as it happens, consists also of a sum of terms premultiplied by $[ab]$, $b[ab]$ and $c[ab]$.

Comparing the $[ab]$ terms on the LHS and RHS, we find an equation in $\lambda$ only (no partials) which is just the 1FF positive hexagon. We then compare the $b[ab]$ terms on the LHS and RHS, and find that the result is just the operator $(1 + \partial_a - \partial_b)$ applied to the 1FF positive hexagon. Similarly we find that the $c[ab]$ terms simply give us the operator $(1 + \partial_a - \partial_c)$ applied to the 1FF positive hexagon. Not surprisingly, it is again true that any $\lambda$ which satisfies the positive hexagon also automatically satisfies the negative hexagon.

We conclude that equation (34) gives a solution to the 2FF hexagon whenever $\lambda$ is a solution to the 1FF hexagon. Moreover, since by factoring out 2FF we get a quotient of Kurlin's quotient, Kurlin's argument still applies to show that the unitarity condition implies that such a solution is also a solution to the pentagon.

### 2.5.2 Simplifying the Positive Hexagon

We go back to the equation (34) giving $\Phi$ in the form:

$$\Phi(a, b) = 1 + [ab]\lambda(a, b) - a[ab]\partial_a\lambda(a, b) - b[ab]\partial_b\lambda(a, b)$$

with $\lambda(a, b)$ a commutative, symmetric power series. Note that this expression involves only the two variables $a$ and $b$, but not $c$. However, the positive hexagon also involves expressions in $(\Phi^{-1})^{132}$ and $\Phi^{312}$, and after the corresponding permutations are given effect, the resulting expressions will involve $a$, $b$ *and* $c$. We would like to get rid of one of these variables, for instance by replacing $c$ by $-a-b$. Lemma 1 tells us we can go this whenever $c$ appears after a commutator, providing we are proceeding modulo 1FF. However this is no longer true modulo 2FF. Instead, we have the following lemma which is valid in $\hat{\mathcal{A}}$, not just in a quotient:

**Lemma 8.** *Given any power series $\alpha(a, b, c) \in \mathcal{A}_3$,*

$$[a, b]\ \alpha(a, b, c) = [ab]\ \alpha(a, b, -a-b) + (a+b+c)[ab]\ \partial_c\alpha(a, b, c) \tag{35}$$

*where the partial is evaluated at $c = -a - b$.*

*Proof.* Using the fact that $a + b + c$ is central, we have

$$(a + b + c)[ab] = [ab](a + b + c)$$

hence

$$[ab]c = [ab](-a - b) + (a + b + c)[ab]$$

hence

$$[ab]c^2 = [ab](-a - b)^2 + 2(a + b + c)[ab](-a - b)$$

and more generally

$$[ab]c^n = [ab](-a-b)^n + n(a+b+c)[ab](-a-b)^{n-1}$$

and indeed

$$[a,b]\ \alpha(a,b,c) = [ab]\ \alpha(a,b,-a-b) + (a+b+c)[ab]\ \partial_c\alpha(a,b,c) \tag{36}$$

where the partial is evaluated at $c = -a-b$. □

Note that in practice we will actually use this lemma to replace $a$ by $-b-c$, as we will later find it more convenient to work with $b$ and $c$.

Hence we will actually take $\Phi$ to have the form:

$$\Phi(a,b) = 1 + [ab]\lambda(-b-c,b) - b[ab](\partial_y\lambda(x,y) - \partial_x\lambda(x,y)) - c[ab]\partial_x\lambda(x,y)$$

where the partials are evaluated at $(x,y) = (-b-c,b)$. In practice, though, we will write this as

$$\Phi(a,b) = 1 + [ab]\lambda(a,b) - b[ab](\partial_b\lambda(a,b) - \partial_a\lambda(a,b)) - c[ab]\partial_a\lambda(a,b)$$

remembering that in fact $a = -b-c$ in expressions premultiplied by a commutator.

We now need to find expressions for $(\Phi^{-1})^{132}$ and $\Phi^{312}$. We already have an expression for $\Phi^{-1}$, namely equation (14). Replacing $w$ by $\lambda$ and its partials, this becomes:

$$\Phi(a,b)^{-1} = 1 - [ab]\lambda(a,b) + a[ab]\partial_a\lambda(a,b) + b[ab]\partial_b\lambda(a,b)$$

Also, as indicated earlier, under the permutation $(123) \to (132)$, chords get permuted as follows:

$$\begin{aligned} a &\to c \\ b &\to b \\ c &\to a \end{aligned}$$

Hence

$$\begin{aligned} (\Phi^{-1})^{132} &= 1 - [cb]\lambda(c,b) + c[cb]\partial_c\lambda(c,b) + b[cb]\partial_b\lambda(b,c) \\ &= 1 + [ab]\lambda(b,c) - b[ab]\partial_b\lambda(b,c) - c[ab]\partial_c\lambda(c,b) \end{aligned}$$

where we have used the symmetry of $\lambda$ and the relation $-[cb] = [ab]$.

Under the permutation $(123) \to (312)$, the chords are permuted as

$$\begin{aligned} a &\to c \\ b &\to a \\ c &\to b \end{aligned}$$

Hence

$$\begin{aligned} \Phi^{312} &= 1 + [ab]\lambda(c,a) - c[ab]\partial_c\lambda(c,a) - a[ab]\partial_a\lambda(a,c) \\ &= 1 + [ab]\lambda(c,a) + b[ab]\partial_a\lambda(a,c) - c[ab](\partial_c\lambda(c,a) - \partial_a\lambda(a,c)) \end{aligned}$$

where we have used the relation $[ca] = [ab]$.

We can now write the positive hexagon equation modulo 2FF:

$$\begin{aligned} e^{b+c} =& \big(1 + [ab]\lambda(a,b) - b[ab](\partial_b\lambda(a,b) - \partial_a\lambda(a,b)) + c[ab]\partial_a\lambda(a,b)\big) \cdot e^b \\ & \cdot \big(1 + [ab]\lambda(c,b) - b[ab]\partial_b\lambda(b,c) - c[ab]\partial_c\lambda(c,b)\big) \cdot \\ & e^c \cdot \big(1 + [ab]\lambda(c,a) + b[ab]\partial_a\lambda(a,c) - c[ab](\partial_c\lambda(c,a) - \partial_a\lambda(a,c))\big) \end{aligned}$$

Linearization still holds in 2FF, so we need to keep only terms that are up to linear in [ab], and we get

$$\begin{aligned} e^{b+c} - e^b e^c =& [ab]\lambda(a,b)e^{b+c} + e^b[ab]\lambda(b,c)e^c + e^b e^c[ab]\lambda(c,a) \qquad (37) \\ & - b[ab](\partial_b\lambda(b,a) - \partial_a\lambda(a,b))e^{b+c} - e^b b[ab]\partial_b\lambda(b,c)e^c \\ & + e^b e^c b[ab]\partial_a\lambda(a,c) + c[ab]\partial_a\lambda(a,b)e^{b+c} - e^b c[ab]\partial_c\lambda(c,b)e^c \\ & - e^b e^c c[ab](\partial_c\lambda(c,a) - \partial_a\lambda(a,c)) \\ =& [ab]\big\{\lambda(a,b)e^{b+c} + \lambda(b,c)e^c + \lambda(c,a)\big\} \qquad (38) \\ & + b[ab]\big\{\lambda(b,c)e^c + \lambda(c,a) - (\partial_b\lambda(b,a) - \partial_a\lambda(a,b))e^{b+c} \\ & - \partial_b\lambda(b,c)e^c + \partial_a\lambda(a,c)\big\} \\ & + c[ab]\big\{\lambda(c,a) + \partial_a\lambda(a,b)e^{b+c} - \partial_c\lambda(c,b)e^c - \partial_c\lambda(c,a) + \partial_a\lambda(a,c)\big\} \end{aligned}$$

where, again, we have throughout $a = -b - c$ whenever $a$ is pre-multiplied by $[ab]$.

### 2.5.3 Simplifying the Triangle Relation

As with the 1FF case, we need to determine $e^{b+c} - e^b e^c$, this time modulo 2FF. The procedure is analogous to the 1FF case. First we note the following expression for the exponential function, which is useful modulo 2FF:

$$e^x = 1 + x + x^2 \, \frac{e^x - x - 1}{x^2}$$

We will prove the following proposition concerning the difference $e^{b+c} - e^b e^c$:

**Proposition 10.** *Modulo 2FF, the triangle difference can be expressed:*

$$\begin{aligned} e^{b+c} - e^b e^c = [a,b]\ &\Big(\frac{e^{b+c}}{ab} + \frac{e^c}{bc} + \frac{1}{ac}\Big) \\ + b[a,b]&\Big(\frac{e^{b+c}}{a^2c} - \frac{e^{b+c}}{b^2c} + \frac{e^c}{b^2c} + \frac{e^c}{bc} + \frac{1}{ac} - \frac{1}{a^2c}\Big) \\ + c[a,b]&\Big(-\frac{e^{b+c}}{a^2b} + \frac{e^c}{bc^2} + \frac{1}{ac} - \frac{1}{bc^2} + \frac{1}{a^2b} \\ &\quad - \frac{1}{ab} + \frac{1}{a^2b} - \frac{1}{bc} - \frac{1}{bc^2}\Big) \end{aligned} \tag{39}$$

The proof relies on a lemma which is the equally trivial, but equally useful 2FF analogue of Lemma 3:

**Lemma 9.** *For $x$ and $y$ any two different elements of $\{a,\ b,\ c\}$, we have the following modulo 2FF:*

$$x^2\frac{1}{x^2} - y^2\frac{1}{y^2} = [xy]\frac{1}{xy} + x[xy]\frac{1}{x^2y} + y[xy]\frac{1}{xy^2}$$

*There are a couple of variants of this equation which will also be useful:*

$$\begin{aligned} x^2\frac{1}{x} - xy\frac{1}{y} &= x[xy]\frac{1}{xy} \\ -(x+y)^2\frac{1}{x+y} + xy\frac{1}{y} + y^2\frac{1}{y} &= x[yx]\frac{1}{(x+y)y} + y[yx]\frac{1}{(x+y)y} \end{aligned}$$

*Proof.* The proofs are straightforward if tedious and are omitted. □

*Proof of Proposition 10.* We plug our expression for the exponential function mod 2FF into the triangle $e^{b+c} - e^b e^c$ and get:

$$
\begin{aligned}
e^{b+c} - e^b e^c = &\Big(1 + (b+c) + (b+c)^2 \ \frac{e^{b+c} - (b+c) - 1}{(b+c)^2}\Big) \\
&- \Big(1 + b + b^2 \ \frac{e^b - b - 1}{b^2}\Big) - \Big(1 + c + c^2 \ \frac{e^c - c - 1}{c^2}\Big) \\
= &(b+c)^2 \ \frac{e^{b+c} - (b+c) - 1}{(b+c)^2} - b^2 \ \frac{e^b - b - 1}{b^2} \ e^c \\
&- b - bc \ \frac{e^c - 1}{c} - 1 - c - c^2 \ \frac{e^c - c - 1}{c^2} \\
= &(b+c)^2 \ \frac{e^{b+c}}{(b+c)^2} - b^2 \ \frac{e^{b+c}}{b^2} \\
&+ b^2 \ \frac{e^c}{b^2} - c^2 \ \frac{e^c}{c^2} \\
&+ b^2 \ \frac{be^c}{b^2} - bc \ \frac{e^c}{c} \\
&- (b+c)^2 \frac{1}{b+c} + bc\frac{1}{c} + c^2\frac{1}{c} \\
&- (b+c)^2 \frac{1}{(b+c)^2} + c^2 \frac{1}{c^2}
\end{aligned}
$$

From here we use various substitutions from the previous lemma. The result then follows from a few additional simple manipulations, which are omitted. □

### 2.5.4 Solution Modulo 2FF

We can now bring together the different components of the hexagon, namely the expression for $e^{b+c} - e^b e^c$ and the expression involving the $\lambda$'s (see equation (37). We get terms that are pre-multiplied by $[ab]$, $b[ab]$ and $c[ab]$:

**Terms in** $[ab]$

$$
\frac{e^{b+c}}{ab} + \frac{e^c}{bc} + \frac{1}{ac} = \lambda(a,b)e^{b+c} + \lambda(c,b)e^c + \lambda(c,a)
$$

Note that this is just the (positive) hexagon equation in 1FF. In the following we will refer to the LHS as $Hex^l$ and to the RHS as $Hex^r$.

**Terms in** $b[ab]$ We have:

$$
\begin{aligned}
\frac{e^{b+c}}{a^2c} - \frac{e^{b+c}}{b^2c} + \frac{e^c}{b^2c} + \frac{e^c}{bc} + \frac{1}{ac} - \frac{1}{a^2c} = &\partial_a\lambda(a,b)e^{b+c} - \partial_b\lambda(a,b)e^{b+c} \\
&+ \lambda(c,b)e^c - \partial_b\lambda(b,c)e^c + \lambda(c,a) + \partial_a\lambda(a,c)
\end{aligned}
$$

I now claim that this $b[ab]$ equation is simply the statement

$$\big[1 + \partial_a - \partial_b\big] Hex^l = \big[1 + \partial_a - \partial_b\big] Hex^r$$

and hence follows from the hexagon. On the RHS, this is easy to check:

$$\begin{aligned}
\big[1 + \partial_a - \partial_b\big] Hex^r =& \lambda(a,b)e^{b+c} + \lambda(c,b)e^c + \lambda(c,a) \\
&+ (\partial_a \lambda(a,b))e^{b+c} + \partial_a \lambda(a,c) \\
&- (\partial_b \lambda(a,b))e^{b+c} - \lambda(a,b)e^{b+c} - (\partial_b \lambda(c,b))e^c \\
&= \partial_a \lambda(a,b)e^{b+c} - (\partial_b \lambda(a,b))e^{b+c} + \lambda(c,b)e^c - (\partial_b \lambda(b,c))e^c \\
&\qquad + \lambda(c,a) + \partial_a \lambda(a,c)
\end{aligned} \tag{40}$$

as required.

Checking the LHS is even simpler. We have

$$Hex^l = \frac{e^{b+c}}{ab} + \frac{e^c}{bc} + \frac{1}{ac}$$

Hence

$$\begin{aligned}
\big[1 + \partial_a - \partial_b\big] Hex^l =& \frac{e^{b+c}}{ab} + \frac{e^c}{bc} + \frac{1}{ac} \\
&+ (-\frac{e^{b+c}}{a^2 b} - \frac{1}{a^2 c}) \\
&- (-\frac{e^{b+c}}{ab^2} + \frac{e^{b+c}}{ab} - \frac{e^c}{b^2 c})
\end{aligned}$$

After a small manipulation involving the $e^{b+c}$ terms and some simplification we get the desired result.

**Terms in** $c[ab]$ Here we have:

$$\begin{aligned}
&-\frac{e^{b+c}}{a^2 b} + \frac{e^c}{bc^2} + \frac{1}{ac} - \frac{1}{bc^2} + \frac{1}{a^2 b} - \frac{1}{ab} + \frac{1}{a^2 b} - \frac{1}{bc} - \frac{1}{bc^2} \\
&\qquad = \partial_a \lambda(a,b)e^{b+c} - \partial_c \lambda(c,b)e^c + \partial_a \lambda(c,a) + \lambda(c,a) - \partial_c \lambda(a,c)
\end{aligned}$$

As with the $b[ab]$ terms, I claim that the $c[ab]$ equation is simply the statement

$$\big[1 + \partial_a - \partial_c\big] Hex^l = \big[1 + \partial_a - \partial_c\big] Hex^r$$

so that it, too, follows from the hexagon. The calculation is similar to the $b[ab]$ case and will not be repeated.

**The 2FF Solution** We have shown that the solution (17) takes the form

$$\Phi(a, b) = 1 + [ab]\lambda(a, b) - a[ab]\partial_a\lambda(a, b) - b[ab]\partial_b\lambda(a, b) \tag{41}$$

where $\lambda$ is a solution of the 1FF equation.

Although this is technically only a solution to the positive hexagon, it is not hard to show that in fact the unitarity condition (8) suffices to insure that the same solution also satisfies the negative hexagon (of course this also follows on general principles from arguments given in [2], alluded to earlier). Moreover, since in 2FF all commutators of commutators are zero, Kurlin's argument (see [9], Proposition 5.10) still applies to show that this unitarity condition also insures that the solution satisfies the pentagon equation.

Thus each solution $\lambda$ to the 1FF equation gives rise to a solution to the 2FF equation according to formula (41).

**Remark 7.** *Singular Solution*

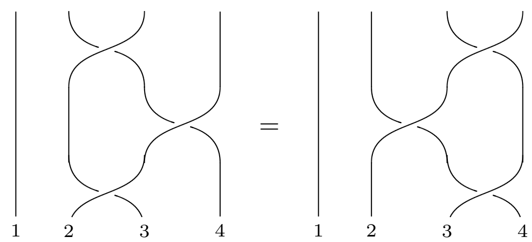
As with the 1FF equation, one can verify that there is also a singular solution $\lambda(x, y) = \frac{1}{xy}$.

# 3 Closed-Form Braidor

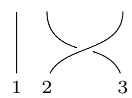
## 3.1 Introduction

In this section we apply the techniques discussed above for the hexagons to solve a different, but related, equation modulo 1FF, namely the braidor equation. A full explanation of this equation and its background is beyond the scope of this paper (see [3], [4] and [12]). However, since this equation is not widely known we provide a brief summary.

The braidor equation can be viewed as the equation one gets if one adds a 'shield' strand on the left of the Reidemeister III move:



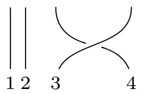

The shield strand is 'inert' in that it is never flipped with any of the other strands. However, the shield strand (but no other strand) can also be doubled. One then asks whether one can associate to any crossing of two strands a chord diagram (or series) $B \in \hat{\mathcal{A}}_3$, which can be represented pictorially as follows:




$\longrightarrow \quad B \in \hat{\mathcal{A}}_3$

We seek such $B \in \hat{\mathcal{A}}_3$ which will satisfy the Reidemeister III equation. Such a $B$ is not necessarily the same as the value $R$ associated to a crossing in the portion of this paper on associators, because in $B$ one allows chords to have endpoints resting on the shield, as well as on the strands being flipped.

One postulates that a crossing of the third and fourth strands in a shielded braid corresponds to $(\Delta 11)B$ as follows:




$$\longrightarrow (\Delta 11)B \in \hat{\mathcal{A}}_4$$

and more generally a crossing of the $i$th and $(i+1)$st strands in a braid corresponds to $(\Delta 1...1)\ldots(\Delta 11)B$ (where there are $(i-2)$ $\Delta$s). Then the Reidemeister III move corresponds to the 'braidor equation':

$$B^{123} \cdot (\Delta 11)B^{1324} \cdot B^{134} = (\Delta 11)B \cdot B^{124} \cdot (\Delta 11)B^{1423} \tag{42}$$

Naturally, one also imposes a 'Reidemeister II' move, which amounts to requiring that $B$ have a power series inverse $B^{-1}$ which corresponds to the negative crossing.

From a slightly different perspective, given a braid on $n$ strands (including an inert shield-strand labeled 1, on the left), we can consider the strands to be parenthesized as follows: $(...((12)3)...)n$. A braidor then corresponds to the transformation: $(12)3 \longrightarrow (13)2$. Similarly, $(\Delta 11)B$ corresponds to the transformation $((12)3)4 \longrightarrow ((12)4)3$, and so forth.

We note that the transformation $(12)3 \longrightarrow (13)2$ can be factored as $(12)3 \xrightarrow{\Phi} 1(23) \xrightarrow{R^{23}} 1(32) \xrightarrow{(\Phi^{-1})^{132}} (13)2$, where $R$ and $\Phi$ are as in Part 2. Consequently, a braidor can be constructed out of an associator as $B = \Phi \cdot R^{23} \cdot (\Phi^{-1})^{132}$. On the other hand, it is at this time an open question as to whether all braidors come from associators.

Accordingly it is legitimate to ask whether one can find a power series of chord diagrams $\lambda(a, b)$, in $\mathcal{A}$ (i.e., on 3 strands), symmetric in $a$ and $b$, such that if we define $\Psi(a, b) := (1 + [a, b]\lambda(a, b))$, put $R^{12} = \exp(a)$ and define $B = \Psi \cdot R^{23} \cdot (\Psi^{-1})^{132}$, the braidor equation is satisfied modulo 1FF. As we make no other assumptions about $\lambda$, *a priori* any resulting $\Psi$ need not be an associator. We note that the assumption that $\lambda$ is symmetric is equivalent to the assumption that $\Psi\Psi^T = 1$ by Proposition 3.

Nevertheless, it turns out that based on the above assumptions, taken modulo 1FF, the braidor equation is equivalent to:

$$Hex^+ + Hex^- = 0 \tag{43}$$

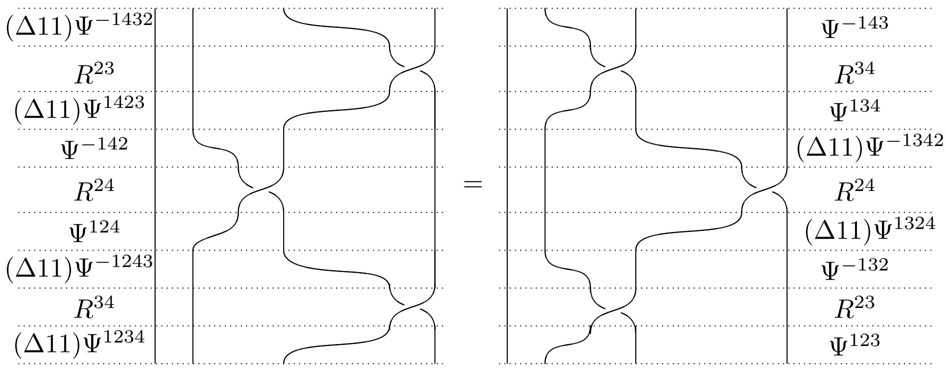
where $Hex^+ = 0$ is the positive hexagon and $Hex^- = 0$ is the negative hexagon. Moreover, we will see that equation (43) can be solved for the underlying $\lambda$, and that there is a 1–1 relation between solutions to the braidor and solutions to the hexagons and pentagon, modulo 1FF.

For now, we show how the braidor equation is simplified mod 1FF.

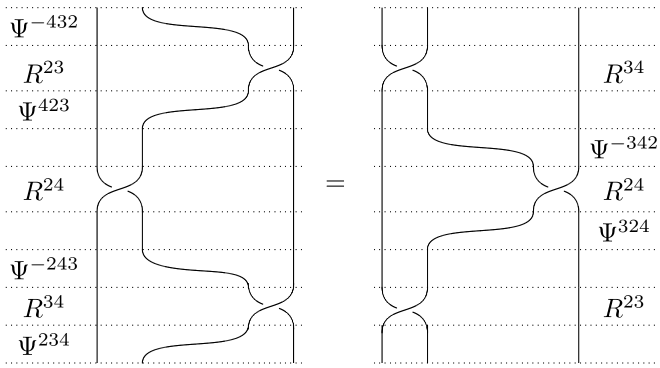
## 3.2 Simplifying the Braidor Equation Modulo 1FF

### 3.2.1 Pictorial Analysis

We begin by pictorially exhibiting the braidor equation, on the basis that $B = \Psi \cdot R^{23} \cdot (\Psi^{-1})^{132}$ - we draw the $\Psi$ as an associator, but will in fact consider it algebraically as only being a series of the form $\Psi(a,b) := (1 + [a,b]\lambda(a,b))$ with $\lambda$ symmetric. The braidor equation then looks like this:




We can easily see that, modulo 1FF, this equation implies that $Hex^+ + Hex^- = 0$ by considering the 'off-shield' equation – the equation which results if we consider only terms with no chord ending on the shield strand (this amounts to considering the quotient in which all such chords are set to zero). Pictorially, we can just drop the shield strand, so we get the following:




If we multiply both sides (at the top) by $\Psi^{432}$, we will see that in the 1FF quotient the result is the equation $Hex^+ + Hex^- = 0$. We will also see that, modulo 1FF, the 'on-shield' portion of the braidor equation which we left out (ie, with chords ending on the shield strand) is automatically satisfied for symmetric $\lambda$, much as the pentagon equation was automatically satisfied for symmetric $\lambda$.

First we show that we have recovered the equation $Hex^+ + Hex^- = 0$ modulo 1FF. After multiplying at the top by $\Psi^{432}$, we consider the equation RHS = 1:

$$R^{23} \cdot \Psi^{324} \cdot R^{24} \cdot \Psi^{-342} \cdot R^{34} \cdot \Psi^{432} = 1$$

We multiply both sides by $R^{-23}$ on the left, invert both sides, then use the unitarity assumption $\Psi \cdot \Psi^T = 1$ to get:

$$R^{23} = \Psi^{234} \cdot R^{-34} \cdot \Psi^{-243} \cdot R^{-24} \cdot \Psi^{423}$$

When we linearize modulo 1FF, the result will be the negative hexagon (written on strands 2,3,4 rather than 1,2,3), except that we have on the left $R^{23} = e^{t^{23}}$ instead of $(1\Delta 1)R^{-23} = e^{-t^{24}-t^{34}}$. However, we will see below that, upon linearizing modulo 1FF, all the off-shield terms are pre-multiplied by the commutator $[\hat{1}] = [t^{23}, t^{34}]$. But modulo 1FF, $[\hat{1}]t^{23} = [\hat{1}](-t^{24} - t^{34})$, and as a result the $R^{23}$ is equivalent to $(1\Delta 1)R = e^{-t^{24}-t^{34}}$.

This shows that on the RHS of the braidor equation (multiplied by $\Psi^{432}$ at the top) we have, modulo 1FF, $-Hex^-$. By similar reasoning, we can show that the LHS gives $Hex^+$ modulo 1FF. Modulo 1FF the equation therefore linearizes to $Hex^+ + Hex^- = 0$.

### 3.2.2 On-Shield Braidor Equation

For the on-shield portion of the braidor equation, we need to write out the braidor equation in its algebraic form and do some computations. We will show that the on-shield equation is automatically trivial modulo 1FF. We write $\psi(a,b) := [a,b]\lambda(a,b)$ and so $\Psi = 1 + \psi$. Then with our assumption $B = \Psi \cdot R^{23} \cdot (\Psi^{-1})^{132}$, we get after linearizing modulo 1FF:

$$\begin{aligned} B &= (1+\psi)\ e^b\ (1+\psi^{-132}) \\ &= e^b + \psi\ e^b + \psi^{-132} \end{aligned}$$

The braidor becomes:

$$\begin{aligned} &\Big(e^b + \psi\ e^b + \psi^{-132}\Big) \cdot \Big(e^d + (\Delta 11)\psi^{1324}\ e^d + (\Delta 11)\psi^{-1342}\Big) \\ &\qquad \Big(e^e + \psi^{134}\ e^e + \psi^{-143}\Big) \\ &= \Big(e^e + (\Delta 11)\psi\ e^e + (\Delta 11)\psi^{-1243}\Big) \cdot \Big(e^d + \psi^{124}\ e^d + \psi^{-142}\Big) \\ &\qquad \Big(e^b + (\Delta 11)\psi^{1423}\ e^b + (\Delta 11)\psi^{-1432}\Big) \end{aligned}$$

Expanding and linearizing modulo 1FF we get:

$$\begin{aligned} &e^b e^d e^e + \psi\ e^{b+d+e} + \psi^{-132} e^{d+e} + (\Delta 11)\psi^{1324}\ e^{d+e} + (\Delta 11)\psi^{-1342} e^e \\ &\qquad + \psi^{134}\ e^e + \psi^{-143} \\ &= e^e e^d e^b + (\Delta 11)\psi\ e^{e+d+b} + (\Delta 11)\psi^{-1243} e^{d+b} + \psi^{124}\ e^{d+b} + \psi^{-142} e^b \\ &\qquad + (\Delta 11)\psi^{1423}\ e^b + (\Delta 11)\psi^{-1432} \end{aligned}$$

We apply the fact that $\psi^{-1} = -\psi$, and collect all terms onto the LHS:

$$\begin{aligned} e^b e^d e^e + \psi\ e^{b+d+e} - \psi^{132} e^{d+e} + (\Delta 11)\psi^{1324}\ e^{d+e} - (\Delta 11)\psi^{1342} e^e \\ + \psi^{134}\ e^e - \psi^{143} - e^e e^d e^b - (\Delta 11)\psi\ e^{e+d+b} + (\Delta 11)\psi^{1243} e^{d+b} \\ - \psi^{124}\ e^{d+b} + \psi^{142} e^b - (\Delta 11)\psi^{1423}\ e^b + (\Delta 11)\psi^{1432} = 0 \end{aligned}$$

We now see that each term (other than the pure exponentials) can be paired off in two specific ways, by collecting the terms into two groupings:

$$\begin{array}{ccccc} & (\Delta 11)\psi^{1432} & \text{———} & -\psi^{143} & \\ \cdot^{\cdot^{\cdot}} & & & & \cdot_{\cdot_{\cdot}} \\ -\psi^{132} e^{d+e} & & & & (\Delta 11)\psi^{1243} e^{d+b} \\ \diagdown & & & & \diagup \\ & (\Delta 11)\psi^{1324} e^{d+e} & \cdots\cdots & -\psi^{124} e^{d+b} & \end{array}$$

and

$$\begin{array}{ccccc} & -(\Delta 11)\psi^{1234} e^{b+d+e} & \text{——} & \psi^{123} e^{b+d+e} & \\ \cdot^{\cdot^{\cdot}} & & & & \cdot_{\cdot_{\cdot}} \\ \psi^{134} e^e & & & & -(\Delta 11)\psi^{1423} e^b \\ \diagdown & & & & \diagup \\ & -(\Delta 11)\psi^{1342} e^e & \cdots\cdots & \psi^{142} e^b & \end{array}$$

Here the dotted lines connect pairs of terms of the form

$$\pm(\Delta 11)\psi^{1ijk} e^u \cdots\cdots \mp\psi^{1jk} e^v$$

where, in the exponential factors, $u$ and $v$ are always different chords. The solid lines represent pairs of the form:

$$\pm(\Delta 11)\psi^{1ijk} e^u \text{ ——————— } \mp\psi^{1ij} e^u$$

where it is noted that the exponential factors here are the same.

We will see that the occurrence of each *dotted-line* pairing allows a transformation of the components of the pairing via Lemmas 5 and 6, such that the resulting components of each *solid-line* pairing can be cancelled against each other – except for the off-shield terms, which give us the (sum of the) hexagon equations.

To begin, we need a formula for the expansion of $(\Delta 11)\Psi$ modulo 1FF (taking $\psi(x, y) = [x, y]\lambda(x, y)$):

**Proposition 11.** *Mod 1FF, we have:*

$$(\Delta 11)\psi = [\hat{2}]\lambda(t^{13}, t^{34}) + [\hat{1}]\lambda(t^{23}, t^{34}) - [\hat{4}]\lambda(t^{24}, t^{34}) + [\hat{4}]\lambda(t^{14}, t^{34})$$

*Proof.* This is the 1FF version of (the first part of) Proposition 9. □

We in fact want a description of all the dotted-line pairs in this form. We can actually do this fairly easily by noting that (ignoring for the moment the signs and exponential factors) we can get from the pairing $-(\Delta 11)\psi^{1234}\ e^{b+d+e} \ldots \psi^{134}\ e^{e}$ to each of the other ones by means of permutations involving strands $2, 3$ and 4. The sequence of permutations is as follows:

$$(\Delta 11)\psi^{1234}\ e^{b+d+e} \cdots\cdots \psi^{134}\ e^{e}$$

Now exchange $(2 \leftrightarrow 3)$ (so $e \leftrightarrow d$ and $b$ is fixed):

$$(\Delta 11)\psi^{1324}\ e^{d+e} \cdots\cdots \psi^{124}\ e^{d+b}$$

Now exchange $(2 \leftrightarrow 4)$ (so $e \leftrightarrow b$ and $d$ is fixed):

$$(\Delta 11)\psi^{1342}\ e^{e} \cdots\cdots \psi^{142}\ e^{b}$$

Now exchange $(3 \leftrightarrow 4)$ (so $d \leftrightarrow b$ and $e$ is fixed):

$$(\Delta 11)\psi^{1432} \cdots\cdots \psi^{132}\ e^{d+e}$$

Now exchange $(2 \leftrightarrow 3)$:

$$(\Delta 11)\psi^{1423}\ e^{b} \cdots\cdots \psi^{123}\ e^{b+d+e}$$

Now exchange $(2 \leftrightarrow 4)$:

$$(\Delta 11)\psi^{1243}\ e^{d+b} \cdots\cdots \psi^{143}$$

Hence we will be able to set out an expansion like the one in Proposition 11, for generic dotted line pairings. To do this, we need the following lemma (whose verification is left to the reader) which explains how permuting strands affects commutators.

**Lemma 10.** *If $\sigma$ is a permutation which sends $(2, 3, 4)$ to $(i, j, k)$, then:*

$$\begin{aligned} \sigma([\hat{1}]) &= (-1)^{\sigma}[\hat{1}] \\ \sigma([\hat{2}]) &= (-1)^{i+\sigma}[\hat{i}] \\ \sigma([\hat{4}]) &= (-1)^{k+\sigma}[\hat{k}] \end{aligned}$$

*where $\sigma$ in the exponents means $sgn(\sigma)$.*

From this lemma, we immediately get the:

**Proposition 12.** *If $\sigma$ is a permutation which sends $(2,3,4)$ to $(i,j,k)$, then:*

$$(\Delta 11)\psi^{1ijk}\ e^u = \Big\{(-1)^{i+\sigma}[\hat{i}]\lambda(t^{1j},t^{jk}) + (-1)^{\sigma}[\hat{1}]\lambda(t^{ij},t^{jk}) - (-1)^{k+\sigma}[\hat{k}]\lambda(t^{ki},t^{jk}) + (-1)^{k+\sigma}[\hat{k}]\lambda(t^{1k},t^{jk})\Big\}e^u$$

*while its dotted-line counterpart is*

$$\psi^{1jk}\ e^v = (-1)^{i+\sigma}[\hat{i}]\lambda(t^{1j},t^{jk})e^v$$

*where $u$ and $v$ are single-chord diagrams (or sums of such diagrams, or the trivial diagram).*

The proof follows from the preceding discussion. We now want to simplify the expansions of the dotted-line pairings, as promised.

**Lemma 11.** *The $[\hat{k}]$ terms in the expansion of $(\Delta 11)\psi^{1ijk}\ e^u$ can be expressed as follows:*

$$\big\{-(-1)^{k+\sigma}[\hat{k}]\lambda(t^{ki},t^{jk})+(-1)^{k+\sigma}[\hat{k}]\lambda(t^{1k},t^{jk})\big\}e^u = \big\{-(-1)^{\sigma}[\hat{4}]\lambda(\overline{t^{ki}},\overline{t^{jk}})+(-1)^{\sigma}[\hat{4}]\lambda(\overline{t^{1k}},\overline{t^{jk}})\big\}e^{\bar{u}}$$

*Proof.* By Lemma 5, since the arguments of the $\lambda$s include only chords involving strand $k$, we can convert the $[\hat{k}]$ to $[\hat{4}]$ and 'bar' the arguments. We pick up factors of $(-1)^k$, which reduces us to sign factors of just $(-1)^\sigma$.

We now consider the exponential $e^u$. We have two $[\hat{4}]$ terms, with opposite signs, and in both of which $\lambda$ is evaluated at arguments involving only $t^{24}$, $t^{34}$ or $t^{14}$ (coming from the $\overline{t^{ki}}, \overline{t^{jk}}$ and $\overline{t^{1k}}$). The constant terms in $\lambda$ (if any) cancel off, and so we are left with $[\hat{4}]$ multiplied by positive powers of $t^{24}$, $t^{34}$ or $t^{14}$, as well as the exponential factor. It follows, from Lemma 6, that we can convert the exponential to its 'barred' version. □

We now consider the $[\hat{i}]$ terms in $(\Delta 11)\psi^{1ijk}\ e^u$ and its dotted-line counterpart.

**Lemma 12.** *The $[\hat{i}]$ term in $(\Delta 11)\psi^{1ijk}\ e^u$ and the counterpart $\psi^{1jk}\ e^v$ differ only by their exponential factors. The difference of these exponential factors has the form:*

$$e^u - e^v = (e^{t^{ij}+t^{ki}} - 1)\cdot\omega$$

*where $\omega$ is a unit (ie a power series with non-zero constant term – and in fact an exponential of some form).*

*Proof.* This can be seen by direct inspection of the dotted line pairings set out earlier. □

**Lemma 13.** *The sum of the $[\hat{i}]$ term in $(\Delta 11)\psi^{1ijk}\ e^u$ and the counterpart $\psi^{1jk}\ e^v$ can be converted to the following form:*

$$(-1)^{i+\sigma}[\hat{i}]\lambda(t^{1j},t^{jk})(e^{t^{ij}+t^{ki}}-1)\cdot\omega = (-1)^{\sigma}[\hat{4}]\lambda(\overline{t^{1j}},\overline{t^{jk}})(e^{\overline{t^{ij}}+\overline{t^{ki}}}-1)\cdot\overline{\omega}$$

*Proof.* The proof is similar to the $[\hat{k}]$ case. First we note that all of the chords in the exponential factor $(e^{t^{ij}+t^{ki}}-1)$ involve strand $i$, so Lemma 5 applies, and we can convert $[\hat{i}]$ to $[\hat{4}]$, replace $(e^{t^{ij}+t^{ki}}-1)$ by its barred form, and pick up factors of $(-1)^i$.

With regard to the $\lambda$s, we note that the exponential factor $(e^{\overline{t^{ij}}+\overline{t^{ki}}}-1)$ includes only positive powers of chords involving strand 4, so Lemma 6 applies to 'bar' the arguments of the $\lambda$s. For the same reason, the $\omega$ also gets barred. □

Combining these results, we see that in the pairing $-(\Delta 11)\psi^{1ijk}\, e^u \ldots \psi^{1jk}\, e^v$ the $[\hat{i}]$ terms have been converted to $[\hat{4}]$ terms and 'barred'. This new $[\hat{4}]$ term in $-(\Delta 11)\psi^{1ijk}\, e^u$ is equal but opposite in sign to one of the $[\hat{4}]$ terms coming from the $[\hat{k}]$s, so we have a cancellation. The result is that the dotted-line pairings are replaced by their reduced forms:

$$(\Delta 11)\psi^{1ijk}\, e^u \longrightarrow (-1)^\sigma[\hat{1}]\lambda(t^{ij},t^{jk})e^u + (-1)^\sigma[\hat{4}]\lambda(\overline{t^{1k}},\overline{t^{jk}})e^{\bar{u}}$$

and

$$\psi^{1jk}\, e^v = \!\!\longrightarrow (-1)^\sigma[\hat{4}]\lambda(\overline{t^{1j}},\overline{t^{jk}})e^{\bar{v}}$$

If we re-express this reduction in terms of the solid-line pairings, we get the proposition:

**Proposition 13.** *The solid-line pairings can be reduced to the following form:*

$$\pm(-1)^\sigma\{[\hat{1}]\lambda(t^{ij},t^{jk})e^u + [\hat{4}]\lambda(\overline{t^{1k}},\overline{t^{jk}})e^{\bar{u}}\} \;\text{———————}\; \mp(-1)^\sigma[\hat{4}]\lambda(\overline{t^{1i}},\overline{t^{ij}})e^{\bar{u}}$$

We now note that $\overline{t^{1k}} = \overline{t^{ij}}$ and $\overline{t^{1i}} = \overline{t^{jk}}$. This, combined with the fact that $\lambda$ is symmetric, shows that the solid-line pairings cancel off, except for the off-shield terms. From the earlier pictorial argument, we know already that all such terms (plus the exponential-only terms) will give us the equality $Hex^+ + Hex^- = 0$, if we can only show that they are pre-multiplied by the commutator $[\hat{1}]$. We turn to this now.

### 3.2.3 Off-Shield Braidor Equation

We will in fact show algebraically that the off-shield equation gives us $Hex^+ + Hex^- = 0$, which as a by-product will confirm that the mostly pictorial proof given earlier was valid as well.

Using Equation (44) and Proposition 13, we first collect the portion of the $(\Delta 11)$ terms that is pre-multiplied by $[\hat{1}]$:

$$\begin{aligned} &-[\hat{1}]\big\{\lambda(b,e)e^{b+d+e} + \lambda(b,d)e^{d+e} + \lambda(e,d)e^{e} \\ &\qquad + \lambda(e,b) + \lambda(d,b)e^{b} + \lambda(d,e)e^{d+b}\big\} \\ &= -[\hat{1}]\big\{\lambda(b,e) + \lambda(e,d)e^{-e} + \lambda(b,d)e^{b} \\ &\qquad \lambda(b,e) + \lambda(e,d)e^{e} + \lambda(b,d)e^{-b}\big\} \end{aligned}$$

where we note that the signs otherwise appearing in the expansion (44) exactly offset the $(-1)^\sigma$ factors from Proposition 13.

The third line contains the $\lambda$ terms for the positive hexagon (shifted to sit on chords 2,3,4), while the fourth line contains the $\lambda$ terms from the similarly shifted negative hexagon. Going back to (44), all that's left is the difference $e^b e^d e^e - e^e e^d e^b$, which we simplify by a procedure analogous to that used in Proposition 6.

We introduce the notation $\bar{e}^x$ defined by the following:

$$e^x = 1 + x\frac{e^x - 1}{x} = 1 + x\bar{e}^x$$

Then:

$$\begin{aligned} e^b e^d e^e &= (1 + b\bar{e}^b)(1 + d\bar{e}^d)(1 + e\bar{e}^e) \\ &= 1 + b\bar{e}^b + d\bar{e}^d + e\bar{e}^e + bd\bar{e}^b\bar{e}^d + be\bar{e}^b\bar{e}^e + de\bar{e}^d\bar{e}^e + bde\bar{e}^b\bar{e}^d\bar{e}^e \end{aligned}$$

After a similar calculation for $e^e e^d e^b$ we get:

$$\begin{aligned} e^b e^d e^e - e^e e^d e^b &= [bd]\bar{e}^b\bar{e}^d + [be]\bar{e}^b\bar{e}^e + [de]\bar{e}^d\bar{e}^e + [be]d\bar{e}^b\bar{e}^d\bar{e}^e \\ &= [be]\bar{e}^b\bar{e}^e(1 + d\bar{e}^d) + [bd]\bar{e}^b\bar{e}^d + [de]\bar{e}^d\bar{e}^e \\ &= -[\hat{1}]\big\{\bar{e}^b\bar{e}^d - \bar{e}^b\bar{e}^e e^d + \bar{e}^d\bar{e}^e\big\} \\ &= -[\hat{1}]\big\{\frac{(e^b-1)}{b}\frac{(e^d-1)}{d} - \frac{(e^b-1)}{b}\frac{(e^e-1)}{e}\, e^d + \frac{(e^d-1)}{d}\frac{(e^e-1)}{e}\big\} \end{aligned}$$

We now rework the last expression to show that it gives the exponential-only portion of the positive and negative hexagons. We first note:

$$\frac{(e^b-1)}{b}\frac{(e^d-1)}{d} = -\frac{(e^b-1)}{bd} + e^d\frac{(e^b-1)}{bd}$$

and

$$\frac{(e^d-1)}{d}\frac{(e^e-1)}{e} = -\frac{(e^e-1)}{de} + e^d\frac{(e^e-1)}{de}$$

while (using the fact that $[\hat{1}](b+e+d) = 0$ by 4T):

$$\begin{aligned} -\frac{(e^b-1)}{b}\frac{(e^e-1)}{e}\, e^d &= -e^d\frac{-(b+e)}{d}\frac{(e^b-1)}{b}\frac{(e^e-1)}{e} \\ &= e^d\frac{(e^b-1)(e^e-1)}{de} + e^d\frac{(e^b-1)(e^e-1)}{bd} \\ &= e^{d+b}\frac{(e^e-1)}{de} - e^d\frac{(e^e-1)}{de} + e^{d+e}\frac{(e^b-1)}{bd} - e^d\frac{(e^b-1)}{bd} \\ &= -\frac{e^{-e}-1}{de} - e^d\frac{(e^e-1)}{de} - \frac{e^{-b}-1}{bd} - e^d\frac{(e^b-1)}{bd} \end{aligned}$$

Adding up the last three results, we get:

$$e^b e^d e^e - e^e e^d e^b = -\frac{(e^b-1)}{bd} - \frac{(e^{-e}-1)}{de} - \frac{(e^e-1)}{de} - \frac{(e^{-b}-1)}{bd}$$

and one can check readily that the first two terms are the exponential-only portion of the positive hexagon, while the last two terms are the exponential-only portion of the negative hexagon.

## 3.3 Solving the Braidor Equation

We have shown that the braidor equation reduces modulo 1FF to $Hex^+ + Hex^- = 0$. We will now show that we can explicitly solve this equation for the underlying function $\lambda$, in a manner very similar to Kurlin's solution of the hexagons. Specifically, we will show that:

**Theorem 3.** *The solutions $\lambda$ to the braidor equation are given implicitly by the property:*

$$\tilde{\lambda}(x,y) := xy\ \lambda(x,y) - 1 \tag{44}$$

*where $\tilde{\lambda}$ is given by:*

$$\tilde{\lambda}(x,y) = \frac{xy\ h(x,y)}{\sinh x \sinh y} \tag{45}$$

*and $h$ is any power series in the variables $x, y, z$ (where $z = -x - y$) such that:*

$$h(x,y) = h(y,z) = h(z,x) \tag{46}$$

*subject only to the constraint:*

$$h(x,0) = -\frac{\sinh x}{x} \tag{47}$$

*Proof.* First, from equations (19) and (26), the equation $Hex^+ + Hex^- = 0$ becomes:

$$\Big\{\lambda(a,b) + e^{-b}\ \lambda(b,c) + e^a\ \lambda(a,c)\ - \frac{1}{ab} - \frac{e^{-b}}{bc} - \frac{e^a}{ac}\Big\}$$
$$+ \Big\{\lambda(a,b) + e^b\ \lambda(b,c) + e^{-a}\ \lambda(a,c)\ - \frac{1}{ab} - \frac{e^b}{bc} - \frac{e^{-a}}{ac}\Big\} = 0$$

Multiplying both sides by $abc$ (and using the symmetry of $\lambda$) we get:

$$2c\ \big(ab\lambda(a,b) - 1\big) + a(e^b + e^{-b})\ \big(bc\lambda(b,c) - 1\big) + b(e^a + e^{-a})\ \big(ac\lambda(a,c) - 1\big) = 0$$

We now plug in the definition (44) and so get:

$$c\ \tilde\lambda(a,b) + a\cosh b\ \tilde\lambda(b,c) + b\cosh a\ \tilde\lambda(a,c) = 0 \tag{48}$$

or equivalently

$$\frac{c}{\cosh b}\ \tilde\lambda(a,b) + a\ \tilde\lambda(b,c) + b\frac{\cosh a}{\cosh b}\ \tilde\lambda(a,c) = 0$$

We now perform the substitution $b \leftrightarrow c$ (which amounts to interchanging the first and second strands) and get:

$$\frac{b}{\cosh c}\ \tilde\lambda(a,c) + a\ \tilde\lambda(b,c) + c\frac{\cosh a}{\cosh c}\ \tilde\lambda(a,b) = 0$$

Now subtract the latter from the former equation, multiply by $\cosh b \cosh c$ and rearrange:

$$c(\cosh c - \cosh a \cosh b)\ \tilde\lambda(a,b) = b(\cosh b - \cosh a \cosh c)\ \tilde\lambda(a,c)$$

We now note the hypertrigonometric equality:

$$\cosh c = \cosh(-c) = \cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b$$

and similarly for $\cosh b$. Hence we get, after also dividing by $abc$:

$$\frac{\sinh a \sinh b}{ab}\ \tilde\lambda(a,b) = \frac{\sinh a \sinh c}{ac}\ \tilde\lambda(a,c)$$

Performing the transformation $a \to b \to c \to a$, we see that we must also have:

$$\frac{\sinh b \sinh c}{bc}\ \tilde\lambda(b,c) = \frac{\sinh a \sinh b}{ab}\ \tilde\lambda(a,b)$$

We therefore see that, defining the new function $h$ by:

$$h(x,y) = \frac{\sinh x \sinh y}{xy}\ \tilde\lambda(x,y)$$

$h$ is a symmetric function and must satisfy the conditions (46).

Conversely, given any function $h$ satisfying these properties, we can define $\tilde\lambda$ via (45). Then we can readily verify that $\tilde\lambda$ solves the modified braidor equation (48):

$$\begin{aligned} c\ \frac{ab\ h(a,b)}{\sinh a \sinh b} + b\ \frac{ac\ h(a,c)}{\sinh a \sinh c}\ \cosh a &= abc\ \frac{h(a,b)}{\sinh a}\ \frac{\sinh c + \cosh a \sinh b}{\sinh b \sinh c} \\ &= abc\ \frac{h(a,b)}{\sinh a}\ \frac{-\sinh a \cosh b}{\sinh b \sinh c} \\ &= -a\cosh b\ \frac{bc\ h(b,c)}{\sinh b \sinh c} \\ &= -a\ \cosh b\ \tilde\lambda(b,c) \end{aligned}$$

as required. Note that in going from the first to second lines we used the fact that

$$\sinh c = -\sinh -(a+b) = -(\sinh a \cosh b + \cosh a \sinh b)$$

Of course our real goal is not to find $\tilde{\lambda}$ satisfying the modified braidor equation, but $\lambda$ satisfying the actual braidor equation. The connection is through the formula (44). However, since we want $\lambda$ to be a power series (in particular, having lowest degree terms of non-negative degree), we see that we must impose an additional constraint on $\tilde{\lambda}$: $\lambda$ will be a power series if and only if $xy\lambda(x,y) = 0$ whenever $xy = 0$, so it suffices to impose the condition $\tilde{\lambda}(x,0) = -1$ (from which it also follows $\tilde{\lambda}(-x,0) = \tilde{\lambda}(0,x) = \tilde{\lambda}(x,-x) = -1$). Through (44) and (45) this becomes the condition:

$$0.\lambda(x,0) - 1 = \frac{x}{\sinh x}\ h(x,0)$$

hence

$$h(x,0) = -\frac{\sinh x}{x}$$

□

This is equivalent to the 'boundary' condition found by Kurlin for his function $h$ in [9], Lemma 4.6. The relation betwen solutions to the braidor and solutions to the hexagons will be explored in the next subsection of the paper.

### 3.4 Relation Between Associators and Braidors

Note that any solution of the positive and negative hexagons taken separately must also be a solution of the 'double hexagon' (43). However, the converse is also true. We show this by comparing the spaces of solutions of the respective equations.

To this end we summarize briefly Kurlin's approach to solving the hexagon equations. Kurlin shows that the hexagon equations are equivalent to certain modified equations expressed in terms of a function $\tilde{f}$ in a form similar to (44). Using the conventions of this paper, the function $\tilde{f}$ is given implicitly by:

$$\tilde{f}(x,y) = xyf(x,y) - 1$$

(in fact, Kurlin gets '+1' on the RHS, due to the use of slightly different conventions - see Remark 3). The modified positive hexagon satisfied by $\tilde{f}$ is:

$$z\tilde{f}(x,y) + e^{-y}x\tilde{f}(y,z) + e^{x}y\tilde{f}(x,z) = 0$$

with a similar modified negative hexagon. Kurlin then shows that if $h_K$ is a symmetric power series satisfying the conditions (46) and the 'boundary' conditions:

$$h_K(x,0) = -\frac{\sinh x}{x}$$

(where again Kurlin actually gets a '+' sign due to the same difference in conventions) then:

$$\tilde{f}(x,y) := \frac{sinh(x+y)}{(x+y)} h(x,y)$$

is a solution of the modified hexagons (and the pentagon). (In fact Kurlin gives conditions on $h_K$ necessary and sufficient to obtain the even and odd parts of $f$, which is equivalent.)

We now relate Kurlin's solutions to ours. Suppose first that we have a function $h_b$ satisfying (46) and (47), and we define $\tilde{\lambda}$ by:

$$\tilde{\lambda}(x,y) = \frac{xy\ h(x,y)}{\sinh x \sinh y}$$

thus getting implicitly a solution $\lambda$ to the braidor equation. Now define $h_K$ by

$$h_K(x,y) := \frac{xy(x+y)}{\sinh x \sinh y \sinh(x+y)}\ h_b(x,y)$$

Then $h_K$ is clearly symmetric and satisfies (46). Moreover,

$$\begin{aligned} h_K(x,0) &= \frac{-x^2}{-\sinh^2 x}\ h_b(x,0) \\ &= \frac{x^2}{\sinh^2 x}\ (-\frac{\sinh x}{x}) \\ &= -\frac{x}{\sinh x} \end{aligned}$$

so $h_K$ satisfies the necessary boundary condition. Finally, putting:

$$\tilde{f}(x,y) = \frac{\sinh(x+y)}{(x+y)}\ h_K(x,y)$$

one verifies readily that $\tilde{f}$ satisfies the modified positive hexagon (and similarly for the modified negative hexagon). Finally, this procedure can clearly be reversed to produce braidors from solutions to the hexagons (and in any event, one already knows on general principles that a braidor can be obtained from an associator).

**Remark 8.** *Relation Between Braidor Equation and Pentagon*

It was mentioned at the beginning of Section 3 that it is an open question as to whether all braidors come from associators. Given the 1–1 relation between braidors and associators modulo 1FF, it is tempting to speculate that the pentagon is somehow implied by the braidor equation (most likely, the part of the braidor that involves the shield strand – we already know that the part of the braidor equation not involving the shield strand gives us $Hex^+ + Hex^- = 0$ modulo 1FF). In fact, though, the results of this paper do not give any evidence in that direction. Rather, we can only observe that the 1FF quotient (or, equivalently, Kurlin's quotient), together with the symmetry of $\lambda$ (or $f$) are sufficient to insure that $\lambda$ and $f$ satisfy the shield portion of the braidor, and the pentagon, respectively.

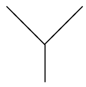
# 4 Concluding Remarks

There are a number of directions in which the work of this paper could be extended or applied. An obvious possible extension is to attempt to extend the results of the 1FF and 2FF quotients to 'nFF' quotients, with $n \in \mathbb{N}$. With regard to applications, one possibility relates to the 2FF quotient, which is related to the kernel of Alexander polynomial. More specifically, there is a standard way to associate to any finite-type invariant of knots, such as the coefficients of the Alexander polynomial, a 'weight system', ie a linear functional from the space of chord diagrams on the circle modulo the 4T relation to $\mathbb{Q}$ (see [1]). The kernel of the Alexander polynomial consists of those chord diagrams on which the weight system corresponding to the Alexander polynomial vanishes. It is known that the Alexander kernel contains the 2FF ideal, and it is thought that it is probably not much bigger than the 2FF, or perhaps the 3FF, ideal. Therefore, the closed-form formula for the associator modulo 2FF, or possibly 3FF, could lead to further insights into the Alexander polynomial.

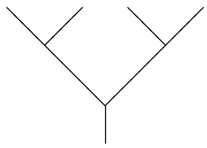
On a related note, one could also attempt to determine the kernel of other finite–type invariants, and consider whether the corresponding quotient also allows a closed–form formula for the associator. Preliminary work in this direction has been carried out in [5] in the case of the HOMFLYPT polynomial.

In a different direction, one can consider quotients which are well-behaved under a suitable class of operations on knots, or knot-like objects. This is related to the 'algebraic knot theory' program initiated by Bar-Natan [6], which considers invariants of knot-like objects which transform 'functorially' with respect to certain operations between such objects. The operations considered include the well-known strand-doubling and connected sum operations, but also strand deletion and 'unzip' operations. A broad class of knot properties can be characterized using these operations, including knot genus, unknotting number and the property of being ribbon. This being the case, invariants which are well-behaved under these operations may shed further light on these properties by allowing one to transform questions about knots into questions which are set in the target space of the invariant, which is presumably simpler and algebraically more tractable.

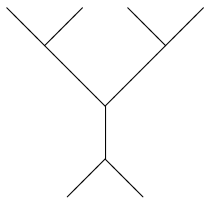
As noted earlier, research by Ng [11] has shown in particular that information about ribbon knots cannot be obtained only from a finite approximation to a knot invariant, so that closed–form solutions will be needed. Here again, though, one is faced with the fact that determination of closed-form formulas is extremely difficult, and so consideration of suitable quotient space is desirable. This leads one to consider quotients which are well-behaved under the allowed operations. To give a hint of the type of quotients this implies, we note that one can pictorially represent the bracket of a Lie algebra as a trivalent graph:



where the three edges represent two inputs and one output. Then, for instance, the quotient explored by Kurlin, namely $\mathcal{L}/\ [[\mathcal{L},\mathcal{L}],[\mathcal{L},\mathcal{L}]]$, is the quotient in which the following diagram, and diagrams containing this diagram, are set to zero:



It turns out that 'internal' properties of a diagram are generally well-behaved under the relevant operations. For instance, one could consider the quotient in which the following diagram, and all diagrams containing this diagram, are set to zero:



The key property of such diagrams for present purposes is that they contain an internal vertex, ie a vertex whose edges lead only to other (trivalent) vertices, and not to endpoints of the diagram. The property of having such internal vertices is well-behaved under the algebraic knot theory operations, and hence leads to a quotient whose exploration would be worthwhile from the algebraic knot theory viewpoint.

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