THE PURE VIRTUAL BRAID GROUP IS QUADRATIC – Peter Lee

Quadraticity

$G = FreeGp\ (X)/N,\ K = \mathbb{Q}G,$

$I_K := ker\ (x \mapsto 1,\ x \in X)$

G is quad. if $gr_I K = \oplus_{p\geq 0} I_K^p / I_K^{p+1}$ is quad.

i.e., graded, generated in degree 1

relations generated in degree 2

$q(gr_I K) := \frac{TV}{\langle R \rangle},\ V := I_K/I_K^2,$

$R := ker\ (V \otimes V \xrightarrow{\mu} I_K^2/I_K^3)$

G is quad. $\iff \mu : q(gr_I K) \twoheadrightarrow gr_I K$ is ISO.

Expansion for a group G

$K \xrightarrow{Z} \widehat{q(gr_I K)}$, $\quad \widehat{gr_I K} \xrightarrow{grZ} \widehat{q(gr_I K)}$, $\quad \widehat{q(gr_I K)} \xrightarrow{\mu} \widehat{gr_I K}$

$grZ \circ \mu$ is identity $\implies \mu$ is ISO.

i.e. Expansion $\implies$ quad.

Braid gp. B_n : Kontsevich integral

[BEER]: PvB_n has no (H.A.) expansion

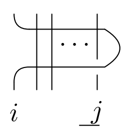
[Lee]: *PvB_n* is quadratic.

PvB_n $=$

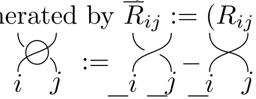
$\langle R_{ij} \rangle_{1\leq i\neq j\leq n}$

$R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij}$ (R III)

$R_{ij}R_{kl} = R_{kl}R_{ij}$ (Comm)

$R_{ij} =$ 

I_K generated by $\overline{R}_{ij} := (R_{ij} - 1)$

$\overline{R}_{ij} :=$ 

R III becomes: $\overline{R}_{ij}\overline{R}_{ik}\overline{R}_{jk} - \overline{R}_{jk}\overline{R}_{ik}\overline{R}_{ij}$

$+\overline{R}_{ij}\overline{R}_{ik} + \overline{R}_{ij}\overline{R}_{jk} + \overline{R}_{ik}\overline{R}_{jk}$

$-\overline{R}_{jk}\overline{R}_{ik} - \overline{R}_{jk}\overline{R}_{ij} - \overline{R}_{ik}\overline{R}_{ij}$

Mod I_F^3 : = (CYB) =

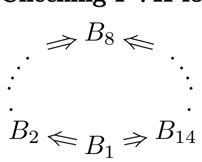
$[\overline{R}_{ij}, \overline{R}_{ik}] + [\overline{R}_{ij}, \overline{R}_{jk}] + [\overline{R}_{ik}, \overline{R}_{jk}] = 0$

Also $[\overline{R}_{ij}, \overline{R}_{kl}] = 0$(comm)

$\mathfrak{pvb}_n := \frac{FreeAlg\ \langle r_{ij} \rangle}{\langle \text{CYB, comm} \rangle}$

where $r_{ij} = \overline{R}_{ij} \mod I_K^2$

Checking PVH for *PvB_n*



Telescopic sum

$(B_2 - B_1) + \dots$

$+(B_{14} - B_1) + \dots = 0$

Arrows $B = B'$ give $(B - B') \in \partial(F.\Upsilon.F)$

Thus we get a syzygy in K

Replace $R_{ij} \mapsto \overline{R}_{ij} + 1$, expand, pass

to ass. graded: get a syz. in $q(gr_I K)$

FACT: all non-trivial syz. in $q(gr_I K)$

arise this way ($\implies PvB_n$ is quad.)

PVH Criterion

$F := FreeAlg.(X),\ I_F := ker\ (x \mapsto 1,\ x \in X),$

$M \subset I_F^2\ s.t.\ K = F/M.$

Let $\Upsilon \xrightarrow{\partial} F$ freely generate M

Claim : • X generates K and $q(gr_I K)$

• Υ generates M and R

PVH Criterion : G is quad. IF:

all 'syzygies' of $q(gr_I K)$

i.e. $ker\ gr(F.\Upsilon.F) \xrightarrow{\partial_q} TV$

are induced from syzygies of K

i.e. $ker\ F.\Upsilon.F \xrightarrow{\partial} F$

Koszulness: if $q(gr_I K)$ is Koszul

e.g. [BEER]: $q(gr\mathbb{Q}PvB_n)$, then

enough to check criterion in deg. 2,3

Due to Hutchings (for B_n), Positselski-Vishik

Proof of PVH for *PvB_n*

View F as: $T\tilde{X},\ \tilde{X} := \{(x-1) : x \in X\}$

F is graded, filtered. Complete F

With $\mathfrak{R} := F.\Upsilon.F$ and deg.$\Upsilon = 2$,

$\mathfrak{R}$ is graded, filtered

$$\begin{array}{ccccccccc}
 & & ker\ \partial & \longrightarrow & ker\ \partial & \xrightarrow{\pi_p^{Syz}} & ker\ gr\partial & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{R}_{\geq p+1} & \longrightarrow & \mathfrak{R}_{\geq p} & \xrightarrow{\pi_p^1} & \mathfrak{R}_p & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow gr\partial & & \\
0 & \longrightarrow & \tilde{X}^{\geq p+1} & \longrightarrow & \tilde{X}^{\geq p} & \xrightarrow{\pi_p^0} & \tilde{X}^p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \frac{\tilde{X}^{\geq p+1}}{\partial\mathfrak{R}_{\geq p+1}} & \xrightarrow{\mu_p} & \frac{\tilde{X}^{\geq p}}{\partial\mathfrak{R}_{\geq p}} & \rightarrow & q(gr_I K)^p & &
\end{array}$$

Snake Lemma LES:

$0 \rightarrow ker\ \partial \xrightarrow{\pi_p^{Syz}} ker\ gr\partial \rightarrow \frac{\tilde{X}^{\geq p+1}}{\partial_K \mathfrak{R}_{\geq p+1}}$

$\xrightarrow{\mu_p} \frac{\tilde{X}^{\geq p}}{\partial_K \mathfrak{R}_{\geq p}} \longrightarrow q(gr_I K)^p \longrightarrow 0$

1. $q(gr_I K)^p \cong \frac{\tilde{X}^{\geq p}}{\partial_K \mathfrak{R}_{\geq p}} / \mu_p \frac{\tilde{X}^{\geq p+1}}{\partial_K \mathfrak{R}_{\geq p+1}},\ \forall p$

2. The compositions $\rho_p : \frac{\tilde{X}^{\geq p}}{\partial_K \mathfrak{R}_{\geq p}} \rightarrow \mu_p(\frac{\tilde{X}^{\geq p}}{\partial_K \mathfrak{R}_{\geq p}})$

$\rightarrow \dots \rightarrow I_K^p$ are surjective,

and are all injective iff the μ_p are all injective

$\iff$ the π_p^{Syz} are all surjective (=PVH)

Thus PVH $\iff I_K^p \cong \frac{\tilde{X}^{\geq p}}{\partial_K \mathfrak{R}_{\geq p}},\ \forall p$

$\implies q(gr_I K)^p \cong I_K^p / I_K^{p+1},\ \forall p$

Koszulness

Let $R_{p,i} := V^{\otimes i} \otimes R \otimes V^{\otimes p-i-2}$

$\oplus_{i<j}(R_{p,i} \cap R_{p,j}) \xrightarrow{d_2} \oplus_i R_{p,i} \xrightarrow{d_1} V^{\otimes p} \rightarrow \mathfrak{pvb}_n^p \rightarrow 0$

For Koszul algebra, e.g. [BEER]: $\mathfrak{pvb}_n$, this is exact

Reduction to Degree 3

For $j > i+1,\ R_{p,i} \cap R_{p,j} = V^{\otimes i} \otimes R \otimes V^{\otimes j-i-2} \otimes R \otimes V^{p-i-j-4}$ Clearly lift to K

$R_{p,i} \cap R_{p,i+1} \cong V^{\otimes i} \otimes (R \otimes V \cap V \otimes R) \otimes V^{\otimes p-i-3}$ So Koszul $\implies$ syzygies det'd in deg. ≤ 3