The Pure Virtual Braid Group is Quadratic

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Abstract

If an augmented algebra K over $\mathbb{Q}$ is filtered by powers of its augmentation ideal I, the associated graded algebra $gr_I K$ need not in general be quadratic: although it is generated in degree 1, its relations may not be generated by homogeneous relations of degree 2. In this paper we give a criterion which is equivalent to $gr_I K$ being quadratic. We apply this criterion to the group algebra of the pure virtual braid group (also known as the quasi-triangular group), and show that the corresponding associated graded algebra is quadratic.

Contents

1 Introduction

This paper will ultimately be concerned with the pure virtual braid groups $\boldsymbol{PvB_n}$, for all $n \in \mathbb{N}$, generated by symbols R_{ij}, $1 \leq i \neq j \leq n$, with relations the Reidemeister III moves (or quantum Yang-Baxter relations) and certain commutativities:

$$R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij} \tag{1}$$
$$R_{ij}R_{kl} = R_{kl}R_{ij}, \tag{2}$$

with i, j, k, l distinct. This group is referred to as the quasi-triangular group $\boldsymbol{QTr_n}$ in [BarEnEtRa]. We will also be concerned with the related algebra $\mathfrak{pvb}_n$, generated by symbols r_{ij}, $1 \leq i \neq j \leq n$, with relations the '6-term' (or 6T) relations, and related commutativities:

$$y_{ijk} := [r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{ik}, r_{jk}] = 0, \tag{3}$$
$$c_{ij}^{kl} := [r_{ij}, r_{kl}] = 0$$

with i, j, k, l distinct. This algebra is the universal enveloping algebra of the quasi-triangular Lie algebra $\mathfrak{qtr}_n$ in [BarEnEtRa].

We will show that PvB_n is a 'quadratic group', in the sense that if its rational group ring $\mathbb{Q}PvB_n$ is filtered by powers of the augmentation ideal I, the associated graded ring $grPvB_n$ is a quadratic algebra: i.e., a graded algebra generated in degree 1, with relations generated by homogeneous relations of degree 2. We note that, in different language, this is the statement that PvB_n has a universal finite-type invariant, which takes values in the algebra $\mathfrak{pvb}_n$.

In [Hutchings], a criterion was given for the quadraticity of the pure braid group. The proof relied on the geometry of braids embedded in $\mathbf{R}^3$. In order to generalize this criterion to all finitely presented groups, we developed an algebraic proof of the criterion. This proof turned out not to rely on the existence of an underlying group, and applies instead to algebras over $\mathbb{Q}$, filtered by powers of an augmentation ideal I. Indeed, this criterion arguably lives naturally in an even broader context, such as perhaps augmented algebras over an operad (or the related 'circuit algebras' of [BN-WKO]), although we do not investigate this broader context here.

Our criterion may be summarized as follows in the case of an augmented algebra K with augmentation ideal I_K. We denote by $gr_I K = \oplus_{m \geq 0} I_K^m / I_K^{m+1}$ the associated graded algebra of K with respect to the filtration by powers of the augmentation ideal, and by $\mathfrak{B}$ the 'blow-up' algebra $\mathfrak{B} := \oplus_{m \geq 0} I_K^m$. We recall that any graded algebra X which is generated in degree 1 has a 'quadratic approximation', namely the graded algebra with the same generators and with ideal of relations generated by the degree 2 relations of X. Let A be the quadratic approximation of $gr_I K$, and $\mathfrak{A}$ be the quadratic approximation of $\mathfrak{B}$. We will see that the generators of $\mathfrak{A}$ surject onto to the generators of A, and that certain canonical spaces of free generators of the relations in $\mathfrak{A}$ and A are isomorphic. It thus makes sense to ask whether $\mathfrak{A}$ and A have the same syzygies - i.e., relations between relations. We show that $gr_I K$ is quadratic if and only if $\mathfrak{A}$ and A do in fact have the same syzygies. Furthermore, if A is Koszul, we show that it is sufficient to check this criterion in degree 3.

In Section 1 of this paper we give a precise statement of this criterion (see Theorem 1). In Section 2 we explain the key step in the proof of the criterion. In Section 3 we supply details of proofs that were omitted in Section 2. In Section 4 we specialize to PvB_n. We present a basis for the quadratic dual algebra $\mathfrak{pvb}_n^!$, and use this basis to compute the syzygies of $\mathfrak{pvb}_n$ and prove that PvB_n satisfies the quadraticity criterion. It follows that PvB_n is quadratic. Although Koszulness of the algebra $\mathfrak{pvb}_n$ was originally established in [BarEnEtRa], we give a different proof in Subsection 4.7. Finally, in Section 5 we point out some possible future avenues of research.

We note that the quadraticity of PvB_n was conjectured in [BarEnEtRa]. As pointed out in section 8.5 of that paper, the quadraticity of PvB_n implies that $H^*(PvB_n) \cong \mathfrak{pvb}_n^!$ as algebras.

After this paper was substantially completed, the result was communicated to Alexander Polishchuk, who pointed out that a theorem somewhat similar to Theorem 1 was obtained in [PosVish] in the context of the cohomology algebra of a nilpotent augmented coalgebra, albeit by very different methods. For this reason we have referred to the criterion in Theorem 1 as the PVH Criterion (with reference to Positselski, Vishik and Hutchings).

1.1 Overview of the PVH Criterion

1.1.1 Group Theoretic Background

Since the classic setting of the PVH criterion is that of group rings, we identify the attributes of group rings which we rely on and will want to see preserved in our generalized context. We recall the follow basic fact:

Proposition 1 (See [MKS], s. 5.15)**.** *If G if given by the short exact sequence*

$$1 \to N \to FG \to G \to 1$$

where FG is a free group generated by symbols $\{g_p : p \in P\}$ and N is a normal subgroup of FG generated by the set $\{r_q : q \in Q\}$, then the rational group ring of G is given by the exact sequence

$$0 \to (N-1) \to \mathbb{Q}FG \to \mathbb{Q}G \to 0$$

where $(N-1)$ is the two-sided ideal in $\mathbb{Q}FG$ generated by $\{(r_q - 1) : q \in Q\}$.

We can clearly restrict the second exact sequence to the exact sequence

$$0 \to (N-1) \to I_{FG} \to I_G \to 0 \tag{4}$$

where I_{FG} and I_G are the augmentation ideals of $\mathbb{Q}FG$ and $\mathbb{Q}G$ respectively.

1.1.2 Generalized Algebraic Setting

By analogy with the above group case, we take K to be an augmented (unital) algebra over $\mathbb{Q}$ with 2-sided augmentation ideal I_K, and F to be the free algebra over $\mathbb{Q}$ with the same generating set as K, with 2-sided augmentation ideal I_F. In particular we assume an exact sequence:

$$0 \longrightarrow I_K \longrightarrow K \xrightarrow{\epsilon} \mathbb{Q} \longrightarrow 0$$

By analogy with the ideal $(N-1)$ in the group context, we let $M \subseteq I_F \subseteq F$ be a 2-sided ideal such that:

$$0 \longrightarrow M \longrightarrow F \longrightarrow K \longrightarrow 0$$
$$0 \longrightarrow M \longrightarrow I_F \longrightarrow I_K \longrightarrow 0$$

are exact. It will be important in what follows that I_F is a free left and right F-module.[1]

We will in fact work with the completions $\hat{K}$ of K (and $\hat{F}$ of F) with respect to the filtrations by powers of their respective augmentation ideals. Our reason for doing this is that, by picking a suitable set of generators for K (and F) and passing to the completions, we claim that we may arrange that $M \subseteq I_F^2$ (see Subsection 3.1), and we will need this in the sequel. Since we always work with the completions, we will simply denote them K and F, without the hat.

[1] When F is the group ring kG (with k a commutative ring and G a free group), the fact that I_F is free as a left-, and as a right-, kG module is due to Fox (for a proof, see for instance [Lam], exercise 1.29). This result can readily be modified to address the case where F is a free algebra (the only distinction being that one no longer needs to deal with inverses of elements).

1.1.3 The Associated Graded Algebra and Its Quadratic Approximation

K is filtered by powers of I_K:

$$\ldots \overset{\iota}{\hookrightarrow} I_K^3 \overset{\iota}{\hookrightarrow} I_K^2 \overset{\iota}{\hookrightarrow} I_K \overset{\iota}{\hookrightarrow} I_K^0 = K \tag{5}$$

where ι are the inclusions.

We denote $\boldsymbol{gr_I K}$ the associated graded of the above filtration. We have $gr_I K \cong \bigoplus_m I_K^m/\iota I_K^{m+1}$. It is clear that $gr_I K$ is generated as an algebra by its degree one piece $\boldsymbol{V} := I_K/I_K^2$, a vector space over $\mathbb{Q}$.

We recall that a graded algebra $L = \oplus_{p\geq 0} L^p$, generated in degree 1 over a ring with identity L^0, may be 'approximated' by a quadratic algebra $q(L)$ with the same generators and with relations generated by the degree 2 relations of L. Specifically, we denote by $T_L L^1$ the tensor algebra on L^1 over the ring L^0: $T_L L^1 := \oplus_{p\geq 0}(L^1)^{\otimes_{L^0} p}$. Moreover, we denote $R_L := ker(L^1 \otimes_{L^0} L^1 \to L^2)$ the kernel of the multiplication map in degree $1+1$; and we denote $\langle R_L\rangle$ the ideal in $T_L L^1$ generated by R_L. Then $q(L) := \frac{T_L L^1}{\langle R_L\rangle}$. We refer to $q(L)$ as the 'quadratic approximation'[2] of L.

Applying this procedure to $gr_I K$, we get the quadratic approximation $A := q(gr_I K)$ of $gr_I K$. Thus, if TV is the rational tensor algebra over V (with tensor products over $\mathbb{Q}$), we let $\langle \boldsymbol{R}\rangle$ be the two-sided ideal in TV generated by the vector subspace $R \subseteq V \otimes_{\mathbb{Q}} V$ of degree two relations of $gr_I K$: i.e., $R := ker(\mu : I_K/I_K^2 \otimes_{\mathbb{Q}} I_K/I_K^2 \to I_K^2/I_K^3)$, where μ is the multiplication in $gr_I K$ induced from multiplication in I_K. Then we define $A := TV/\langle R\rangle$. We will denote by A^m the m-th graded piece of A.

We use the following notation for the (free generators of the) relations in A:

$$R_{m,j} := V^{\otimes_{\mathbb{Q}} j} \otimes_{\mathbb{Q}} R \otimes_{\mathbb{Q}} V^{\otimes_{\mathbb{Q}}(m-j-2)}$$

and:

$$R_m = \bigoplus_{j=1}^{m-2} R_{m,j}$$

We have obvious maps $\partial_A : R_m \to V^{\otimes_{\mathbb{Q}} m}$ which are (sums of the) summand-wise inclusions.

We note that since A has the same generators and the same quadratic relations as $gr_I K$, there is always a surjection $A \twoheadrightarrow gr_I K$. Quadraticity of $gr_I K$ is thus equivalent to the fact that this surjection is an isomorphism $A^m \cong I_K^m/I_K^{m+1}$, for all m. We will often use this alternative definition of quadraticity.

1.1.4 The Blow-Up Algebra and the Global Quadratic Approximation

There is a second graded algebra canonically associated to the pair (K, I_K), referred to as the blow-up algebra $\mathfrak{B}$. It is defined as $\mathfrak{B} := \oplus_{m\geq 0} I_K^m$, where

[2] This terminology is due to Dror Bar-Natan.

$I_K^0 := K$. Clearly, $\mathfrak{B}$ is generated as a K-algebra by its degree 1 piece, I_K.

We note that the blow-up algebra $\mathfrak{B}$ and the associated graded $gr_I K$ are related by the fact that $gr_I K = \mathfrak{B}/\iota\mathfrak{B} = \oplus_{p\geq 0}\mathfrak{B}^m/\iota\mathfrak{B}^{m+1}$.

Like $gr_I K$, $\mathfrak{B}$ has a canonically associated quadratic approximation, which we denote $\mathfrak{A}$, which has the same generators and whose ideal of relations is generated by the degree 2 relations of $\mathfrak{B}$. We often refer to $\mathfrak{A}$ as the global quadratic approximation.

More precisely, we start with the tensor algebra $T_K I_K := \oplus_{m\geq 0} I_K^{\otimes_K m}$, where $I_K^{\otimes 0} := K$. By analogy with R, we define $\mathfrak{R} := ker(\mu_K : I_K \otimes_K I_K \to I_K^2)$, and by analogy with $R_{m,j}$ and R_m, we define:

$$\mathfrak{R}_{m,j}^K := I_K^{\otimes_K j} \otimes_K \mathfrak{R} \otimes_K I_K^{\otimes_K (m-j-2)}$$

and

$$\mathfrak{R}_m^K := \bigoplus_j \mathfrak{R}_{m,j}^K \tag{6}$$

There is an obvious map $\partial_K : \mathfrak{R}_m \to I_K^{\otimes_K m}$ (induced by the inclusion $\mathfrak{R} \hookrightarrow I_K^{\otimes_K 2}$). Then we define $\mathfrak{A}^m := \frac{I_K^{\otimes_K m}}{\partial_K \mathfrak{R}_m}$, and $\mathfrak{A} := \frac{T_K I_K}{\langle \mathfrak{R} \rangle} = \oplus_{m\geq 0}\mathfrak{A}^m$.

We introduce a 'contraction' map $\mu_i : I_K^{\otimes_K m} \to I_K^{\otimes_K m-1}$ which is multiplication of components i and $i+1$ in the tensor product. Since we are tensoring over K, $\mu_i = \mu_j$ for all i, j, so we often refer to the contraction as simply μ_K.

By construction, $\partial_K(\mathfrak{R}_m) \subseteq ker\ \mu_K$. In fact, that inclusion is an equality:

Proposition 2. $\partial_K(\mathfrak{R}_m^K) = ker\ \mu_K$.

Proof. Deferred to Subsection 3.4. □

Thus the maps μ_K induce maps (which we will denote $\mu_{\mathfrak{A}}$): $\mu_{\mathfrak{A}} : \mathfrak{A}^p \to \mathfrak{A}^{p-1}$ for all $p \geq 2$, and we also agree to denote by $\mu_{\mathfrak{A}}$ the inclusion $I_K \hookrightarrow K$. We get a sequence:

$$\mathfrak{A}^p \xrightarrow{\mu_{\mathfrak{A}}} \mathfrak{A}^{p-1} \xrightarrow{\mu_{\mathfrak{A}}} \dots \xrightarrow{\mu_{\mathfrak{A}}} I_K \tag{7}$$

We denote the composition by $\mu_{\mathfrak{A}}^p$ (for $p \geq 2$), and agree that $\mu_{\mathfrak{A}}^1 = id_{I_K}$ and $\mu_{\mathfrak{A}}^0 = id_K$. Thus the $\mu_{\mathfrak{A}}^p$ for $p \geq 0$ give us a surjection $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$. As with $gr_I K$ one may ask whether $\mathfrak{B}$ is quadratic, i.e. whether the surjection $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$ is actually an isomorphism. We will see below that $gr_I K$ is quadratic if and only if $\mathfrak{B}$ is.

1.1.5 Relation Between the Two Quadratic Approximations

We may fit the concepts introduced above into the following picture:

$$
\begin{array}{ccc}
gr_I K & \xleftarrow{\oplus_p \bullet^p \,/\iota \bullet^{p+1}} & \mathfrak{B} \\
\Big\downarrow {\scriptstyle q(-)} & & \Big\downarrow {\scriptstyle q(-)} \\
gr_{\mathfrak{A}} K & \xleftarrow[\oplus_p \bullet^p \,/\mu_{\mathfrak{A}} \bullet^{p+1}]{} & \mathfrak{A}
\end{array}
$$

where we define $gr_{\mathfrak{A}} K := \oplus_{p \geq 0} \mathfrak{A}^p / \mu_{\mathfrak{A}} \mathfrak{A}^{p+1}$. The following Proposition says that this is a commutative diagram:

Proposition 3.

$$gr_{\mathfrak{A}} K = q(gr_I K) = A$$

as graded vector spaces.

The proof of this proposition will follow from Lemma 2 below.

Also, we can fit the sequences (5) and (7) into a commutative diagram:

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathfrak{A}^3 & \xrightarrow{\mu_{\mathfrak{A}}} & \mathfrak{A}^2 & \xrightarrow{\mu_{\mathfrak{A}}} & \mathfrak{A}^1 & \xrightarrow{\mu_{\mathfrak{A}}} & \mathfrak{A}^0 \\
& \Big\downarrow {\scriptstyle \mu^3_{\mathfrak{A}}} & & \Big\downarrow {\scriptstyle \mu^2_{\mathfrak{A}}} & & \Big\| {\scriptstyle \mu^1_{\mathfrak{A}}} & & \Big\| {\scriptstyle \mu^0_{\mathfrak{A}}} \\
\cdots \longrightarrow & I_K^3 & \overset{\iota}{\twoheadrightarrow} & I_K^2 & \overset{\iota}{\twoheadrightarrow} & I_K & \overset{\iota}{\twoheadrightarrow} & K
\end{array}
$$

such that the 'associated gradeds' of each row are $gr_{\mathfrak{A}} = q(gr_I K) = A$ (for the top row) and $gr_I K$ (for the bottom row). In these circumstances we refer to to the top row as a 'proto-filtration' of K.

In Theorem 1 we will see that, if the PVH Criterion is met, then the proto-filtration (7) coincides with the filtration (5), i.e. the vertical maps are isomorphisms. Hence their associated graded algebras coincide, and hence A is quadratic.

1.1.6 The PVH Criterion

The generators of A^p and $\mathfrak{A}^p$ are related by an obvious projection $F^0 : I_K^{\otimes_K m} \twoheadrightarrow (I_K/I_K^2)^{\otimes_{\mathbb{Q}} m} = V^{\otimes_{\mathbb{Q}} m}$, with kernel given by $ker\ F^0 = I_K.I_K^{\otimes_K m}$, as per the following proposition:

Proposition 4.

$$I_K^{\otimes_K m} / I_K.I_K^{\otimes_K m} \cong V^{\otimes_{\mathbb{Q}} m}$$

Here $I_K.I_K^{\otimes_K m}$ means $I_K^{\otimes_K m}$ with the left-most component multiplied (on the left) by I_K.

Proof. Postponed to Subsection 3.3. □

Hence we then have the following commutative diagram, with exact diagonals:

$$\begin{array}{ccccc} ker\ \partial^{Ind} & & \xleftarrow{\ F^{Syz}\ } & & ker\ \partial_K \\ & \searrow & & \swarrow & \\ & & \mathfrak{R}_m^K & & \\ & {\scriptstyle \partial_K} \swarrow & & \searrow {\scriptstyle \partial^{Ind}} & \\ I^{\otimes_K m} & & \xrightarrow{\ F^0\ } & & (I_K/I_K^2)^{\otimes_{\mathbb{Q}} m} \end{array}$$

where ∂^{Ind} is the composition and F^{Syz} is the map induced from F^0 on kernels.

We are now in a position to state our criterion for quadraticity of K (with notation as in the above diagram, and with the assumptions in Subsection 1.1.2):

Proposition 5 (PVH Criterion I)**.** *K is quadratic if and only if F^{Syz} is surjective (and hence an isomorphism) for all $m \geq 2$.*

This generalizes a result first obtained in [Hutchings], where K was the group ring of the pure braid group (see also [BNStoi]). We give the proof in Section 2.

We can get a more precise and readily verifiable criterion for $m \geq 3$ if we first verify the criterion in degree 2. To set this up, we state the:

Proposition 6. *Let $\{y_q : q \in Q\}$ be a minimal set of generators for M as a two-sided F-module. Suppose the $\{y_q + I_F^3 : q \in Q\}$ are linearly independent in $(M + I_F^3)/I_F^3$. Then, for all $m \geq 2$, we have an isomorphism:*

$$F^1 : \mathfrak{R}_m \xrightarrow{\sim} R_m$$

as vector spaces over $\mathbb{Q}$. Moreover, $\partial^{Ind} = \partial_A \circ F^1$, and hence $ker\ \partial^{Ind} \cong ker\ \partial_A$ consists of the syzygies of the quadratic algebra A.

Remark 1. *The linear independence of the $\{y_q + I_F^3 : q \in Q\}$ can be viewed as the PVH Criterion in degree 2. Indeed, $\partial_A : R \hookrightarrow V^{\otimes_{\mathbb{Q}} 2}$ is the inclusion, so $ker\ \partial^{Ind} \cong ker\ \partial_A = 0$ and F^{Syz} is automatically surjective.*

Theorem 1 (PVH Criterion II)**.** *Under the assumptions in Proposition 6, $gr_I K$ is quadratic if and only if $ker\ \partial_K \cong ker\ \partial_A$ is an isomorphism for all $m \geq 3$, i.e. informally iff 'the quadratic algebras A and $\mathfrak{A}$ have the same syzygies'.*

If A is Koszul,[3] [4] then we need only check that this isomorphism holds in

[3] The statement about Koszulness relies on results about Koszul algebras which have only been developed for graded algebras whose graded components are finitely generated over the ground ring. Hence, for purposes of this part of the theorem, we assume the algebra K to be finitely generated, which is sufficient to ensure that A^m is a finite dimensional $\mathbb{Q}$-vector space for all m.

[4] In fact, A need only be 2-Koszul, i.e. its Koszul complex need only be exact up to homological degree 2 inclusive.

degree 3.

The proof of Proposition 6 and Theorem 1 is deferred to Subsection 3.5.

A closely related theorem appeared in [PosVish], in the context of the cohomology algebra of a nilpotent augmented coalgebra (see the 'Main Theorem' of that paper).

1.2 How the PVH Criterion is Useful

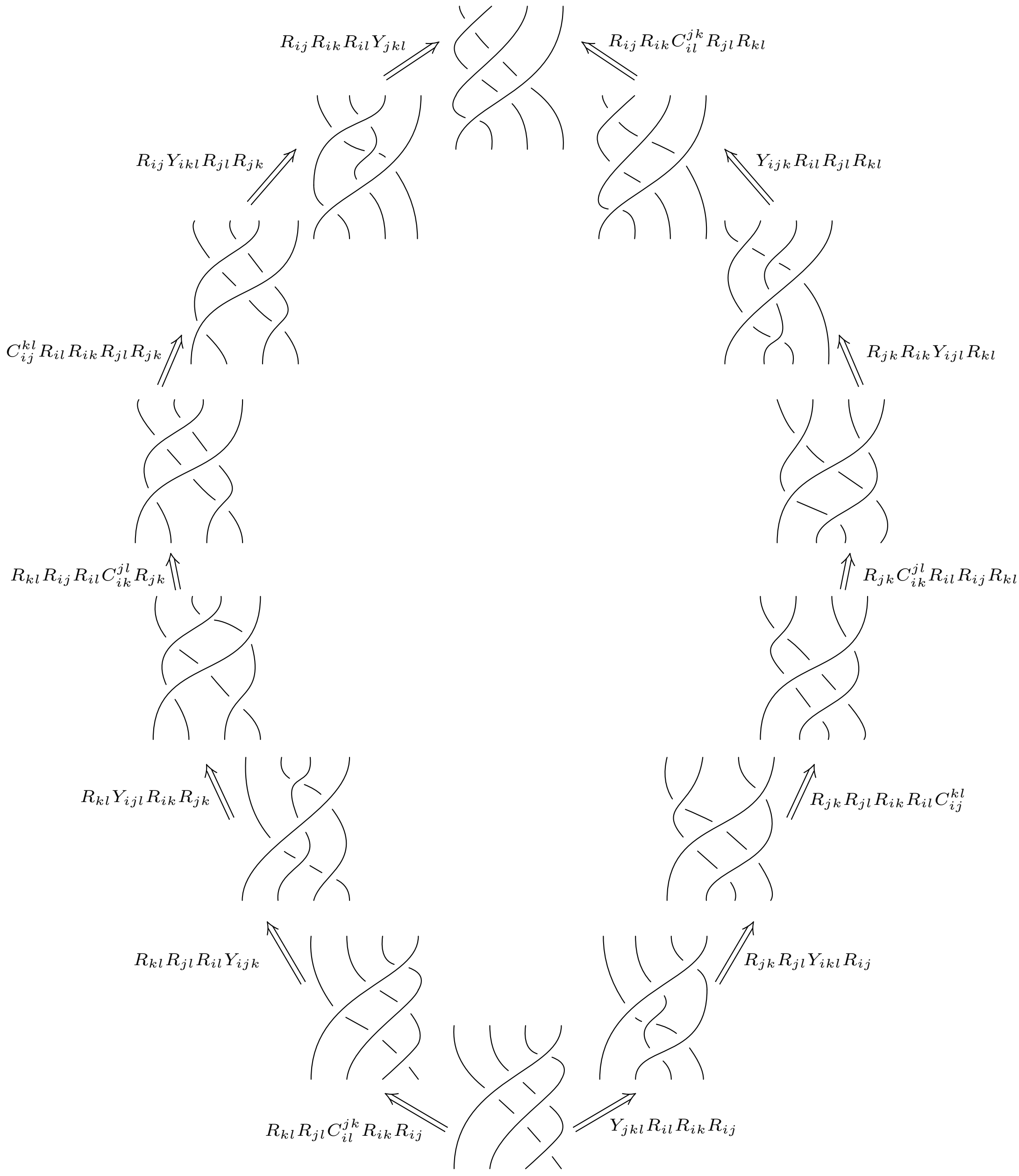
Assuming the requirements of Theorem 1 are met, we can conclude that $gr_I K$ is quadratic if, informally, the quadratic algebra A and its global analogue $\mathfrak{A}$ have the same syzygies.

It is often the case that the syzygies of a quadratic algebra can be determined quite explicitly, using quadratic duality. Essentially, if the quadratic algebra A is Koszul, then the syzygies are generated by $A^{!3}$ (i.e. the degree 3 part of the quadratic dual $A^!$ of A). Thus the problem of comparing syzygies is reduced to the finite, computable problem of determining a basis for $A^{!3}$ and checking whether the resulting syzygies of A^3 also hold in $\mathfrak{A}^3$.

In the context of PvB_n, it was shown in [BarEnEtRa] that $\mathfrak{pvb}_n$ is Koszul (a different proof is provided in Section 4.7 of this paper), so we only need to check the PVH Criterion in degree 3.

If we take K to be the group ring of PvB_n and I_K its augmentation ideal, it is possible to interpret the spaces $\mathfrak{A}^m$ as spaces of 'm-singular virtual braids' – essentially virtual braids with m 'semi-virtual' double points (subject to a certain equivalence relation) - see [GPV]. One knows certain syzygies that are satisfied by such semi-virtual braids, particularly the syzygy known as the Zamolodchikov tetrahedron:[5]

[5] The picture builds on an xy-pic template due to Aaron Lauda – see [Lau]. Another picture is at [BN2].



(The notation will be clarified in Subsection 4.2.)

In the second part of this paper, we will find a basis for the quadratic dual algebra $\mathfrak{pvb}^!_n$, and in particular for $\mathfrak{pvb}^{!3}_n$. We will then check 'by hand' that the corresponding degree 3 syzygies of A are also satisfied by $\mathfrak{A}$. There are a number of ‘trivial’ syzygies in A which are trivially seen to hold in $\mathfrak{A}$ as well. There is also a family of non-trivial syzygies in A, which turn out to correspond to the ‘Zamolodchikov’ syzygy alluded to above (this is explained in Subsection 4.4.1). This will allow us to conclude that $gr_I \mathbb{Q}PvB_n \cong \mathfrak{pvb}_n$.

1.3 Acknowledgements

It is my great pleasure to thank my Ph.D. supervisor Dror Bar-Natan for the endless patience and encouragement he has shown me as I carried out this research. Although he has certainly contributed directly to the clarification and simplification of many points, more importantly his ongoing critical feedback and support were essential to completing the project.

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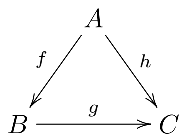
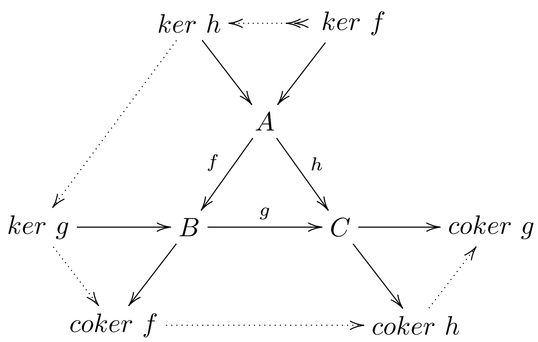
2 Key to the PVH Criterion

We recall a simple but useful fact from homological algebra. Given modules A, B, and C over a ring, suppose with have a commutative triangle:



If we add kernels and cokernels wherever possible, and induced maps, we get the following commutative diagram:



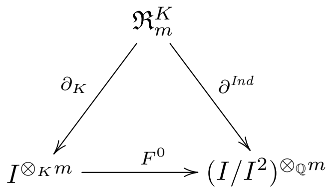
Then we get the following ‘Hexagon Lemma’, which is a special case of the Snake Lemma:

Lemma 1. *With notation as above, the following sequence is exact:*

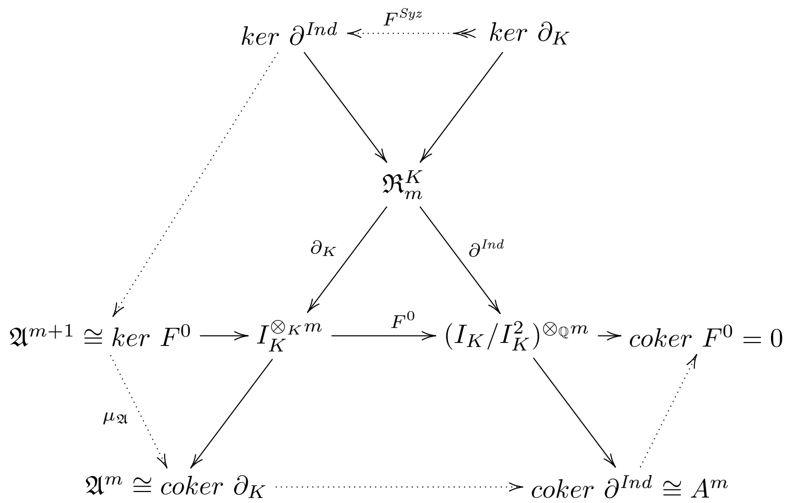
$$0 \to ker\ f \to ker\ h \to ker\ g \to coker\ f \to coker\ h \to coker\ g \to 0 \quad (8)$$

Proof. Easy diagram chase. □

We apply the Hexagon Lemma to the commutative triangle:



(see Subsection 1.1.6). Adding kernels, cokernels and induced maps, we get the diagram:



Lemma 2. *The following sequence is exact:*

$$0 \to ker\ \partial_K \xrightarrow{F^{Syz}} ker\ \partial^{Ind} \to \mathfrak{A}^{m+1} \xrightarrow{\mu_{\mathfrak{A}}} \mathfrak{A}^m \longrightarrow A^m \longrightarrow 0 \quad (9)$$

Proof. The only point that does not follow from the Hexagon Lemma is that *coker* $\partial^{Ind} \cong A^m$. Although this is true in general, we omit the proof as it follows easily in the situations of greatest interest to us, where the requirements of Proposition 6 are met. In such cases, we have $F^1 : \mathfrak{R}_m \xrightarrow{\sim} R_m$, $\partial^{Ind} = \partial_A \circ F^1$ and *coker* $\partial_A \cong \frac{V^{\otimes_{\mathbb{Q}} m}}{\partial_A(R_m)} \cong A^m$, as desired. □

We note that Proposition 3 follows immediately from the last 4 terms of the above exact sequence.

Lemma 3. *For every $m \in \mathbb{N}$, F^{Syz} is surjective if and only if $\mu_{\mathfrak{A}} : \mathfrak{A}^{m+1} \to \mathfrak{A}^m$ is injective.*

Proof. Clear from the long exact sequence. □

It is clear that the $\mu_{\mathfrak{A}} : \mathfrak{A}^{m+1} \to \mathfrak{A}^m$ are injective for all m if and only if the compositions $\mathfrak{A}^{m+1} \to \mathfrak{A}^m \to \cdots \to I_K^m$ are also injective for all m. Since the compositions are always surjective, injectivity is equivalent to isomorphism. This proves Proposition 5.

3 Postponed Proofs

3.1 Eliminating Linear Relations

We mentioned in Subsection 1.1.2 that we work with the completions $\hat{K}$ of K (and $\hat{F}$ of F) because, by picking a suitable set of generators for K (and F) and passing to the completions, we may arrange that $M \subseteq I_F^2$ (and we will use this in the sequel).

To prove this claim, we let $\{x_p : p \in P\}$ be a set of generators for the algebra K, so that $\{\bar{x}_p := (x_p - 1) : p \in P\}$ is a set of generators for I_K as a left- or right-sided ideal in K. The images of the $\{\bar{x}_p : p \in P\}$ in the vector space (I_K/I_K^2) generate that space, so the images of some subset $\{\bar{x}_p : p \in S \subseteq P\}$ form a basis. Thus the $\{\bar{x}_p : p \in P-S\}$ may be expressed as linear combinations of the $\{\bar{x}_p : p \in S\}$ modulo elements of I_K^2. More generally, we may replace any polynomial involving the $\{\bar{x}_p : p \in P - S\}$ by a polynomial involving only $\{\bar{x}_p : p \in S\}$, modulo elements in higher powers of I_K. It therefore follows that the $\{\bar{x}_p : p \in S\}$ generate the completion, and we may drop the $\{\bar{x}_p : p \in P-S\}$ from our list of generators.

We note in particular that in the case where K is the group algebra of some group G, the generators of K as an algebra would normally include, not only the group generators, but also their inverses. Moreover, the relations ideal M would include relations derived from the group laws for the generators. Thus if a is a generator of the group, and b its inverse, we have the group law $ab = 1$ which gives, under the substitution $a \mapsto \bar{a}+1$, $b \mapsto \bar{b}+1$, where $\bar{a} := (a-1)$ and $\bar{b} := (b-1)$, the relation $\bar{a} + \bar{b} + \bar{a}\bar{b} = 0$, which is not in I_F^2. However using the relation $\bar{b} = -\bar{a} - \bar{a}\bar{b}$ we can replace all occurrences of $\bar{b}$ by $-\bar{a}$, provided we are working in the completion of K. So in the case of group algebras we will take as generators only the group generators and we omit the group law relations from M.

Coming back to the case of a general K, we can also see that $M \subseteq I_F^2$. Indeed, (I_F/I_F^2) and (I_K/I_K^2) are now vector spaces with bases having the same number of elements, and hence are isomorphic. However it is also clear that:

$$
\begin{aligned}
(I_K/I_K^2) &= \frac{I_F/M}{I_F^2/(M\cap I_F^2)} \\
&= \frac{I_F/M}{(I_F^2+M)/M} \\
&= \frac{I_F}{I_F^2+M} \\
&= \frac{(I_F/I_F^2)}{(I_F^2+M)/I_F^2} \\
&= \frac{(I_F/I_F^2)}{M/(M\cap I_F^2)}
\end{aligned}
$$

so we must have $M/(M\cap I_F^2)=0$, i.e. $M\subseteq I_F^2$.

3.2 Some Algebraic Preliminaries

In this subsection we record two algebraic lemmas which are used in the sequel. The reader may initially want only to skim the statement of the lemmas and then pass on to the next subsection.

Lemma 4. *Let C be an algebra and, for $i=1,2$, let $A_i\subseteq B_i$ be 2-sided C-modules. Then the obvious map $B_1\otimes_C B_2 \longrightarrow \frac{B_1}{A_1}\otimes_C\frac{B_2}{A_2}$ induces an isomorphism:*

$$\frac{B_1\otimes_C B_2}{A_1\otimes_C B_2+B_1\otimes_C A_2}\cong\frac{B_1}{A_1}\otimes_C\frac{B_2}{A_2}$$

with an obvious extension to the case $i=1\ldots n$, $n\in\mathbb{N}$.

Proof. Straightforward. □

Lemma 5. *Let C be an augmented algebra over $\mathbb{Q}$ with augmentation ideal J, and $H\subseteq L$ be 2-sided C-modules such that:*

$$JL\subseteq H\supseteq LJ$$

Then L/H is a trivial 2-sided C-module.

Corollary 1. *I_K^m/I_K^{m+1} is a trivial 2-sided A-module.*

Proof. Follows from Lemma 5, with C replaced by A, J by I_K, L by I_K^m and H by I_K^{m+1}. □

Proof of Lemma 5. The left C-action $C\otimes_{\mathbb{Q}}(L/H)\longrightarrow(L/H)$ is 0 on $J\otimes_{\mathbb{Q}}(L/H)$, hence factors through $\frac{C\otimes_{\mathbb{Q}}(L/H)}{J\otimes_{\mathbb{Q}}(L/H)}$, and:

$$\frac{C\otimes_{\mathbb{Q}}(L/H)}{J\otimes_{\mathbb{Q}}(L/H)}\cong(C/J)\otimes_{\mathbb{Q}}(L/H)\cong\mathbb{Q}\otimes_{\mathbb{Q}}(L/H)\cong(L/H)$$

The same is true for the right action, and the result follows.

□

3.3 Proof of Proposition 4

Proof of Proposition 4. We have:

$$\frac{I_K^{\otimes_K m}}{I_K.I_K^{\otimes_K m}} \cong \frac{I_K^{\otimes_K m}}{\sum I_K^{\otimes_K p} \otimes_K I_K^2 \otimes_K I_K^{\otimes_K m-p-1}} \cong (I_K/I_K^2)^{\otimes_K m} \tag{10}$$

by Lemma 4.

It is clear that we can replace the $\otimes_K$ in (10) by $\otimes_{\mathbb{Q}}$ since, by Corollary 1, (I_K/I_K^2) is a trivial 2-sided K-module. $\square$

3.4 Proof of Proposition 2

Before moving on to the proof of Proposition 2, we introduce some notation. Recall that μ_K denotes the product $I_K^{\otimes_K m+1} \longrightarrow I_K.I_K^{\otimes_K m}$. By analogy, we will denote by μ_{FF} the product

$$\mu_{FF} : I_F^{\otimes_F m+1} \longrightarrow I_F.I_F^{\otimes_F m} \tag{11}$$

Proof of Proposition 2. [6]

We note that μ_{FF} is a homomorphism of 2-sided F-modules (where F acts by left-multiplication in the left component of $I_F \otimes_F I_F$, and by right-multiplication in the right component). It is clear that μ_{FF} is surjective; moreover, since I_F is a free F-module, the functor $I_F \otimes_F -$ preserves the inclusion $I_F \hookrightarrow F$, and so μ_{FF} is also injective, hence an isomorphism.

Since $M \subseteq I_F^2$, we can define:

$$\mathfrak{R}^F := \mu_{FF}^{-1}(M) \subseteq I_F \otimes_F I_F$$

and get $M = \mu_{FF}(\mathfrak{R}^F)$.

Let us define:

$$\mathfrak{R}^F_{m,p} := I_F^{\otimes_F p} \otimes_F \mathfrak{R}^F \otimes_F I_F^{\otimes_F m-p-2}$$

and

$$\mathfrak{R}^F_m := \sum_p \mathfrak{R}^F_{m,p} \subseteq I_F^{\otimes_F m+1}$$

Similarly, let us define:

$$M_{m,p} := I_F^{\otimes_F p} \otimes_F M \otimes_F I_F^{\otimes_F m-p-1}$$

and

$$M_m := \sum_p M_{m,p} \subseteq I_F^{\otimes_F m}$$

[6] With thanks to Dror Bar-Natan for considerably simplifying the original proof.

Then it is again clear that, since I_F is a free F-module, the isomorphism $M \overset{\mu_{FF}}{\cong} \mathfrak{R}^F$ extends to isomorphisms (for all m, p):

$$M_{m-1,p} \cong \mathfrak{R}^F_{m,p}$$

and hence

$$\sum_p M_{m-1,p} \cong \sum_p \mathfrak{R}^F_{m,p} \tag{12}$$

Now consider the following commutative diagram:

$$\begin{array}{ccc}
\sum_p \mathfrak{R}^F_{m,p} & \xrightarrow[\sim]{\mu^{res}_{FF}} & \sum_p M_{m-1,p} \\
\downarrow & & \downarrow \\
I_F^{\otimes_F m} & \xrightarrow[\sim]{\mu_{FF}} & I_F.I_F^{\otimes_F m-1} \\
\downarrow{\scriptstyle \pi_m} & & \downarrow{\scriptstyle \pi^{res}_{m-1}} \\
I_K^{\otimes_K m} & \xrightarrow{\mu_K} & I_K.I_K^{\otimes_K m-1}
\end{array}$$

where $\pi_m : I_F^{\otimes_F m} \twoheadrightarrow \frac{I_F^{\otimes_F m}}{M_m} = I_K^{\otimes_K m}$ (by Lemma 4), and the superscript $\bullet^{res}$ denotes the restriction of the relevant map. Note with regard to the top right corner that $ker\ \pi^{res}_{m-1} = M_{m-1} \cap I_F.I_F^{\otimes_F m-1} = M_{m-1}$ since $M \subseteq I_F^2$. Thus the right column is exact by definition. Then we have:

$$\begin{aligned}
ker\ \mu_K = \pi_m({\mu_{FF}}^{-1}(ker\ \pi^{res}_{m-1})) &= \pi_m(\sum_p {\mu_{FF}}^{-1} M_{m-1,p}) \\
= \pi_m(\sum_p \mathfrak{R}^F_{m,p}) &= \partial_K \mathfrak{R}_m
\end{aligned}$$

where interchanging the $\sum$ and μ_{FF}^{-1} in the second equality is justified by (12). Thus we conclude that $\partial_K(\mathfrak{R}_m) = ker\ \mu_K$ for all m as required.

□

3.5 Proof of Proposition 6 and Theorem 1

We first need the following lemma.

Lemma 6. *We have $R \cong M/(I_F^3 \cap M) \cong (M + I_F^3)/I_F^3$. Moreover, $(M + I_F^3)/I_F^3$ is a trivial 2-sided F-module (with F acting by left multiplication on the left component, and by right multiplication on the right component).*

Proof. To see this, note that:

$$\frac{I_K^2}{I_K^3} \cong \frac{I_F^2/M}{I_F^3/(M \cap I_F^3)} \cong \frac{I_F^2/M}{(I_F^3 + M)/M} \cong \frac{I_F^2}{I_F^3 + M} \cong \frac{I_F^2/I_F^3}{(I_F^3 + M)/I_F^3}$$

The lemma will follow once we see, with regard to the last numerator, that $I_F^2/I_F^3 \cong (I_K/I_K^2) \otimes_{\mathbb{Q}} (I_K/I_K^2)$, as follows:

$$\frac{I_F^2}{I_F^3} \overset{\mu_{FF}}{\underset{\sim}{\longleftarrow}} \frac{I_F \otimes_F I_F}{I_F.I_F^{\otimes_F 2}} \cong \frac{I_F^{\otimes_F 2}/M_2}{I_F.I_F^{\otimes_F 2}/M_2} \cong \frac{I_K \otimes_K I_K}{I_K.I_K^{\otimes_K 2}} \cong (I_K/I_K^2)^{\otimes_{\mathbb{Q}} 2}$$

where for third equivalence we used Lemma 4 and for the last equivalence we used Proposition 4.

The triviality of the F-action is clear from the fact that $(I_F^3 + M)/I_F^3 \subseteq I_F^2/I_F^3$, and the fact that the latter is a trivial F-module, which easily follows from Lemma 5.

□

Proof of Proposition 6. Recalling that M is generated as a 2-sided F-module by $\{y_q : q \in Q\}$ (see the statement of Proposition 6), let us write $\mathcal{Y}_F := \{Y_q := {\mu_{FF}}^{-1}(y_q) : q \in Q\}$. Then $\mathcal{Y}_F$ generates both $\mathfrak{R}^F/M_2$ and $(M + I_F^3)/I_F^3$ as F-modules; but since both $\mathfrak{R}^F/M_2$ and $(M+I_F^3)/I_F^3$ are trivial 2-sided F-modules, $\mathcal{Y}_F$ in fact generates both as 2-sided $\mathbb{Q}$-modules. Therefore we have surjections

$$\mathfrak{R} \cong \mathfrak{R}^F/M_2 \overset{\pi_K}{\twoheadleftarrow} \mathbb{Q}\mathcal{Y}_F \overset{\pi_R}{\twoheadrightarrow} (M + I_F^3)/I_F^3 \cong R$$

Under our assumption that the $\{y_q + I_F^3 : q \in Q\}$ are linearly independent, π_R is clearly injective, hence an isomorphism.

Similarly, note that because μ_{FF} is an isomorphism, and $y_q + I_F^3 = \mu_{FF}(Y_q) + \mu_{FF}(I_F.I_F^{\otimes_F 2}) = \mu_{FF}(Y_q + I_F.I_F^{\otimes_F 2})$, the linear independence of the $\{y_q + I_F^3 : q \in Q\}$ is equivalent to the linear independence of $\{Y_q + I_F.I_F^{\otimes_F 2} : q \in Q\}$. Now since $M_2 \subseteq I_F.I_F^{\otimes_F 2}$, it follows a fortiori that the $\{Y_q + M_2 : q \in Q\}$ are linearly independent. Hence π_K is also injective, hence an isomorphism. Thus it follows that $F^1 := \pi_R \circ \pi_K^{-1}$ gives an isomorphism $\mathfrak{R} \xrightarrow{\sim} R$ as $\mathbb{Q}$-modules, as required.

We now show that $\mathfrak{R}_m \cong R_m$ as $\mathbb{Q}$-modules. This largely reduces to the following lemma:

Lemma 7. *If L is a trivial 2-sided K-module, then $I_K^{\otimes_K p} \otimes_K L \otimes_K I_K^{\otimes_K q} \cong (I_K/I_K^2)^{\otimes_{\mathbb{Q}} p} \otimes_{\mathbb{Q}} L \otimes_{\mathbb{Q}} (I_K/I_K^2)^{\otimes_{\mathbb{Q}} q}$.*

Proof. Suppose $i, i' \in I_K$ and $l \in L$. Then $ii' \otimes_K l = i \otimes_K i'l = 0$. Thus $I_K \otimes_K L \cong (I_K/I_K^2) \otimes_K L$, and both components are trivial K-modules. Therefore $I_K \otimes_K L \cong (I_K/I_K^2) \otimes_{\mathbb{Q}} L$, and this extends readily to the desired result. □

Now with $L = \mathfrak{R}$, we get:

$$\begin{aligned}\mathfrak{R}_{m,p} \cong I_K^{\otimes_K p} \otimes_K \mathfrak{R} \otimes_K I_K^{\otimes_K q} &\cong (I_K/I_K^2)^{\otimes_{\mathbb{Q}} p} \otimes_{\mathbb{Q}} \mathfrak{R} \otimes_{\mathbb{Q}} (I_K/I_K^2)^{\otimes_{\mathbb{Q}} q} \\ &\cong (I_K/I_K^2)^{\otimes_{\mathbb{Q}} p} \otimes_{\mathbb{Q}} R \otimes_{\mathbb{Q}} (I_K/I_K^2)^{\otimes_{\mathbb{Q}} q} \cong R_{m,p}\end{aligned}$$

hence

$$\mathfrak{R}_m = \bigoplus \mathfrak{R}_{m,p} \cong \bigoplus R_{m,p} \cong R_m$$

as $\mathbb{Q}$-modules.

Lastly, we check that $\partial^{Ind} = F \circ \partial_K = \partial_A \circ F^1$. More precisely, we need to check that $F \circ \partial_K = \partial_A \circ \pi_R \circ \pi_K^{-1}$, or equivalently that $F \circ \partial_K \circ \pi_K = \partial_A \circ \pi_R$. So let $Y_q \in \mathcal{Y}_F$, then:

$$\begin{aligned} F \circ \partial_K \circ \pi_K(Y_q) &= F \circ \partial_K((Y_q + M_2)/M_2) = F((Y_q + M_2)/M_2) \\ &= \frac{(Y_q + M_2)/M_2 + I_F.I_F^{\otimes_F 2}/M_2}{I_F.I_F^{\otimes_F 2}/M_2} = \frac{(Y_q + I_F.I_F^{\otimes_F 2})/M_2}{I_F.I_F^{\otimes_F 2}/M_2} \\ &= \frac{Y_q + I_F.I_F^{\otimes_F 2}}{I_F.I_F^{\otimes_F 2}} = \partial_A \circ \pi_R(Y_q) \end{aligned}$$

This extends immediately to the summands $\mathfrak{R}_{m,p}$ and $R_{m,p}$ of $\mathfrak{R}_m$ and R_m respectively, as needed. □

Proof of Theorem 1. The first claim in Theorem 1 follows from Proposition 5 and Remark 1. We deal with the restriction to degree 3 for the Koszul case in the next subsection. □

3.6 Some Reminders About Quadratic Duality

3.6.1 Basics

In this subsection we briefly review the theory of quadratic algebras to the extent needed to prove the final claim in Theorem 1, and to cover material that will be needed later (but skipping proofs). The reader who is not familiar with this theory can find a quick overview in [Froberg2] or [Hille], or more extensive treatment in [Pol] and [Kraehmer]; the original source is [Priddy].[7]

We start with the quadratic algebra A which is given by $A = TV/\langle R\rangle$ (in the notation of Subsection 1.1.3). The quadratic dual algebra $A^!$ is defined as $A^! := TV^*/\langle R^\perp\rangle$, where V^* is the linear dual vector space and $R^\perp \subseteq V^* \otimes V^*$ is the annihilator of R.

One indication of the usefulness of the concept of quadratic duality is that the degree 2 part of the dual algebra catalogues the relations of the original algebra (this is true for all quadratic algebras). More generally, the Koszul complex provides a readily computable 'candidate' resolution for A, which is an actual resolution precisely when A is Koszul. In particular the degree 3 part of the dual provides at least a candidate catalogue of the relations among the relations of the original algebra (i.e. syzygies) - and more generally, the degree m part of the dual provides a candidate catalogue of the relations among relations among ...

[7] As noted in footnote 3, we rely on results about Koszul algebras which have only been developed for graded algebras whose graded components are finitely generated over the ground ring. Hence, wherever we rely on Koszulness of A, we assume the algebra K to be finitely generated. This is sufficient to ensure that A^m is finitely generated over $\mathbb{Q}$.

$((m-1)$ times) of the original algebra (which we will call the level m syzygies). Moreover, there are specific maps from the degree m part of the dual into the space of level m syzygies. The statement that a quadratic algebra is Koszul is equivalent to the statement that the dual algebra not only provides a candidate catalogue of the syzygies of all levels, but an actual, complete catalogue of those syzygies. For purposes of this paper, it is only the level 3 syzygies that are important.

More specifically, if we define $\Delta^!_{1,1} : A^{!2*} \to V \otimes V$ as the dual to multiplication $V^* \otimes V^* \to A^{!2}$, then in fact $\Delta^!_{1,1}$ is an isomorphism:

$$\Delta^!_{1,1} : \; A^{!2*} \xrightarrow{\sim} \; R \tag{13}$$

Thus $A^{!2}$ catalogues the degree 2 relations of A and the map $\Delta^!_{1,1}$ sends a basis of $A^{!2}$ to a basis of R (see (22) and (23) below, in the case of $\mathfrak{pvb}^!_n$).

In the same vein, $A^{!3}$ catalogues all relations between relations of A, in degree three[8] - in other words, $A^{!3} \cong (R \otimes V \cap V \otimes R)$ (see [Pol], proof of Theorem 4.4.1). More specifically, if $\Delta^!_{2,1}$ is dual to the multiplication: $A^{!2} \otimes V^* \twoheadrightarrow A^{!3}$, then the map

$$(\Delta^!_{1,1} \otimes 1) \circ \Delta^!_{2,1} : \quad A^{!3*} \hookrightarrow \; R \otimes V \subseteq X^3_1 \tag{14}$$

is actually an isomorphism $A^{!3*} \xrightarrow{\sim} (R \otimes V \cap V \otimes R)$ which maps a basis for $A^{!3}$ to a basis for the degree 3 syzygies (viewed as a subspace of X^3_1).

Similarly, if $\Delta^!_{1,2}$ is dual to the multiplication: $V^* \otimes A^{!2} \to A^{!3}$, the map:

$$(1 \otimes \Delta^!_{1,1}) \circ \Delta^!_{1,2} : \quad A^{!3*} \hookrightarrow \; R \otimes V \subseteq X^3_2 \tag{15}$$

is an isomorphism $A^{!3*} \xrightarrow{\sim} (R \otimes V \cap V \otimes R)$, and maps a basis for $A^{!3}$ to a basis for the degree 3 syzygies (viewed as a subspace of X^3_2).

A priori, $A^{!3}$ need not generate the (level 3) syzygies of A in degrees higher than 3. However, if A is Koszul then indeed $A^{!3}$ *does* generate the (level 3) syzygies of A in all degrees, as we will explain further in the next subsection.

3.6.2 The Role of Koszulness

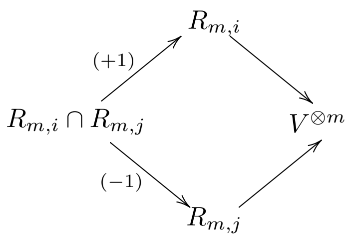
We will make use of the following theorem, which follows from [Pol], Theorem 2.4.1 (p.29) and Proposition 1.7.2 (p.16), to which the reader is referred for proofs.

[8]Note that since R is a vector space over $\mathbb{Q}$, there are no relations within R, so level 3 syzygies must have at least degree 3 in the generators of A. Given a level 3, degree 3 syzygy, we can also get level 3 syzygies of higher degree by pre- or post-multiplying all terms in the syzygy by monomials in the generators, although level 3 syzygies of higher degree need not all arise in this way (except when the algebra is Koszul).

Theorem 2. *Koszulness*[9] *of the algebra A implies exactness of the sequence:*

$$\bigoplus_{i<j}(R_{m,i}\cap R_{m,j}) \xrightarrow{\partial_{Syz}} \bigoplus_{i} R_{m,i} \xrightarrow{\kappa} V^{\otimes m} \tag{16}$$

In (16), the direct sums are external, and the maps are induced from the following diagram:



where the left diagonals are multiplication by the indicated factors, and the right diagonals are the inclusions.

Note that we can decompose $\bigoplus_{i<j}(R_{m,i}\cap R_{m,j})$ as follows:

$$\bigoplus_{i<j}(R_{m,i}\bigcap R_{m,j}) = \bigoplus_{i}(R_{m,i}\cap R_{m,i+1}) \oplus \bigoplus_{i+1<j}(R_{m,i}\cap R_{m,j})$$

The syzygies $\bigoplus_{i+1<j}(R_{m,i}\cap R_{m,j})$ are 'trivial' in the sense that they arise from the obvious fact that non-overlapping relations commute. This fact remains true at the global level, so that these 'trivial' syzygies also trivially satisfy the PVH Criterion.

The more interesting syzygies are the $(R_{m,i}\cap R_{m,i+1})$. From the review given in the previous subsection, we have $(R_{m,i}\cap R_{m,i+1}) \cong V^{\otimes i}\otimes(X_1^3\cap X_2^3)\otimes V^{\otimes m-i-2} \cong V^{\otimes i}\otimes A^{!3}\otimes V^{\otimes m-i-2}$. This makes clear that the PVH Criterion need only be checked in degree 3 in the Koszul case.

4 The Quadraticity of PvB_n

4.1 Overview

We now turn to PvB_n. Our goal being to establish that PvB_n is quadratic using the PVH Criterion, we will follow the following steps:

- Check that the preliminary requirements (as per Subsection 1.1.2) for applying the PVH Criterion are met.
- Find the infinitesimal syzygies. We will use the fact that $\mathfrak{pvb}_n$ is Koszul, and that accordingly the infinitesimal syzygies are essentially given by $\mathfrak{pvb}_n^{!3}$. (The Koszulness of $\mathfrak{pvb}_n^!$ was first established in [BarEnEtRa], and

[9]As per footnote 4, A need only be 2-Koszul, i.e. its Koszul complex need only be exact up to homological degree 2 inclusive.

we give an alternative proof in subsection 4.7). After finding a basis for $\mathfrak{pvb}_n^!$, and in particular for $\mathfrak{pvb}_n^{!3}$, we will see that finding the infinitesimal syzygies becomes a fairly straightforward calculation.

- Find the global syzygies corresponding to the Zamolodchikov tetrahedron, and compute the induced infinitesimal syzygies.

- Check that global syzygies induce all of the infinitesimal syzygies, confirming that the PVH Criterion is met.

4.2 Terminology and Preliminary Requirements for PVH Criterion

We denote by $\mathbb{Q}PvB_n$ and $\mathbb{Q}F$ the rational group ring of PvB_n and the rational free group ring on the same generators, respectively. Their respective augmentation ideals are denoted I_K and I_F.

Consistent with the discussion in Subsection 3.1, we take $\mathbb{Q}PvB_n$ to be completed with respect to the filtration by powers of the augmentation ideal, so that we can eliminate the inverses of group generators, and the linear relations corresponding to the group laws, from our presentation for $\mathbb{Q}PvB_n$ as an algebra.

Given the presentation for PvB_n in Section 1, the augmentation ideal I_K is generated as a 2-sided $\mathbb{Q}PvB_n$-module by the set $\tilde{X} := \{\overline{R}_{ij} := (R_{ij} - 1) : 1 \leq i \neq j \leq n\}$. As per Corollary 1, I_K/I_K^2 is a trivial 2-sided $\mathbb{Q}PvB_n$-module, and hence is a vector space generated by $\tilde{X}$. It is straightforward to check that the elements of $\tilde{X}$ (modulo I_K^2) are linearly independent (i.e. $\mathbb{Q}\tilde{X} \cap I_K^2 = 0$), and hence in fact form a basis of I_K/I_K^2. The $\tilde{X}$ (modulo I_K^2) correspond to the generators $\{r_{ij}\}$ for $\mathfrak{pvb}_n$ from the presentation (3).

As per Subsection 1.1.1, specifically (4), the relations in I_F are given by the 2-sided ideal M in F generated by

$$Y'_{ijk} := R_{ij}R_{ik}R_{jk}R_{ij}^{-1}R_{ik}^{-1}R_{jk}^{-1} - 1$$
$$C_{ij}^{kl\prime} := R_{ij}R_{kl}R_{ij}^{-1}R_{kl}^{-1} - 1$$

for $1 \leq i, j, k, l \leq n$, and i, j, k, l all distinct. Equivalently, M is generated (as 2-sided F-ideal) by

$$Y_{ijk} := R_{ij}R_{ik}R_{jk} - R_{jk}R_{ik}R_{ij} \tag{17}$$
$$C_{ij}^{kl} := R_{ij}R_{kl} - R_{kl}R_{ij} \tag{18}$$

As per Lemma 6, the relations in $\mathfrak{pvb}_n$ are generated by $R \cong (M + I_F^3)/I_F^3$. Thus to obtain R, we make the substitution $R_{ij} \mapsto (\overline{R}_{ij} + 1)$ throughout the Y_{ijk} and C_{ij}^{kl}, and drop all terms of degree 3 (or higher) in the $\overline{R}_{ij}$. We obtain the quadratic relators $\{y_{ijk}; c_{ij}^{kl}\}$ for $\mathfrak{pvb}_n$ (see (3)), up to replacing the $\{\overline{R}_{ij}\}$ by the $\{r_{ij}\}$.

Since PvB_n is a finitely presented group, the requirements for applicability of the PVH Criterion, as we have developed it, essentially reduce to (see Subsection 1.1.2):

- Check that the ideal of relations $M \subseteq I_F$ actually satisfies $M \subseteq I_F^2$.
- Check that the generators (17) and (18) for M satisfy the requirement that $\{Y_{ijk} + I_F^3, C_{ij}^{kl} + I_F^3\}$ are linearly independent in $(M + I_F^3)/I_F^3$ (as required by Theorem 1).

The first requirement amounts to checking that the relators obtained above for R (i.e. (3)) are all quadratic in the r_{ij}, which is clearly true. We note that this is essentially due to the fact that the relators for PvB_n all have degree 0 in each of the generators of PvB_n, so that after performing the substitution $R_{ij} \mapsto (\overline{R}_{ij} + 1)$ and expanding in terms of the $\overline{R}_{ij}$, all constant terms and terms linear in the $\overline{R}_{ij}$ cancel out. This proves the first requirement.

The second requirement is essentially a matter of checking that the $\{y_{ijk}; c_{ij}^{kl}\}$ are linearly independent. There are several ways to do this - one slightly fancy way to do it is to use the isomorphism

$$\Delta^!_{1,1} : \mathfrak{pvb}_n^{!2^*} \xrightarrow{\sim} R$$

which we recalled in (13), and note that $\Delta^!_{1,1}$ takes a basis of $\mathfrak{pvb}_n^{!2}$ precisely to the relators $\{y_{ijk}; c_{ij}^{kl}\}$ of $\mathfrak{pvb}_n$ (we compute this in (22) and (23) below).

4.3 Finding the Infinitesimal Syzygies

As a preliminary matter we recall the definition of $\mathfrak{pvb}_n^!$ and exhibit its relations. As noted in Subsection 1.1.3, $\mathfrak{pvb}_n$ is defined as $\mathfrak{pvb}_n = TV/\langle R\rangle$ (where $V = I_K/I_K^2$ and R were obtained in Subsection 4.2). The quadratic dual algebra $\mathfrak{pvb}_n^!$ is defined as $\mathfrak{pvb}_n^! := TV^*/\langle R^\perp\rangle$, where V^* is the linear dual vector space and $R^\perp \subseteq V^* \otimes V^*$ is the annihilator of R.

From these definitions, one readily finds that $\mathfrak{pvb}_n^!$ is the exterior algebra generated by the set $\{r_{ij}^* : 1 \le i \ne j \le n\}$, subject to the relations:

$$r_{ij}^* \wedge r_{ik}^* = r_{ij}^* \wedge r_{jk}^* - r_{ik}^* \wedge r_{kj}^* \tag{19}$$

$$r_{ik}^* \wedge r_{jk}^* = r_{ij}^* \wedge r_{jk}^* - r_{ji}^* \wedge r_{ik}^* \tag{20}$$

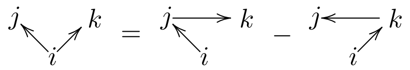
$$r_{ij}^* \wedge r_{ji}^* = 0 \tag{21}$$

where the indices i, j, k are all distinct.

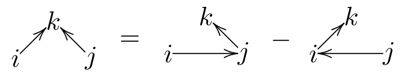
4.3.1 A Basis for $\mathfrak{pvb}_n^!$

In this section we will identify a basis for the dual algebra $\mathfrak{pvb}_n^!$. We state the result for all degrees, although we only actually need $\mathfrak{pvb}_n^{!3}$.

We note that monomials in $\mathfrak{pvb}_n^!$ may be interpreted as directed graphs, with vertices given by the integers $[n] := \{1, \ldots, n\}$, and edges consisting of all ordered pairs (i, j) such that r_{ij} is in the monomial. We thus get a graphical depiction of the above relations:



(Pruning V)



(Pruning A)

$$i \rightleftarrows j \qquad = \qquad 0 \tag{No Loop}$$

We note that there is a sign indeterminacy in the graphs, in that for instance the LHS of (Pruning V) can equally refer to $\pm r_{ij} \wedge r_{ik}$. We will only use the graphs when the signs are immaterial.

Theorem 3. *The algebra $\mathfrak{pvb}_n^!$ has a basis consisting exactly of the monomials corresponding to 'chain gangs', i.e. unordered partitions of $[n]$ into ordered subsets.*

Corollary 2. *The degree k component of $\mathfrak{pvb}_n^!$ has dimension $L(n, n-k)$, where the 'Lah number' $L(n, n-k)$ denotes the number of unordered partitions of $[n]$ into $(n-k)$ ordered subsets.*

Proof. Clear from the theorem, since it is easy to see that a chain gang on the index set $[n]$ with $(n-k)$ chains must have exactly k arrows (and correspond to a basis monomial of degree k). □

We note that it was already proved in [BarEnEtRa] that the dimensions of the graded components of $\mathfrak{pvb}_n^!$ are given by the numbers of unordered partitions of $[n]$ into ordered subsets (although a basis for $\mathfrak{pvb}_n^!$ was not provided).

We postpone the proof until Subsection 4.5. However, the idea of the proof is straightforward, i.e. show that a basis is given by all monomials whose graphical representation has no **A-joins** or **V-joins** (by which we mean the diagrams in the LHS of the relations (Pruning A) and (Pruning V), respectvely) and no loops:

- One first shows that if a tree has an A-join or a V-join, we can replace it by a sum of trees in which the particular join is replaced by an oriented segment of length 2, using either (Pruning A) or (Pruning V). Eventually we are left with a sum of oriented chains.
- One must then show that these oriented chains are linearly independent.
- Next one shows that all monomials whose graph contains a loop (oriented or not) are 0: it turns out that loops of length greater than 2 can be reduced progressively to loops of length 2, and then the resulting graph is 0 either by (No Loops) or by anti-commutativity.

Remark 2. *We will see that directed chains of length 3 are in a 1-1 correspondence with certain (level 3) syzygies of the global algebra $\mathfrak{A}$ - specifically one Zamolodchikov tetrahedron for each ordering of a particular choice of 4 of the n* *strands in PvB_n. An arrow from index 'i' to index 'j' means strand 'i' remains above strand 'j' throughout the syzygy. Although not relevant for our purposes, this correspondence between oriented chains of length m and level m syzygies holds for syzygies of all levels. These higher level global syzygies correspond to generalizations of the Zamolodchikov tetrahedron, and most likely correspond in some sense to generators of the cohomology of PvB_n.*

Remark 3. *If in the basis given above one includes only generators r_{ij} with $i < j$, we obtain a basis for the algebra $\mathfrak{pfb}^!_n$. This basis is different from the basis given in [BarEnEtRa]. The basis given here is more useful for purposes of applying the PVH criterion, because of the fact that directed chains of length 3 correspond to syzygies arising from the Zamolodchikov tetrahedron.*

Remark 4. *If, as in the previous remark, we again consider the implied basis for $\mathfrak{pfb}^!_n$, we see that the (No Loop) relation and the exclusion of monomials whose graph contains a loop are irrelevant. We are left with a rule that says that a basis of $\mathfrak{pfb}^!_n$ is given by all monomials whose graph does not contain an A-join or a V-join. The exclusion of A-joins and V-joins amounts to specifying a quadratic Gröbner basis for the ideal of relations in $\mathfrak{pfb}^!_n$. By a theorem of [Yuz], this gives a proof that the algebra $\mathfrak{pfb}^!_n$ (and its dual $\mathfrak{pfb}_n$) is Koszul. Unfortunately the given basis for $\mathfrak{pvb}^!_n$, as opposed to $\mathfrak{pfb}^!_n$, does not prove Koszulness, since the no-loop exclusion corresponds to Gröbner basis elements of arbitrarily high degree (i.e. of degree equal to the length of the loop). In subsection 4.7 we give an alternative basis for $\mathfrak{pvb}^!_n$, from which the Koszulness of $\mathfrak{pvb}^!_n$ can be deduced.*

4.3.2 The Infinitesimal Syzygies

One can readily compute that the isomorphism $\Delta^!_{1,1}$ acts on basis elements of $\mathfrak{pvb}^{!2*}_n$ as follows:[10]

$$\Delta^!_{1,1} : r_{ij} \wedge r_{jk} \mapsto [r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{ik}, r_{jk}] \tag{22}$$

$$r_{ij} \wedge r_{kl} \mapsto [r_{ij}, r_{kl}] \tag{23}$$

Indeed one can in fact view (Pruning A) and (Pruning V) as giving the only elements of V^* that do not multiply freely in $\mathfrak{pvb}^{!2}_n$, and then (since $\Delta^!_{1,1}$ is dual to the product in $\mathfrak{pvb}^{!2}_n$) the above result is immediate.

As noted following (14) and (15), the maps $(1\otimes\Delta^!_{1,1})\circ\Delta^!_{1,2}$ and $(\Delta^!_{1,1}\otimes 1)\circ\Delta^!_{2,1}$ are isomorphisms and give the inclusion of $\mathfrak{pvb}^{!3}_n$ into X^3_1 and X^3_2 respectively. It follows that the image of the map $\kappa \circ \partial_{Syz}$ of $X^3_1 \cap X^3_2$ into $V^{\otimes 3}$ (see (16)) is given by

$$\kappa \circ \partial_{Syz}(X^3_1 \cap X^3_2) = \left[(\Delta^!_{1,1} \otimes 1) \circ \Delta^!_{2,1} - (1 \otimes \Delta^!_{1,1}) \circ \Delta^!_{1,2}\right](\mathfrak{pvb}^{!3}_n) \tag{24}$$

[10]Instead of writing r^{**}_{ij} for elements of $\mathfrak{pvb}^{!2*}_n$, we write r_{ij}.

It is easy to see that there are three types of basis element in $\mathfrak{pvb}_n^{!3}$, corresponding to three types of chain gang with three edges:

- $r_{ij} \wedge r_{jk} \wedge r_{kl}$ with i, j, k, l all distinct;
- $r_{ij} \wedge r_{jk} \wedge r_{st}$ with i, j, k, s, t all distinct;
- $r_{ij} \wedge r_{kl} \wedge r_{st}$ with i, j, k, l, s, t all distinct.

We first deal with the first type of basis element. We will show that in this case the first term of (24) is given by:[11]

$$\begin{aligned}
\Delta^!_{2,1}(r_{ij} \wedge r_{jk} \wedge r_{kl}) = &-(\Delta^!_{1,1} \otimes 1)(r_{ij} \wedge r_{jk}) \otimes (-r_{il} - r_{jl} - r_{kl}) \\
&+ (\Delta^!_{1,1} \otimes 1)(r_{ij} \wedge r_{jl}) \otimes (-r_{ik} - r_{jk} + r_{kl}) \\
&- (\Delta^!_{1,1} \otimes 1)(r_{ik} \wedge r_{kl}) \otimes (-r_{ij} + r_{jk} + r_{jl}) \\
&+ (\Delta^!_{1,1} \otimes 1)(r_{jk} \wedge r_{kl}) \otimes (r_{ij} + r_{ik} + r_{il}) \\
&- (\Delta^!_{1,1} \otimes 1)(r_{ij} \wedge r_{kl}) \otimes (r_{ik} + r_{il} + r_{jk} + r_{jl}) \\
&+ (\Delta^!_{1,1} \otimes 1)(r_{ik} \wedge r_{jl}) \otimes (r_{ij} + r_{il} - r_{jk} + r_{kl}) \\
&+ (\Delta^!_{1,1} \otimes 1)(r_{il} \wedge r_{jk}) \otimes (-r_{ij} - r_{ik} + r_{jl} + r_{kl})
\end{aligned}$$

We defer a more detailed justification of the above calculation to Subsection 4.6. Now using (22) and (23), we get:

$$\begin{aligned}
(\Delta^!_{1,1} \otimes 1) \circ \Delta^!_{2,1}(r_{ij} \wedge r_{jk} \wedge r_{kl}) = &-y_{ijk} \otimes (-r_{il} - r_{jl} - r_{kl}) + y_{ijl} \otimes (-r_{ik} - r_{jk} + r_{kl}) \\
&- y_{ikl} \otimes (-r_{ij} + r_{jk} + r_{jl}) + y_{jkl} \otimes (r_{ij} + r_{ik} + r_{il}) \\
&- c^{kl}_{ij} \otimes (r_{ik} + r_{il} + r_{jk} + r_{jl}) + c^{jl}_{ik} \otimes (r_{ij} + r_{il} - r_{jk} + r_{kl}) \\
&- c^{jk}_{il} \otimes (r_{ij} + r_{ik} - r_{jl} - r_{kl}) \qquad (25)
\end{aligned}$$

In (25) the tensor products are the tensor products in the tensor algebra TV, so we drop them. Furthermore, $(1 \otimes \Delta^!_{1,1}) \circ \Delta^!_{1,2}(r_{ij} \wedge r_{jk} \wedge r_{kl})$ is the same, but with the tensor components flipped. Putting the two together gives:

$$\begin{aligned}
(\Delta^!_{1,1} \otimes 1) \circ \Delta^!_{2,1}(r_{ij} \wedge r_{jk} \wedge r_{kl}) = &-[y_{ijk}, (-r_{il} - r_{jl} - r_{kl})] + [y_{ijl}, (-r_{ik} - r_{jk} + r_{kl})] \\
&- [y_{ikl}, (-r_{ij} + r_{jk} + r_{jl})] + [y_{jkl}, (r_{ij} + r_{ik} + r_{il})] \\
&- [c^{kl}_{ij}, (r_{ik} + r_{il} + r_{jk} + r_{jl})] + [c^{jl}_{ik}, (r_{ij} + r_{il} - r_{jk} + r_{kl})] \\
&- [c^{jk}_{il}, (r_{ij} + r_{ik} - r_{jl} - r_{kl})] \qquad (26)
\end{aligned}$$

We will see that these syzygies are induced (via the map F^{Syz}) from global syzygies in the next subsection.

[11] Again, we write r_{ij} instead of r^{**}_{ij} for elements of $\mathfrak{pvb}_n^{!3*}$.

This leaves the two remaining types of degree 3 basis element in $\mathfrak{pvb}_n^{!3}$. It is fairly straightforward to compute that they correspond, respectively, to the relations:

$$y_{ijk} r_{st} = r_{st} y_{ijk}$$

and

$$c_{ij}^{kl} r_{st} = r_{st} c_{ij}^{kl}$$

which are clearly satisfied also at the global level.

4.4 Global Syzygies and the PVH Criterion

4.4.1 The Global Syzygies

We now display (a sum of) elements of $I_K^{\otimes_K 3}$ which specify a syzygy of $\mathfrak{A}^3$ and which project via F^0 (see notation in Theorem 5) to the syzygy (26) in $\mathfrak{pvb}_n$. The syzygy corresponds to the standard syzygy in the braid group (i.e. the Zamolodchikov tetrahedron pictured in Subsection 1.2), which can be written:

$$\begin{gathered}
Y_{jkl} R_{il} R_{ik} R_{ij} + R_{jk} R_{jl} Y_{ikl} R_{ij} + R_{jk} R_{jl} R_{ik} R_{il} C_{ij}^{kl} \\
+ R_{jk} C_{ik}^{jl} R_{il} R_{ij} R_{kl} + R_{jk} R_{ik} Y_{ijl} R_{kl} + Y_{ijk} R_{il} R_{jl} R_{kl} + R_{ij} R_{ik} C_{il}^{jk} R_{jl} R_{kl} \\
- R_{ij} R_{ik} R_{il} Y_{jkl} - R_{ij} Y_{ikl} R_{jl} R_{jk} - C_{ij}^{kl} R_{il} R_{ik} R_{jl} R_{jk} \\
- R_{kl} R_{ij} R_{il} C_{ik}^{jl} R_{jk} - R_{kl} Y_{ijl} R_{ik} R_{jk} - R_{kl} R_{jl} R_{il} Y_{ijk} - R_{kl} R_{jl} C_{il}^{jk} R_{ik} R_{ij}
\end{gathered}$$

where again

$$\begin{aligned}
Y_{ijk} &= R_{ij} R_{ik} R_{jk} - R_{jk} R_{ik} R_{ij} \\
C_{ij}^{kl} &= R_{ij} R_{kl} - R_{kl} R_{ij}
\end{aligned} \tag{27}$$

This calculation was illustrated in Subsection 1.2.

The calculation may be explained as follows. The illustration shows 14 braids $\{B_i\}_{i=1,\dots,14}$ around its perimeter. These are linked by arrows labeled by various multiples of the moves Y_{ijk} or C_{ij}^{kl}. As per (27), if we attach the labels $B_1, B_2, \dots$ starting at the bottom braid and proceeding clockwise around the perimeter, the arrows correspond to differences $(B_2 - B_1), \dots, (B_8 - B_7)$ up the left side of the diagram, and to differences $(B_{14} - B_1), \dots, (B_8 - B_9)$ around the right side. But clearly the telescopic sums on the left and right both give $B_8 - B_1$, so we get a syzygy which we wrote down above.

At this point the expression above should be viewed as an element of I_F. To write this syzygy as a sum of terms in $I_K^{\otimes_K 3}$, we proceed as in Subsection 4.2 and make the substitution $R_{ij} \to (\overline{R}_{ij} + 1)$ throughout. One finds that all terms of degree 0, 1 or 2 in the $\overline{R}_{ij}$ cancel, so in fact the syzygy lives in I_F^3.

Recall that we denote π_3 the projection $I_F^{\otimes_F 3} \twoheadrightarrow I_K^{\otimes_K 3}$ (see Subsection 3.4). If we denote by μ_{FF}^3 the composition $I_F^{\otimes_F 3} \xrightarrow{\mu_{FF}} I_F^{\otimes_F 2} \xrightarrow{\mu_{FF}} I_F$, then applying $\pi_3 \circ (\mu_{FF}^3)^{-1}$ we get an element of $I_K^{\otimes_K 3}$ (which we will not write down in full).

This syzygy induces an infinitesimal syzygy which we obtain by dropping all but the lowest degree terms in the $\overline{R}_{ij}$ (this corresponds to applying the map F^{Syz}, which is just the restriction of the map F^0 from Subsection 1.1.6). After reorganizing, we get:

$$
\begin{aligned}
&[\tilde{Y}_{jkl}, \overline{R}_{ij} + \overline{R}_{ik} + \overline{R}_{il}] - [\tilde{Y}_{ikl}, -\overline{R}_{ij} + \overline{R}_{jk} + \overline{R}_{jl}] + [\tilde{Y}_{ijl}, -\overline{R}_{ik} - \overline{R}_{jk} + \overline{R}_{kl}] \\
&\qquad - [\tilde{Y}_{ijk}, -\overline{R}_{il} - \overline{R}_{jl} - \overline{R}_{kl}] \\
&\quad - [\tilde{C}^{kl}_{ij}, \overline{R}_{ik} + \overline{R}_{il} + \overline{R}_{jk} + \overline{R}_{jl}] + [\tilde{C}^{jl}_{ik}, \overline{R}_{ij} + \overline{R}_{il} - \overline{R}_{jk} + \overline{R}_{kl}] \\
&\qquad - [\tilde{C}^{jk}_{il}, \overline{R}_{ij} + \overline{R}_{ik} - \overline{R}_{jl} - \overline{R}_{kl}]
\end{aligned}
$$

where the $\tilde{C}^{kl}_{ij}$ and $\tilde{Y}_{ijk}$ are the same as the c^{kl}_{ij} and y_{ijk} in (3), except the r_{ij} are replaced by the $\overline{R}_{ij}$. By inspection, we see that this coincides with the infinitesimal syzygy (26). Hence we have confirmed that all of the infinitesimal syzygies are covered by global syzygies.

4.5 Proof of the Basis for $\mathfrak{pvb}^!_n$

We will follow the outline of the proof provided in Subsection 4.3.1.

We will say that a pair of vertices in a forest graph is **unordered** if there is not an oriented sequence of edges from one of them to the other. We define the **defect** of a tree as the number of unordered pairs of vertices in the graph, and the defect of a forest as the sum of the defects of its components.

Then chain gangs (unordered partitions of $[n]$ into ordered subsets) are exactly the forests with 0 defect. Moreover, in the pruning moves the A- and V-joins have defect 1, while the remaining terms have defect 0.

We will refer to a relation formed by adding to each of the terms in either (Pruning A) or (Pruning V) exactly the same additional edges, without ever forming a loop, as a **multiple** of the original relation. Note that the graphs representing the multiple need not be connected. The defect function has the following 'multiplicativity' property on forests:

Lemma 8. *In each multiple of either (Pruning A) or (Pruning V), the term which is built from the term in the original relation containing a join has defect strictly larger than the other terms.*

The proof is deferred to the end of this subsection.

Proof of Theorem 3. We follow the plan of proof given following the statement of Theorem 3.

The multiplicativity property of the defect makes clear that all forests can be expressed in terms of (sums of) chain gangs: If a forest contains an A- or V-join, then using either (Pruning A) or (Pruning V) we can replace it with a sum of forests with strictly lower defect. Iterating, we get a forest with 0 defect, i.e. a chain gang.

Now we show that chain gangs are linearly independent modulo the relations in $\mathfrak{pvb}^!_n$. The proof is a variation on the standard diamond lemma proof, which we briefly recall. By a **reduction**, we mean specifically replacing the LHS of either pruning relation, or a multiple thereof, by the RHS.

We will show that reducing a defect to 0 will produce the same chain gangs regardless of the sequence of reductions chosen, by induction on the size of the defect. This is clearly true when we start with a forest with defect 1, since there is only one way to reduce such a forest.

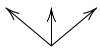
Suppose the claim is true for all forests of defect $\leq m$. Let us consider a forest of defect $m+1$, and suppose there are two possible reductions, called (a) and (b). Then applying either (a) or (b) gives a (sum of) new forests, which we call A and B respectively, each of defect $\leq m$.

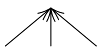
Suppose (a) and (b) (or the pruning relations of which they are multiples) involve changes to pairs of edges that do not overlap. Then it is still possible to apply reduction (a) to B, and reduction (b) to A. Doing so, we obtain the same forest C of defect $\leq m-1$, since the result of applying non-overlapping reductions clearly does not depend on the order they are applied.

Alternatively, suppose (a) and (b) (or the pruning relations of which they are multiples) involve changes to pairs of edges that do overlap. We will see that we can find further reductions (a'), (a") and (b'), (b") such that applying the sequence (a)-(a')-(a") or (b)-(b')-(b") leads to the same (sum of) forests C, of defect $\leq m-2$.[12]

Either way, we know by induction that all reduction sequences from A give the same results, and similarly for B, and since they have a common reduction sequence going through C, we see that A and B both give the same (sum of) forests of defect 0. Hence all reductions of the original forest must give the same (sum of) chain gangs.

We now deal with the case where reductions (a) and (b) involve pairs of edges that overlap, and exhibit the reductions (a'), (a") and (b'), (b"). By inspection of the A- and V-joins, the following three types of overlap can arise (up to sign):


(X)


(Y)


(Z)

In each case we have only shown the edges involved in the reductions.

Case (Z) is dealt with as follows (a star over a wedge $\overset{*}{\wedge}$ indicates the join which is being reduced - hence to make the following more legible we have dropped the * from elements $r^*_{ij} \in \mathfrak{pvb}^!_n$):

$$\begin{aligned} r_{ij} \overset{*}{\wedge} r_{kj} \wedge r_{kl} &= r_{ik} \wedge r_{kj} \overset{*}{\wedge} r_{kl} + r_{ij} \wedge r_{ki} \overset{*}{\wedge} r_{kl} \\ &= r_{ik} \wedge r_{kj} \wedge r_{jl} - r_{ik} \wedge r_{kl} \wedge r_{lj} - r_{ij} \overset{*}{\wedge} r_{il} \wedge r_{ki} + r_{kl} \wedge r_{li} \wedge r_{ij} \\ &= r_{ik} \wedge r_{kj} \wedge r_{jl} - r_{ik} \wedge r_{kl} \wedge r_{lj} - r_{ki} \wedge r_{ij} \wedge r_{jl} + r_{ki} \wedge r_{il} \wedge r_{lj} - r_{kl} \wedge r_{li} \wedge r_{ij} \end{aligned}$$

[12] In fact the reductions (a') and (b') may really involve two reductions, applicable to different terms.

while on the other hand

$$\begin{aligned} r_{ij} \wedge r_{kj} \overset{*}{\wedge} r_{kl} &= r_{ik} \wedge r_{kj} \overset{*}{\wedge} r_{kl} + r_{ki} \overset{*}{\wedge} r_{kl} \wedge r_{ij} \\ &= r_{ik} \wedge r_{kj} \wedge r_{jl} - r_{ik} \wedge r_{kl} \wedge r_{lj} + r_{ki} \wedge r_{il} \overset{*}{\wedge} r_{ij} - r_{kl} \wedge r_{li} \wedge r_{ij} \\ &= r_{ik} \wedge r_{kj} \wedge r_{jl} - r_{ik} \wedge r_{kl} \wedge r_{lj} + r_{ki} \wedge r_{il} \wedge r_{lj} - r_{ki} \wedge r_{ij} \wedge r_{jl} - r_{kl} \wedge r_{li} \wedge r_{ij} \end{aligned}$$

Since the results of the two calculations are the same, we see that regardless of which join we reduce first, there is a further sequence of reductions that leads to the same (signed sum of) trees, which is what we needed.[13]

Cases (X) and (Y) are dealt with similarly - we simply note that

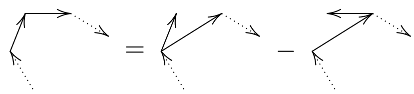
$$\begin{aligned} r_{ij} \overset{*}{\wedge} r_{ik} \wedge r_{il} = -r_{ij} \wedge r_{jl} \wedge r_{lk} + r_{ij} \wedge r_{jk} \wedge r_{kl} + r_{il} \wedge r_{lj} \wedge r_{jk} \\ + r_{ik} \wedge r_{kl} \wedge r_{lj} - r_{ik} \wedge r_{kj} \wedge r_{jl} - r_{il} \wedge r_{lk} \wedge r_{kj} = r_{ij} \wedge r_{ik} \overset{*}{\wedge} r_{il} \end{aligned}$$

and

$$\begin{aligned} r_{il} \overset{*}{\wedge} r_{jl} \wedge r_{kl} = r_{ij} \wedge r_{jk} \wedge r_{kl} - r_{ik} \wedge r_{kj} \wedge r_{jl} + r_{ki} \wedge r_{ij} \wedge r_{jl} \\ - r_{ji} \wedge r_{ik} \wedge r_{kl} + r_{jk} \wedge r_{ki} \wedge r_{il} - r_{kj} \wedge r_{ji} \wedge r_{il} = r_{il} \wedge r_{jl} \overset{*}{\wedge} r_{kl} \end{aligned}$$

and leave the details of the calculations for the reader.

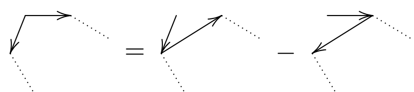
The next step in the proof is to show that all graphs with loops are 0. Let us start by considering oriented loops. Using (Pruning V), we can reduce oriented loops of length greater than 2 to (sums of) oriented loops of shorter length:



Once again we note that there is a sign indeterminacy in the above graphical representation, which does not affect the outcome as we do not rely on any cancelation of terms and the specific coefficients do not matter.

Once we are down to oriented loops of length 2, these are 0 by (No Loops).

Now we consider unoriented loops. For loops containing a V-join, we can use (Pruning V) to reduce loops of length greater than 2 to (sums of) loops of shorter length:



The case of unoriented loops containing an A-join, rather than a V-join, is similar. Although the result of any such reduction may or may not be unoriented, we can still continue reducing the length of the loops using either the

[13] As per the previous footnote, note that reductions (a') and (b'), indicated by the stars in the RHS of the first lines, actually involve two reductions, applicable to separate terms.

oriented or unoriented procedure. Once we are down to loops of length 2, these are 0 either by (No Loops) or by anti-commutativity.

Thus, if we follow the above procedure, we can reduce all loops to 0. It is also clear from the above that even if we followed a different sequence of pruning moves we would never reduce loops to a sum of diagrams including trees, since a pruning move can never break a loop. □

We have completed the proof of Theorem 3, subject to proving multiplicativity of the defect function, i.e. Lemma 8. We do this now.

By a '**vertex in a relation**' we will mean a vertex which is an endpoint of at least one edge in the graphs corresponding to the terms of the relation. It is fairly clear this is a well-defined notion (and in particular that the number of vertices in a relation is constant over all terms in a relation).

By the '**join term**' in a multiple of a pruning relation, we mean the term that was built by adding edges to the term in the original pruning relation which contained an A- or V-join.

Proof of Lemma 8. We proceed by induction on the number of vertices in a relation. We claim that if (x) and (y) are vertices in the new relation, and there is a directed chain of edges from (x) to (y) in the join term, then there is also a directed chain of edges from (x) to (y) (in the same direction) in the other terms of the relation. Hence:

1. When we form a multiple of (Pruning A) or (Pruning V) by adding edges, in that multiple each vertex is no more ordered (with respect to other vertices) in the join-term than in the non-join terms.

2. However, in each relation, there is at least one pair of vertices which is unordered in the join-term, but is ordered in the other terms, namely the unordered paid in the original pruning relation.

So we can conclude that the join-term in the new relation must have strictly highest defect.

The above claim is easily verified in the original relations (Pruning A) and (Pruning V). We now assume the claim has been proved whenever there are up to m edges in a relation; we take a relation with m edges and add a further edge. There are three cases.

Case I: Two New Vertices. If the added edge forms a separate component in the new graphs, then clearly the defect will have increased by the same amount in all terms of the relation.

Case II: One Old, One New Vertex. So let us suppose that the added edge has one vertex (a) already in the relation, and one new vertex (b). It is clear that the orderliness of pairs of vertices not including (b) is unchanged.

Now suppose that (c) is any other vertex in the relation. If (b) and (c) are ordered in the new join term, say with a directed chain from (b) to (c), this chain must go through (a) since vertex (b) was not previously the endpoint of any edge. Thus there was also a directed chain from (a) to (c) in the join-term of the old relation, hence by induction there were directed chains from (a) to (c) in the non-join terms in the old relation. It follows that there is also a directed chain from (b) to (c) in the new non-join terms. The case of a directed chain from (c) to (b) in the new join-term is similar.

If (b) and (c) are unordered in the new join term, there is nothing to prove. Letting (c) range over all other vertices in the relation proves Case II.

Case III: Two Old Vertices. All that is left is to consider the case where the new edge links two existing vertices in the relation. Because we assume that the added edge does not create a loop, it follows that the edge must be linking two formerly disconnected components of the graphs underlying the relation. We assume the new edge links existing vertices (a) and (b). It is clear that the orderliness of pairs of vertices already within the same component in the old relation is unchanged. So we take (c) and (d) to be to be any two vertices in the component of (a) and (b) respectively. We can assume without loss of generality that either (a) $\neq$ (c) or (b) $\neq$ (d) (because if (a) = (c) and (b) = (d) then that pair is joined by the new edge and hence ordered in all terms of the relation).

The reasoning is similar to Case II. If (c) and (d) are ordered in the new join term, say with a directed chain from (c) to (d), this chain must go through (a) and (b) since we assume there are no loops. Thus the new edge must be oriented (a) to (b); moreover, there must also have been directed chains from (c) to (a) and from (b) to (d) in the join-term of the old relation. By induction, there were directed chains from (c) to (a) and from (b) to (d) in the non-join terms in the old relation. It follows that there is also a directed chain from (c) to (d) in the new non-join terms. The case of a directed chain from (d) to (c) in the new join-term is similar.

Finally, if (c) and (d) are unordered in the new join term, there is nothing to prove. Letting (c) and (d) range over all other vertices in the relation proves Case III. □

4.6 Justification of the Co-Product Formulas

We give here a summary of the action of the product $m^! : \mathfrak{pvb}_n^{!2} \otimes V^* \to \mathfrak{pvb}_n^{!3}$ in terms of the 'directed chains' basis for the respective spaces. The verifications are routine and we will leave them to the reader.

$$
\begin{aligned}
r^*_{ij} \wedge r^*_{jk} \otimes r^*_{il} &\mapsto r^*_{il} \wedge r^*_{lj} \wedge r^*_{jk} - r^*_{ij} \wedge r^*_{jl} \wedge r^*_{lk} + r^*_{ij} \wedge r^*_{jk} \wedge r^*_{kl} \\
r^*_{ij} \wedge r^*_{jk} \otimes r^*_{jl} &\mapsto -r^*_{ij} \wedge r^*_{jl} \wedge r^*_{lk} + r^*_{ij} \wedge r^*_{jk} \wedge r^*_{kl} \\
r^*_{ij} \wedge r^*_{jk} \otimes r^*_{kl} &\mapsto r^*_{ij} \wedge r^*_{jk} \wedge r^*_{kl} \\
r^*_{ij} \wedge r^*_{jk} \otimes r^*_{li} &\mapsto r^*_{li} \wedge r^*_{ij} \wedge r^*_{jk} \\
r^*_{ij} \wedge r^*_{jk} \otimes r^*_{lj} &\mapsto -r^*_{il} \wedge r^*_{lj} \wedge r^*_{jk} + r^*_{li} \wedge r^*_{ij} \wedge r^*_{jk} \\
r^*_{ij} \wedge r^*_{jk} \otimes r^*_{lk} &\mapsto r^*_{ij} \wedge r^*_{jl} \wedge r^*_{lk} - r^*_{il} \wedge r^*_{lj} \wedge r^*_{jk} + r^*_{li} \wedge r^*_{ij} \wedge r^*_{jk}
\end{aligned}
$$

and

$$
\begin{aligned}
r_{ij}^* \wedge r_{kl}^* \otimes r_{ik}^* &\mapsto -r_{ij}^* \wedge r_{jk}^* \wedge r_{kl}^* + r_{ik}^* \wedge r_{kj}^* \wedge r_{jl}^* - r_{ik}^* \wedge r_{kl}^* \wedge r_{lj}^* \\
r_{ij}^* \wedge r_{kl}^* \otimes r_{ki}^* &\mapsto r_{kl}^* \wedge r_{li}^* \wedge r_{ij}^* - r_{ki}^* \wedge r_{il}^* \wedge r_{lj}^* + r_{ki}^* \wedge r_{ij}^* \wedge r_{jl}^* \\
r_{ij}^* \wedge r_{kl}^* \otimes r_{il}^* &\mapsto -r_{ij}^* \wedge r_{jk}^* \wedge r_{kl}^* + r_{ik}^* \wedge r_{kj}^* \wedge r_{jl}^* - r_{ik}^* \wedge r_{kl}^* \wedge r_{lj}^* \\
&\quad - r_{ki}^* \wedge r_{ij}^* \wedge r_{jl}^* + r_{ki}^* \wedge r_{il}^* \wedge r_{lj}^* \\
r_{ij}^* \wedge r_{kl}^* \otimes r_{li}^* &\mapsto r_{kl}^* \wedge r_{li}^* \wedge r_{ij}^* \\
r_{ij}^* \wedge r_{kl}^* \otimes r_{jk}^* &\mapsto -r_{ij}^* \wedge r_{jk}^* \wedge r_{kl}^* \\
r_{ij}^* \wedge r_{kl}^* \otimes r_{kj}^* &\mapsto r_{kl}^* \wedge r_{li}^* \wedge r_{ij}^* - r_{ki}^* \wedge r_{il}^* \wedge r_{lj}^* + r_{ki}^* \wedge r_{ij}^* \wedge r_{jl}^* \\
&\quad + r_{ik}^* \wedge r_{kl}^* \wedge r_{lj}^* - r_{ik}^* \wedge r_{kj}^* \wedge r_{jl}^* \\
r_{ij}^* \wedge r_{kl}^* \otimes r_{jl}^* &\mapsto -r_{ij}^* \wedge r_{jk}^* \wedge r_{kl}^* + r_{ik}^* \wedge r_{kj}^* \wedge r_{jl}^* - r_{ki}^* \wedge r_{ij}^* \wedge r_{jl}^* \\
r_{ij}^* \wedge r_{kl}^* \otimes r_{lj}^* &\mapsto r_{kl}^* \wedge r_{li}^* \wedge r_{ij}^* - r_{ki}^* \wedge r_{il}^* \wedge r_{lj}^* + r_{ik}^* \wedge r_{kl}^* \wedge r_{lj}^*
\end{aligned}
$$

Again the verification of the formulae for the dual map $\Delta^!_{2,1}$ is tedious but routine and is left to the (beleaguered) reader.

4.7 Proof of the Koszulness of $\mathfrak{pvb}^!_n$

As indicated in Remark 4, the basis given in Theorem 3 will not in itself lead to a proof of Koszulness because the explicit exclusion of monomials whose graphical representation contains a loop corresponds to Gröbner basis elements of arbitrarily high degree. In contrast, standard theorems on Gröbner bases only tell us that (under mild assumptions) algebras with quadratic Gröbner bases are Koszul.

So we will exhibit a different basis for $\mathfrak{pvb}^!_n$, consisting of all monomials not containing certain length two subwords, which corresponds to the specification of a quadratic Gröbner basis for $\mathfrak{pvb}^!_n$. We will see that, by a result of Yuzvinsky [Yuz] (see also [ShelYuz]), $\mathfrak{pvb}^!_n$(and hence also $\mathfrak{pvb}_n$) is Koszul.

To begin with, given any finite subset $I \subseteq \mathbb{N}$ (which we order numerically), we will define two kinds of graph with vertices indexed by I - we will call these Down graphs and Up graphs. We will then show how to combine Down and Up graphs to get graphs (which we will call Up-Down graphs) which correspond to a new basis for $\mathfrak{pvb}^!_n$, of the desired form (i.e. corresponding to the specification of a quadratic Gröbner basis for $\mathfrak{pvb}^!_n$).

We will also see that the Down and Up graphs, respectively, catalogue bases for:

- the algebra $\mathfrak{pfb}^!_n$, which is quadratic dual to the quadratic approximation for the flat virtual braid group PfB_n (i.e. the quadratic dual to the universal enveloping algebra of the triangular Lie algebra $\mathfrak{tr_n}$ in [BarEnEtRa]); and

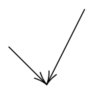
- the algebra $\mathfrak{pb}^!_n$, quadratic dual to the quadratic approximation for the pure braid group PB_n.

However, we do not have a coherent explanation for why bases for $\mathfrak{pfb}^!_n$ and $\mathfrak{pb}^!_n$ should fit together in this way to produce bases of $\mathfrak{pvb}^!_n$. See Remark 8 below.

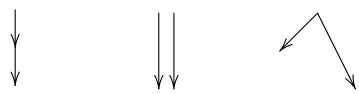
4.7.1 Down Graphs and $\mathfrak{pfb}^!_n$

A **Down tree** on the index set $I = \{i_1, \ldots, i_m\} \subseteq \mathbb{N}$ (with smallest index i_1) consists of a 'tuft' of directed edges $\{(i_2, i_1), \ldots, (i_m, i_1)\}$. (The graph is non-planar in that all orderings of the edges incident to a particular vertex are considered equivalent.) This corresponds to allowing all trees built with directed edges with *decreasing* indices (i.e. edges (i, j) with $i > j$) by

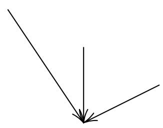
- allowing the following subgraphs:



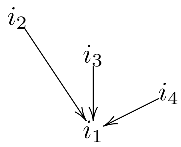
- excluding the following three subgraphs:



where in all cases the relative heights of the endpoints indicate the relative ordering of the indices (in particular, the middle subgraph has a doubled edge: $\{(i, j), (i, j)\}$). We declare by way of convention that a Down tree on on index set with one element is the empty graph. An example of a Down graph is the following:



i.e.



where $i_2 > i_3 > i_4 > i_1$.

Note that because of the last two types of excluded graph, we needn't have explicitly restricted ourselves to trees, as these exclusions prevent the formation of (ordered or unordered) loops in the graph (recall also that Down graphs are built only with directed edges with decreasing indices). Thus we have an exclusion rule, of degree 2 in the number of edges, which effectively eliminates loops. In particular the obstacle to proving Koszulness due to the presence of non-quadratic Gröbner basis elements no longer arises.

We now define a Down forest on an index set I partitioned as $I = S_1 \sqcup \cdots \sqcup S_u$ to be the union of the Down trees on the subsets S_i.

Remark 5. *The monomials corresponding to Down forests induced by unordered partitions of $[n] = \{1, 2, \dots, n\}$ (using the correspondence explained in subsection 4.3.1) form a basis for the algebra $\mathfrak{pfb}_n^!$ (as a skew-commutative algebra* [14]*). Indeed, it is easy to see that Down forests are in bijective correspondence with the 'reduced monomials with disjoint supports' which were proved in [BarEnEtRa], Proposition 4.2, to form a basis of $\mathfrak{pfb}_n^! = U(\mathfrak{tr_n})^!$ (with the minor difference that the edges in [BarEnEtRa] had increasing indices). Also, the above excluded subgraphs correspond to the excluded monomials implied by the Gröbner basis given in [BarEnEtRa], Corollary 4.3, for $U(\mathfrak{tr_n})^!$ (subject to always writing generators with increasing indices, using the relation $r_{i,j} = -r_{j,i}$). The fact that these Gröbner basis elements are quadratic allowed [BarEnEtRa] to conclude that $\mathfrak{pfb}_n^!$ is Koszul.*

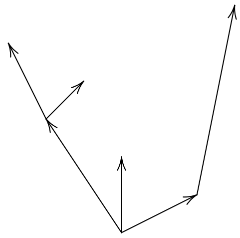
4.7.2 Up Graphs and $\mathfrak{pb}_n^!$

An **Up tree** on the index set $I = \{i_1, \dots, i_m\} \subseteq \mathbb{N}$ (with $i_1 < \cdots < i_m$) consists of all trees built with directed edges with *increasing* indices by

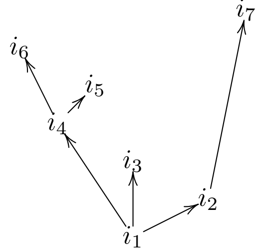
- allowing the following two subgraphs:

- excluding the following two subgraphs:

where, again, in all cases the relative heights of the endpoints indicate relative ordering of the indices. Furthermore, the graphs are again non-planar in that all orderings of the edges incident to a particular vertex are considered equivalent; also, we again declare by way of convention that a Up tree on an index set with one element is the empty graph. An example of a Up tree is the following:



i.e.



where $i_1 < \cdots < i_7$.

[14] See [Mikha] for more on such bases.

As with Down graphs we needn't have explicitly restricted ourselves to trees, since one effect of the excluded subgraphs is to prevent the formation of (ordered or unordered) loops in the graph. Again, the obstacle to proving Koszulness due to the presence of non-quadratic Gröbner basis elements has been avoided.

We now define an Up forest as a union of Up trees (with disjoint index sets).

Proposition 7. *The Up trees on a given index set I with m elements (in which all indices belong to at least one edge) are in bijective correspondence with the cyclic orderings of $[m] = \{1, \ldots, m\}$, or equivalently the orderings of I starting with the smallest index. This number is clearly $(m-1)!$.*

Proof. It is fairly easy to see that Up trees are what is called 'recursive' - i.e. non-planar rooted trees with vertices labeled by distinct numbers, where the labels are strictly increasing as move in the direction of the arrows. It is a classical result that there are $(m-1)!$ of these on an index set of size m. One way to see it is to place the root at the bottom of the picture with the edges pointing up, and order the children of each node by increasing size toward the left (we can do this since the trees are non-planar, i.e. the children of each node are unordered): see the sample Up tree above. Now thicken all the edges into ribbons (which are kept flat to the plane with no twisting). Finally, starting at the root go along the outside edge of the ribbon graph in a clockwise direction writing down each index the first time it is reached. The result is an ordering of the m indices starting with the smallest (in the case of the sample Up tree above we get $(i_1, i_4, i_6, i_5, i_3, i_2, i_7)$, and there are clearly $(m-1)!$ of these. It is easy to see that this procedure gives the required bijection. $\square$

Corollary 3. *The Up forests on an index set I are in bijective correspondence with the unordered partitions of I into cyclically ordered subsets.*

Remark 6. *The monomials corresponding to Up forests induced by unordered partitions of $[n] = \{1, 2, \ldots, n\}$) into cyclically ordered subsets form a basis for the algebra $\mathfrak{pb}_n^!$. Indeed, it is easy to see that Up forests are in bijective correspondence with the basis elements for $\mathfrak{pb}_n^!$ given in [Yuz], see also [Arnold] and [ShelYuz]. Also, the above excluded subgraphs correspond to the excluded monomials implied by the Gröbner basis given in [Yuz]. The fact that these Gröbner basis elements are quadratic allowed [ShelYuz] to conclude that $\mathfrak{pb}_n^!$ is Koszul.*

4.7.3 Up-Down Graphs and $\mathfrak{pvb}_n^!$

To define Up-Down graphs we first need the concept of an ordered 2-step partition (essentially due to [BarEnEtRa][15]). Namely given $n \in \mathbb{N}$ and $[n] := \{1, 2, \ldots, n\}$, first take an unordered partition of $[n]$ as $[n] = S_1 \sqcup \cdots \sqcup S_l$ where the sets S_i are cyclically ordered (and let m_i denote the minimal element of S_i). Second, take an unordered partition of the set $\mathcal{M} := \{m_i : i = 1, \ldots, l\}$ of minimal elements into distinct unordered subsets, $\mathcal{M} = M_1 \sqcup \cdots \sqcup M_k$.

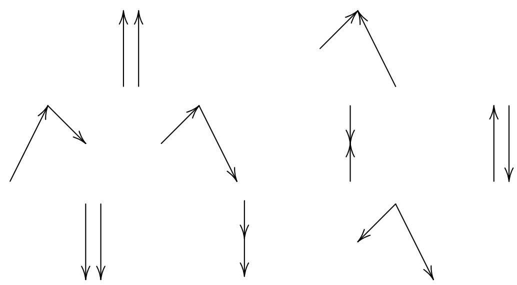
[15] See the proof of Corollary 4.6, (iii). Our *ordered* 2-step partitions differ from their '2-step partitions' in that our underlying sets S_i are cyclically ordered and theirs are unordered.

Now suppose given such an ordered 2-step partition of $[n]$. First we form the Down forest on the index set $\mathcal{M}$ with the given partition. Next we form the Up tree on each of the (cyclically ordered) sets S_i. The resulting graph is called an **Up-Down forest** on the index set $[n]$. An Up-Down forest is uniquely determined by a particular ordered 2-step partition, and conversely.

Theorem 4. *The monomials corresponding to Up-Down graphs on ordered 2-step partitions of* $[n] = \{1, \ldots, n\}$ *form a basis of* $\mathfrak{pvb}^!_n$.

Remark 7. *It is not hard to see that the Up-Down graphs on* $[n]$ *consist exactly of the red-black graphs corresponding to the monomials referred to in Proposition 4.5 of [BarEnEtRa].*[16] *These monomials are shown in that proposition to form a basis of a certain algebra* QA^0_n *related to* $\mathfrak{pvb}^!_n$*: namely, after making a certain change of basis to* $\mathfrak{pvb}^!_n = U(\mathfrak{qtr_n})^!$*, [BarEnEtRa] show that a certain filtration is defined on* $\mathfrak{pvb}^!_n$*. Then* QA^0_n *is the quadratic approximation to the associated graded of* $\mathfrak{pvb}^!_n$ *with respect to that filtration. The given basis for* QA^0_n *is then used to find the Hilbert series and to prove the Koszulness of* QA^0_n*, which in turn lead to the Hilbert series and Koszulness of* $\mathfrak{pvb}^!_n$*. It is interesting that the same collection of (Up-Down) graphs can be used to index a basis of* $\mathfrak{pvb}^!_n$ *itself and show directly that it is Koszul, as we shall see next.*

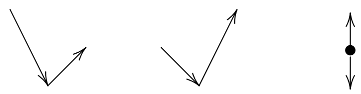
Proposition 8. *The algebra* $\mathfrak{pvb}^!_n$ *has a basis consisting of all monomials whose graph does not contain any of the following as subgraphs (again the relative heights of the endpoints indicate relative ordering of the indices, and the graphs are non-planar, so that the all edges (incoming or outgoing) incident to a particular vertex may be represented in any order without changing the graph):*



Note that the first row consists exactly of the excluded subgraphs for Up graphs. Thus if, in a graph corresponding to a basis monomial for $\mathfrak{pvb}^!_n$, we look only at the subgraph of upward arrows, we see that this subgraph must be an Up graph (and all Up graphs may arise).

The second row of excluded subgraphs features 'mixed' subgraphs, in that they each involve both an up arrow and a down arrow. It is clear that the non-excluded (= permitted) mixed subgraphs must be the following:

[16]The Down and Up graphs correspond respectively to red and black graphs in the terminology used in the definition of 2-step partition immediately prior to Proposition 4.5 of [BarEnEtRa].



As is readily seen, the effect of these excluded and non-excluded mixed subgraphs is to ensure that different Up trees (which use only up arrows) which are connected to each other by down arrows are in fact only connected to each other by down arrows between their minimal elements.

Finally, the last row of excluded subgraphs involve only downward pointing arrows, and consist precisely of the excluded subgraphs for Down graphs. Thus a graph which excludes all the subgraphs listed in the proposition will be an Up-Down graph, and conversely. Thus Theorem 4 follows from Proposition 8.

Proof of the Proposition. To begin with we linearly order the generators $\{r_{ij} : 1 \leq i \neq j \leq n\}$ of $\mathfrak{pvb}_n^!$ using the numerical order of the indices, i.e. $r_{ij} > r_{kl} \iff (i > k)$ or $(i = k$ and $j > l)$. Then, given a wedge product of generators, we first order the generators in the product in increasing order, and then we linearly order such monomials first by length and then lexicographically (we also agree that $u > 0$ for all non-zero u). This ordering (which we refer to as the lexicographical ordering) is multiplicative in the sense that if u, v, w are wedge products such that $u > v$ and $uw \neq 0$ then $uw > vw$.

We define a set $S^{(2)}$ of 'illegal' degree 2 monomials, consisting of those degree 2 monomials which can be expressed as linear combinations of 'smaller' monomials (with respect to the lexicographical ordering) using the defining relations of $\mathfrak{pvb}_n^!$. The set $S^{(2)}$ cannot be read off directly from the relations in the form (19), (20) and (21) as some of these have the same maximal terms. However one readily finds that those relations can be put in the following equivalent form (where $1 \leq i < j < k \leq n$):

$$r_{ik} \wedge r_{jk} = r_{ij} \wedge r_{jk} - r_{ji} \wedge r_{ik} \tag{28}$$
$$r_{kj} \wedge r_{ji} = r_{ji} \wedge r_{ik} - r_{ji} \wedge r_{jk} - r_{ji} \wedge r_{ki} \tag{29}$$
$$r_{ki} \wedge r_{kj} = r_{ki} \wedge r_{ij} - r_{ji} \wedge r_{ik} + r_{ji} \wedge r_{jk} + r_{ji} \wedge r_{ki} \tag{30}$$
$$r_{ik} \wedge r_{kj} = r_{ij} \wedge r_{jk} - r_{ij} \wedge r_{ik} \tag{31}$$
$$r_{jk} \wedge r_{ki} = r_{ji} \wedge r_{ik} - r_{ji} \wedge r_{jk} \tag{32}$$
$$r_{ij} \wedge r_{kj} = r_{ij} \wedge r_{jk} - r_{ij} \wedge r_{ik} - r_{ki} \wedge r_{ij} \tag{33}$$

as well as the relations (21). Each relation now has a distinct maximal term, and these have been collected on the LHS above. Thus $S^{(2)}$ consists of the union of the sets:

$$\{r_{jk} \wedge r_{ik},\ r_{kj} \wedge r_{ji},\ r_{kj} \wedge r_{ki} : 1 \leq i < j < k \leq n\} \tag{34}$$
$$\{r_{ik} \wedge r_{kj},\ r_{jk} \wedge r_{ki},\ r_{ij} \wedge r_{kj},\ 1 \leq i < j < k \leq n\} \tag{35}$$
$$\{r_{ij} \wedge r_{ji},\ r_{ij} \wedge r_{ij} : 1 \leq i \neq j \leq n\} \tag{36}$$

These monomials are readily seen to correspond with the excluded diagrams of the Proposition. The Proposition will be proved if we can show that the set $\overline{S}$ of monomials which do not contain any of the excluded 2-letter monomials $S^{(2)}$ (even after re-ordering of the generators forming the monomial) comprise a basis for $\mathfrak{pvb}^!_n$.

The proof of this fact is in the following two steps:

- show that the set $\overline{S}$ generates $\mathfrak{pvb}^!_n$; and
- show that $\overline{S}$ has the same number of elements in each degree as the basis for $\mathfrak{pvb}^!_n$ given in Theorem 3 (which implies that the elements of $\overline{S}$ are linearly independent, and hence form a basis).

The fact that $\overline{S}$ generates $\mathfrak{pvb}^!_n$ is easy, since if we have a monomial which contains (possibly after reordering its factors) an excluded 2-letter monomial, we can replace the monomial by a sum of terms in which the excluded 2-letter monomial is replaced by a smaller, legal 2-letter monomial. It is clear that all of these terms are strictly smaller than the original monomial with respect to the lexicographical ordering, because of the multiplicative property of that ordering. Hence, repeating if necessary, we must eventually reach a sum of terms none of which contains an excluded 2-letter submonomial, even after reordering of its factors - i.e. a sum of terms belonging to $\overline{S}$.

The fact that $\overline{S}$ has the same number of elements in each degree as the basis for $\mathfrak{pvb}^!_n$ given in Theorem 3 is also straightforward. Let us consider again the procedure described above for creating Up-Down graphs:

- First, take an unordered partition of $[n]$ into some number $l \leq n$ of cyclically ordered subsets (and form the unique Up graphs determined by the cyclically ordered subsets) - the number of ways of doing this is $s(n, l)$, where $s(-,-)$ denotes (unsigned) Stirling numbers of the first kind. It is easy to see that the resulting Up forests have $(n - l)$ arrows, so that the resulting monomials have degree $(n - l)$. We let m_i denote the minimal element of cycle C_i for $i - 1, \ldots, l$.
- Second, take an unordered partition of $\mathcal{M} := \{m_i : i = 1, \ldots, l\}$ as $\mathcal{M} = M_1 \sqcup \cdots \sqcup M_k$, where the M_i are unordered, and form the unique Down graph determined by this partition of $\mathcal{M}$. The number of ways of doing this is $S(l, k)$, where $S(-,-)$ denotes (unsigned) Stirling numbers of the second kind. It is easy to see that the resulting Down forests have $(l - k)$ arrows, so that the resulting monomials have degree $(l - k)$.

It is clear that the resulting Up-Down graph will have $(n-k) = (n-l)+(l-k)$ arrows, and hence will correspond to a degree $(n - k)$ monomial.

Thus if $\bar{S}^{n-k}$ denotes the monomials in $\bar{S}$ of degree $(n - k)$ we find:

$$\dim \bar{S}^{n-k} = \sum_{l=k}^{n} s(n, l)S(l, k) = L(n, k) = \dim A^{!(n-k)}$$

For the last equality we used Corollary 2, and for the second-last equality we used the so-called Lah-Stirling identity:

$$L = sS$$

where L, s and S are infinite-dimensional lower-triangular matrices whose (n,k)-th entries are, respectively, $L(n,k)$ (Lah numbers of Corollary 2), $s(n,k)$ and $S(n,k)$. See [Riordan].

This completes the proof. $\square$

Corollary 4. *The algebra* $\mathfrak{pvb}_n^!$ *(and hence also* $\mathfrak{pvb}_n$*) is Koszul.*

Proof. The fact that the monomials $\overline{S}$ not containing any of the 2-letter monomials $S^{(2)}$ form a basis for $\mathfrak{pvb}_n^!$ means that the equations (28)-(33) and (21) (whose leading terms are the $S^{(2)}$) constitute a Gröbner basis for $\mathfrak{pvb}_n^!$ (as a skew-commutative algebra - see [Mikha]). This Gröbner basis is quadratic, and hence by a result of [Yuz] [17], $\mathfrak{pvb}_n^!$ is Koszul. $\square$

Remark 8. $\mathfrak{pvb}_n^!$ *as a 'Product' of the families* $\mathfrak{pfb}_n^!$ *and* $\mathfrak{pb}_n^!$

Given the correspondence between Down forests and $\mathfrak{pfb}_n^!$, and between Up forests and $\mathfrak{pb}_n^!$, identified in Remarks 5 and 6, Theorem 4 suggests that the family of all $\mathfrak{pvb}_n^!$ (parametrized by n) may be some kind of 'product' of the families of the $\mathfrak{pfb}_n^!$ and $\mathfrak{pb}_n^!$. Indeed, one could express the Lah-Stirling identity above in the form:

$$\dim \mathfrak{pvb}_n^{!n-k} = L(n,k) = \sum_l s(n,l)S(l,k) = \sum_l \dim \mathfrak{pb}_n^{!(n-l)} \dim \mathfrak{pfb}_l^{!(l-k)}$$

As pointed out in [BarEnEtRa], $\mathfrak{pb}_n$ may be viewed as a quotient of $\mathfrak{pvb}_n$ by $\mathfrak{pfb}_n$. However, this does not explain why one might be able to view $\mathfrak{pvb}_n^!$ as the kind of product of the families $\mathfrak{pb}_n^!$ and $\mathfrak{pfb}_n^!$ suggested by the Lah-Stirling identity.

5 Final Remarks

5.1 Other Groups

One could seek to apply the PVH Criterion to determine whether other groups are quadratic. One group that comes to mind is the pure cactus group Γ, as developed for instance in [EHKR]. As a prerequisite, one would need to have a presentation for the pure cactus group, and to show that the quadratic approximation to the associated graded of $\mathbb{Q}\Gamma$ with respect to the filtration by powers of the augmentation ideal (i.e. the universal enveloping algebra of the holonomy Lie algebra of Γ) is Koszul.

[17] Theorem 6.16.

5.2 Generalizing the PVH Criterion

As mentioned in the Introduction, the PVH Criterion arguably lives naturally in a broader context than we have explored here, such as perhaps augmented algebras over an operad (or the related 'circuit algebras' of [BN-WKO]).

In a different direction, one could try to generalize the criterion to deal with filtrations of an algebra by powers of an ideal other than an augmentation ideal. A particular case of this deals with groups that exhibit a 'fibering'. For instance the virtual braid group vB_n fits into an exact sequence:

$$1 \to PvB_n \to vB_n \to S_n \to 1$$

where S_n is the symmetric group. (Similar sequences exist for the braid group and the cactus group.) In such cases it is more interesting to consider the ideal corresponding to the kernel of the induced homomorphism $\mathbb{Q}vB_n \to \mathbb{Q}S_n$, rather than the augmentation ideal of $\mathbb{Q}vB_n$. The extension of the PVH Criterion to cover these particular ideals should not be too difficult.

5.3 Further Applications of the Global Quadratic Approximation

At least if $\mathfrak{B}$ is quadratic, the algebra $\mathfrak{A}$ appears to carry the same information as K (and when K is the group ring of some group G, as G itself). The advantage of $\mathfrak{A}$ is that it is a 'quadratic algebra', in that it is a graded algebra generated in degree 1 and with ideal of relations generated by homogeneous relations of degree 2. Thus one may be able to bring the theory of quadratic algebras (including quadratic duality and Koszulness) to bear on the study of $\mathfrak{A}$, and indirectly of K or G.

One may for instance ask what 'Koszulness' of the quadratic algebra $\mathfrak{A}$ means (assuming it makes sense). It seems such Koszulness may have implications for computing the cohomology of the group G, when $K = \mathbb{Q}G$ (see Remark 2).

More generally, one might attempt to prove results about K by working with the quadratic dual algebra $\mathfrak{A}^!$. The dual algebra is often easier to work with, as seen for example by the fact that we were able, in this paper, to find 2 bases for $\mathfrak{pvb}_n^!$ and thereby prove Koszulness, whereas no basis is as yet known for $\mathfrak{pvb}_n$ itself.

In this vein, one could seek to relate the module categories of $\mathfrak{A}$ and its quadratic dual, by analogy with [BeilGS] and [Floystad] in which these categories are shown to be equivalent (under more restrictive conditions, in particular on the ground ring). One is often interested in determining whether there is an algebra map $K \to A$ which induces an isomorphism in homology (or, alternatively, a filtered algebra map $K \to A$ whose associated graded is the identity). If $\mathfrak{B}$ is quadratic, one could instead ask whether there is an algebra map $\mathfrak{A} \to A$ with such a property. We note that such a map would induce a particular structure of $\mathfrak{A}$-module on A. Hence one may be able to obtain information about the existence of such a structure (and indirectly the existence of the desired map) by instead studying $\mathfrak{A}^!$-module structures.

In attempting to pursue such questions, one immediately runs into the fact that $\mathfrak{A}$ is an algebra over the ground ring $\mathfrak{A}^0$ which in general need not be semi-simple, whereas thus far most of the results concerning quadratic duality and Koszulness apply to graded rings over a semi-simple ring (or even a field). This is true in particular of the papers [BeilGS] and [Floystad].[18]

At this point, these questions remain speculative.

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[18] These results usually also require that the graded components of the algebra be finitely generated over the ground ring. This would in general be more easily managed, by for instance assuming that the algebra K or group G is finitely generated.

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