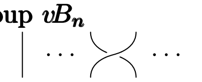
PURE VIRTUAL / FLAT BRAIDS AND REPRESENTATION STABILITY – Peter Lee

Virtual Braid Group vB_n

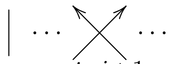
Generators: σ_i :



$i \quad i{+}1$

$(i=1,\dots,(n-1))$

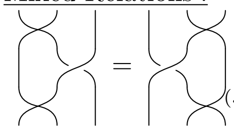
s_i :



$i \quad i{+}1$

w/ braid group and S_n relations

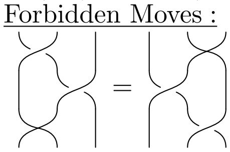
Mixed Relations :



$s_i\sigma_{i+1}^{\pm 1}s_i$
$=s_{i+1}\sigma_i^{\pm 1}s_{i+1}$
$(s_i,\sigma_j)=1,\ |i-j|\geq 2$

$i \quad i{+}1\ i{+}2 \qquad i \quad i{+}1\ i{+}2$

Forbidden Moves :



'Overcrossings commute' ('OC'), and 'UC'

$i \quad i{+}1\ i{+}2 \qquad i \quad i{+}1\ i{+}2$

Welded BG: $wB_n = \Sigma_n^+ := vB_n/\mathrm{OC}$

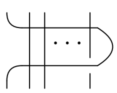
PvB_n =

$\underline{\langle R_{ij}\rangle_{1\leq i\neq j\leq n}}$

$R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij}$ (R III)

$R_{ij}R_{kl} = R_{kl}R_{ij}$ (Comm)

R_{ij} =



$i \qquad j$

$PwB_n = P\Sigma_n := PvB_n/$'pure OC'

$PfB_n := PvB_n/\ (R_{ij}R_{ji} = 1)$

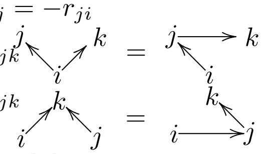
Cohomology

$H_n^k = \Lambda^\bullet\langle r_{ij}\rangle$ subject to $1\leq i\neq j\leq n,\ r_{ij} = -r_{ji}$

$r_{ij}\wedge r_{ik} = r_{ij}\wedge r_{jk}$

$r_{ij}\wedge r_{ik} = r_{ij}\wedge r_{jk}$

$[r_{ij}, r_{kl}] = 0$



Basis: Partitions of $[n]$

Action of S_n : $\sigma : r_{ij}\mapsto r_{\sigma i,\sigma j}, \quad \sigma\in S_n$

Basic fact: $H_n^{n-1} = Alt_n$ as S_n-modules

Theorem: Let $H_n^k := H^k(PfB_n,\mathbb{Q})$. As S_n-modules,
$H_n^k = \bigoplus_\alpha \mathrm{Ind}_{\prod_i S_i\wr S_{\alpha_i}}^{S_n} \bigotimes_i Alt_i\wr(-1)^{i-1}$
where $\alpha : \sum i\alpha_i = n$ and $l(\alpha) := \sum\alpha_i = n-k,$ and
$(-1)^{i-1} := Alt_i$ or 1_i according to parity of $(i-1)$

Corollary: H_n^k is an FI#-module, and is uniformly representation stable, in meaning of [CF], [CEF]

Plethystic Formulation: $\mathrm{ch}(V) =$ Frob. characteristic of V

$$\boxed{\mathrm{ch}H_n^k = \sum_\beta\prod_t v_{t,\beta_t}[e_{t+1}]\cdot h_{n-k-l(\beta)} \qquad \text{(Pleth)}}$$

- $\beta :\ \sum t\beta_t = k,\ l(\beta)\leq n-k$
- $e_p := s_{(1^p)},\ h_p := s_{(p)}$ the Schur functions
- $v_{t,\beta_t} = e_{\beta_t}$ or h_{β_t} if t is odd or even

Other groups: (Pleth) works for PB_n, $PwB_n = P\Sigma_n$ and PvB_nif we replace e_{t+1} by $\mathrm{ch}H_n^{n-1}$.

Limits on Multiplicities (by calculating plethysms)

- $V(\lambda) := V_{(\lambda_0\geq\lambda_1\geq\dots)}$ occurs in H_n^k only if $\sum_{i\geq 1}\lambda_i\geq k$
- $V(p) := V_{(p_0\geq p)}$ never occurs in H_n^k, for $p\geq 0$, except for 1 copy of $V(1)$ in degree 1

Sketch of Proof of Theorem

Whenever $\mathcal{S}\vdash[n]$, let $\bar{\mathcal{S}}$ be the induced partition of n
If $\alpha\vdash n,\ S_n$ acts transitively on $\{\mathcal{S} : \mathcal{S}\vdash[n],\ \bar{\mathcal{S}} = \alpha\}$
Let $H_n^{\mathcal{S}} = \langle m_{\mathcal{S}}\rangle$,
($m_{\mathcal{S}}$ = basis element on $\mathcal{S}$)
Then $H_n^k = \bigoplus_\alpha\bigoplus_{\mathcal{S}} H_n^{\mathcal{S}}$
($\alpha\vdash n,\ l(\alpha) = n-k$, and $\bar{\mathcal{S}} = \alpha$)
$\quad = \bigoplus_\alpha \mathrm{Ind}_{\mathrm{Stab}\mathcal{S}_\alpha}^{S_n} H_n^{\mathcal{S}_\alpha} = \bigoplus_\alpha \mathrm{Ind}_{\prod_i S_i\wr S_{\alpha_i}}^{S_n} \otimes_i H_n^{\mathcal{S}_i}$
($\mathcal{S}_\alpha\vdash[n]$ by consecutive subintervals of non-decreasing length, $\mathcal{S}_i$ the subpartitions with blocks of size i)
Write $m_{\mathcal{S}} = \prod_i m_{\mathcal{S}_i} = \prod_i m_1^i\wedge\dots\wedge m_{\alpha_i}^i$
Claim $H_n^{\mathcal{S}_i}\cong Alt_i\wr(-1)^{i-1}$
Note, if $Alt_i\wr(-1)^{i-1} = \langle u_i\rangle$ as vector space,
then $(\sigma_1,\dots,\sigma_{\alpha_i},\tau)\cdot u_i = \epsilon_i u_i, \quad \epsilon_i = (-1)^{\sum_l|\sigma_l|+(i-1)|\tau|}$
Also, $(\sigma_1,\dots,\sigma_{\alpha_i},\tau)\cdot(m_1^i\wedge\dots\wedge m_{\alpha_i}^i)$
$\quad = \sigma_{\tau(1)}m_{\tau(1)}\wedge\dots\wedge\sigma_{\tau(\alpha_i)}m_{\tau(\alpha_i)} = \epsilon_i m_{\mathcal{S}_i}$

Sketch of Proof of Stability (adapted from [Church-Farb])

There is a bijection: $\left\{\begin{array}{c}\text{Partitions of } n\\ \text{into } (n-k) \text{ parts}\\ \text{with } j \text{ parts } >1\end{array}\right\} \leftrightsquigarrow \left\{\begin{array}{c}\text{Partitions}\\ \text{of } k \text{ into } j \text{ parts}\end{array}\right\}$

(i.e., subtract/add 1 from/to each part)

Note: $S_1\wr S_{\alpha_1}\cong S_{n-k-j}$ acts trivially on H_n^k, hence

$H_n^k = \bigoplus_{1\leq j\leq n-k}\bigoplus_\alpha \mathrm{Ind}_{(\prod_{i>1}S_i\wr S_{\alpha_i})\times S_{n-k-j}}^{S_n}(\bigotimes_i Alt_i\wr(-1)^{i-1})\otimes 1_{n-k-j}$

where $\alpha : \sum_i i\alpha_i = n,\ \alpha_1 = n-k-j$

$$= \bigoplus_{1\leq j\leq n-k}\bigoplus_\beta \mathrm{Ind}^{S_n}_{\underbrace{(\prod S_{t+1}\wr S_{\beta_t})}_{H_\beta}\times S_{n-k-j}}\underbrace{(\bigotimes_t Alt_{t+1}\wr(-1)^t)}_{V_\beta}\otimes 1_{n-k-j}$$

where $\beta : \sum_t t\beta_t = k,\ l(\beta) = j\leq k$. Hence, for $n\geq 2k$,

$H_n^k = \bigoplus_{1\leq j\leq k}\bigoplus_\beta \mathrm{Ind}_{H_\beta\times S_{n-k-j}}^{S_n} V_\beta\otimes 1_{n-k-j}$. So [Hemmer] applies.