# Lecture #1 – Tuesday, September 9, 2003

## SETS

- non-example: $A = \{\text{paintings that are beautiful}\}$
- example: $A = \{\text{natural numbers divisible by } 4\}$

### Notation

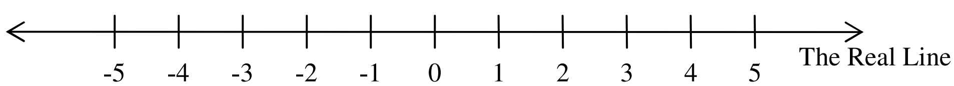
- $4 \in A$ – "4 is in $A$"
- $3 \notin A$ – "4 is not in $A$"

- $B = \{\text{natural numbers divisible by } 7\}$
  - $\begin{aligned} A \cup B &= \{\text{natural numbers divisible by 4 or 7}\} \\ &= \{4,7,8,12,14,...\} \end{aligned}$ – "union of $A$ and $B$"
  - $A \cap B = \{\text{natural numbers divisible by 4 and 7}\}$ – "intersection of $A$ and $B$"

## FACTS ABOUT REAL NUMBERS

- **N** = natural numbers = $\{1,2,3,4,...\}$
- **Z** = integers = $\{...,-3,-2,-1,0,1,2,3,...\}$
- **Q** = rational numbers = $\{1,2,3,4,...\}$
- **R** = real numbers = all this and more (+ irrational numbers)

### Geometrically




- any point on the line represents a real number

### Intervals

- $[-1,2] = \{x \in \mathbf{R} : -1 \le x \le 2\}$
- $(-1,2) = \{x \in \mathbf{R} : -1 < x < 2\}$
- $[0, \infty)$

### Ordering (Inequalities)

- If $a$, $b$ are real numbers, then exactly one of the following is true: $\begin{cases} a < b \\ a = b \\ a > b \end{cases}$

### Important Properties

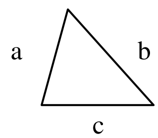
- If $a < b$ and $c > 0$, then $ac < bc$
- If $a < b$ and $c < 0$, then $ac > bc$

### Distance/Absolute Values

- $|a| = \text{distance from } a \text{ to } 0 = \begin{cases} a, \text{if } a \geq 0 \\ -a, \text{if } a < 0 \end{cases}$
- $|a-b| = \text{distance between } a \text{ and } b$

### Triangle Inequality

- $|a+b| \leq |a| + |b|$
- analogous to triangle theorem from geometry




  - $c \leq a+b$ c

# Lecture #2 – Thursday, September 11, 2003

## TRIANGLE INEQUALITY

- $|a+b| \leq |a| + |b|, a, b \in \mathbf{R}$

### Proof (A Proof By Cases)

1) If $a, b$ both $\geq 0$
   - Then $a+b \geq 0$
   - $|a+b| = a+b = |a| + |b|$
2) If $a > 0, b < 0, |a| \geq |b|$
   - Then $a+b \geq 0$
   - $|a+b| = a+b$
     $|a| + |b| = a-b$
   - Because $b < 0$, so $a+b < a-b$
3) If
   - Then $a+b \geq 0$
   - $|a+b| = a+b = |a| + |b|$
4) $a > 0, b < 0, |a| < |b|$
5)
6)

# Review of Inequalities

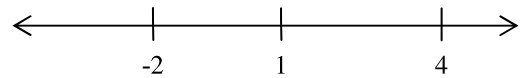
## Solve

1)
$$2x+3\le 6$$
$$2x\le 3$$
$$x\le \frac{3}{2}$$
$$x\in\left(-\infty,\frac{3}{2}\right]$$

2)
$$-2x+3\le 5$$
$$-2x\le 2$$
$$x\ge -1$$
$$x\in[-1,\infty)$$

3) $(x-1)(x-4)(x+2)>0$

Where is it = 0? $x=1,4,-2$

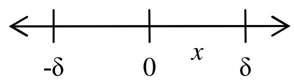



| | | | | |
|---|---|---|---|---|
| $(x-1)$ | - | - | + | + |
| $(x-4)$ | - | - | - | + |
| $(x+2)$ | - | + | + | + |
| Product | - | + | - | + |

$\therefore x\in(-2,1)\cup(4,\infty)$

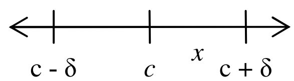
## Inequalities With Absolute Values

- $|x|<\delta$



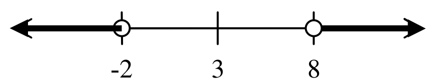

– it means that $-\delta<0<\delta$, $x\in(-\delta,\delta)$

- $|x-c|<\delta$



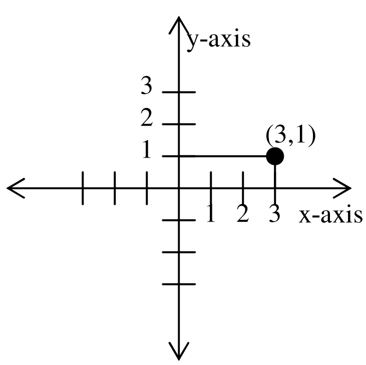

– it means that $c-\delta<x<c+\delta$, $x\in(c-\delta,c+\delta)$

- $|x-3|>5$
  $x-3>5$
  $x-3<-5$
  $\therefore(-\infty,-2)\cup(8,\infty)$

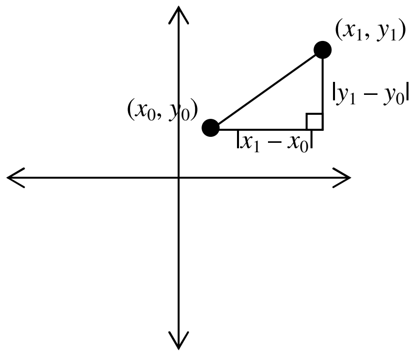



# Coordinate Geometry

## Rectangular Coordinates



### Pythagorean Theorem




$$d = \sqrt{|x_1 - x_0|^2 + |y_1 - y_0|^2}$$

### Line Equations

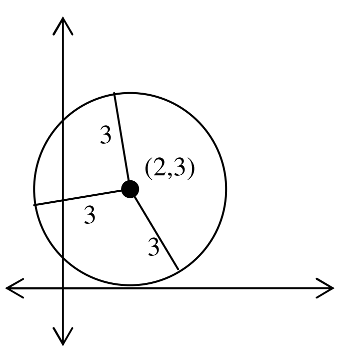
- $y - y_0 = m(x - x_0)$ – "point-slope" form
- $y = mx + b$ – "y-intercept" form
- $ax + by + c = 0$ – standard form

- If $\begin{cases} y = m_1 x + b_1 \\ y = m_2 x + b_2 \end{cases}$,
    - Two lines are parallel if $m_1 = m_2$
    - Two lines are perpendicular if $m_1 m_2 = -1$

### Conic Sections

- $ax^2 + by^2 + cx + dy + e = 0, a, b, c, d, e$ are constants
- Get ellipses, hyperbolas, parabolas

- $(x-2)^2 + (y-3)^2 = 9$
  $\sqrt{(x-2)^2 + (y-3)^2} = 3$




## FUNCTIONS

- A rule ("black box") that takes an input and produces a <u>single</u> output

### Examples

- $f(x) = x^2$
  - Domain of *f* is the set of legitimate inputs – $\text{domain}(f) = (-\infty, \infty)$
  - Range of *f* is the set of legitimate outputs – $\text{range}(f) = [0, \infty)$

- $f(x) = \sqrt{9 - x^2}$
  - $\text{domain}(f) = [-3, 3]$
  - $\text{range}(f) = [0, 3]$

### Piecewise Defined Functions

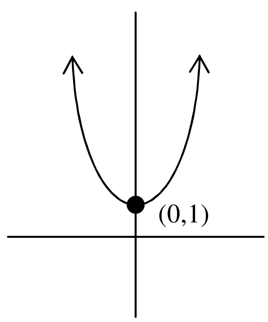
- $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

## Graphs and Functions

- $\text{graph}(f) = \{(x, y) : x \in \text{dom}(f), y = f(x)\}$

### Example

$f(x) = x^2 + 1$
$\text{domain}(f) = (-\infty, \infty)$
$range(f) = [1, \infty)$




## Some Basic Functions

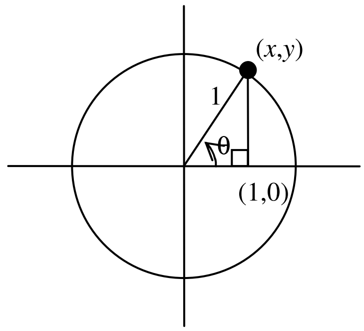
### Polynomials

- $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$
  - $a_i$ are constants
  - $\text{domain}(f) = (-\infty, \infty)$

### Rational Functions

- $f(x) = \dfrac{P(x)}{Q(x)}$, where $P, Q$ are polynomials
  - $\text{domain}(f) = \{x \in \mathbf{R} : Q(x) \neq 0\}$

## TRIGONOMETRIC FUNCTIONS




$\sin\theta = y$

$\cos\theta = x$

$\tan\theta = \dfrac{y}{x}$

$\csc\theta = \dfrac{1}{y}$

$\sec\theta = \dfrac{1}{x}$

$\cot\theta = \dfrac{x}{y}$

### Example of a Trigonometric Identity

- $\sin^2\theta + \cos^2\theta = 1$, because $x^2 + y^2 = 1$

## COMBINING FUNCTIONS

- If $f$, $g$ are functions
    - "sum" – $(f+g)(x) = f(x) + g(x)$
    - "difference" – $(f-g)(x) = f(x) - g(x)$
    - "product" – $(f \cdot g)(x) = f(x) \cdot g(x)$
    - Warning: domain is $\text{domain}(f) \cap \text{domain}(g)$

### Example

- Given $f(x) = \sqrt{1+x}$ and $f(x) = \sqrt{1+x}$, $g(x) = \sqrt{-x+5}$
    - $\text{domain}(f) = [-1, \infty)$<br>$\text{domain}(g) = (-\infty, 5]$
    - $\text{domain}(f+g) = [-1, 5]$

## COMPOSITION OF FUNCTIONS (TWO "BLACK BOXES" IN SUCCESSION)

- $(f \circ g)(x) = f(g(x))$
- Warning: $f \circ g \neq g \circ f$

### Examples

- Given $f(x) = x + 3$ and $g(x) = x^2$
    - $(f \circ g)(x) = x^2 + 3$
    - $(g \circ f)(x) = (x+3)^2$
    - $\text{domain}(f \circ g) = \{x \in \text{domain}(g) : g(x) \in \text{domain}(f)\}$

- Given $f(x) = \sqrt{x-2}$ and $g(x) = \sqrt{x}$
    - $\text{domain}(f \circ g) = [4, \infty)$

# Lecture #3 – Tuesday, September 16, 2003

## Introduction To Proofs: "Proofs Are Your Friends"

### Mathematical Rigor: To What End?

- Moral: Need rigor to obtain a correct, elegant solution which doesn't come from brute force

### Proofs: To What End?

- Informally, a proof is a guarantee that a claim is true

### Definitions

- Proposition – a sentence that is either true or false
- Theorem – a proposition that is guaranteed by a proof
- Proof – showing a theorem follows logically from the set of axioms – ex: B follows logically from A if B is true in all possible worlds where A is true

### Examples

1) The acceleration of a rigid body is proportional to force applied.
    - False is real world
2) For all integers *n*, if $n > 2$, there are no positive integers *a*, *b*, *c* such that $a^n + b^n = c^n$
    - True
3) For all non-negative integers *n*, $n^2 + n + 41$ is prime
    - False – $n = 41$ would make it false (counterexample)

### Notation

- $\forall$ – "for all" – universal quantifier
- $\exists$ – "there exist", "for some" – existential quantifier

| $P$ | $Q$ | $\neg Q$<br>not $Q$ | $P \wedge Q$<br>$P$ and $Q$ | $P \vee Q$<br>$P$ or $Q$ | $P \Rightarrow Q$<br>if $P$, then $Q$ | $P \Leftrightarrow Q$<br>$P$ iff $Q$ | $\neg Q \Rightarrow \neg P$ |
|---|---|---|---|---|---|---|---|
| False | False | True | False | False | True | True | True |
| False | True | True | False | True | True | False | True |
| True | False | False | False | True | False | False | False |
| True | True | False | True | True | True | True | True |

## I: Proof By Enumeration

### Example 1

- Given: Roses are red and violets are blue.
- Prove: Roses are red.
- Proof: $4^{th}$ line from the table ( $P \wedge Q$ )

### Example 2

- Given: If it rains today, I'll eat my hat. It rains today.
- Prove: I'll eat my hat.

- Proof: 4th line from the table ( $P \Rightarrow Q$ )

# Lecture #4 – Thursday, September 18, 2003

## II: Proof By Contrapositive ("Indirect Proof")

### Example 1

- If John is at work, then he's logged in. ( $P \Rightarrow Q$ )
- If John is not logged in, then he's not at work. ( $\neg Q \Rightarrow \neg P$ )
- Warning: Converse of $P \Rightarrow Q$ is $Q \Rightarrow P$
- Moral: Proving contrapositive is logically equivalent to proving the original statement

### Example 2

Prove: For any integer $n$, if $n^2$ is even, then $n$ is even.

Proof:

- Reformulate: If $n$ is odd, then $n^2$ is odd.

1) If $n$ is odd, then $n = 2a + 1, a \in \mathbf{Z}$
2) If $n = 2a + 1$, then $n^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1$
3) Because $a \in \mathbf{Z}$, $2a^2 + 2a \in \mathbf{Z}$
4) Because $2a^2 + 2a \in \mathbf{Z}$, $n^2$ is odd

QED

### Example of A Non-Proof

Prove: The "Theorem" $1 = -1$

Proof:

- $1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1} \cdot \sqrt{-1} = \left(\sqrt{-1}\right)^2 = -1$ QED

- But $\sqrt{a \cdot b} \neq \sqrt{a} \cdot \sqrt{b}$ for all $a$ and $b$!
- Moral: Justify each step.

### Other Classical Errors

- Divide both sides of an equation by a variable – $ax = bx \Rightarrow a = b$ – what if $x = 0$ ?
- Divide both sides of an inequality by a variable – $ax < bx \Rightarrow a < b$ – what if $x \leq 0$ ?

## III: Proof By Contradiction

- Want to show (WTS): $P$
- Idea: Assume $P$ is not true, then show that a contradiction occurs when combining it with the axioms.

### Example

Prove: $\sqrt{2}$ is irrational.

Proof:

1) Assume that $\sqrt{2}$ is rational. That means $\exists a, b \in \mathbf{Z}$ such that $\sqrt{2} = \frac{a}{b}$, where $a$, $b$ have no common factors.
2) Then $2 = \frac{a^2}{b^2}$, so $2b^2 = a^2$
3) That means $a^2$ is even. So by theorem above, $a$ is even.
4) $a = 2k$ for some $k \in \mathbf{Z}$, so $a^2 = 4k^2$.
5) So $2b^2 = 4k^2$, so $b^2 = 2k^2$.
6) So $b^2$ is even by theorem above.
7) 2 divides both $a$ and $b$.
8) Contradiction: $a$, $b$ have no common factors. Thus, $\sqrt{2}$ is irrational.

- lemma(s) – smaller theorems used to prove bigger theorems

## IV: Prove By Cases

- Warning: It only works if there are a finite number of cases.

### Example

Prove: The Triangle Inequality $|a+b| \le |a| + |b|, a, b \in \mathbf{R}$

Proof:

- Case 1: $a, b \ge 0 \ldots$
- Case 2: $a, b < 0 \ldots$
- Case 3: $a \ge 0, b < 0, |a| \ge |b| \ldots$
- Case 4: $a \ge 0, b < 0, |a| < |b| \ldots$

## V: Proof By Induction

- What do you do when there are infinitely many cases?
- Example: Universally quantified statements over $\mathbf{N}$ – i.e. "for all $n \in \mathbf{N} \ldots$"

### Principle of Induction

- If you can prove that: some property holds for $n = 0$ ("base case") and if the property holds for $n = k$, it holds for $n = k + 1$ ("inductive step"), then the property holds for all $n \in \mathbf{N}$.

### Example

Prove: $\forall n \in \mathbf{N}$, $n^3 - n$ is divisible by 3.

Definition: For all $a, b \in \mathbf{Z}$, "$a$ divides $b$" ($a \mid b$) iff $\exists q \in \mathbf{Z}, b = a \cdot q$

Proof:

- "Base Case": If $n = 0$, the $0^3 - 0 = 0$. $3 \mid 0$ because $0 = 3 \cdot 0$
- "Inductive Step": Suppose it is true for $n = k$ so $3 \mid k^3 - k$ and $k^3 - k = 3q$ for some $q \in \mathbf{Z}$.

- Now WTS: $3 \mid (k+1)^3 - (k+1)$

  $(k+1)^3 - (k+1)$
  $= k^3 + 3k^2 + 3k + 1 - k - 1$
  $= k^3 - k + 3k^2 + 3k$
  $= 3q + 3(k^2 + k)$

  So, by definition of divisibility, $3 \mid (k+1)^3 - (k+1)$

QED

### Non-Proof Example

"Theorem": All iMacs are the same colour.
Prove: All $n \in \mathbf{N}$, a set of iMacs of size $n$ is monochromatic.

Proof:

- "Base Case": $n = 1$, then every set of one iMacs is one colour
- "Inductive Step": Suppose true for $n = k$, WTS for $n = k + 1$
  Let $S = \{i_1, i_2, \ldots i_{k+1}\}$
  $S_1 = \{i_1, i_2, \ldots i_k\}$, $S_2 = \{i_2, i_3, \ldots i_{k+1}\}$
  $S_1 \cap S_2 = \{i_2 \ldots i_k\}$
- Since $S_1$, $S_2$ are monochromatic and they intersect in $S_1 \cap S_2$, so $S$ is monochromatic. So all iMacs are monochromatic.

QED

- But the "Inductive Step" fails for $n = 2$

## VI: Strong Induction

- Same as Induction, except $n = k$ is replaced with "if the property holds for $n \le k$ …"

### Example (Half of "Fundamental Theorem of Arithmetic")

Prove: Any natural number $n$, for $n > 1$, can be written as a product of primes

Proof (Attempt with Induction):

- "Base Case": $n = 2$. $n$ is a prime.
- "Inductive Step": Suppose that $n = k$, $k$ can be written as a product of primes, WTS for $n = k + 1$
  STUCK – primes that divide $k$ don't divide $k + 1$

Proof (Attempt with Strong Induction):

- "Base Case": $n = 2$. $n$ is a prime.
- "Inductive Step": Suppose that $n \le k$, $k$ can be written as a product of primes, WTS for $n = k + 1$
  - Case 1: $n = k + 1$ is prime.
  - Case 2: $n = k + 1$ is not prime, so $\exists a, b \in \mathbf{N}, 1 < a, b < k + 1$ so that $k + 1 = a \cdot b$. Because $a, b \le k$, so $a \cdot b$ can be written as a product of primes. Therefore, $k + 1$ is a product of primes
  - Therefore, by strong induction, it is true for all $n \in \mathbf{N}$.

QED

# Lecture #5 – Tuesday, September 23, 2003

## WHY CALCULUS?

- Bascially, because the study of natural phenomena involves:
    - rates of change
    - non-linear phenomena – calculus allows for linear approximation

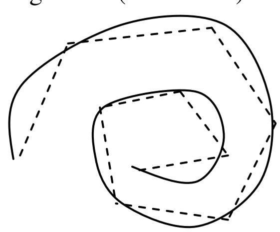
## LIMITS

### Basic Idea

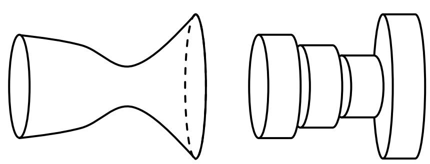
1) A (precise) way to take successive approximations and get an <u>exact</u> answer
2) Start with things you can easily compute (ex: lengths of line segments, area or rectangles) and build up to computations for more complicated objects

### Pictures

1) Length of a (non-linear) curve:



2) Volume of a solid:

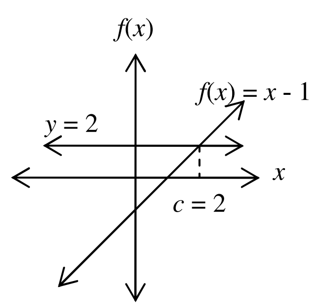


### The Question

What happens to a function value $f(x)$ as $x$ approaches a fixed number $c$?

### Example

- $f(x) = x - 1$
  $c = 2$
- The function value then approaches 1 – $\lim_{x \to 1}(x-1) = 1$




### Notation

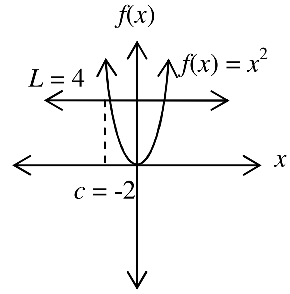
$\lim_{x \to c} f(x) = L$

### Definition (Attempt 1)

- $f(x)$ gets closer to $L$ as $x$ gets close $c$

### Example

- $f(x) = x^2$
  $c = -2$

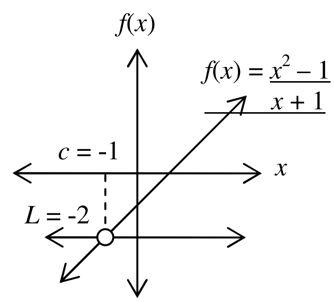



### Technical Note

Doesn't matter whether the function is <u>defined</u> at $x = c$ or not – ask only about values of $x$ close to $c$, but not equal $c$

### Example

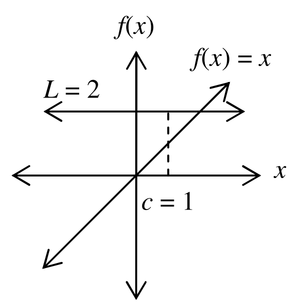
- $f(x) = \dfrac{x^2 - 1}{x + 1}$
  $c = -1$
- $\lim\limits_{x \to -1} \dfrac{x^2 - 1}{x + 1} = -2$




### Example

Why is something <u>not</u> the limit?

- $f(x) = x$
  $c = 1$
  $L = 2$

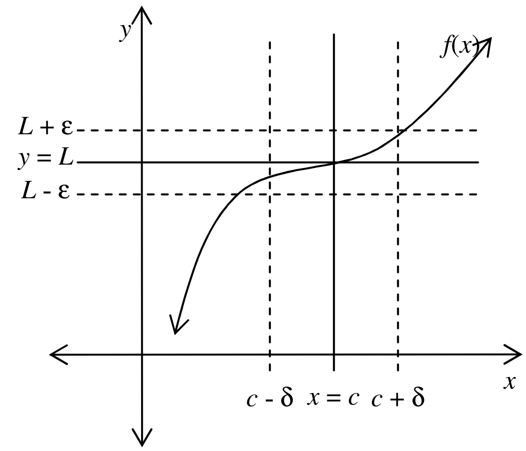



| $x$ | $\lvert f(x) - 2 \rvert$ |
|---|---|
| 1.5 | 0.499 |
| 1.25 | 0.249 |
| 1.1 | 0.099 |

- What went wrong? $\lvert f(x) - 2 \rvert > 0.5$ if $\lvert x - 1 \rvert < 0.5$

### Definition (Attempt 2)

- $\lim_{x \to c} f(x) = L$ if no matter what distance ε (on y-axis) I pick, if I say $|f(x) - L| < \varepsilon$, then you should be able to find another distance δ (on x-axis) so that you can guarantee that if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$




# Lecture #6 – Thursday, September 25, 2003

## LIMITS (CONTINUED)

### Definition (Attempt 3 and Final): Epsilon-Delta Definition of Limits

- Let $f(x)$ be defined in some interval of the form $(c - p, c) \cup (c, c + p)$ for some $p > 0$
- We say $\lim_{x \to c} f(x) = L$ if for all $\varepsilon > 0$, there exist a $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - L| < \varepsilon$ – $\forall \varepsilon > 0, \exists \delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

### Example

$f(x) = 2x + 1$

$c = 1$

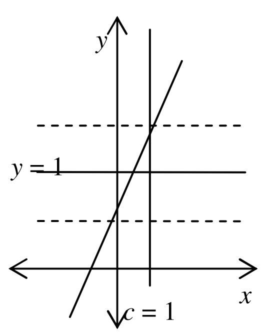
$L = 3$

$|(2x+1) - 3| < 1$

$|2x - 2| < 1$

$2|x - 1| < 1$

$|x - 1| < \frac{1}{2}$




| If ε is equal to… | Can choose δ to be… |
|---|---|
| $\varepsilon = 1$ | $\delta = \frac{1}{2}$ |
| $\varepsilon = \frac{1}{10}$ | $\delta = \frac{1}{20}$ |
| Any ε | $\delta = \frac{\varepsilon}{2}$ |

### Technical Note

- Warning: The limit of the function at $x = c$ can be different from the function value $f(c)$
- A limit exists exactly when both left hand side and right hand side limits exist and they are equal
- Another way that limits might not exist – ex: $f(x) = \frac{1}{x}, c = 0$

### Definition

Left hand limit:

- Assume $f$ is defined on some interval $(c-p,c)$ for $p>0$
- Then $\lim_{x\to c^-} f(x)=L$ if $\forall\varepsilon>0,\exists\delta>0$ such that $c-\delta<x<c\Rightarrow|f(x)-L|<\varepsilon$

Right hand limit:

- Assume $f$ is defined on some interval $(c,c+p)$ for $p>0$
- Then $\lim_{x\to c^+} f(x)=L$ if $\forall\varepsilon>0,\exists\delta>0$ such that $c<x<c+\delta\Rightarrow|f(x)-L|<\varepsilon$

### Example

Prove (using ε-δ) that $\lim_{x\to 2} x^2=4$

- Let $\varepsilon>0$
- Want: $|x^2-4|<\varepsilon$ – our task is to find a δ so that $|x-2|<\delta$ then $|x^2-4|<\varepsilon$ holds
- Algebra: $(x^2-4)=(x+2)(x-2)$
- Trick: From now on, assume we'll take $\delta<1$
  That means we can assume $|x-2|<1$ so $1<x<3$
- That means $\begin{matrix}|x+2|<|x|+2\\|x+2|<5\end{matrix}$, so $|x^2-4|=|x+2||x-2|<5|x-2|$
- Choose $\delta=\min\left(\frac{\varepsilon}{5},1\right)$
- Check: If $\delta<\frac{\varepsilon}{5}$, then $|x^2-4|<5|x-2|<5\left(\frac{\varepsilon}{5}\right)<\varepsilon$

Q.E.D.

### Example

Prove: $\lim_{x\to 4}\sqrt{x}=2$

- Want: For $\varepsilon>0$, $|\sqrt{x}-2|<\varepsilon$ for small enough $|x-4|$
- By the properties of the square root function, we already know $\delta<4$ – so $0<x<8$
- Algebra: $(x-4)=(\sqrt{x}+2)(\sqrt{x}-2)$
  $|x-4|=|\sqrt{x}+2||\sqrt{x}-2|$
- $2<|\sqrt{x}+2|$ because $x>0$

- $\left|\sqrt{x}-2\right| < \frac{1}{2}|x-4|$
- Choose: $\delta = \min(4, 2\varepsilon)$
- Check: If $|x-4| < \delta$, then $\left|\sqrt{x}-2\right| < \frac{1}{2}|x-4| < \frac{1}{2}(2\varepsilon) = \varepsilon$

Q.E.D

### Example

Prove: $\lim_{x\to 2}\frac{1}{x^2} = \frac{1}{4}$

- Want: For $\varepsilon > 0$, want $\left|\frac{1}{x^2}-\frac{1}{4}\right| < \varepsilon$ for $x$ close enough to 2 (ie: $|x-2|$)
- We already know $\delta < 2$, $0 < x < 4$
- Algebra: $\frac{1}{x^2}-\frac{1}{4} = \frac{4-x^2}{4x^2} = \frac{(2+x)(2-x)}{4x^2}$
- $\left|\frac{1}{x^2}-\frac{1}{4}\right| = \left|\frac{(2+x)(2-x)}{4x^2}\right| = \frac{|(2+x)(2-x)|}{|4x^2|} = \frac{|(2+x)|}{|4x^2|}|x-2|$
- Make additional requirement: $\delta < 1$, so $1 < x < 3$
- $|2+x| < 2+|x| < 2+3 < 5$

  $\left|\frac{1}{4x^2}\right| < \frac{1}{4}$ because $x > 1$
- Therefore, $\frac{|(2+x)|}{|4x^2|} < \frac{5}{4}$
- So $\left|\frac{1}{x^2}-\frac{1}{4}\right| < \frac{5}{4}|x-2|$
- Choose $\delta = \min\left(1, \frac{4}{5}\varepsilon\right)$
- Check: If $|x-2| < \delta$, then $\left|\frac{1}{x^2}-\frac{1}{4}\right| < \frac{5}{4}|x-2| < \varepsilon$

# Lecture #7 – Tuesday, September 30, 2003

## COMPUTATIONS WITH LIMITS

### Building Blocks

1) $\lim\limits_{x \to c} x = c$
2) $\lim\limits_{x \to 0} |x| = c$
3) $\lim\limits_{x \to c} k = k$

### Theorem 1

- If $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = M$
    - $\lim\limits_{x \to c} (f(x) + g(x)) = L + M$
    - $\lim\limits_{x \to c} (\alpha f(x)) = \alpha \cdot L$
    - $\lim\limits_{x \to c} (f(x) \cdot g(x)) = L \cdot M$

### Proof 1

Given $\varepsilon > 0$, we want to find $\delta > 0$ so that if $|x - c| < \delta$ then $|f(x) + g(x) - (L + M) < \varepsilon|$

- Algebra: $|f(x) + g(x) - L - M| = |(f(x) - L) + (g(x) - M)| \le |f(x) - L| + |g(x) - M|$
- We know that $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = M$
- If $|f(x) - L| < \frac{\varepsilon}{2}$ and $|g(x) - M| < \frac{\varepsilon}{2}$, then we're in business!
- For $\frac{\varepsilon}{2} > 0$, we can find $\delta_1$ so that $|f(x) - L| < \frac{\varepsilon}{2}$ if $|x - c| < \delta_1$, and we can find $\delta_2$ so that $|g(x) - M| < \frac{\varepsilon}{2}$ if $|x - c| < \delta_2$
- To make sure both are true, we pick $\delta = \min(\delta_1, \delta_2)$
- Check: If $|x - c| < \delta \le \delta_1, \delta_2$, then $|f(x) + g(x) - (L + M)| \le |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Q.E.D.

### Corollaries (theorems that follows (almost) immediately)

1) $\lim\limits_{x \to c} (f(x) - g(x)) = L - M$
2) $\lim\limits_{x \to c} (\alpha_1 \cdot f_1(x) + \alpha_2 \cdot f_2(x) + \ldots + \alpha_n \cdot f_n(x)) = \alpha_1 L_1 + \alpha_2 L_2 + \ldots \alpha_n L_n$ if $\lim\limits_{x \to c} f_i(x) = L_i$
3) $\lim\limits_{x \to c} (f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x)) = L_1 \cdot L_2 \cdot \ldots \cdot L_n$
4) If $P(x) = \text{polynomial} = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$, then $\lim\limits_{x \to c} P(x) = P(c) = a_n (c)^n + a_{n-1} (c)^{n-1} + \ldots + a_0$

### Example

- $\lim\limits_{x \to 2} 2x^2 + 10x = 28$

- Now what happens if the polynomial is in the denominator?

**Example**

- $\lim_{x \to 2} \frac{1}{2x^2+1} = \frac{1}{9}$

**Theorem 2**

- If $\lim_{x \to c} g(x) = M, M \neq 0$, then $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M}$
    - Proof: For $\varepsilon > 0$ we want $\delta > 0$ so that $\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \varepsilon$
    - Algebra: $\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M - g(x)}{g(x)M}\right| = \left|\frac{1}{g(x)M}\right| |g(x) - M|$
    - Choose $\delta_1$ so that $|g(x) - M| < \frac{|M|}{2}$. If $x$ is in the region, then

$$\left|\frac{1}{g(x)M}\right| |g(x) - M| < \left|\frac{g(x) - M}{\frac{|M|^2}{2}}\right| = \frac{2}{M^2} |g(x) - M|$$

    - If I choose $\delta_1$ so that $|g(x) - M| < \left(\frac{M^2}{2}\right)\varepsilon$
    - Check: If $|x - c| < \delta$, then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \frac{1}{\frac{|M|^2}{2}} |g(x) - M|$$

$$< \frac{1}{\frac{|M|^2}{2}} \cdot \varepsilon \frac{|M|^2}{2} = \varepsilon$$

# Lecture #8 – Thursday, October 2, 2003

**Examples**

- $\lim_{x \to 3} \frac{1}{2x^2+1} = \frac{1}{19}$
- $\lim_{x \to -1} \frac{1}{x^3} = -1$

**Theorem 3**

- If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M, M \neq 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$

- Proof: $\frac{f(x)}{g(x)} = f(x)\cdot\frac{1}{g(x)}$

$$\begin{aligned}\lim_{x\to c}\frac{f(x)}{g(x)} &= \lim_{x\to c} f(x)\cdot\frac{1}{g(x)}\\ &= \lim_{x\to c} f(x)\cdot\lim_{x\to c}\frac{1}{g(x)}\\ &= L\cdot\frac{1}{M} = \frac{L}{M}\end{aligned}$$

**Example**

- $\lim_{x\to 2}\frac{2x^2+x}{x^2-3} = \frac{10}{1} = 10$

- <u>The hard case</u>: What happens if the denominator approaches 0?

**Example**

- $\lim_{x\to 1}\frac{1}{x-1}$ DNE
- $\lim_{x\to 1}\frac{x^2-1}{x-1} = 2$

**Theorem 4**

- If $\lim_{x\to c} f(x) = L, L \neq 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c}\frac{f(x)}{g(x)}$ does not exist
  - Proof: By contradiction
    - Suppose $\lim_{x\to c}\frac{f(x)}{g(x)} = M$

      $$\lim_{x\to c} f(x) = \lim_{x\to c}\frac{f(x)}{g(x)}\cdot g(x)$$

    - Then $$= \lim_{x\to c}\left(\frac{f(x)}{g(x)}\right)\cdot\lim_{x\to c} g(x)$$ $$= M\cdot 0 = 0$$
    - Contradiction because $\lim_{x\to c} f(x) \neq 0$. So $\lim_{x\to c}\frac{f(x)}{g(x)}$ DNE.

**Moral**

If you're asked to do a limit computation $\frac{f(x)}{g(x)}$, you should proceed in several steps:

1) See if the denominator is equal to 0.
2) If NO, plug it in at $x = c$. Done.
   If YES, check if numerator limit is 0.
3) If NO, then the limit does not exist. Done.
   If YES, do algebra to simplify the expression and go back to Step 1.

**Example**

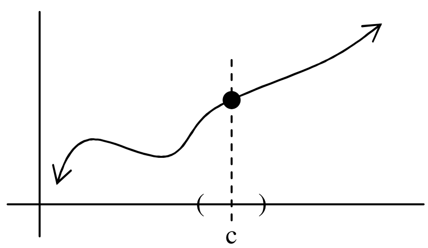
- $\lim_{x\to 4}\frac{x-2}{\sqrt{x-2}}$ DNE (by Theorem 4)
- $\lim_{x\to 4}\frac{x^2+4}{x+2}=2$

## CONTINUITY

**Idea**

- A function is continuous if you can draw the graph "without lifting your pencil"

**Example**

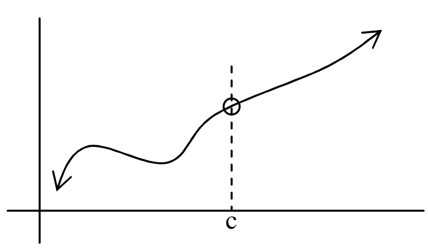



**Definition**

- Suppose *f* is defined on $(c-p, c+p)$. The function *f* is continuous at *c* if:
    - The limit $\lim_{x\to c} f(x)$ exist
    - the limit $\lim_{x\to c} f(x)=f(c)$

- How can a function <u>not</u> be continuous?
    - Case A: The limit exist, but $\neq$ $f(c)$
    - Case B: The limit doesn't exist

**Example of Case A**

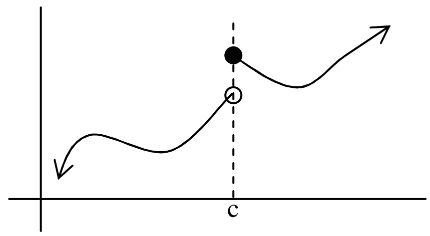



- "removable discontinuity"

**Example of Case B – "Essential Discontinuity"**

- Different ways it could happen

- The one-side limits exist, but they don't agree




  - One of the side limits exist, but only one
    $$f(x)=\begin{cases}0, x\le 0\\ \sin\left(\frac{1}{x}\right), x>0\end{cases}$$
  - Neither of the one sided limits exist
    $$f(x)=\begin{cases}0, x\in \mathbf{Q}\\ 1, x\notin \mathbf{Q}\end{cases}$$

### Examples

- Functions that are continuous:
  - Polynomials: $\lim_{x\to c} P(x)=P(c)$
  - Absolute value (by Building Blocks): $\lim_{x\to c}|x|=|c|$
  - Square root: $\lim_{x\to c}\sqrt{x}=\sqrt{c}$ for $c>0$

### Theorem 1

- If $f$ and $g$ are both continuous at $c$, then $f+g$, $f-g$, $\alpha f$, $f\cdot g$, $\frac{f}{g}$ if $g(c)\neq 0$ are continuous.

### Example

- $h(x)=\sqrt{x}+3|x|+\frac{1}{x+1}$ is continuous at any $x>0$

### Another Definition For Continuity

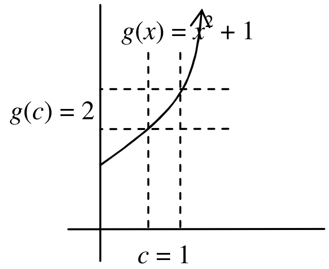
- $f$ is continuous at $c$ if $\forall\varepsilon>0$, $\exists\delta>0$ such that if $|x-c|<\delta$ then $|f(x)-f(c)|<\varepsilon$

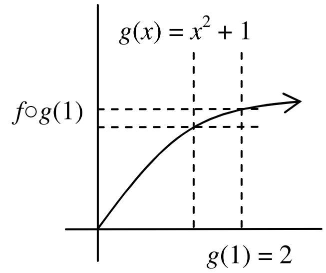
### Theorem 2

- If $g(x)$ is continuous at $c$ and $f$ is continuous at $g(c)$, then $f\circ g$ is continuous at $c$

### Example

$g(x) = x^2 + 1$
$f(x) = \sqrt{x}$
$f \circ g = \sqrt{x^2 + 1}$



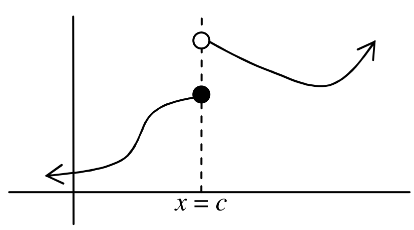

- Proof: Let $\varepsilon > 0$. We want to find $\delta > 0$ so that if $|x - c| < \delta$ then $|f(g(x)) - f(g(c))| < \varepsilon$
    - Do one function at a time
    - Choose $\delta_1 > 0$ so that if $|t - g(c)| < \delta_1$, then $|f(t) - f(g(c))| < \varepsilon$ because *f* is continuous at $g(c)$
    - Choose $\delta > 0$ so that if $|x - c| < \delta$, then $|g(x) - g(c)| < \delta_1$ because *g* is continuous at *c*
    - Check: If $|x - c| < \delta$ then $|g(x) - g(c)| < \delta_1$ then $|f(g(x)) - f(g(c))| < \varepsilon$

### Example

- If $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$, $h(x) = \dfrac{1}{x}$ (continuous, $c \neq 0$)
  $k(x) = h \circ g \circ f(x)$
- $= \dfrac{1}{\sqrt{x^2 + 1}}$ is also continuous for any *c*

### Detecting Discontinuities

- Definition: *f* is continuous from the left if $\lim_{x \to c^-} f(x) = f(c)$; *f* is continuous from the right if $\lim_{x \to c^+} f(x) = f(c)$



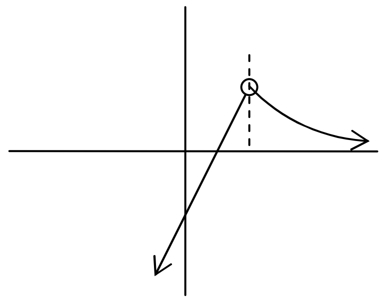

### Another Way To Say A Function Is Continuous

- *f* is continuous at $x = c$ iff
    - $\lim_{x \to c^-} f(x)$ exists and is equal to $f(c)$
    - $\lim_{x \to c^+} f(x)$ exists and is equal to $f(c)$

**Example**

Determine the discontinuities of the function:

$$g(x)=\begin{cases} 2x-1 \text{ if } x<1 \\ 0 \text{ if } x=1 \\ \dfrac{1}{x^2} \text{ if } x>1 \end{cases}$$

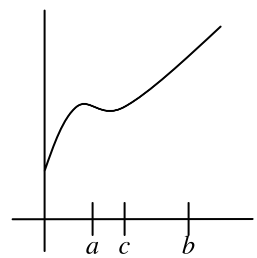


- Certainly the function is continuous at any point in $(-\infty,1)$ or any point in $(1,\infty)$. So the only point in question is $c=1$.
- $\lim\limits_{x\to 1^-} g(x)=\lim\limits_{x\to 1^-} 2x-1=1$ and $\lim\limits_{x\to 1^+} g(x)=\lim\limits_{x\to 1^+}\dfrac{1}{x^2}=1$ which is not equal to $g(1)=0$. So this function is not continuous. $c=1$ is removable discontinuity.

**Last Definition (For Now) For Continuity Of Fuctions**

- Let $(a,b)$ be an interval where *f* is defined.
- Definition: *f* is continuous on $(a,b)$ if *f* is continuous at <u>every</u> point *c* in $(a,b)$

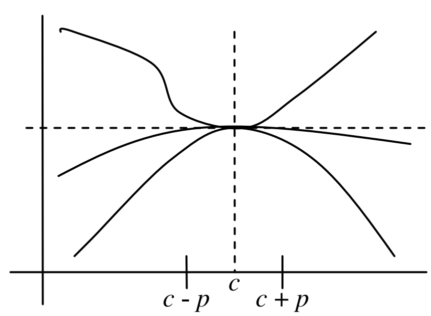
- Let $[a,b]$ be an interval where *f* is defined.
- Definition: *f* is continuous on $[a,b]$ if *f* is continuous on $(a,b)$ and it's continuous from the right at *a* and continuous from the left at *b*




# Lecture #9 – Tuesday, October 7, 2003

## THEOREM "THE PINCHING THEOREM"

- Let $p>0$. Suppose that for all *x* such that $0<|x-c|<p$, we have $h(x)\le f(x)\le g(x)$.
- If $\lim\limits_{x\to c} h(x)=L=\lim\limits_{x\to c} g(x)$, then $\lim\limits_{x\to c} f(x)=L$

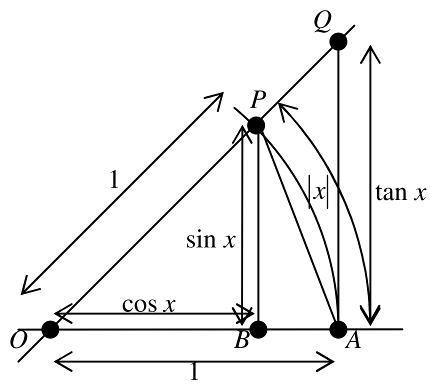



- Proof: Let $\varepsilon>0$. Want: $|f(x)-L|<\varepsilon$ for $|x-c|$ small enough.
  - Choose $\delta_1$ so that $|h(x)-L|<\varepsilon$ if $|x-c|<\delta_1 \Rightarrow L-\varepsilon<h(x)<L+\varepsilon$
  - Choose $\delta_2$ so that $|g(x)-L|<\varepsilon$ if $|x-c|<\delta_2 \Rightarrow L-\varepsilon<g(x)<L+\varepsilon$
  - If we choose $\delta=\min(\delta_1,\delta_2)$, then $L-\varepsilon<h(x)\le f(x)\le g(x)<L+\varepsilon \Rightarrow |f(x)-L|<\varepsilon$

Q.E.D.

## APPLICATION: SINE AND COSINE

### First A Figure




- First task: Compute $\lim_{x \to 0} \sin x = 0$ (1)
  - Proof: From the figure, for $x$ small enough $|\sin x| < |x|$
    - So in fact, $-x < \sin < x$ for $x$ small
    - So by Pinching Theorem, $\lim_{x \to c}(-x) = 0$ and $\lim_{x \to c}(x) = 0$, so $\lim_{x \to 0} \sin x = 0$

Q.E.D.

- Because (by Pythagorean Theorem) $\sin^2 x + \cos^2 x = 1$, $\cos x = \sqrt{1 - \sin^2 x}$ for $x$ small
- So because of the theorem about composing continuous functions,

  $\lim_{x \to 0} \cos x = \lim_{x \to 0} \sqrt{1 - \sin^2 x} = \sqrt{\lim_{x \to 0}\left(1 - \sin^2 x\right) = 1}$ (because $\lim_{x \to 0} \sin x = 0$ )
- Therefore, $\lim_{x \to 0} \cos x = 1$ (2)

- Now what about $\left.\begin{array}{l} \lim_{x \to c} \sin x = \sin c \\ \lim_{x \to c} \cos x = \cos c \end{array}\right\}$ sine and cosine are continuous
  - Proof of (3):
    - $\lim_{x \to c} \sin x$ is the same as $\lim_{h \to 0} \sin(c + h)$
    - Now, we can use the addition formulas: $\sin(c + h) = \sin c \cosh + \cos c \sinh$
    - $$\begin{aligned} \lim_{h \to 0} \sin(c + h) &= (\sin c)\lim_{h \to 0} \cosh + (\cos c)\lim_{h \to 0} \sinh \\ &= \sin c \end{aligned}$$

Q.E.D.

# Lecture #10 – Thursday, October 9, 2003

- In fact, <u>all</u> the trigonometric functions are continuous (where defined): this follows from the theorem about quotients of continuous functions being continuous.

## TWO MORE IMPORTANT LIMITS

- $\lim_{x\to 0}\frac{\sin x}{x}=1$ (1) and $\lim_{x\to 0}\frac{1-\cos x}{x}=0$ (2)
- Note: For both of these, if you "plug in", you get "$\frac{0}{0}$" – need a subtler argument

### Proof of (1)

- Refer to Figure
- Area of $\Delta OBP=\frac{1}{2}(\sin x)(1)=\frac{\sin x}{2}$
- Area of sector $OAP=\frac{x}{2}$
- Area of $\Delta OAQ=\frac{\tan x}{2}$
- $\frac{\sin x}{2}<\frac{x}{2}<\frac{\tan x}{2}\Rightarrow \sin x<x<\frac{\sin x}{\cos x}\Rightarrow\frac{\sin x}{x}<1<\frac{\sin x}{x}\cdot\frac{1}{\cos x}$
- Two inequalities: $\frac{\sin x}{x}<1$ and $1<\frac{\sin x}{x}\cdot\frac{1}{\cos x}$
  $\cos x<\frac{\sin x}{x}$
- So $\cos x<\frac{\sin x}{x}<1$ – true for $x$ "small" and $x>0$ or $x<0$
- By Pinching Theorem, we can conclude that $\lim_{x\to 0}\frac{\sin x}{x}=1$ because $\lim_{x\to 0}\cos x=1$ and $\lim_{x\to 0}1=1$.

Q.E.D.

### Proof of (2)

- $\frac{1-\cos x}{x}=\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}=\frac{1-\cos^2 x}{x(1+\cos x)}=\frac{\sin^2 x}{x(1+\cos x)}=\frac{\sin x}{x}\cdot\frac{\sin x}{(1+\cos x)}$
- Therefore $\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{(1+\cos x)}=0$

Q.E.D.

- In fact, $\lim_{x\to 0}\frac{\sin(ax)}{ax}=0$ and $\lim_{x\to 0}\frac{1-\cos(ax)}{ax}=0$

### Examples of Application

1) $\lim_{x\to 0}\frac{\sin(5x)}{4x}=\lim_{x\to 0}\frac{\sin(5x)}{4x}\cdot\frac{5}{5}=\lim_{x\to 0}\frac{\sin(5x)}{5x}\cdot\frac{5}{4}=\frac{5}{4}\lim_{x\to 0}\frac{\sin(5x)}{5x}=\frac{5}{4}$

2) $\lim_{x\to 0}\frac{2x^2+x}{\sin x}=\lim_{x\to 0}\frac{x(2x+1)}{\sin x}=\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}(2x+1)=\frac{1}{\lim_{x\to 0}\frac{x}{\sin x}}\cdot\lim_{x\to 0}(2x+1)=1$

$$\lim_{x\to 0}\frac{x^2}{\sec x-1}=\lim_{x\to 0}\frac{x^2}{\frac{1}{\cos x}-1}=\lim_{x\to 0}\frac{x^2}{\frac{1-\cos x}{\cos x}}=\lim_{x\to 0}\frac{x^2\cos x}{1-\cos x}=\lim_{x\to 0}\frac{x^2\cos x(1+\cos x)}{(1-\cos x)(1+\cos x)}$$

3) $$=\lim_{x\to 0}\frac{x^2\cos x+x^2\cos^2 x}{\sin^2 x}=\lim_{x\to 0}\frac{x^2}{\sin^2 x}\cdot\cos x+\lim_{x\to 0}\frac{x^2}{\sin^2 x}\cdot\cos^2 x$$

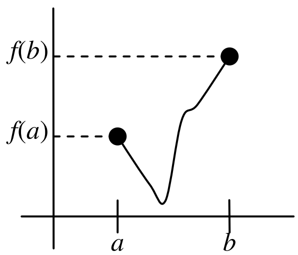
$$=\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2\cos x+\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2\cos^2 x=2$$

## Two Very Important Theorems About Continuous Functions

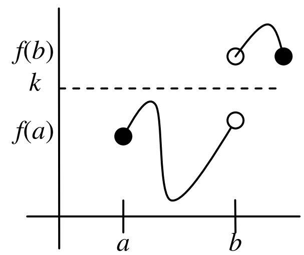
- Intermediate Value Theorem
- Extreme Value Theorem
- Understanding of the proof is going to require the Least Upper Bound Axioms

### Idea For Intermediate Value Theorem (IVT)

- A continuous function has a graph which is "an unbroken curve"




continuous



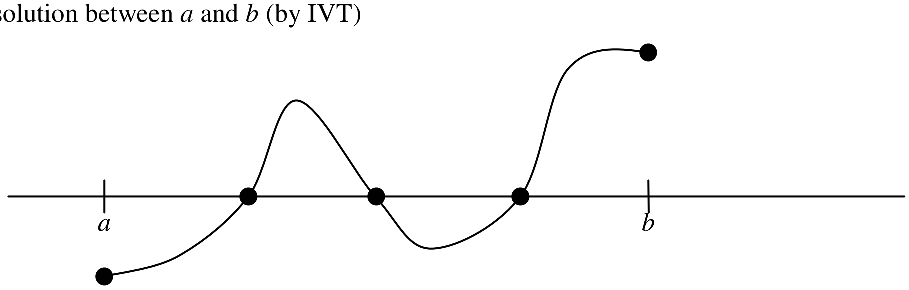

non-continuous

## Theorem: Intermediate Value Theorem

- If $f$ is continuous on $[a,b]$ and $k$ is any number between $f(a)$ and $f(b)$, then there is a value $c$, $a<c<b$ so that $f(c)=k$

### Application

- Locating the zeros of a function
- Suppose $f$ so continuous on $[a,b]$ and $f(a)<0$ and $f(b)>0$, then there is a solution $c$ to $f(c)=0$ solution between $a$ and $b$ (by IVT)



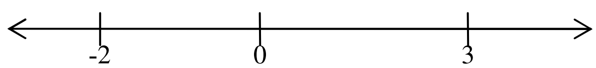
**Example**

$f(x)=\cos\left(\frac{\pi}{2}x\right)-x^2$ on $[0,1]$

- Evaluate at $x=0$: $\cos\left(\frac{\pi}{2}\cdot 0\right)-(0)^2=1>0$
- Evaluate at $x=1$: $\cos\left(\frac{\pi}{2}\cdot 1\right)-(1)^2=-1<0$
- IVT says there is a solution to $\cos\left(\frac{\pi}{2}x\right)-x^2=0$ between $[0,1]$
- Now evaluate at $x=\frac{1}{2}$: $f\left(\frac{1}{2}\right)=\cos\left(\frac{\pi}{4}\right)-\left(\frac{1}{2}\right)^2\approx 0.45>0$
- Now IVT says there is root between $\frac{1}{2}$ and 1.
- Can keep going and get an <u>approximation</u> to root – "bisection method"

**Application: Solving Inequalities**

- We were implicitly using IVT before!
- Solve the inequality $x(x+2)(x-3)>0$
    - Find the zeros: 0, -2, 3



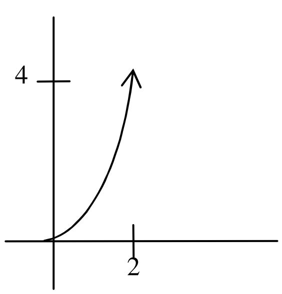

    - If $a$, $b$ are 2 zeros of $P(x)=x(x+2)(x-3)$, then the IVT is saying that on $(a,b)$, $P(x)$ has to be all positive or negative – because if it's positive $f(a)>0, a\in(a,b)$, $f(b)<0, a\in(a,b)$ then there is a zero between $a$ and $b$; that's a contradiction since you've already found all the zeros

## BOUNDEDNESS AND EXTREME VALUES

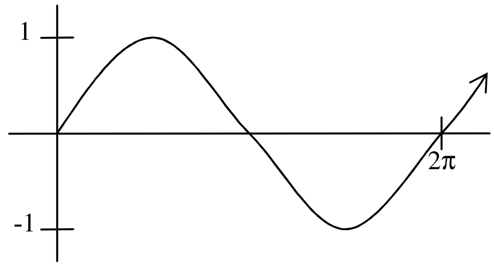
- Let $f$ be a function, defined on an interval $I$
- Definition: $f$ is called bounded on $I$ if there are constants $k$ and $K$ so that $k<f(x)<K$ for all $x\in I$

**Example**

- $f(x)=x^2$ on $[0,2]$



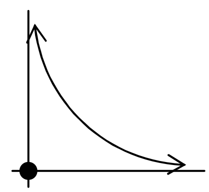
- $f(x) = \sin x$ on $[0, 2\pi]$




- Definition: If $f$ is not bounded it is called <u>unbounded</u> (on $I$)

**Example**

- $g(x) = \begin{cases} \dfrac{1}{x^2} \text{ on } (0, \infty) \\ 0 \text{ at } x = 0 \end{cases}$



    - $g(x)$ is <u>unbounded</u> on $(0, \infty)$
    - $g(x)$ is <u>bounded</u> on $[1, \infty)$

- Note: A function can have a <u>maximum value</u> or a <u>minimum value</u> or <u>both</u> or <u>neither</u> on an interval $I$

**Example**

- $g(x) = \begin{cases} \dfrac{1}{x^2} \text{ on } (0, \infty) \\ 0 \text{ at } x = 0 \end{cases}$
    - On $[0, \infty)$: unbounded, no maximum, has minimum
    - On $[1, \infty)$: bounded, has maximum, no minimum
    - On $(1, \infty)$: bounded, no maximum, no minimum

- Note: This has nothing to do with continuity

**Example**

- $f(x) = \begin{cases} 1 \text{ if } x \in \mathbf{Q} \\ -1 \text{ if } x \notin \mathbf{Q} \end{cases}$ is bounded, has maximum, has minimum on $(-\infty, \infty)$

## THEOREM: EXTREME VALUE THEOREM

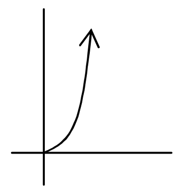
- If a function is <u>continuous</u> on a <u>bounded</u>, <u>closed</u> interval $[a, b]$, then the function takes on both a maximum value $M$ and a minimum value $m$ – i.e. $m \le f(x) \le M$ on $[a, b]$ ($m$ and $M$ are called the "extreme values")

# Lecture #11 – Tuesday, October 14, 2003

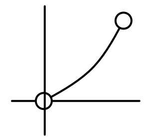
- Warning: Need all three of the assumptions

### Counter-Examples

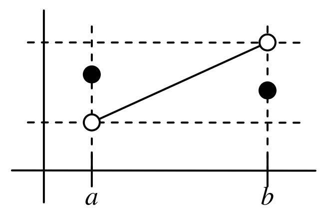
1) The interval must be bounded
   $f(x) = x^2$ on $[0, \infty)$



2) The interval must be closed
   $f(x) = x^2$ on $(0,1)$



3) The function must be continuous




### Idea of IVT

- A continuous function "takes intervals to intervals"

### Idea of IVT and EVT Together

- A continuous function "takes bounded closed intervals to bounded closed intervals"

- Proofs of IVT and EVT require an understanding of the "Least Upper Bound Axiom" of the real number

## LEAST UPPER BOUNDS

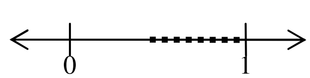
- Let $S$ be a nonempty set of real numbers

### Example

- $S_1 = (-\infty, 0)$

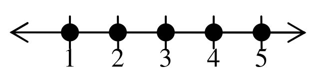

- $T_1 = [0, \infty)$

- $S_2 = \left(\frac{1}{2}, \frac{2}{3}, \ldots, \frac{n}{n+1}, \ldots\right)$




- $T_2 = [1, \infty)$

- $S_3 = (1,2,3,\ldots, n, \ldots)$




- $T_1 = \phi$

- $S_4 = \left(\left|\frac{x}{(x+1)(x+3)}\right| : 3 \le x \le 5\right)$
- Definition: A number $M$ is an upper bound for $S$ if $x \le M$ for any $x \in S$

- Note: Not all sets <u>have</u> upper bounds – those with upper bounds in the example are $S_1$, $S_2$, $S_4$
- Definition: If $S$ has an upper bound, we say $S$ is bounded above
- Now let's think about the set of possible upper bounds for $S$ (call it $T$)
- What is the <u>smallest possible</u> upper bound? – in the Example, for $S_1$, the smallest is 0; for $S_2$, the smallest is 1

### Definition

- If $S$ is nonempty set of real numbers that is bounded above, the <u>least upper bound</u> (lub) of $S$ is an upper bound that is less than or equal to any upper bound for $S$

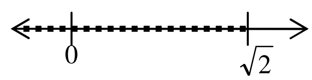
## LEAST UPPER BOUND AXIOM

- <u>Every</u> nonempty set of real numbers that has an upper bound has a <u>least</u> upper bound

- If you think this is "obvious", it's NOT TRUE, for example, the rational numbers!

### Example

- $S = \{x < \sqrt{2}, x \text{ is rational}\}$




- Let $T$ = the set of possible rational upper bound for $S$ = $\{x > \sqrt{2} : x \text{ is rational}\}$
- Has no least "element"

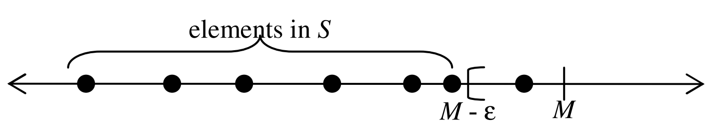
### Back To Real Numbers: Examples

1) $\text{lub}(-4,-1) = -1$
2) $\text{lub}\left\{-1, -\frac{1}{8}, -\frac{1}{27}, \ldots, -\frac{1}{n^3}, \ldots\right\} = 0$
3) $\text{lub}\{x \in R : x^2 < 3\} = \sqrt{3}$

# Lecture #12 – Thursday, October 16, 2003

### Theorem

- If $M$ is the least upper bound of set $S$ and $\varepsilon > 0$ is any positive number, then there is an element $s$ in $S$ such that $M - \varepsilon < s \le M$




- Idea: The elements in $S$ get arbitrarily close to $M$

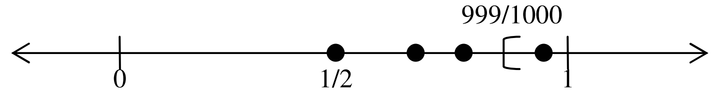
### Proof

- Let $\varepsilon > 0$. Then since $M$ is an upper bound, any element $x$ of $S$ satisfy $x < M$
- So we only need to find $s \in S$ so that $M - \varepsilon < s$
- We'll argue by contradiction. Suppose for all $s \in S$, $s \le M - \varepsilon$
  - But that means $M - \varepsilon$ is an upper bound for $S$
  - But $M - \varepsilon < M$ and $M$ is supposed to be the lub: contradiction

- So there must be $s$, $M-\varepsilon < s \le M$

### Example

$S=\left\{\frac{1}{2},\frac{2}{3},\ldots,\frac{n}{n+1},\ldots\right\}$




- Take $\varepsilon=\frac{1}{1000}$. Is it true that there is an element of $S$ such that $\frac{999}{1000}<s<1$? Yes: $s=\frac{1000}{1001},\frac{9999}{10000}$

## Lower Bound

### Definition

- Let $S$ be a nonempty set of real numbers, then we say $m$ is a lower bound of $S$ if $\forall x\in S, m\le x$ – if $S$ has a lower bound, then $S$ is bounded below
- If $S$ is a nonempty set of real numbers bounded below, then the greatest lower bound (glb) is a lower bound for $S$ that is greater than or equal to any other lower bound for $S$

### Theorem: Existence of Greatest Lower Bound

- Every nonempty set of $S$ of **R** bounded below has a greatest lower bound

### Theorem

- If $m$ is the glb of $S$, $\varepsilon>0$, then $\exists$ an element $s\in S$ such that $m\le s<m+\varepsilon$

### Proofs

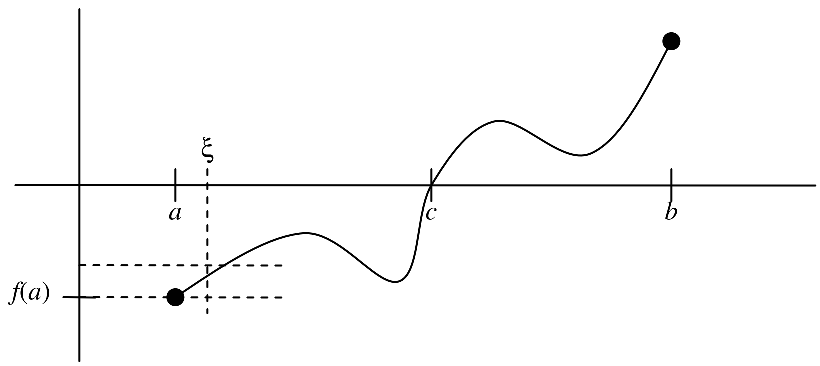
- Use what you know for least upper bounds

## Proof of Intermediate Value Theorem

### Lemma

- Let $f$ be continuous on $[a,b]$. If $f(a)<0<f(b)$ (or $f(a)>0>f(b)$) then $\exists$ c, $a<c<b$ such that $f(c)=0$

**Picture**




**Proof**

- Suppose $f(a) < 0 < f(b)$ (The other situation can be proved similarly)
- Let $S = \{\xi : f \text{ is negative on } [\mathrm{a}, \xi]\}$
- By continuity of *f*, *S* is not empty
- Because $f(b) > 0$, *b* is an upper bound for *S*. That means *S* has a least upper bound: call it *c*
- Want: $f(c) = 0$ (we'll argue that it can't be >0 and can't be <0)
    - Suppose $f(c) > 0$
        - By continuity of *f* at *c*, there is an interval $(c - \varepsilon, c + \varepsilon)$ where *f* is >0
        - Then because $f\left(c - \frac{\varepsilon}{}\right) > 0$, $c - \frac{\varepsilon}{2}$ is also an upper bound for *S*, but *c* is the <u>least</u> upper bound. Contradiction.
    - Suppose $f(c) > 0$
        - By continuity of *f* at *c*, *f* is <0 on an interval $(c - \varepsilon, c + \varepsilon)$
        - So $c + \frac{\varepsilon}{2}$ is in *S*, but *c* is an upper bound
        - Contradiction because $c + \frac{\varepsilon}{2} > c$
    - So $f(c)$ is neither <0 or >0, so $f(c) = 0$

Q.E.D.

**Theorem: IVT**

- If *f* is continuous on $[a, b]$, and *k* is any value between $f(a)$ and $f(b)$, then ∃ *c*, $a < c < b$, such that $f(c) = k$

**Proof**

- Suppose $f(a) < k < f(b)$ (The other case is similar)
- Consider $g(x) = f(x) - k$
- Then $g(a) < 0$, $g(b) > 0$ so *g* is continuous on $[a, b]$
- By the Lemma, there is a *c*, $a < c < b$ so that $g(c) = 0$
- But then if $g(c) = 0 = f(c) - k$, then $f(c) = k$

Q.E.D.

## PROOF OF EXTREME VALUE THEOREM

### Lemma

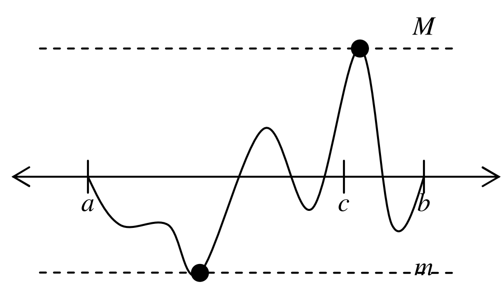
- If $f$ is continuous on $[a,b]$, then $f$ is bounded on $[a,b]$

### Proof

- Let $S = \{x \in \mathbf{R} : x \in [a,b] \text{ and } f \text{ is bounded on } [a,x]\}$
    - $S$ is bounded above by definition of $S$: $b$ is an upper bound
    - $S$ is not empty: can take $x = a$
- Let $c = \text{lub}\, S$ . Want: $c = b$
    - Because $b$ is an upper bound and $c$ is the <u>least</u> upper bound, then $c \le b$ .
    - We must rule out the possibility that $c < b$ .
    - Suppose $c < b$
        - By continuity of $f$ at $c$, $f$ is bounded on some interval $(c-\varepsilon, c+\varepsilon)$.
        - Because $c$ is the lub, $f$ is bounded on $\left[a, c-\dfrac{\varepsilon}{2}\right]$. $f$ is also bounded on $\left[c-\dfrac{\varepsilon}{2}, c+\dfrac{\varepsilon}{2}\right]$ $\Rightarrow f$ is bounded on $\left[a, c+\dfrac{\varepsilon}{2}\right]$ $\Rightarrow$ $c+\dfrac{\varepsilon}{2}$ is in $S$.
        - But $c$ is an upper bound for $S$, so contradiction.
        - So $c = b$ .
- Want: $f$ is bounded on $[a,b]$.
    - Because $f$ is bounded at $b$, there is an interval $[b-\delta, b]$ where $f$ is bounded.
    - Because $b$ is lub of $S$, $f$ is bounded on $\left[a, b-\dfrac{\varepsilon}{2}\right]$.
    - So $f$ is bounded on $[a,b]$.

### Theorem: EVT

- If $f$ is continuous on $[a,b]$, then $f$ achieves both a maximum $M$ and minimum $m$ value on $[a,b]$.




### Proof

- By Lemma, $f$ is <u>bounded</u>.
- Definition: $M = \text{lub}\{f(x) : x \in [a,b]\}$
- Want: $\exists\, c$ such that $f(c) = M$ .
    - Suppose not. Then $f(x) \ne M$ for all $x \in [a,b]$.
    - Then $M - f(x) \ne 0$ for all $x \in [a,b]$ (in fact $M - f(x) > 0$ for all $x \in [a,b]$).
    - Define $g(x) = \dfrac{1}{M - f(x)}$ . So $g$ is continuous on $[a,b]$.
    - Because $M$ is lub, $M - f(x)$ can be made arbitrarily small, and $g(x)$ can be made arbitrarily large.

  - So $g$ is continuous on $[a,b]$, but not bounded. This contradicts the Lemma.
- Theforefore, there <u>is</u> a $c$ such that $f(c)=M$ .

Q.E.D.