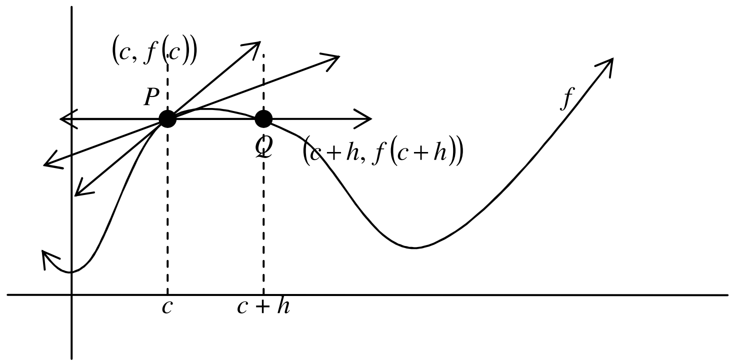
Lecture #12 – Thursday, October 16, 2003

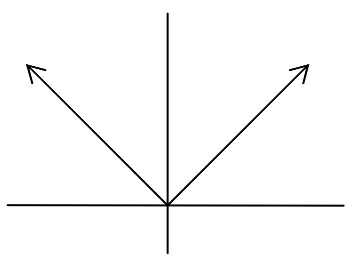
DIFFERENTIATION



- What is the slope of the secant line $\overleftrightarrow{PQ}$? $\dfrac{f(c+h)-f(c)}{(c+h)-c} = \dfrac{f(c+h)-f(c)}{h}$.
- Idea: Get the slope of the tangent line as a limit of slopes of secant lines.
- The slope of the tangent line at $x = c$ "ought" to be $\lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$.

Example

$f(x) = |x|$ at $x = 0$



- This DOESN'T have a well defined tangent line

Definition

- f is differentiable at $x = c$ if the limit $\lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$ exists. If it does, we call it the derivative of f at c and we denote it by $f'(c)$.

Geometrically

- $f'(c)$ is the slope of the tangent line going through $(c, f(c))$.
- What is the equation for tangent line? $y - f(c) = f'(c)(x - c)$

Lecture #13 – Tuesday, October 21, 2003

Example

- For function $f(x) = x^2$, the derivative of *f* at $c = 2$ is $f'(2) = \lim_{h \to 0} \frac{(2+h)^2 - (2)^2}{h} = \lim_{h \to 0}(4+h) = 4$.
- The derivative of *f* is itself a function – for $f(x) = x^2$ repeat the same calculation for <u>any</u> value of *c*.
- $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{(c+h)^2 - (c)^2}{h} = \lim_{h \to 0} \frac{c^2 + 2ch + h^2 - c^2}{h} = \lim_{h \to 0}(2c+h) = 2c$
- At any fixed value of *x*, $f'(x) = 2x$.

Definition

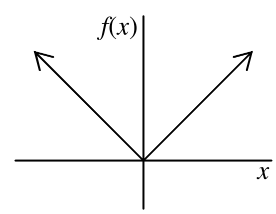
- The <u>derivative of *f*</u> is a function, denoted f', and $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, if it exists.

Terminology

- To <u>differentiate</u> a function is to find the derivative.
- Notice: The function *f* has to be defined in the interval $(x - \delta, x + \delta)$ in order for $f'(x)$ to be defined.

Example

- Actually, even if *f* is <u>continuous</u> on $(x - \delta, x + \delta)$, it doesn't mean $f'(x)$ is defined.
- Consider $f(x) = |x|, c = 0$



Theorem

- If *f* is differentiable at *x*, the *f* is continuous.
- "Being differentiable is 'better' than being continuous."
- Proof:
  - Because *f* is differentiable at *x*, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$
  - $\lim_{h \to 0}(f(x+h) - f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot h = f'(x) \cdot 0 = 0$
  - So *f* is continuous.

DIFFERENTIATION RULES

Building Blocks

- If $f(x) = c$ (a constant function), then $f'(x) = 0$ for all *x*.
- If $f(x) = x$, then $f'(x) = 1$ for all *x*.

Theorem: Sums and Scalar Multiples

- Let f, g be differentiable at x and α a constant.
- Then $(f+g)$ and αf are differentiable, then
  - $(f+g)'(x) = f'(x) + g'(x)$
  - $(\alpha f)'(x) = \alpha f'(x)$
- Proof:
  - $$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\ &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\ &= f'(x)+g'(x)\end{aligned}$$

Example

- If $f(x) = 10x$, $f'(x) = 10$

Theorem: Differences and Linear Combinations

- From the Sums and Scalar Multiples rule,
  - $(f-g)'(x) = f'(x) - g'(x)$
  - $(\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n)'(x) = \alpha_1 \cdot f_1'(x) + \alpha_2 \cdot f_2'(x) + \ldots + \alpha_n \cdot f_n'(x)$

Theorem: Product Rule

- If f and g are differentiable at x, then $f \cdot g$ is differentiable and $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- Proof:
  - $$\begin{aligned}(f\cdot g)'(x) &= \lim_{h\to 0}\frac{(f\cdot g)(x+h)-(f\cdot g)(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h)-f(x)\cdot g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h)-f(x)\cdot g(x+h)+f(x)\cdot g(x+h)-f(x)\cdot g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot g(x+h)+\lim_{h\to 0}f(x)\cdot\frac{g(x+h)-g(x)}{h}\end{aligned}$$
    Because f, g are differentiable, they are continuous.
    $$= f'(x)g(x)+f(x)g'(x)$$

Theorem: Power Rule

- Using the Product Rule, we derive the Power Rule.
- For $n > 0, n \in \mathbf{Z}$, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
- Proof (by induction):
  - True for $k = 1$
  - Assume $\left(x^k\right)' = kx^{k-1}$. Prove for $k+1$:

- $\left(x^{k+1}\right)' = \left(x^k \cdot x\right)' = kx^{k-1} \cdot x + x^k \cdot 1 = kx^k + x^k = (k+1)x^k$

Examples

- $f(x) = x^2 + 10x^3 - 3x^5$
  $f'(x) = 2x + 30x^2 - 15x^4$

Lecture # 14 – Thursday, October 23, 2003

Theorem: Reciprocal Rule

- If g is differentiable at x, $g(x) \neq 0$, then $\frac{1}{g}$ is also differentiable, and $\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{g(x)^2}$.
- Proof:
  - $$\begin{aligned}\lim_{h\to 0} &= \frac{\frac{1}{g(x+h)} + \frac{1}{g(x)}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{g(x) - g(x+h)}{g(x+h)\cdot g(x)}\right] \\ &= \lim_{h\to 0}\left[\frac{1}{g(x+h)\cdot g(x)}\right]\cdot\left[-\frac{g(x+h) - g(x)}{h}\right] \\ &= \frac{1}{g(x)^2}\left(-g'(x)\right) = \frac{g'(x)}{g(x)^2}\end{aligned}$$

Theorem: General Power Rule (for any exponent)

- For $n < 0$, $\left(x^n\right)' = nx^{n-1}$.
- Proof:
  - Let $p(x) = x^n = \frac{1}{g(x)}$, and let $g(x) = x^{-n}, -n \text{ is positive}$.
  - By previous Power Rule, we know $g'(x) = (-n)x^{-n-1}$.
  - So $p'(x) = \frac{nx^{-n-1}}{x^{-2n}} = nx^{-n-1} \cdot x^{-2n} = nx^{n-1}$.
- Last case: $n = 0$.
  - $p(x) = x^0 = 1$ – constant.
  - $p'(x) = 0x^{0-1} = 0$.

Theorem: Quotient Rule

- If f is differentiable at x and g is differentiable at x, $g(x) \neq 0$, the $\frac{f}{g}$ is differentiable at x and

  $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$.

- Proof:
  - Use the fact that $\frac{f}{g} = f \cdot \frac{1}{g}$.

$$\left(\frac{f}{g}\right)'(x) = \left(f \cdot \frac{1}{g}\right)'(x)$$

- $$\begin{aligned} &= f'(x) \cdot \frac{1}{g}(x) + f(x) \cdot \left(\frac{1}{g}\right)'(x) \\ &= \frac{f'(x)}{g(x)} + \frac{-f(x)g'(x)}{g(x)^2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \end{aligned}$$

Example

1) Find where the tangent line to the graph of $y = x^2 - \frac{16}{x}$ is horizontal.
   - Tangent line horizontal $\Rightarrow$ derivative = 0.
   - What is y'? $y' = 2x + \frac{16}{x^2}$
   - So $2x + \frac{16}{x^2} = 0 \Rightarrow \frac{x^3 + 8}{x^2} = 0 \Rightarrow x = -2$.
   - So the tangent line is horizontal at the point $(-2,12)$.

2) Find derivative: $f(x) = \dfrac{x^3 + 1}{2 + \frac{3}{x}}$ at $x = 3$.
   - $$f'(x) = \frac{3x^2 \cdot \left(2 + \frac{3}{x}\right) - \left(x^3 + 1\right)\left(-3x^{-2}\right)}{\left(2 + \frac{3}{x}\right)^2}$$

Leibniz Notation

- y is a "function" of x – we used to write $y = f(x)$.
  - y = "function"
  - x = "variable"
- Derivative of y with respect to (w.r.t.) x = "$\frac{dy}{dx}$".
- Note: Could also be $\frac{du}{ds}$, $\frac{dt}{dy}$.

Higher Derivatives

- Second derivative of f is $(f')' = f''$.

- Third derivative of f is $\left(\left(f'\right)'\right)' = f'''$.
- The r^{th} derivative of f is $f^{(r)}$.
- In the Leibniz notation, $\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}$, $\frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d^3y}{dx^3}, \ldots,$

Examples

1) Find the 3rd derivative of $y = 3x^3 + 2x + 15$.
   - $\frac{dy}{dx} = 9x^2 + 2$
   - $\frac{d^2y}{dx^2} = 18x$
   - $\frac{d^3y}{dx^3} = 18$

2) Find $\frac{d^2}{dx^2}\left[\left(x^2 - 3x\right)\cdot\frac{d}{dx}\left[x + x^{-1}\right]\right]$

$$\frac{d^2}{dx^2}\left[\left(x^2 - 3x\right)\cdot\frac{d}{dx}\left[x + x^{-1}\right]\right]$$
$$= \frac{d^2}{dx^2}\left[\left(x^2 - 3x\right)\left(1 - x^{-2}\right)\right]$$

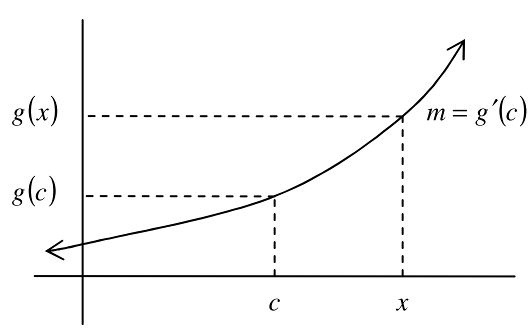
- $$= \frac{d^2}{dx^2}\left[x^2 - 1 - 3x + 3x^{-1}\right]$$

$$= \frac{d^2}{dx^2}\left[2x - 3 - 3x^{-2}\right]$$
$$= 2 + 6x^{-3}$$

Chain Rule

- $(f \circ g)(x) = f'(g(x)) \cdot g'(x)$

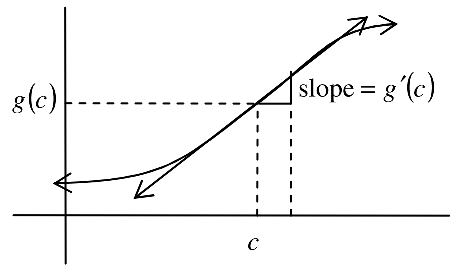
DERIVATIVE AS A RATE OF CHANGE

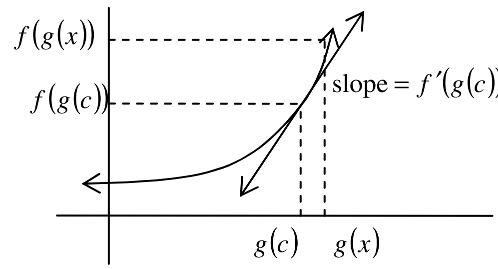


- $g'(c) \approx \frac{g(x) - g(c)}{x - c}$
- $g(x) - g(c) \approx g'(c)(x - c)$

IDEA BEHIND THE CHAIN RULE

- Suppose we have $g'(c)$ and $f'(g(c))$. We want to ask about the rate of change of composition $f \circ g$.





- For the Chain Rule, we want to know if x is close to c, what is $f(g(x)) - f(g(c))$?
- What we <u>know</u>:
  - Change $x - c \ \rightarrow \ g'(c)(x-c) \ \rightarrow \ g(x) - g(c)$ changing.
  - Change $g(x) - g(c) \ \rightarrow \ f'(g(c))(g(x) - g(c)) \ \rightarrow \ f(g(x)) - f(g(c))$ change in f.
  - Put together: change $x - c \ \rightarrow \ f'(g(c)) \cdot g'(c)(x-c)$.

- In Leibniz notation, the Chain Rule looks like:
  - Suppose $\left.\begin{array}{l} y = f(u) \\ u = g(x) \end{array}\right\} y = (f \circ g)(x)$
  - The Chain Rule says: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

Example: Checking the Chain Rule

- $\left.\begin{array}{l} y = u \\ u = x^2 \end{array}\right\} y = x^6$
- $\dfrac{dy}{dx} = 6x^5$
- $\dfrac{dy}{du} = 3u^2$, $\dfrac{du}{dx} = 2x^2$. So the Chain Rule says $6x^5 = 3\left(x^2\right)^2 \cdot 2x = 6x^5$

- Warning: When using the Chain Rule, have to substitute u as a function of x.

- Another way to write the Chain Rule:
  - $y = f(u)$, $u = u(x)$
  - $\dfrac{d}{dx}\left[f(u(x))\right] = f'(u(x)) \cdot \dfrac{du}{dx}$

Example

- $\dfrac{d}{dx}\left[\left(x^3 - x + 1\right)^{100}\right]$
- $f(u) = u^{100}$, $f'(u) = 100u^{99}$
- $u(x) = x^3 - x + 1$, $u'(x) = 3^2 - 1$
- $\dfrac{d}{dx}\left[\left(x^3 - x + 1\right)^{100}\right] = 100\left(x^3 - x + 1\right)^{99}\left(3x^2 - 1\right)$

- By the way, can "keep going" – ex: $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{ds}\cdot\frac{ds}{dx}$

Example

- $y=\left[1+(2+3x)^5\right]^3$
- $\left.\begin{array}{l} s(x)=2+3x \\ u=1+s^5 \\ y=u^3 \end{array}\right\} \frac{dy}{dx}=3\left[1+(2+3x)^5\right]^2\cdot 5(2+3x)^4\cdot 3$

Theorem: Chain Rule

- If g is differentiable at x, f is differentiable at $g(x)$, then $f\circ g$ is differentiable at x, and $(f\circ g)'(x)=f'(g(x))\cdot g'(x)$.

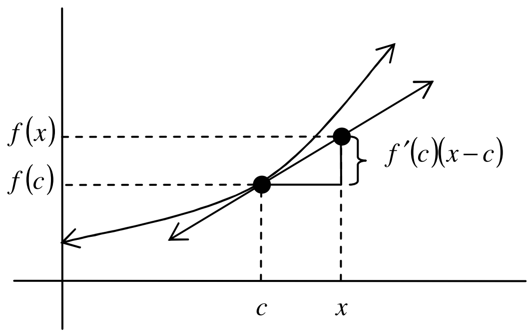
- Reminder: Another equivalent way define derivate of f at x is $f'(x)=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$.

- Idea of proof:
    - We want to compute $\frac{f(g(t))-f(g(x))}{t-x}=\frac{f(g(t))-f(g(x))}{g(t)-\overline{g}(x)}\cdot\frac{g(t)-\overline{g}(x)}{t-x}$
    - Problems:
        - $\frac{f(g(t))-f(g(x))}{g(t)-\overline{g}(x)}$ is not quite the definition of $f'(g(x))$.
        - $\frac{g(t)-\overline{g}(x)}{t-x}$ might not be defined if $g(t)=g(x)$.

- Real proof:
    - Define $F(g)=\begin{cases}\frac{f(y)-f(g(x))}{y-g(x)} \text{if } y\neq g(x) \\ f'(g(x)) \text{if } y=g(x)\end{cases}$
    - Because the definition of F and f is differentiable at $g(x)$, $\lim_{y\to g(x)}F(y)=F(g(x))=f'(g(x))$.
    - Notice: For all $t\neq x$, $\frac{f(g(t))-f(g(x))}{t-x}=F(g(x))\cdot\left[\frac{g(t)-g(x)}{t-x}\right]$
    - $F\circ g$ is continuous at x because g is continuous (since differentiable), as we take $\lim t\to x$, LHS $=(f\circ g)'(x)$, RHS $=f'(g(x))\cdot g'(x)$.

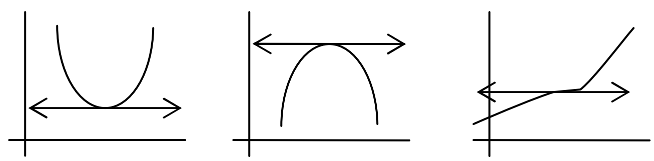
Lecture #15 – Tuesday, October 28, 2003

MORE ON DERIVATIVES AS A RATE OF CHANGE



- The derivative (the slope of the tangent line) has an interpretation as the "instantaneous" rate of change of $f(x)$ at $x = c$.

- Note:
  - If $f'(x) > 0$, *f* is increasing around *x*.
  - If $f'(x) < 0$, *f* is decreasing around *x*.
  - If $f'(x) = 0$, *f*'s rate of change is 0 around *x*.



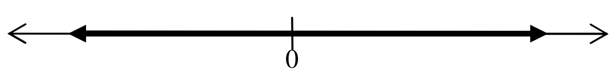
Example

Find the rate of change of the area *A* of a circle which respect to radius *r*, when the radius is 3.

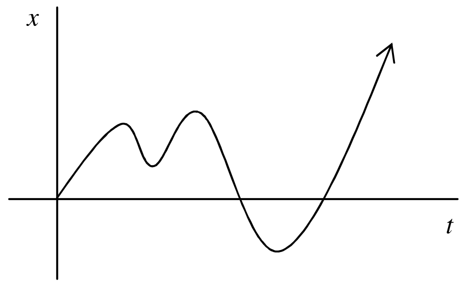
- $A(r) = \pi r^2$
- $\dfrac{dA}{dr} = 2\pi r$
- When $r = 3$, $\dfrac{dA}{dr} = 6\pi$.

Common Applications: Position *x*(*t*) as a function of time

- If $x(t)$ is a position as a function of time:

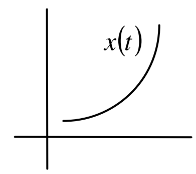
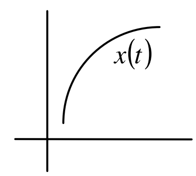
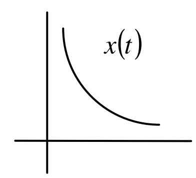
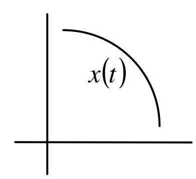


- The way we usually draw it:



- $\text{velocity}(t) = \text{rate of change of } x(t)$ – $v(t) > 0$ means moving left, $v(t) < 0$ means moving right.
  $v(t) = x'(t)$
- $\text{acceleration}(t) = \text{rate of change of } v(t)$
  $a(t) = v'(t) = x''(t)$
- $\text{speed}(t) = |v(t)|$

- Using *x*, *v*, *a*, can describe motion in detail:

$v(t)$	$a(t)$	
+	+	
+	-	
-	+	
-	-	

Example: Free Fall In a Gravitational Field

- Galileo figured out the trajectory of an object in free fall: $x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$.
  - g = gravitational constant ≈ 32 ft/sec
  - v_0 and y_0 are constants – depend on how the object is moving at $t = 0$.
  - What is the position at $t = 0$? $x(0) = x_0$ – position of object at $t = 0$.

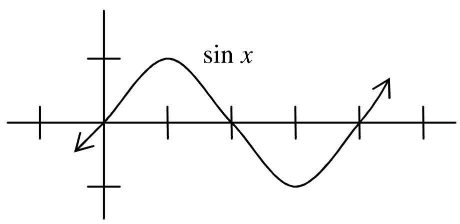
- What is the velocity at $t=0$? $x'(t)=-gt+v_0$ and $x'(0)=v_0$ – velocity of object at $t=0$.

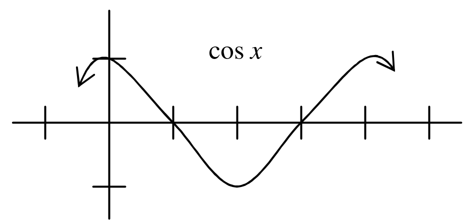
Example

1) What is the equation of the motion if $x_0=4\text{ft}$, $v_0=0$? When do the ball hit the ground?
   - $x(t)=-16t^2+4$.
   - At what time is $x(t)=0$? Solve for *t*: $t=\frac{1}{2}$.

2) What is the equation of the motion if $x_0=0$, $v_0=64\text{ft/sec}$? When do the ball hit the ground?
   - $x(t)=-16t^2+64t$.
   - At what time is $x(t)=0$? $x(t)=0=-16t^2+64t \Rightarrow t=4$

- To get to the more interesting examples, we will need derivatives of trig functions.





- $\frac{d}{dx}[\sin x]=\cos x$
- $\frac{d}{dx}[\cos x]=-\sin x$

Lecture #16 – Thursday, October 30, 2003

Differentiating The Trigonometric Functions

Basic Formulas

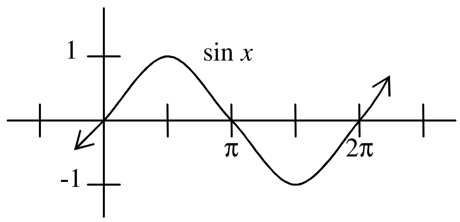
1) $\frac{d}{dx}[\sin x]=\cos x$
   - Proof: Fix *x*.

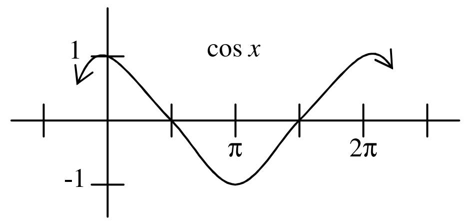
$$\begin{aligned}&\lim_{h\to o}\frac{\sin(x+h)-\sin(x)}{h}\\&=\lim_{h\to o}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\&=\lim_{h\to o}\left[\sin x\frac{(\cos(h)-1)}{h}+\cos x\frac{\sin(h)}{h}\right]\\&=\cos x\end{aligned}$$

2) $\frac{d}{dx}[\cos x]=-\sin x$

- Proof: Fix x.
$$\lim_{h\to o}\frac{\cos(x+h)-\cos(x)}{h}$$
$$=\lim_{h\to o}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$
$$=\lim_{h\to o}\cos x\frac{\cos(h)-1}{h}-\lim_{h\to o}\sin x\frac{\sin(h)}{h}$$
$$=-\sin x$$

- If you ever forget the formulas, just think about the graphs.





The Other Trig Formulas

- These formulas are derived from $\frac{d}{dx}[\sin x]=\cos x$ and $\frac{d}{dx}[\cos x]=-\sin x$ by using the Quotient or Reciprocal Rule.
- $\frac{d}{dx}[\tan x]=\frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]=\frac{\cos x\cdot\cos x+\sin x\cdot\sin x}{(\cos x)^2}=\frac{1}{(\cos x)^2}=\sec^2 x$

$$\frac{d}{dx}[\sin x]=\cos x \qquad \frac{d}{dx}[\csc x]=-\csc x\cot x$$
$$\frac{d}{dx}[\cos x]=-\sin x \qquad \frac{d}{dx}[\sec x]=\sec x\tan x$$
$$\frac{d}{dx}[\tan x]=\sec^2 x \qquad \frac{d}{dx}[\cot x]=-\csc^2 x$$

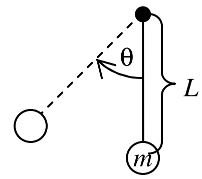
- Of course, you can use the Chain Rule with the trig functions.

Examples

1) $\frac{d}{dx}\left[(1+\tan x)^3\right]=3(1+\tan x)^2\cdot\left(\sec^2 x\right)$
2) $\frac{d}{dx}\left[\cos\left(1+x+x^2\right)\right]=-\sin\left(1+x+x^2\right)\cdot(1+2x)$
3) $\frac{d}{dx}[\tan(\sec x)]=\sec^2(\sec x)\cdot\sec x\tan x$

Example: Mass on a pendulum

The motion of the pendulum is given by $\theta(t) = a\sin(\omega t + \phi)$ (a, ω, ϕ are constants that depend on L, m, and initial conditions).



- $\theta'(t) = \omega a\cos(\omega t + \phi)$
- $\theta''(t) = -\omega^2 a\sin(\omega t + \phi)$
- $\theta(t)$ satisfy the <u>differential equation</u> – $\theta'' = -w^2\theta$.

IMPLICIT DIFFERENTIATION

Example

Suppose we're thinking about the points in $\mathbf{R}^2$ satisfying $xy = 1$. We want to know the slope of tangent line at $x = 2$, $y = \frac{1}{2}$.

- Solution 1: Solve explicitly for *y* in terms of *x* – i.e. think of *y* as a function of *x*.
  - $y = \frac{1}{x}$
  - $\frac{dy}{dx} = -\frac{1}{x^2}$
  - So the slope at $\left(2, \frac{1}{2}\right)$ is $-\frac{1}{2}$.

- Solution 2: Don't bother to solve for *y*.
  - You know, in principle, you could solve for *y*, so you treat the equation $xy = 1$ as a relationship between functions.
  - $xy = 1$. So $x \cdot y(x) = 1$.
  - If two functions are equal, their derivatives are equal.
  - $\left.\begin{array}{l}\text{LHS}: 1\cdot y(x) + x\cdot\frac{dy}{dx} \\ \text{RHS}: 0\end{array}\right\} y(x) + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y(x)}{x} = -\frac{1}{x^2}$

- In general, when you do implicit differentiation, do the following:
  - Work with the equation (relationship between *x* and *y*) directly.
  - Differentiate LHS and RHS and these derivatives must be equal. Treat *y* as a function of *x* (which is, in principle).
  - Solve for $\frac{dy}{dx}$ – in general, it will be an expressing in <u>both</u> *x* and *y*.

Example

Think about the curve in $\mathbf{R}^2$ specified by $x^3 + y^3 - 3xy = 0$. Find $\frac{dy}{dx}$ for a point on this curve.

- LHS: Treating $y = y(x)$ as a function of *x*.
- Differentiating, $3x^2 + 3y^2\frac{dy}{dx} + \left(-3y + (-3x)\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx}\left(3y^2 - 3x\right) = 3y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$

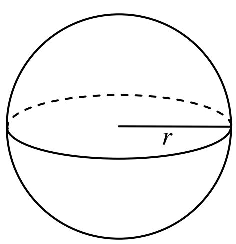
- Using implicit differentiation, we can prove the Rational Power Rule: $\frac{d}{dx}\left[x^{\frac{p}{q}}\right] = \frac{p}{q}x^{\left(\frac{p}{q}-1\right)}$, $p, q \in \mathbf{Z}$.
- Proof:
  - Let $y = x^{\frac{p}{q}} \Rightarrow y^q = x^p$.
  - Implicit differentiation: $qy^{q-1}\frac{dy}{dx} = px^{p-1}$ (by usual Power Rule for **Z**).
  - $\frac{dy}{dx} = px^{p-1} \cdot \frac{1}{q}y^{1-q} = \frac{p}{q}x^{p-1}\left(x^{\frac{p}{q}}\right)^{1-q} = \frac{p}{q}x^{p-1}x^{\frac{p}{q}}x^{-p} = \frac{p}{q}x^{\frac{p}{q}-1}$

More On Rates of Change (Related Rates)

- Variation on the same theme:
  - We'll have two variables related by an equation, both are functions of *t*.
  - When we differentiate, we'll get a relationship between the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Example

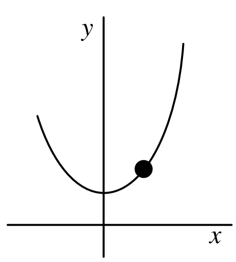
The volume of spherical balloon is related to radius. If the radius is expanding uniformly at 2cm/min, how fast is the volume changing when radius is 5cm?



$$\left.\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ V &= V(t) \\ r &= r(t) \end{aligned}\right\} \frac{dV}{dt} = 4\pi r^2 \frac{dt}{dr} = 4\pi(5)^2(2) = 200\pi\text{cm}^3/\text{min}$$

Example

Bead moving along a wire which is on the curve $y = x^2 + 2$. Given $\frac{dx}{dt}$, how fast is the height changing when $x = 2$?

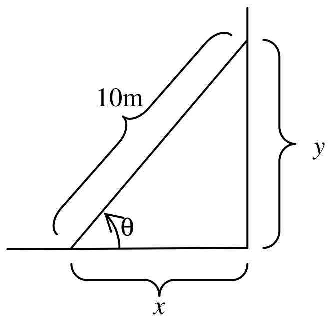


Steps:

1) Draw picture.
2) Know: $\frac{dx}{dt}$.
3) Want: $\frac{dy}{dt}$.
4) What is the relationship? $y = x^2 + 2$.
5) Implicitly differentiate: $\frac{dy}{dt} = 2x\frac{dx}{dt} = 4$.

Example

Draw at the bottom of the ladder, $\frac{dx}{dt} = \frac{1}{2}$ m/min . How fast is θ changing when the top of ladder is 8m from the floor?



- Know: $\frac{dx}{dt}$.
- Want: $\frac{d\theta}{dt}$.
- Relationship: $\cos\theta = \frac{x}{10}$

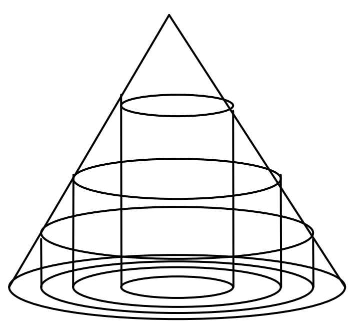
$$-\sin\theta \frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt} \Rightarrow -\frac{8}{10}\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{1}{16}.$$

Lecture #17 – Tuesday, November 4, 2003

Mean Value Theorem and Applications

Preview

1) Optimization problems: These show up in economics, physics, etc.
    - Example: If a cylinder is circumscribed in a cone, then what is the maximum volume possible of such a cylinder?

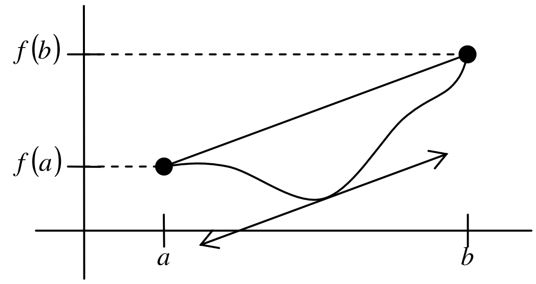


2) Graph sketching:
    - Example: Sketch the graph of the function $f(x) = \frac{1}{4}x^4 - 2x^2 + \frac{7}{4}$.
    - Idea: Use qualitative the right information (ex: is the function increasing or decreasing, etc.). Some things you evaluate exactly (ex: x, y intercepts, max/min values).

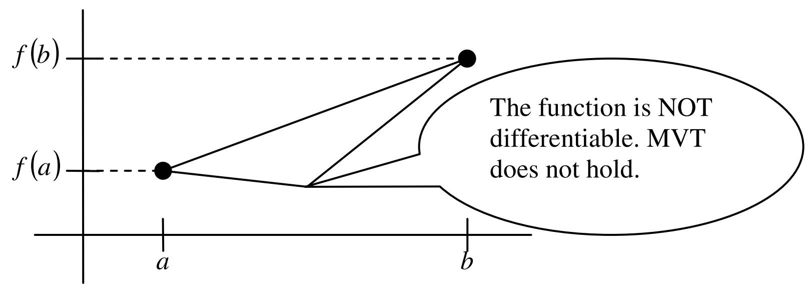
The Mean Value Theorem

- This is a theorem relating the derivative of a function with its end-point values $f(a)$ and $f(b)$.

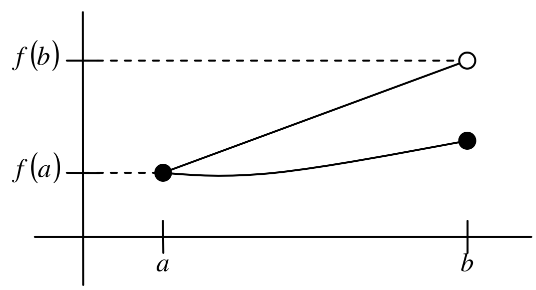
- If a theorem is differentiable on (a,b) and *f* is continuous on $[a,b]$, there is at least one point *c* in (a,b) so that
$$f'(c)=\frac{f(b)-f(a)}{b-a}$$
$$= \text{slope of line between } (a, f(a)) \text{ and } (a, f(b))$$
.



- Warning: The function <u>must</u> be differentiable on (a,b)



- Warning: The function must also be continuous on $[a,b]$



Proving the Mean Value Theorem

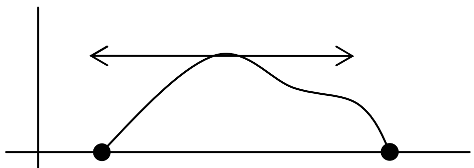
Lemma

- Let *f* be differentiable at *x*.
- If $f'(x_0)>0$, then $f(x_0-h)<f(x_0)<f(x_0+h)$ for *h* sufficiently small, $h>0$.
- If $f'(x_0)<00$, then $f(x_0-h)>f(x_0)>f(x_0+h)$ for *h* sufficiently small, $h>0$.
- Proof:
  - We know that $\lim\limits_{k\to 0}\frac{f(x_0+k)-f(x_0)}{k}=f'(x_0)>0$.
  - Idea: Use $f'(x_0)$ as your ε.
  - $\exists\delta>0 \text{ s.t. } \forall 0<|k|<h \text{ then } 0\le\left|\frac{f(x_0+k)-f(x_0)}{k}-f'(x_0)\right|<f'(x_0)$, so
$$-f'(x_0)<\frac{f(x_0+k)-f(x_0)}{k}-f'(x_0)<f'(x_0).$$
  - Add $f'(x_0)$: $0<\frac{f(x_0+k)-f(x_0)}{k}<2f'(x_0)$.
  - If $0<k<\delta$, then $0<f(x_0+k)-f(x_0)\Rightarrow f(x_0+k)>f(x_0)$.
  - If $-\delta<k<0$, then $0>f(x_0+k)-f(x_0)\Rightarrow f(x_0+k)<f(x_0)$.

Q.E.D.

Rolle's Theorem

- Basically, the MVT with $g(a) = g(b) = 0$.



- Let g be differentiable on (a,b), continuous on $[a,b]$, $g(a) = 0 = g(b)$.
- Then there is at least one c in (a,b) where $g'(c) = 0$.
- Proof:
    - If $g \equiv 0$, then $g'(c) = 0$ for all $c \in (a,b)$.
    - If $g \neq 0$, somewhere it has to be positive or negative.
    - Suppose g is positive somewhere on (a,b).
        - If g is positive, then because g is continuous on $[a,b]$, by the EVT g has to take a max and it takes a max at a point $c \in (a,b)$.
        - Want to show: $g'(c) = 0$.
        - By the lemma, if $g'(c) > 0$ or $g'(c) < 0$, $(c, g(c))$ is not a max. Contradiction.

Q.E.D.

Lecture #18 – Thursday, November 6, 2003

Example: Application of Rolle's Theorem

Show that $f(x) = 2x^3 + 5x - 10$ has at most one root.

- Suppose it has 2 roots, $a < b$. $f(a) = f(b) = 0$.
- So Rolle's Theorem says $\exists$ c, $f'(c) = 0$.
- $f'(x) = 6x^2 + 5 > 0$, never zero! Contradiction!

Mean Value Theorem

- Let f be continuous on $[a,b]$, differentiable on (a,b), then there exists a c in (a,b) so that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
- Proof: Use Rolle's Theorm
    - Idea: Subtract off a linear function to "make" f into a function that satisfies the requirement for Rolle's Theorem.
    - Note: The equation of the line through $(a, f(a))$ and $(b, f(b))$ is $y = \dfrac{f(b) - f(a)}{b - a}(x - a) + f(a)$.
    - Let $g(x) = f(x) - \left[\dfrac{f(b) - f(a)}{b - a}(x - a) + f(a)\right]$.
    - Notice: $g(b) = g(a) = 0$.
    - By Rolle's Theorem, there is a c in (a,b) where $g'(c) = 0 = f'(c) - \dfrac{f(b) - f(a)}{b - a}$.

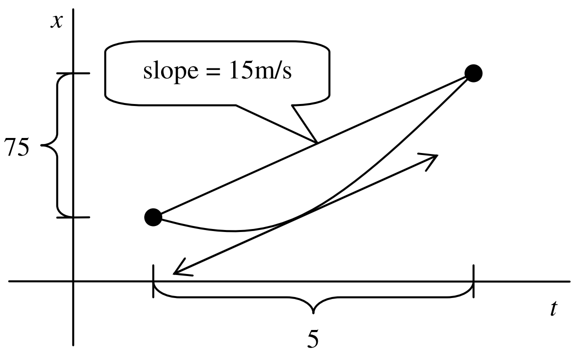
- So this means that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Q.E.D.

Examples: Applications of MVT

1) Suppose *f* is continuous on $[a, b]$. You know $f(1) = 3$ and that in $(1,3)$, $1 \le f'(x) \le 3$. What is the maximum value of $f(3)$?
    - The maximum possible $f(3) = 9$.
    - We know by MVT that there is a *c* in $(1,3)$ such that $1 \le f'(c) = \dfrac{f(3) - 3}{2} \le 3 \Rightarrow 5 \le f(3) \le 9$.
    - Max possible = 9, min possible = 5.

2) Jury duty problem. The facts: She travels 75m in 5sec. The claim: She was going under speed limit of 40km/hr ≅ 11.1m/s.

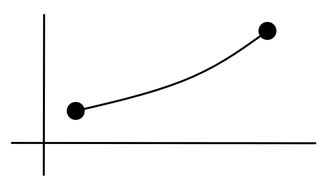


- Somewhere between when she hit the brakes and *t* = 5s.
- She had to be going 15m/s which is already over 40km/hr.

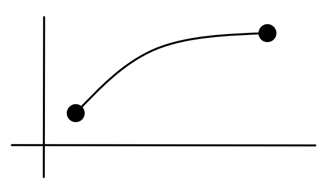
Increasing and Decreasing Functions

Definition

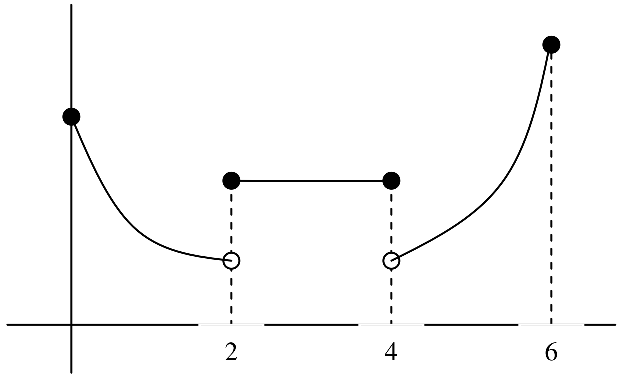
- A function *f* is increasing on an interval *I*. If $\forall x_1, x_2$ in *I*, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.



- A function *f* is decreasing on an interval *I*. If $\forall x_1, x_2$ in *I*, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.



Example



Theorem

- Let *f* be differentiable on open interval *I*.
    - If $f'(x) > 0$ for all $x \in I$, then *f* is increasing on *I*.
    - If $f'(x) < 0$ for all $x \in I$, then *f* is decreasing on *I*.
    - $f'(x) = 0$ for all $x \in I$, then *f* is constant on *I*.

Proof

- Let $x_1 < x_2$ on *I*. Then by MVT, $\exists\, c$, $x_1 < c < x_2$ such that $0 < f'(c) < \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.
- Multiply by $(x_2 - x_1)$: $0 < f(x_2) - f(x_1) \Rightarrow f(x_2) > f(x_1)$.

Q.E.D.

- Actually, works for close intervals.

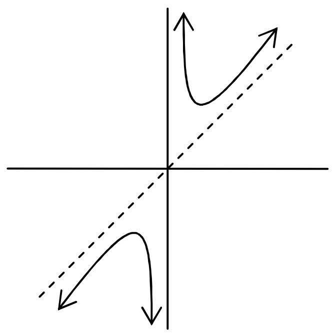
Theorem

- Let *f* be continuous on $[a,b]$, differentiable on (a,b).
    - If $f'(x) > 0$ on all of (a,b), then *f* is increasing on $[a,b]$.
    - If $f'(x) < 0$ on all of (a,b), then *f* is decreasing on $[a,b]$.
    - If $f'(x) = 0$ on all of (a,b), then *f* constant on $[a,b]$.

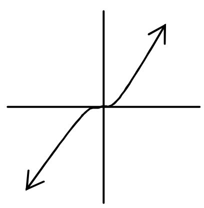
- Note: We already know that if $f(x) = \text{constant}$ on *I*, $f'(x) = 0$ on *I*. This theorem states the converse, so *f* is constant on *I* $\Leftrightarrow$ $f'(x) = 0$ on *I*.

Example: Finding intervals of increase/decrease

- $f(x) = x + \dfrac{1}{x}$
- $f'(x) = 1 - \dfrac{1}{x^2}$
- Increasing when $1 - \dfrac{1}{x^2} > 0 \Rightarrow 1 > \dfrac{1}{x^2} \Rightarrow x^2 > 1 \Rightarrow x \in (-\infty, -1] \cup [1, \infty)$.
- Decreasing when $x^2 < 1 \Rightarrow x \in [-1, 0) \cup (0, 1]$.

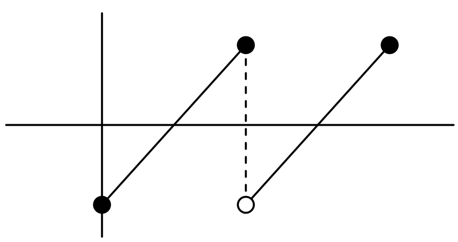


- Warning: Just because the function is increasing (decreasing) on I does <u>not</u> mean the derivative is >0 (<0) on I!



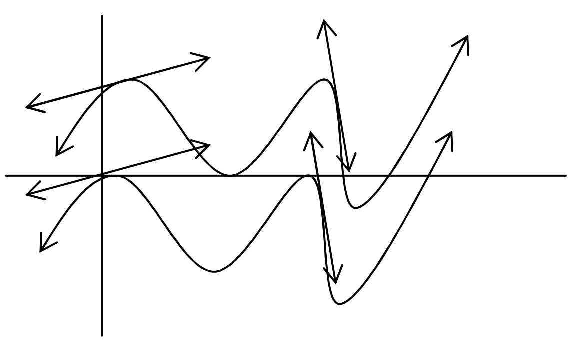
  - At 0, $f'(x)=0$, but it is increasing everywhere.

- Warning: f needs to be differentiable!



AN ASIDE: (WE'LL COME BACK TO THIS NEXT TERM…)

- Notice: If f and g differ by a constant, so $f(x)=g(x)+c$, then $f'(x)=g'(x)$.



- The Theorem (Corollary) then says:
  If f, g are differentiable on I and if $f'(x)=g'(x)$ on I, then $f(x)=g(x)+c$ on I.
  - Proof:
    - $h(x)=f(x)-g(x)$.
    - $h'(x)=0$ on I.
    - So $h(x)\equiv c \Rightarrow f=g+c$.

Q.E.D.

Example

Find a function *f* so that $f' = 2xh2 + 10x + 2$ and $f(0) = 7$.

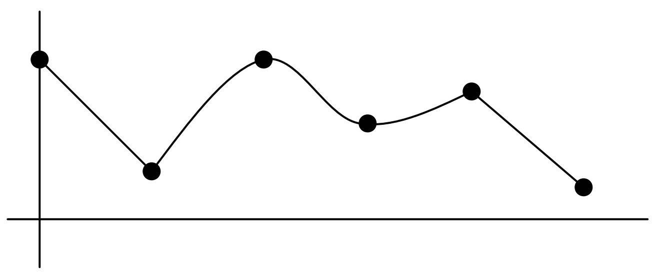
- A candidate $g(x)$ for f': $g(x) = \frac{2}{3}x^3 + 5x^2 + 2x$.
- This satisfies $g' = f'$. So $f(x) = \frac{2}{3}x^3 + 5x^2 + 2x + 7$. So $c = 7$ to satisfy $f(0) = 7$.

Back To Main Theme: Local Extreme Values

Definition

- A function *f* takes on a <u>local maximum</u> at *c* if $f(x) \le f(c)$ for *x* sufficiently close to *c* ($\exists \delta > 0$ s.t. $x \in (c - \delta, c + \delta)$).
- A function *f* takes on a <u>local minimum</u> at *c* if $f(x) \ge f(c)$ for *x* sufficiently close to *c*.

Example



- Terminology: Local maximum/minimum are called local extrema.

Theorem

- If *f* takes on a local extreme value at $c \in$ interior of *I*, then either $f'(c) = 0$ or $f'(c)$ DNE.
- Proof:
  - Suppose $f'(c)$ exist.
  - Then if $f'(c) > 0$, by lemma from Rolle's Theorem, $\exists$ interval $(c - \delta, c + \delta)$ so that for $c - \delta < x_1 < c < x_2 < c + \delta$, $f(x_1) > f(c) < f(x_2)$ so *c* is not extreme value.
  - Ditto if $f'(c) < 0$. So $f'(c) = 0$.

Q.E.D.

Lecture #19 – Tuesday, November 11, 2003

Definition

- Let *c* be in $\mathrm{dom}(f)$ where $f'(c) = 0$ or $f'(c)$ DNE. The *c* is called a critical point.
- Idea: Critical points are candidates for local extreme values.

Examples

1) $f(x) = -x^2 + 2$
   - $f'(x) = -2x$
   - The critical point is at $c = 0$.

2) $f(x) = |x - 2|$
   - The critical point is at $c = 2$.

3) $f(x) = x^3 + 1$
   - $f(x) = 3x^2$
   - The critical point is at $c = 0$.

- Once you have a list of critical points, what to do? Figure out whether it's a local max, local min, or neither.

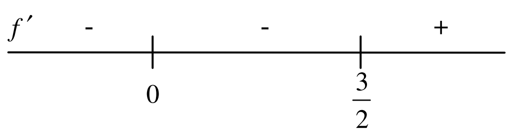
Theorem: "First Derivative Test"

- Suppose *c* is a critical point for *f*, *f* is continuous at *c*. If $\exists \delta > 0$ such that:
  - $f'(x) > 0$ for all $x \in (c - \delta, c)$, $f'(x) < 0$ for all $x \in (c, c + \delta)$, then *c* is a local max.
  - $f'(x) < 0$ for all $x \in (c - \delta, c)$, $f'(x) > 0$ for all $x \in (c, c + \delta)$, then *c* is a local min.
  - $f'(x)$ is the same sign on both sides, then it's not a local extreme value.

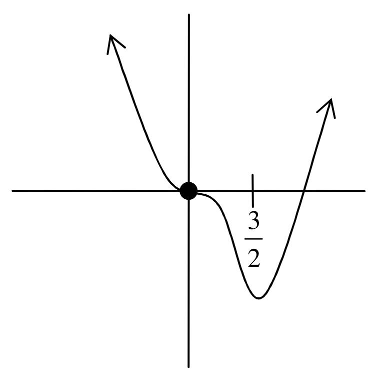
Example

$f(x) = x^4 - 2x^3$. Find local min's and max's.

- $f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3)$
- Critical points: 0, $\frac{3}{2}$.



-



Theorem: "Second Derivative Test"

- Suppose $f'(c) = 0$ and $f''(c)$ <u>exists</u>.
  - If $f''(c) > 0$, then $f(c)$ is a local min.
  - If $f''(c) < 0$, then $f(c)$ is a local max.

- Proof: Case $f''(c) > 0$
  - Remember the Lemma for Rolle's Theorm.
  - Then $\exists \delta > 0$ for $c - \delta < x_1 < c < x_2 < c + \delta$. So $f'(x_1) < f'(c) < f'(x_2)$.
  - Since $f'(c) = 0$, so $f'(x_1) < 0$ and $f'(x_2) > 0$.
  - FDT tells us *c* is a local min.

Q.E.D.

Example

Find local max/min of $f(x) = 2x^3 - 3x^2 - 12x + 5$.

- $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$.
- Critical points: $c = 2, -1$.
- $f'(x) = 12x - 6$
- So 2 is a local min and -1 is a local max.

Lecture #20 – Thursday, November 13, 2003

Definition: Endpoint Extrema

- If c is an endpoint of a domain of f, f has an <u>endpoint maximum</u> at c if $f(c) \geq f(x)$ for x close enough to c and x in $\text{dom}(f)$.
- If c is an endpoint of a domain of f, f has an <u>endpoint minimum</u> at c if $f(c) \leq f(x)$ for x close enough to c and x in $\text{dom}(f)$.

Absolute Extrema

Definition

- A function f has an <u>absolute maximum</u> at d if $f(d) \geq f(x)$ for <u>all</u> $x \in \text{dom}(f)$.
- A function f has an <u>absolute minimum</u> at d if $f(d) \leq f(x)$ for <u>all</u> $x \in \text{dom}(f)$.

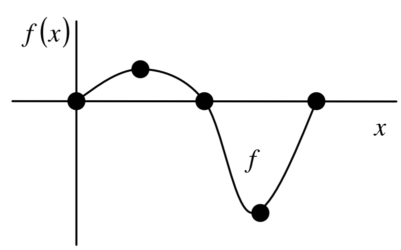
Summary

- For f defined on a closed bounded $[a, b]$, to find the absolute extrema:
    - Find critical numbers in interior $c_1, c_2, c_3, \ldots, c_n$.
    - Calculate $f(a), f(c_1), f(c_1), \ldots, f(c_n), f(b)$.
    - Largest/smallest from above is the absolute maximum/minimum.

Example

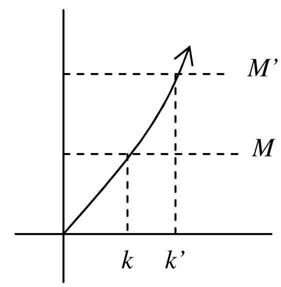
$f(x) = \sin x - \sin^2 x$, domain = $[0, 2\pi]$.

- $f'(x) = \cos x - 2\sin x \cos x$
- Solve: $0 = \cos x - 2\sin x \cos x \Leftrightarrow 0 = \cos x(1 - 2\sin x)$. So $\cos x = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ or $\sin x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$.
- Calculate:
    - $f(0) = 0$
    - $f\left(\dfrac{\pi}{6}\right) = \dfrac{1}{4}$ – absolute maximum
    - $f\left(\dfrac{\pi}{2}\right) = 0$



- $f\left(\frac{5\pi}{6}\right)=\frac{1}{4}$ – absolute maximum
- $f\left(\frac{3\pi}{2}\right)=-2$ – absolute minimum
- $f(2\pi)=0$

Definition/Notation for Unbounded Domain



1) "As $x\to\infty$, $f(x)\to\infty$".
   - As x increases without (upper) bound, $f(x)$ becomes arbitrarily large.
   - For <u>any</u> positive $M>0$, $\exists k>0$ so that if $x\ge k$, then $f(x)\ge M$.
2) "As $x\to\infty$, $f(x)\to-\infty$".
   - For <u>any</u> positive $M<0$, $\exists k>0$ so that if $x\ge k$, then $f(x)\le M$.
3) "As $x\to-\infty$, $f(x)\to\infty$".
   - For <u>any</u> positive $M>0$, $\exists k<0$ so that if $x\le k$, then $f(x)\ge M$.
4) "As $x\to-\infty$, $f(x)\to-\infty$".
   - For <u>any</u> positive $M<0$, $\exists k<0$ so that if $x\le k$, then $f(x)\le M$.

- The point: If (1) or (3) happens, no absolute maximum! If (2) or (4), no absolute minimum!
- Warning: Not all functions fit into (1) to (4) – ex: $f(x)=\sin x$.
- But: For polynomial $f(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_0, a_n\ne 0, n\ne 0$ one of (1) to (4) <u>does</u> happen, and you determine it by looking only at the "leading term" a_nx^n.

Example

$f(x)=3x^4-5x^3+10x$

- As $x\to\infty$, $f(x)\to\infty$. As $x\to-\infty$, $f(x)\to\infty$. So no absolute maximum.

Final Summary: Finding Absolute Extrema

1) Find critical numbers in interior.
2) If yo have endpoints, check for endpoint extrema.
3) Look at critical numbers and identify local max's and min's.
4) If domain is unbounded, look at "behaviour at infinity".
5) Evaluate f at local min's or max's and endpoint extrema to find the absolute max/min, if it exists.

Example

$f(x)=6\sqrt{x}-x\sqrt{x}$, $\operatorname{dom}(f)=[0,\infty)$.

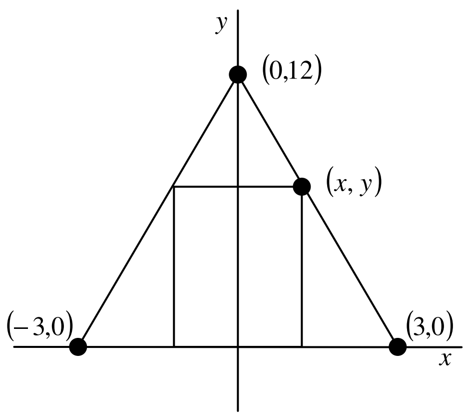
- $f'(x)=3x^{-\frac{1}{2}}-\frac{3}{2}x^{\frac{1}{2}}=3\left(\frac{1}{\sqrt{x}}-\frac{\sqrt{x}}{2}\right)=3\left(\frac{2-x}{2\sqrt{x}}\right)=0\Leftrightarrow\frac{1}{\sqrt{x}}=\frac{\sqrt{x}}{2}\Leftrightarrow x=2$

- FDT: To the left of $x=2$, $f'>0$. To the right of $x=2$, $f'<0$. So by FDT, $x=2$ is a <u>local max</u>.
- What is the behaviour as $x\to\infty$? Because $-x\sqrt{x}\to-\infty$ as $x\to\infty$, $f(x)\to-\infty$ as $x\to\infty$. Give up for absolute min.
- Evaluate: $f(0)=0$, $f(2)=4\sqrt{2}$. So $4\sqrt{2}$ is the absolute max.

OPTIMIZATION PROBLEMS

Baby Example

An isosceles triangle has a base of 6 units and height of 12 units. Suppose we have to inscribe a rectangle. what are the dimensions of such a rectangle of maximum area?



- Area $=2xy$
- Constraint: (x,y) has to be on the line connecting $(0,12)$, $(3,0)$: $y=-4x+12$.

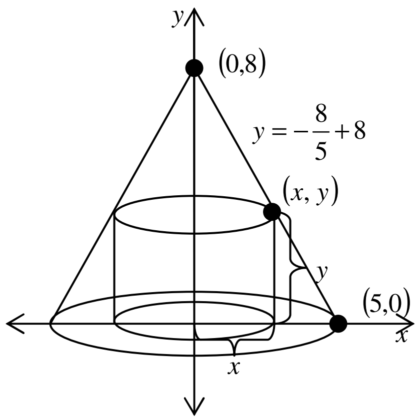
- Replacing y: Area $=A(x)=2x(-4x+12)=8(-x^2+3x)$.
- Key: Now Area is a function of <u>one</u> variable!
- For the problem, $\operatorname{dom}(A)=[0,3]$.
- $A'(x)=8(-2x+3)=0\Rightarrow x=\dfrac{3}{2}$.
- $A''(x)=8(-2)=-16<0$. So it's a local max by SDT.
- $A(0)=A(3)=0$, $A\left(\dfrac{3}{2}\right)=8\left(-\dfrac{9}{4}+\dfrac{9}{2}\right)=18$.
- So the dimensions are 3×6.

In General

1) Draw picture.
2) What do you want to maximize/minimize? Find a formula for it.
3) Use the constraints in the problem to write the formula from (2) in terms of one variable only.
4) Determine the relevant domain for the problem.
5) Determine the absolute max/min as before.

Example

Suppose we're to inscribe a cylinder in a right circular cone height 8, base radius 5. Find the dimensions of the cylinder, maximizing volume.



- Want to maximize the volume.
- $\text{Volume} = \pi x^2 y$.
- $V(x) = \pi x^2\left(-\frac{8}{5}x+8\right) = \frac{8}{5}\pi\left(-x^3+5x^2\right).$
- Relevant domain: $[0,5]$.
- $V'(x) = \frac{8}{5}\pi\left(-3x^2+10x\right) = \frac{8}{5}\pi x(-3x+10) = 0 \Rightarrow x = \frac{10}{3}$.
- $V''(x) = \frac{8}{5}\pi(-6x+10)$. $V''\left(\frac{10}{3}\right) < 0$, which is local max by SDT.
- $V(0) = V(5) = 0$, $V\left(\frac{10}{3}\right) = \frac{800}{27}\pi$. So the absolute max is $\frac{800}{27}\pi$.
- So the dimensions are: radius $\frac{10}{3}$, height $\frac{8}{3}$.

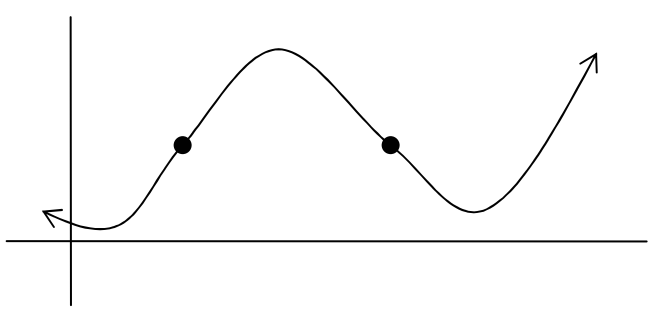
Lecture #21 – Tuesday, November 18, 2003

CONCAVITY

Discussion of Concavity: About Second Derivative

- Rough example: From economics, the cost function for producing *x* widgets $c(x)$, $\frac{dc}{dx}(100)$ = cost of producing the next "widget". The second derivative is measuring how fast $\frac{dc}{dx}$ is changing.

Picture



Definition: Concavity

Let *f* be differentiable on an open interval *I*. Then *f* is <u>concave up</u> if f' is increasing on *I*, and <u>concave down</u> if f' is decreasing on *I*.

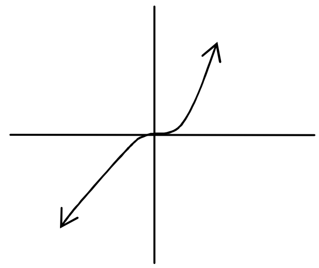
Definition: Point of Inflection

Let *f* be continuous at *c*. Then *c* is a <u>point of inflection</u> if $\exists\delta > 0$ such that *f* is concave in one sense on $(c-\delta, c)$, and *f* is concave in the other sense on $(c, c+\delta)$.

Example

$f(x) = x^3$

- $f'(x) = 3x^2$, and 0 is a point of inflection.



Theorem

Let *f* be <u>twice</u> differentiable on open interval *I*.

1) If $f''(x) > 0$, then *f* is concave up.
2) If $f''(x) < 0$, then *f* is concave down.

Proof

- The same as when we proved that if $f'(x) > 0$ on *I*, then *f* is increasing. Now use $g = f'$.

- How to find points of inflection? Come up with a list of <u>possible</u> points of inflection.

Theorem

If $(c, f(c))$ is a point of inflection, then either:

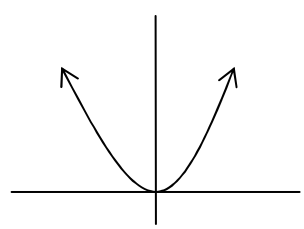
1) $f''(c) = 0$
2) $f''(c)$ DNE

- This is analogue of the theorem for local extreme values.

- Warning: Just as critical points don't need to be local extrema, just because $f''(c) = 0$ does not mean $(c, f(c))$ is a point of inflection.

Example

- $f(x) = x^4$
- $f'(x) = 4x^3$
- $f''(x) = 12x^2$
- At $x = 0$, $f''(c) = 0$, but it's <u>not</u> a point of inflection.



Proof

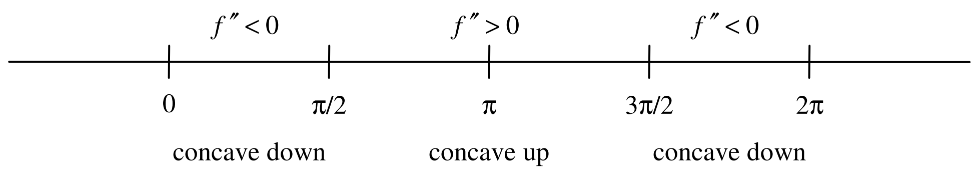
- Assume $(c, f(c))$ is a point of inflection. Assume *f* is concave up on the left and concave down on the right.
- By definition, f' is increasing on the left on $(c-\delta, c)$, f' is decreasing on the right on $(c, c+\delta)$.
- Assume f'' exists. Then f' is continuous. So $f''(x) > 0$ on $(c-\delta, c]$ and $f''(x) < 0$ on $[c, c+\delta)$. Therefore, $f''(c) = 0$.
- The only other possibility is that $f''(c)$ DNE.

Q.E.D.

Example

Find intervals of concavity and points of inflection for $f(x) = x + \cos x$ on $[0, 2\pi]$.

- $f'(x) = 1 - \sin x$, $f''(x) = -\cos x$.
- Possible points of inflection: $x = \frac{\pi}{2}, \frac{3\pi}{2}$.
- Look at intervals:



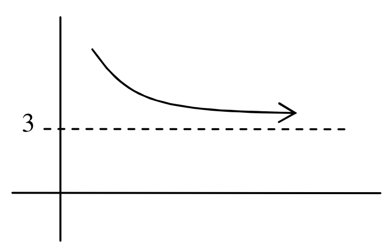
Lecture #22 – Thursday, November 20, 2003

Horizontal Asymptotes

- "Limits of the function at ±∞".
- Now we care about the behaviour of *f*, regardless of weather it goes to ±∞.

Example

- $f(x) = \dfrac{3x+1}{x}$.
- We want to say "$\lim_{x\to\infty} f(x) = 3$".



Definition

$\lim_{x\to\infty} f(x) = L$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ so that if $x > k$, then $|f9x - L| < \varepsilon$. If this is true, then the line $y = L$ s called the <u>horizontal asymptote</u>.

Example

Prove that $\lim_{x\to\infty} \frac{3x+1}{x} = 3$.

- Given $\varepsilon > 0$, we need to find $k > 0$ so that if $x > k$, we want $\left|\frac{3x+1}{x} - 3\right| < \varepsilon$.
- $\left|\frac{3x+1}{x} - 3\right| < \varepsilon \Leftrightarrow \left|\frac{3x+1-3x}{x}\right| < \varepsilon \Leftrightarrow \left|\frac{1}{x}\right| < \varepsilon \Leftrightarrow \frac{1}{x} < \varepsilon \Leftrightarrow x > \frac{1}{\varepsilon}$.
- Choose $k = \frac{1}{\varepsilon}$.

Q.E.D.

- Notice: $\lim_{x\to\infty} \frac{3x+1}{x} = \lim_{x\to\infty} \frac{x\left(3+\frac{1}{x}\right)}{x}$.

Examples: Techniques for finding horizontal asymptotes

1) $\lim_{x\to\infty} \frac{x+1-\sqrt{x}}{x^2+2x+1} = \lim_{x\to\infty} \frac{x\left(1+\frac{1}{x}-\frac{1}{\sqrt{x}}\right)}{x^2\left(1+\frac{2}{x}+\frac{1}{x^2}\right)} = 0$.

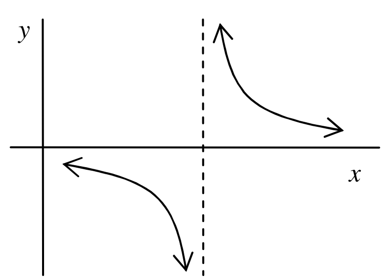
2) $\lim_{x\to\infty} \frac{2x^3+3x+7}{3x^3+10x-5} = \lim_{x\to\infty} \frac{x^3\left(2+\frac{3}{x^2}+\frac{7}{x^3}\right)}{x^3\left(3+\frac{10}{x^2}-\frac{5}{x^3}\right)} = \frac{2}{3}$

General Technique

- For $R(x) = \frac{a_n x^n + \ldots + a_0}{b_k x^k + \ldots + b_0}$, $\lim_{x\to\pm\infty} R(x) = \begin{cases} \pm\infty \text{ if } n > k \\ \frac{\mathrm{a}}{\mathrm{b}} \text{ if } n = k \\ 0 \text{ if } n < k \end{cases}$.

Vertical Asymptotes

- Now, let's consider the possibility that $\lim_{x\to c} f(x) = \pm\infty$ <u>or</u> $\lim_{x\to c^{\pm}} f(x) = \pm\infty$.



Definition

The line $x = c$ is a vertical asymptote for *f* if any of these happen:

1) $\lim_{x \to c} = \pm\infty$
2) $\lim_{x \to c^+} = \pm\infty$
3) $\lim_{x \to c^-} = \pm\infty$

Precise Definition (For 2)

$\forall N > 0, \exists \delta > 0$ so that if $c < s < c + \delta$ then $f(x) > N$.

Example

Prove that $\lim_{x \to 1^+} \frac{1}{x-1} = +\infty$.

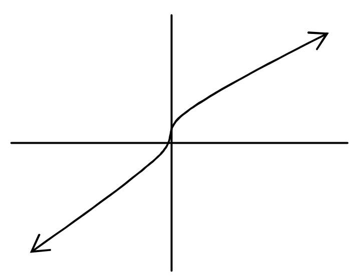
- Given $N > 0$, we want $\frac{1}{x-1} > N$ if $1 < x < 1 + \delta$ for some δ.
- Since $1 < x < 1 + \delta$, then $x - 1 < \delta$. Choose $\delta = \frac{1}{N}$.
- Now, if $x - 1 < \delta = \frac{1}{N}$ means $\frac{1}{x-1} > N$.

Q.E.D.

VERTICAL TANGENTS AND CUSPS

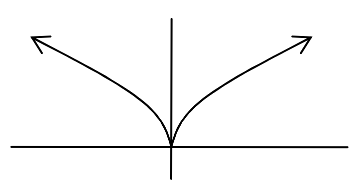
Example

- $y = x^{\frac{1}{3}}$
- Example of vertical tangent because the derivative "becomes ∞".



Example

- $y = x^{\frac{2}{3}}$
- Example of vertical cusp.



Definition

Graph of *f* has a vertical tangent at $(c, f(c))$ if $\lim_{x \to c} f'(x) = \pm\infty$.

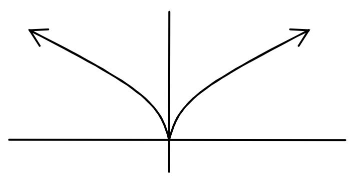
Definition

Graph of f has a vertical cusp at $(c, f(c))$ if:

- $\lim_{x\to c^+} f'(x) = \infty$, $\lim_{x\to c^-} f'(x) = -\infty$
- $\lim_{x\to c^+} f'(x) = -\infty$, $\lim_{x\to c^-} f'(x) = \infty$

Example

- $f(x) = x^{\frac{2}{3}}$



- Double check $(0,0)$ is a vertical cusp.
- We need to compare the slopes of f close to 0.
- $f(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3}\cdot\frac{1}{\sqrt[3]{x}}$.
- $\lim_{x\to 0^+} \frac{2}{3}\cdot\frac{1}{\sqrt[3]{x}} = +\infty$, $\lim_{x\to 0^-} \frac{2}{3}\cdot\frac{1}{\sqrt[3]{x}} = -\infty$.
- So $(0,0)$ is a vertical cusp.

CURVE SKETCHING

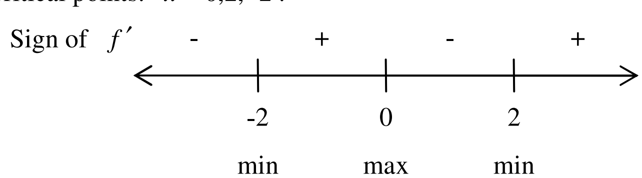
Procedure

1) Find domain of f, behaviour at endpoints and $\pm\infty$.
2) Intercepts (both y and x).
3) Symmetry and/or periodicity.
4) Use f': look for critical points, FDT, local extrema.
5) Use f'': concavity and points of inflection.
6) Plot relevant points and sketch.

Example

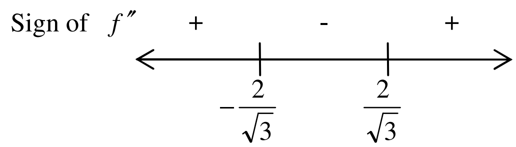
Sketch the graph of $f(x) = \frac{1}{4}x^4 - 2x^2 + \frac{7}{4}$.

- Domain of $f = (-\infty, \infty)$.
- Look at: $\lim_{x\to\infty} = +\infty$, $\lim_{x\to-\infty} = +\infty$.
- y-intercept: $\left(0, \frac{7}{4}\right)$.
- x-intercept: Solve $\frac{1}{4}x^4 - 2x^2 + \frac{7}{4} = 0 \Leftrightarrow x^4 - 8x^2 + 7 = 0 \Leftrightarrow (x^2-7)(x^2-1) = 0$. So x-intercepts are at $x = 1, -1, \sqrt{7}, -\sqrt{7}$.
- Symmetry: $f(x)$ is even, so it is enough to graph f on $[0, \infty)$.
- Use f': $f'(x) = x^3 - 4x = x(x^2-4) = x(x+2)(x-2)$.
    - Critical points: $x = 0, 2, -2$.

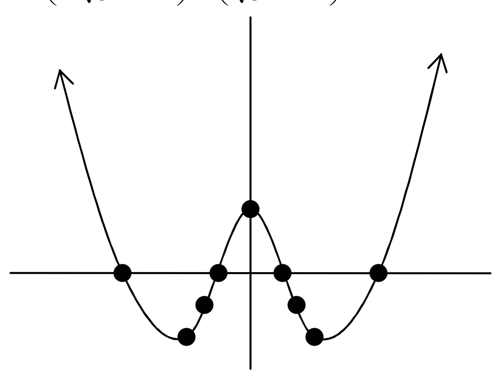


- Local minimums at $\left(2,-\frac{9}{4}\right)$ and $\left(-2,-\frac{9}{4}\right)$.
- Local maximum at $\left(0,\frac{7}{4}\right)$.

- Use f'': $f''(x)=3x^2-4=3\left(x^2-\frac{4}{3}\right)=3\left(x+\frac{2}{\sqrt{3}}\right)\left(x-\frac{2}{\sqrt{3}}\right)$.
  - Possible points of inflection at $x=\frac{2}{\sqrt{3}},-\frac{2}{\sqrt{3}}$



  - Points of inflection: $\left(-\frac{2}{\sqrt{3}},-\frac{17}{36}\right)$, $\left(\frac{2}{\sqrt{3}},-\frac{17}{36}\right)$.



Example

Sketch the graph of $f(x)=\frac{x^2-3}{x^3}=x^{-1}-3x^{-3}$.

- Domain of f = $(-\infty,0)\cup(0,\infty)$.
  - $\lim_{x\to0^+}\frac{x^2-3}{x^3}=-\infty$, $\lim_{x\to0^-}\frac{x^2-3}{x^3}=+\infty$.
- Look for critical points: $f'(x)=-x^{-2}+9x^{-4}=\frac{9-x^2}{x^4}$.
  - Careful: Critical points are at $x=3,-3$, and <u>not</u> $x=0$ because 0 not in domain.

Example

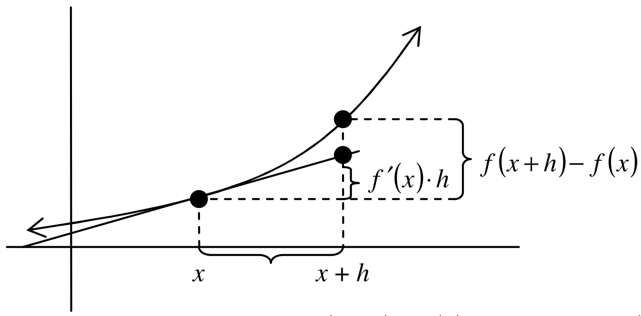
$f(x)=\sin 2x-2\sin x$

- Domain: $(-\infty,\infty)$.
- No asymptotes or limits as $x\to\pm\infty$.
- f is odd and periodic.
- Its enough to graph it on $[0,\pi]$.

Lecture #23 – Tuesday, November 25, 2003

NEWTON-RAPHSON METHOD

- This is a way of approximate solutions to equations of the form $f(x)=0$.
- "Math Fact": General method for finding roots, for example to polynomial equations, is not known, so approximation methods are required.
- Reminder: Geometric interpretation for derivatives is "rate of change".



- The Point: As h gets small, $f(x+h)-f(x)$ is close to $f'(x)\cdot h$.
  - $f(x+h)-f(x)$: <u>increment</u> of f from x to $x+h$ (Δf)
  - $f'(x)\cdot h$: <u>differential</u> of f at x with increment h (df)
  - $\Delta f \cong df$ for small h

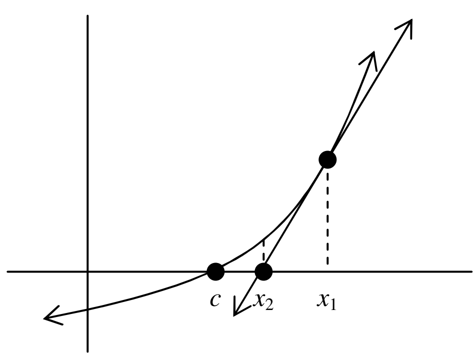
Example

Approximate $\sqrt{2}$.

- In this situation, $f(x)=\sqrt{x}$ and $x+h=2$.
- $f(x+h)-f(x)\cong f'(x)\cdot h \Leftrightarrow f(x+h)\cong f'(x)\cdot h+f(x)$.
- Let's pick $x=\dfrac{16}{9}$.
- $f'(x)=\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{x}}$. So $f\left(\dfrac{16}{9}\right)=\sqrt{\dfrac{16}{9}}=\dfrac{4}{3}$ and $f'\left(\dfrac{16}{9}\right)=\dfrac{1}{2}\cdot\dfrac{3}{4}=\dfrac{3}{8}$.
- $f\left(\dfrac{16}{9}\right)+f'\left(\dfrac{16}{9}\right)\cdot\left(2-\dfrac{16}{9}\right)=\dfrac{4}{3}+\dfrac{3}{8}\cdot\dfrac{2}{9}=\dfrac{17}{12}\approx 1.417$.

Newton-Raphson Method

- A little more sophisticated – use differentials at each step of the algorithm.

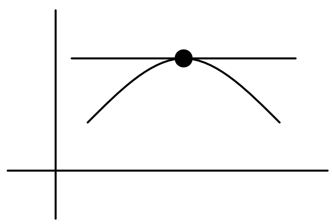


- Algorithm in "picture":
  - Start with some point x_1.
  - Draw tangent to f at $(x_1, f(x_1))$ and find the x-intercept.
  - Call that x_2.
  - Repeat as necessary.

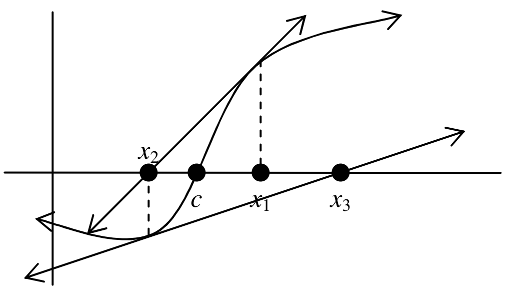
What Is x_2?

- The equation for the tangent line is $y - f(x_1) = f'(x_1)(x_2 - x_1)$.
- At the x-intercept x_2, we must have:
  $$0 - f(x_1) = f'(x_1)(x_2 - x_1) = f'(x_1)x_2 - f'(x_1)x_1 \Rightarrow x_2 = \frac{f'(x_1)x_1 - f(x_1)}{f'(x_1)} = x_1 - \frac{f(x_1)}{f'(x_1)}.$$
- So in general: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

Various Pitfalls

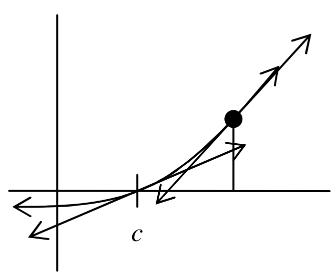


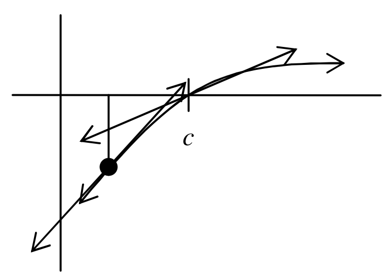
- At any x_n, if $f'(x_n) = 0$, tough luck!
- No x-intercept.



- Concavity issue: x_3 is further away from c than x_2!

- Fact (without proof): If f is twice differentiable and if $f(x) \cdot f''(x) > 0$ on an interval joining x to c, then Newton-Raphson method works.





Lecture #24 – Thursday, November 27, 2003

Example

Approximate $\sqrt{2}$.

- Take $f(x) = x^2 - 2$. Find a root.
- Use IVT: $f(0) = -2 < 0$, $f(2) = 2 > 0$.

- Take derivative: $f'(x) = 2x$.
- Take $x_1 = 2$: $x_2 = 2 - \frac{2}{4} = \frac{3}{2}$.
- $x_3 = \frac{3}{2} - \frac{1/4}{3} = \frac{3}{2} - \frac{1}{12} = \frac{17}{12}$.
- $x_4 = \frac{17}{12} - \frac{f\left(\frac{17}{12}\right)}{f'\left(\frac{17}{12}\right)} = \frac{17}{12} - \frac{1}{144} \cdot \frac{6}{17} = \frac{3462}{144 \cdot 17} \approx 1.41422$.

L'HÔPITAL'S RULES

- These are rules that help you evaluate limits that are of the form "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$".

Theorem: L'Hôpital's Rules

Suppose $f(x)$ and $g(x)$ both have limit 0 as x approaches c^+, c^-, c, ∞, or $-\infty$. If the limit of $\frac{f'(x)}{g'(x)} = L$ approaches the same value (c^+, c^-, c, ∞, $-\infty$), then the limit $\frac{f(x)}{g(x)} = L$.

- Note: L is allowed to be "$\pm\infty$" – i.e. if $\frac{f'(x)}{g'(x)}$ has an asymptote, then so does $\frac{f(x)}{g(x)}$.

Examples

1) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{\frac{1}{2} \cdot \frac{1}{\sqrt{x}}}{1} = \frac{1}{4}$.

2) $\lim_{x \to 1} \sqrt{\frac{1 - x^2}{1 - x^3}} = \sqrt{\lim_{x \to 1} \frac{-2x}{-3x^2}} = \sqrt{\frac{2}{3}}$.

- WARNING: You <u>must</u> check at all times that you <u>can</u> apply the l'Hôpital's Rule – i.e. you have a "$\frac{0}{0}$". In particular, be careful when you apply l'Hôpital's Rule several times.
- To prove l'Hôpital, we'll need Cauchy MVT.

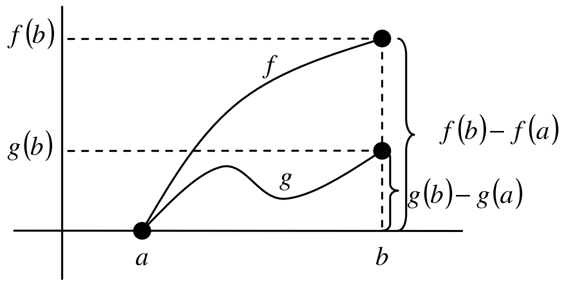
Theorem: Cauchy Mean Value theorem

Suppose f, g are differentiable on (a,b) and continuous on $[a,b]$. If $g' \neq 0$ for all points in (a,b), then there exists r in (a,b) so that $\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

- Notice: If $g(x) = x$, the Cauchy MVT is the MVT – $g'(x) = 1$, so $f'(r) = \frac{f(b) - f(a)}{b - a}$.

- Notice: this is promising; it says something about the ratios of derivatives relating to ratios with the functions.

Picture



- Assume $f(a)=g(a)=0$.
- Somewhere between (a,b), the ratio of the instantaneous rates of change has to equal tot eh average rates of change.

Proof

- The proof will use Rolle's Theorem.
- Let's rewrite: $\dfrac{f'(r)}{g'(r)}=\dfrac{f(b)-f(a)}{g(b)-g(a)} \Leftrightarrow f'(r)(g(b)-g(a))-g'(r)(f(b)-f(a))=0$.
- Want: Some r where $f'(r)(g(b)-g(a))-g'(r)(f(b)-f(a))=0$, but LHS looks like a derivative.
- Consider the function $G(x)=(f(x)-f(a))(g(b)-g(a))-(g(x)-g(a))(f(b)-f(a))$.
- So $G(a)=G(b)=0$. G is continuous and differentiable because f and g are.
- Rolle's Theorem says there is an r in (a,b) such that $G'(r)=0=f'(r)(g(b)-g(a))-g'(r)(f(b)-f(a))$. So $f'(r)(g(b)-g(a))=g'(r)(f(b)-f(a))$.
- Since $f'(r)(g(b)-g(a))\neq 0$ and $g'(r)(f(b)-f(a))\neq 0$, we can divide. We get $\dfrac{f'(r)}{g'(r)}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$.

Q.E.D.

Proof of L'Hôpital (case $\to c^+$)

- Assume $\lim_{x\to c^+} f(x)=\lim_{x\to c^+} f(x)=0$, $\lim_{x\to c^+}\dfrac{f'(x)}{g'(x)}=L$.
- So f and g are differentiable on some $(c,c+h)$. Also $g'\neq 0$.
- Cauchy MVT says $\exists\, r_h$ so that $\dfrac{f'(r)}{g'(r)}=\dfrac{f(c+h)-f(c)}{g(c+h)-g(c)}=\dfrac{f(c+h)}{g(c+h)}$.
- Taking limit as $h\to 0$, $\lim_{h\to 0}\dfrac{f'(r_h)}{g'(r_h)}=\lim_{h\to 0}\dfrac{f(c+h)}{g(c+h)}$.
- LHS limit is L, because r_h is in $(c,c+h)$. So RHS limit is also L.

Q.E.D.

Example

$$\lim_{x\to 0}\frac{\cos x-1+\frac{x^2}{2}}{x^4}\underset{\text{l'H}}{=}\lim_{x\to 0}\frac{-\sin x+x}{4x^3}\underset{\text{l'H}}{=}\lim_{x\to 0}\frac{-\cos x+1}{12x^2}\underset{\text{l'H}}{=}\lim_{x\to 0}\frac{\sin x}{24x}\underset{\text{l'H}}{=}\lim_{x\to 0}\frac{\cos x}{24}=\frac{1}{24}$$

Lecture #25 – Tuesday, December 2, 2003

Examples

1) $\lim\limits_{x\to 0}\dfrac{\sqrt{a+x}-\sqrt{a-x}}{x}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}\cdot\frac{1}{\sqrt{a+x}}+\frac{1}{2}\cdot\frac{1}{\sqrt{a-x}}}{1}=\dfrac{1}{\sqrt{a}}$.

2) $\lim\limits_{x\to 0^+}\dfrac{\sqrt{x}}{\sin\sqrt{x}+\sqrt{x}}=\lim\limits_{t\to 0^+}\dfrac{t}{\sin t+t}=\lim\limits_{t\to 0^+}\dfrac{1}{\cos t+1}=\dfrac{1}{2}$

L'Hôpital For "Infinity Over Infinity"

Suppose $\lim\limits_{\substack{x\to c\\ c^+\\ c^-\\ \infty\\ -\infty}} f(x)=\infty$ and $\lim\limits_{\substack{x\to c\\ c^+\\ c^-\\ \infty\\ -\infty}} g(x)=\infty$. If $\lim\limits_{\substack{x\to c\\ c^+\\ c^-\\ \infty\\ -\infty}}\dfrac{f'(x)}{g'(x)}=L$, then $\lim\limits_{\substack{x\to c\\ c^+\\ c^-\\ \infty\\ -\infty}}\dfrac{f(x)}{g(x)}=L$.

Examples

1) $\lim\limits_{x\to\infty}\dfrac{\sqrt{1+x^2}}{x^2}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{2}\cdot\frac{1}{\sqrt{1+x^2}}\cdot 2x}{2x}=0$.

2) $\lim\limits_{x\to\infty} x\cdot\sin\left(\dfrac{1}{x}\right)$

   - There are two ways to rewrite it as a quotient: $\lim\limits_{x\to\infty}\dfrac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$ or $\lim\limits_{x\to\infty}\dfrac{x}{\frac{1}{\sin\left(\frac{1}{x}\right)}}$.
   - Could also do $\lim\limits_{x\to\infty} x\cdot\sin\left(\dfrac{1}{x}\right)=\lim\limits_{x\to\infty}\dfrac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}=\lim\limits_{y\to 0}\dfrac{\sin(y)}{y}=0$.

3) Limits of the form "$\infty-\infty$": $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}(\tan x-\sec x)=\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\left(\dfrac{\sin x-1}{\cos x}\right)=\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\left(\dfrac{\cos x}{-\sin x}\right)=0$.

More Curve Sketching

Example

Sketch the graph of $f(x)=\dfrac{x}{x^2-1}$.

- $\text{domain}(f)=(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.
- Behaviour at endpoints:
  - $\lim\limits_{x\to-\infty}\dfrac{x}{x^2-1}=0$ "coming from below".

- $\lim\limits_{x\to\infty} \dfrac{x}{x^2-1} = 0$ "coming from above".
- $\lim\limits_{x\to -1^-} \dfrac{x}{x^2-1} = \lim\limits_{x\to -1^-} \dfrac{x}{x-1}\cdot\dfrac{1}{x+1} = -\infty$, $\lim\limits_{x\to -1^+} \dfrac{x}{x^2-1} = \lim\limits_{x\to -1^+} \dfrac{x}{x-1}\cdot\dfrac{1}{x+1} = \infty$.
- $\lim\limits_{x\to 1^-} \dfrac{x}{x^2-1} = \lim\limits_{x\to 1^-} \dfrac{x}{x+1}\cdot\dfrac{1}{x-1} = -\infty$, $\lim\limits_{x\to 1^+} \dfrac{x}{x^2-1} = \lim\limits_{x\to 1^+} \dfrac{x}{x+1}\cdot\dfrac{1}{x-1} = \infty$.

- Use f': $f'(x) = \dfrac{-\left(x^2+1\right)}{\left(x^2-1\right)^2}$.
  - Since $f'(x) \neq 0$ for all $x \in \text{domain}(f)$, there are no critical points.

- Use f'': $f''(x) = \dfrac{2x\left(x^2+1\right)}{\left(x^2-1\right)^3}$.

f''	-		+		-		+
	down	-1	up	0	down	1	up

Sketch: