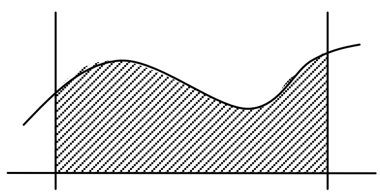
Lecture #26 – Tuesday, January 6, 2004

INTEGRATION: THE OTHER HALF OF CALCULUS

- If it was so useful to take derivatives (approximations, related rates, optimization, etc.), it will be just as useful to know how to "undo" a derivative.
    - ex: Given velocity and/or acceleration, can find position function.
    - Differential equations and solutions – ex: $F = m\dfrac{d^2x}{dt^2}$, $\dfrac{dC}{dt} = bC, b \text{ constant}$.
- Somehow, "the whole is more than the sum of the parts" because integration has a geometric interpretation which is powerful in its own right as an "area under a curve" and makes the relationship between differentiation and integration very powerful.

GEOMETRY OF THE INTEGRAL

Picture



y

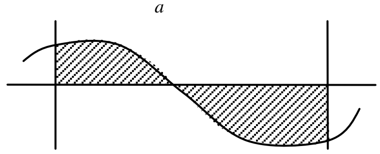
- What is the area under the curve $y = f(x)$?
- Notation: Area under $y = f(x)$ from $x = a$ to $x = b$ is $\int_a^b f(x)dx$ = the (definite) integral of f from a to b.

Properties of the Integral

1) $\int_a^a f(x)dx = 0$.

2) $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ (assume $a < b < c$).

3) If f is "below the x-axis", then $\int_a^b f(x)dx = -[\text{area from } x = a \text{ to } x = b \text{ between x-axis and graph of } f]$.



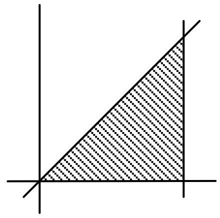
- Note: If , then $\int_a^b f(x)dx = \mathbf{I} - \mathbf{II}$.

4) $\int_b^a f(x)dx = -\int_a^b f(x)dx$ (assume $a < b$).

A CRUCIAL IDEA

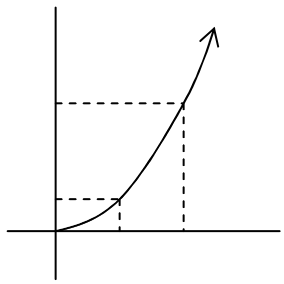
- "Mantra: The integral is itself a function".

Example



- Consider $f(x) = x$.
- $F(t) = \int_0^t f(x)dx = \int_0^t xdx$ = area under the line $y = x$ from $x = 0$ to $x = t$
- If $t = 1$, Area $= \frac{1\times 1}{2} = \frac{1}{2}$.
- If $t = 2$, Area $= \frac{2\times 2}{2} = 2$.
- If $t = 3$, Area $= \frac{3\times 3}{2} = \frac{9}{2}$.
- So, at any value, $\int_0^t f(x)dx = \frac{t^2}{2}$.

Graph of integral $F(t)$:



Preview

- The "area function" F is going to have the property that $F' = f$.

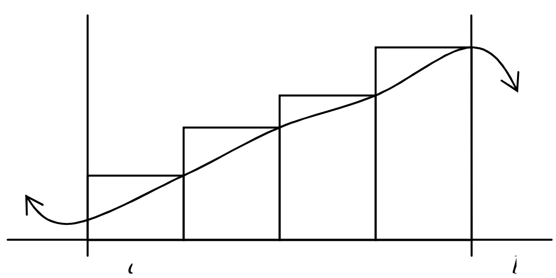
Lecture #27 – Thursday, January 8, 2004

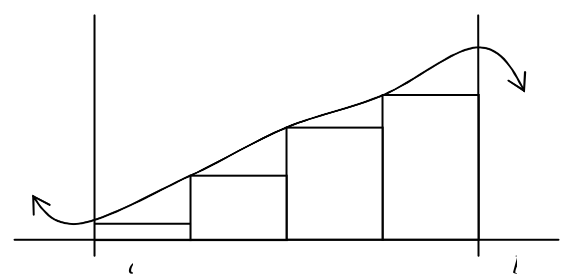
Geometric Intuition

- The integral of f from a to b is "the area under the curve of f from a to b".
- But now, how to compute? Different approaches:
    - Upper and lower sums.
    - Riemann sums.
    - Fundamental Theorem of Calculus.

UPPER AND LOWER SUMS: A ROUGH GUIDE

- Area under curve = Ω.





- Maximum area = $\sum_{i=1}^{4} M_i(x_i - x_{i-1})$, where M_i =

Minimum area = $\sum_{i=1}^{4} m_i(x_i - x_{i-1})$, where m_i =

maximum value of f on $[x_{i-1}, x_i]$. minimum value of f on $[x_{i-1}, x_i]$.

- The point: $\sum_{i=1}^{4} M_i(x_i - x_{i-1}) \le \Omega \le \sum_{i=1}^{4} m_i(x_i - x_{i-1})$. We can make these approximations better by taking more intervals.

Definition

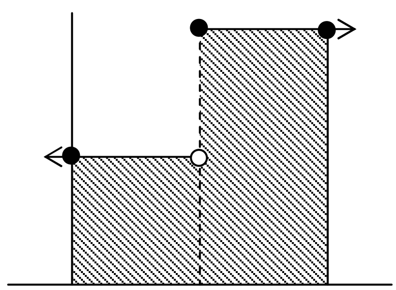
A partition P of $[a,b]$ is a list $P = \{x_0 = a, x_1, x_2, ..., x_N = b\}, x_0 < x_1 < x_2 < ... < x_N$.

$\sum_{i=1}^{4} M_i(x_i - x_{i-1})$ is denoted $L_f(P)$ and $\sum_{i=1}^{4} m_i(x_i - x_{i-1})$ is denoted $U_f(P)$, and $L_f(P) \le \Omega \le U_f(P)$.

- Intuitively, we'll make P smaller so that "$L_f(P) \to \Omega$" and "$U_f(P) \to \Omega$".
- More precisely, define L = least upper bound of $L_f(P)$, and U = greatest lower bound of $U_f(P)$ (if these exists…). Now if $L = U$, then $\Omega = \int_a^b f(x)dx = L = U$ and "f is integrable on $[a,b]$".
- Fact: If f is continuous on $[a,b]$, it is integrable.

Example

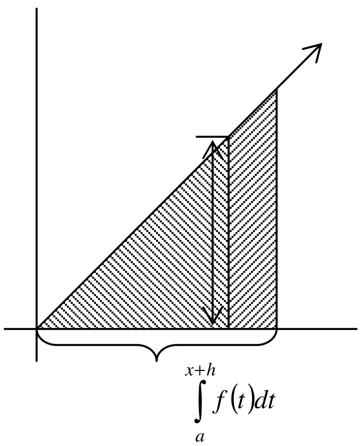
- f doesn't have to be continuous to be integrable.

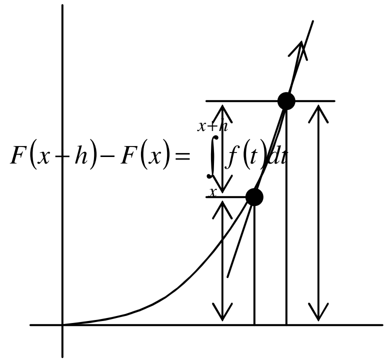


- Clearly, the area is 3/4. Here, f is integrable, but is not continuous.

TOWARDS THE "FUNDAMENTAL THEOREM OF CALCULUS"

- Recall: Why are area under a curve are related to derivates?





$F(x)=\int_a^x f(t)dt$

= area under f
from a to x

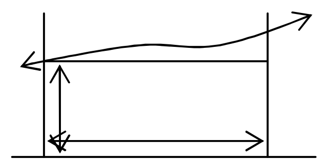
$F(x+h)=\int_a^{x+h} f(t)dt$

Theorem

Let f be continuous on $[a,b]$. Then F defined on $[a,b]$ as $F(x)=\int_a^x f(t)dt$ is differentiable, with $F'(x)=f(x)$.

Proof

- Compute $F'(x)$: $F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$.
    - $F(x+h)-F(x)=\int_x^{x+h} f(t)dt$ (see picture).
    - Intuitively, $F(x+h)-F(x)\approx f(x)\cdot h$.
    - $F'(x)=\lim_{h\to 0^+}\frac{F(x+h)-F(x)}{h}\approx \lim_{h\to 0^+}\frac{f(x)\cdot h}{h}=\lim_{h\to 0^+} f(x)=f($



- Cleaning up the "≈":
    - Let m_h be the minimum value of f on $[x,x+h]$ and let M_h be the maximum value of f on $[x,x+h]$.
    - Then $m_h h\le \int_x^{x+h} f(t)dt\le M_h h\Rightarrow \frac{f(x)}{h}\le\frac{\int_x^{x+h} f(t)dt}{h}\le\frac{f(x)h}{h}, h\ne 0$.
    - Since $\lim_{h\to 0} m_h = f(x)=\lim_{h\to 0} M_h$,

      $$\lim_{h\to 0}\frac{m_h h}{h}=\frac{f(x)h}{h}\le\lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h}\le\lim_{h\to 0}\frac{m_h h}{h}=\frac{f(x)h}{h}\Rightarrow f(x)\le\lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h}\le f(x).$$
    - So, by Pinching Theorem, $\lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h}=f(x)$. Therefore, $F'(x)=f(x)$.

Examples

1) Consider $F(x)=\int_0^x t(t-1)dt$ and $f(t)=t(t-1)$. What are the critical points of F on $(0,\infty)$?
   - $F'(x)=f(x)=x(x-1)$. So the critical point in $(0,\infty)$ is at $x=1$.

2) Given $F(x)=\int_0^{x^2} t(t-1)^2\,dt$, what are the intervals of increase and decrease of F on $(0,\infty)$?
   - Hint: Consider $G(u)=\int_0^u t(t-1)^2$ and $u(x)=x^2$.
   - $F'(x)=G'(u(x))\cdot u'(x)$ (chain rule). Since $G'(u)=f(u)=u(u-1)^2$ and $u'(x)=2x$, so $F'(x)=x^2\left(x^2-1\right)^2\cdot 2x=2x^3\left(x^2-1\right)^2$.
   - Therefore, F is increasing on $(0,1)$ and $(1,\infty)$, with critical point at $x=1$.

COMPUTATIONS OF INTEGRALS

- The theorem proven above says that the integral F is <u>an</u> "antiderivative" of f.

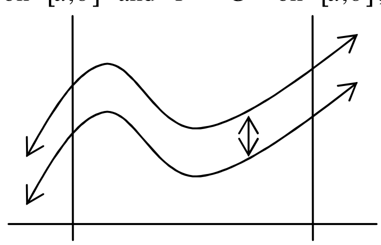
Definition

Let f be continuous on $[a,b]$. A function G is called the antiderivative of f if:

1) G is also continuous on $[a,b]$.
2) $G'(x)=f(x)$ on $[a,b]$.

So $F(x)=\int_a^x f(t)dt$ is an antiderivative of f.

- <u>Recall</u>: If F, G are both continuous on $[a,b]$ and $F'=G'$ on $[a,b]$, then $F(x)=G(x)+C$.



FUNDAMENTAL THEOREM OF CALCULUS

Theorem

Let f be continuous on $[a,b]$. If G is <u>any</u> antiderivative of $f(x)$, then $\int_a^b f(x)dx=G(b)-G(a)=G(x)\Big|_a^b$.

Proof

- We already know that $F(x)=\int_a^x f(t)dt$ is an antiderivative of $f(x)$, so $F(x)=G(x)+C$, where C is a constant.
- $\int_a^b f(t)dt = F(b) = F(b) - F(a) = (G(b)+C) - (G(a)+C) = G(b) - G(a)$.

Q.E.D.

Examples

1) Calculate $\int_1^3 x^2 dx$.
   - Let $f(x)=x^2$, $G(x)=\frac{x^3}{3}$.
   - So $\int_1^3 x^2 dx = G(3)-G(1) = \frac{27}{3}-\frac{1}{3}=\frac{26}{3}$.

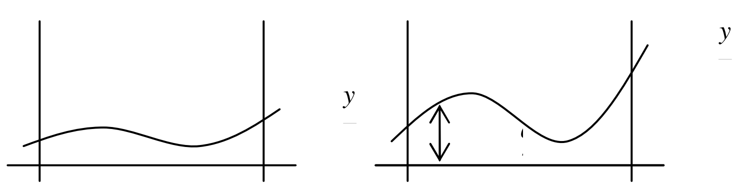
2) Calculate $\int_0^\pi \sin x dx$.
   - Let $f(x)=\sin x$, $G(x)=-\cos x$.
   - So $\int_0^\pi \sin x dx = -\cos(x)\big|_0^\pi = -\cos(\pi)+\cos(0)=1+1=2$.

SOME BASIC ANTIDERIVATIVES

Function f	Antiderivative G
$x^r, r \neq -1$	$\frac{x^{r+1}}{r+1}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$

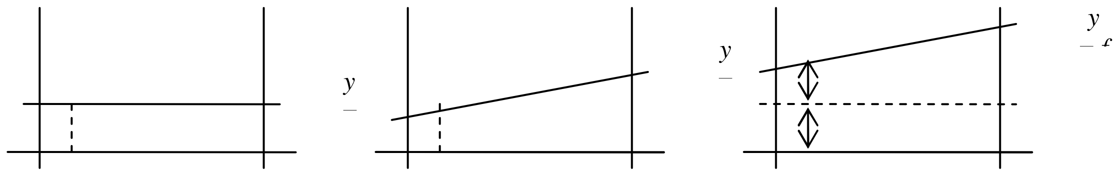
SOME GENERAL RULES

1) $\int_a^b \alpha f(x)dx = \alpha\int_a^b f(x)dx$ – "can pull out a constant".



   - All the rectangles for the approximation on the LHS get multiplied by α.

2) $\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$ – "can separate sums".

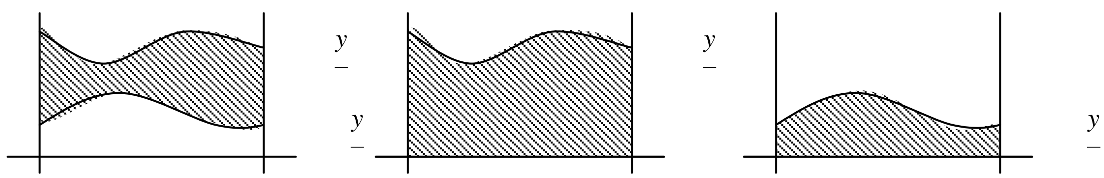


Example

Evaluate $\int_1^4 \left(2x^3 + 3x\right)dx$.

- $\int_1^4 \left(2x^3 + 3x\right)dx = 2\int_1^4 \left(x^3\right)dx + 3\int_1^4 (x)dx = 2\left(\left.\frac{x^4}{4}\right|_1^4\right) + 3\left(\left.\frac{x^2}{2}\right|_1^4\right) = 2\left(\frac{256}{4} - \frac{1}{4}\right) + 3\left(\frac{16}{2} - \frac{1}{2}\right) = \frac{255}{2} - \frac{45}{2} = 105$.

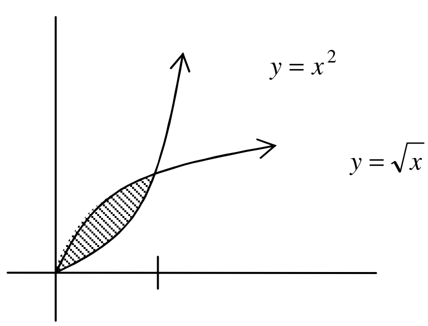
Another Way to Compute Areas



- $\Omega = \int_a^b \left(f(x) - g(x)\right)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
- Can now compute areas <u>between curves</u>.

Example

Sketch the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$. Find the area.



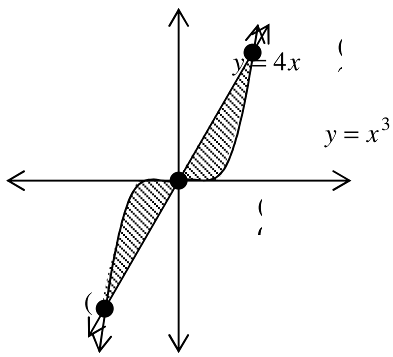
$$\Omega = \int_0^1 \left(\sqrt{x} - x^2\right)dx = \int_0^1 \left(x^{\frac{1}{2}} - x^2\right)dx$$

$$= \left.\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right)\right|_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) - (0 - 0) = \frac{1}{3}$$

Lecture #29 – Thursday, January 15, 2004

Example

Sketch the region bounded by the curves: $y = 4x$, $y = x^3$. Compute the area.



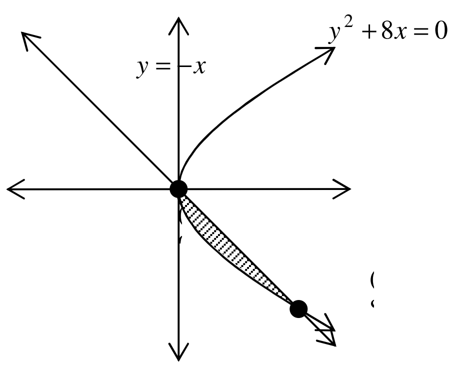
- The points of intersection occur when $4x = x^3 \Leftrightarrow x(x+2)(x-2) = 0$.

$$\begin{aligned}\text{The Area} &= \int_{-2}^{0}\left(x^3 - 4x\right)dx + \int_{0}^{-2}\left(4x - x^3\right)dx \\ &= \left(\frac{x^4}{4} - 2x^2\right)\Bigg|_{-2}^{0} + \left(2x^2 - \frac{x^4}{4}\right)\Bigg|_{0}^{2} \\ &= ((0-0)-(4-8)) + ((8-4)-(0-0)) \\ &= 8\end{aligned}$$

- Question: $\int_{-2}^{2}\left(x^3 - 4x\right)dx$.

Example

Sketch the region bounded by $y^2 + 8x = 0$, $x + y = 0$. Compute the area.



- The point of intersection occurs when $y^2 = 8(-y) \Rightarrow y = -8, x = 8$.

$$\text{The Area} = \int_{0}^{8}\left[-x - \left(-\sqrt{8}\cdot\sqrt{x}\right)\right]dx = \int_{0}^{8}\left[2\sqrt{2}x^{\frac{1}{2}} - x\right]dx$$

$$= \left(2\sqrt{2}\cdot\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2}\right)\Bigg|_{0}^{8} = \frac{128}{3} - \frac{96}{3} = \frac{32}{3}$$

INDEFINITE INTEGRALS

- Before continuing to more powerful techniques for integration, "rephrase" the Fundamental Theorem of Calculus: $\int_a^b f(x)dx = F(b) - F(a) = F(x)\big|_a^b$, where F is an antiderivative of f.
- In order to convey the information that F is an antiderivative of $f(x)$, can write: $\int f(x)dx = F(x) + C$. Here, C is an arbitrary constant. This is called the <u>indefinite integral</u>.

Examples

1) $\int x^2 dx = \frac{x^3}{3} + C$.
2) $\int \sin x dx = -\cos x + C$.

- Just as for definite integrals, indefinite integrals have linearly properties:
  $\int (\alpha f(x) + \beta g(x)) dx = \alpha \int f(x) dx + \beta \int g(x) dx$, α, β are constants.

Example

There is a particle moving in (x, y)-plane. Describe position as $(x(t), y(t))$. Suppose you know: $x'(t) = t - 1$, $y'(t) = t^2$, and at $t = 2$ the particle is at $\left(3, \frac{3}{8}\right)$. Where is the particle 3 seconds later?

- $x(t) = \int x'(t) dt = \int (t-1) dt = \frac{t^2}{2} - t + C_1$, $y(t) = \int y'(t) dt = \int t^2 dt = \frac{t^3}{3} + C_2$.
- $x(2) = 3 = \frac{2^2}{2} - 2 + C_1 \Rightarrow C_1 = 3$, $y(2) = \frac{3}{8} = \frac{2^3}{3} + C_2 \Rightarrow C_2 = 0$.
- $x(t) = \frac{t^2}{2} - t - 3$, $y(t) = \frac{t^3}{3}$.
- So, at $t = 5$, the particle is at $\left(\frac{21}{2}, \frac{125}{3}\right)$.

THE CHAIN RULE APPLIED TO INTEGRALS

- Recall: $\frac{d}{dx}[F(g(x))] = F'(g(x)) \cdot g'(x)$.
- Suppose we know F is an antiderivative of f. Then the Chain Rule says $\frac{d}{dx}[F(g(x))] = f(g(x)) \cdot g'(x)$. This says $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$. So $\int f(g(x)) \cdot g'(x) dx = F(g(x)) + C, F' = f$.

U-Substitution

- Let $u = g(x)$, so $u' = g(x)$, and "$du = g'(x)dx$".
- Now, $\int f(g(x)) \cdot g'(x) dx = \int f(u) du = F(g(x)) = F(u)$.

Example

1) $\int \cos(x^2 + 2) \cdot 2x dx$.
   - Let $u = g(x) = x^2 + 2$, so $g'(x) = 2x$.
   - Let $f(u) = \cos u$, so $F(u) = \sin u$.
   - So, $\int \cos(x^2 + 2) \cdot 2x dx = \sin(x^2 + 2) + C$.

2) $\int \sin x \cos dx$.

- Let $u = \cos x$, so $du = -\sin x dx$.
- So $\int \sin x \cos dx = \int u(-du) = -\frac{u^2}{2} + C = -\frac{\cos^2 x}{2} + C$.

3) $\int \frac{\sin x dx}{\sqrt[3]{1+\cos x}}$.

- Let $u = 1 + \cos x$, so $du = -\sin x dx$.
- So $\int \frac{\sin x dx}{\sqrt[3]{1+\cos x}} = \int -\frac{du}{\sqrt[3]{u}} = -\frac{3}{2} u^{\frac{2}{3}} + C = -\frac{3}{2}(1+\cos x)^{\frac{2}{3}} + C$.

DEFINITE INTEGRALS AND CHAIN RULE

Theorem

If *f*, *g'* are continuous, then $\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$.

Proof

$$\int_a^b f(g(x)) \cdot g'(x) dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)) = F(u)\Big|_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u) du .$$

Examples

1) Evaluate $\int_0^{\frac{\pi}{2}} \sin^3 x \cos x dx$.

- Let $u = \sin x = g(x)$, $du = \cos x dx$.
- So $\int_0^{\frac{\pi}{2}} \sin^3 x \cos x dx = \int_0^1 u^3 du = \frac{u^4}{4}\Big|_0^1 = \frac{1}{4}$.

2) Evaluate $\int_0^{\sqrt{3}} x^5 \sqrt{x^2+1} dx$.

- Let $u = x^2 + 1 = g(x)$, $du = 2x dx$.
- So

$$\int_0^{\sqrt{3}} x^5 \sqrt{x^2+1} dx = \int_0^{\sqrt{3}} \frac{x^4}{2} \sqrt{x^2+1}(2x) dx = \int_0^{\sqrt{3}} \frac{(x^2)^2}{2} \sqrt{x^2+1}(2x) dx = \int_1^4 \frac{(u-1)^2}{2} u^{\frac{1}{2}} du$$

$$= \frac{1}{2}\int_1^4 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}} \right) du = \frac{1}{2}\left[\left(\frac{2}{7} u^{\frac{7}{2}} - \frac{4}{5} u^{\frac{5}{2}} + \frac{2}{3} u^{\frac{3}{2}} \right)\Big|_1^4 \right] = \frac{848}{105} .$$

Lecture # 30 – Tuesday, January 20, 2004

Definite Integrals: Useful Properties

1) Suppose f is defined and continuous on $(-a, a)$.
   - If f is an odd function, $\int_{-a}^{a} f(x)dx = 0$.
   - If f is an even function, $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.
   - Note: This is useful for trigonometric functions, and useful for doing Fourier Analysis.

2) Chain Rule with integrals.
   - Recall: $\frac{d}{dx}\left[\int_{a}^{x} f(t)dt\right] = f(x)$.
   - If $u = u(x)$ is differentiable, then $\frac{d}{dx}\left[\int_{a}^{u(x)} f(t)dt\right] = f(u(x))\cdot u'(x) = f(u)\frac{du}{dx}$.
   - Proof: Let $F(u) = \int_{a}^{u} f(t)dt$. From FTC, $F'(u) = f(u)$. So $\frac{d}{dx}[F(u(x))] = F'(u)\frac{du}{dx} = f(u)\frac{du}{dx}$.

3) If $f(x) \geq 0$ on $[a, b]$, then $\int_{a}^{b} f(x)dx \geq 0$.

4) If $f(x) > 0$ on $[a, b]$, then $\int_{a}^{b} f(x)dx > 0$.

5) If $f(x) \leq g(x)$ on $[a, b]$, then $\int_{a}^{b} f(x)dx \leq \int_{a}^{b} g(x)dx$.

6) If $f(x) < g(x)$ on $[a, b]$, then $\int_{a}^{b} f(x)dx < \int_{a}^{b} g(x)dx$.

7) "Triangle Inequality" for integrals:
   - Recall: $|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|$.
   - In approximations for integrals, $|\text{area}(R_1) + \text{area}(R_2) + \ldots + \text{area}(R_N)| \leq |\text{area}(R_1)| + |\text{area}(R_2)| + \ldots + |\text{area}(R_N)|$. So LHS is the absolute value of $\int_{a}^{b} f(x)dx$, RHS is the area under $|f(x)|$.
   - In terms of integrals, then $\left|\int_{a}^{b} f(x)dx\right| \leq \int_{a}^{b} |f(x)|dx$.

- Proof: If $-\left|f(x)\right| \le f(x) \le \left|f(x)\right|$, then $-\int_a^b \left|f(x)\right| dx \le \int_a^b f(x)dx \le \int_a^b \left|f(x)\right| dx \Rightarrow \left|\int_a^b f(x)dx\right| \le \int_a^b \left|f(x)\right| dx$.

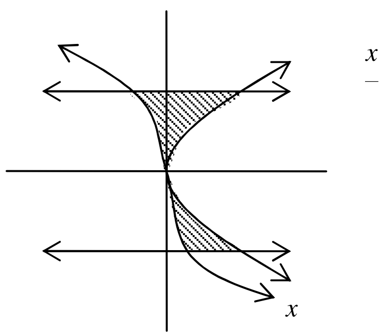
8) "Intermediate Value Theorem" for integrals:
   - If f is continuous on $[a, b]$, take m = minimum value of f on $[a, b]$, and M = maximum value of f on $[a, b]$. Then $\int_a^b m dx \le \int_a^b f(x)dx \le \int_a^b M dx \Leftrightarrow m(b-a) \le \int_a^b f(x)dx \le M(b-a)$.

MORE TECHNIQUES OF INTEGRALS: USING VARIABLE *Y* INSTEAD OF *X*

Idea

Estimate areas using <u>horizontal</u> stripes.

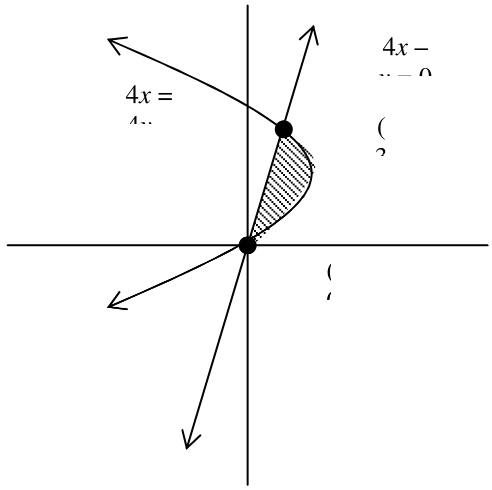
Example



- You may use variable y according to region.

Example

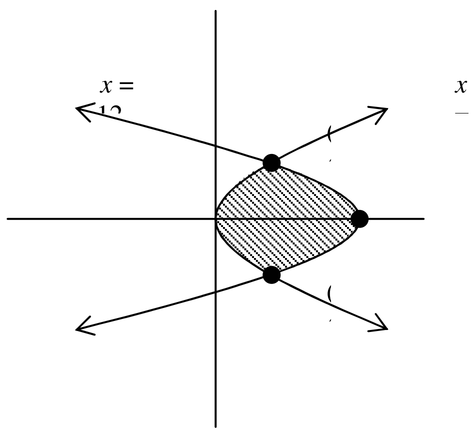
Sketch the region bounded by the curves and compute the area: $\begin{cases} 4x = 4y - y^2 \Leftrightarrow x = y - \dfrac{y^2}{4} \\ 4x - y = 0 \Leftrightarrow x = \dfrac{1}{4}y \end{cases}$.



- Points of intersection: $y = 4y - y^2 \Rightarrow 0 = 3y - y^2 \Rightarrow 0 = y(3-y)$. So $(0,0)$ and $\left(\frac{3}{4}, 3\right)$.
- $$\text{Area} = \int_0^3 \left[\left(y - \frac{y^2}{4}\right) - \frac{y}{4}\right] dy = \left(\frac{y^2}{2} - \frac{y^3}{12} - \frac{y^2}{8}\right)\Bigg|_0^3 = \frac{27}{24}.$$

Example

Sketch the region bounded by the curves and compute the area: $\begin{cases} x = y^2 \\ x = 12 - 2y^2 \end{cases}$.



- Points of intersection: $y = \pm 2$, $x = 4$.
- $$\text{Area} = \int_{-2}^{2} \left(12 - 2y^2 - y^2\right) dy = 2\int_0^2 \left(12 - 3y^2\right) dy = 6\int_0^2 \left(4 - y^2\right) dy.$$ $$= 6\left(4y - \frac{y^3}{3}\right)\Bigg|_0^2 = 6\left(8 - \frac{8}{3}\right) = 32$$

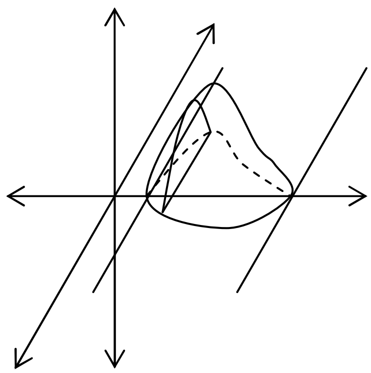
Lecture #31 – Thursday, January 22, 2004

VOLUMES

- Warning: Many possible scenarios…

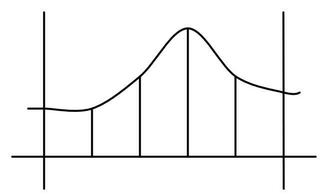
PART 1: VOLUME BY "PARALLEL CROSS SECTION"

Picture



- Suppose we have a solid between $x = a$ and $x = b$, with cross-sectional area $A(x)$ (assume $A(x)$ is continuous), then Volume = $\int_a^b A(x)dx$.

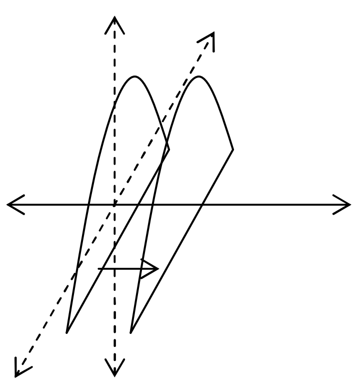
Proof



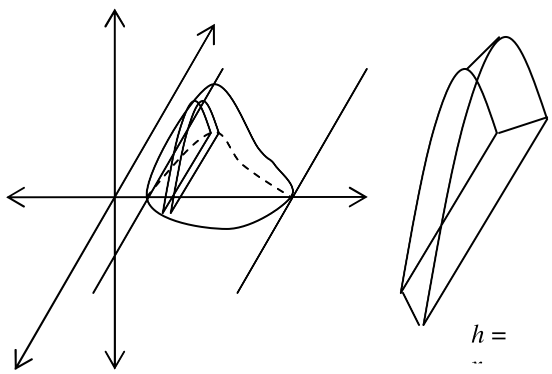
- Recall how we computed areas.
- Area$\Omega \approx \sum_{i=1}^{n}(\text{area of rectangles } R_i) = \sum_{i=1}^{n} f(x_1^*)(x_i - x_{i-1})$.
- As $x_i - x_{i-1} \to 0$, the upper and lower sums converge.
  So $\Omega \to \int_a^b f(x)dx$.
- <u>Idea</u>: Integrate the function that gives height.
- Idea: For volumes, we do the same thing, but replace (area of rectangle R_i) with (volume of cross-section C_i).

Definition

A <u>parallel cross section</u> is given by the translation of a plane region Ω along an axis perpendicular to Ω.



To Compute Areas

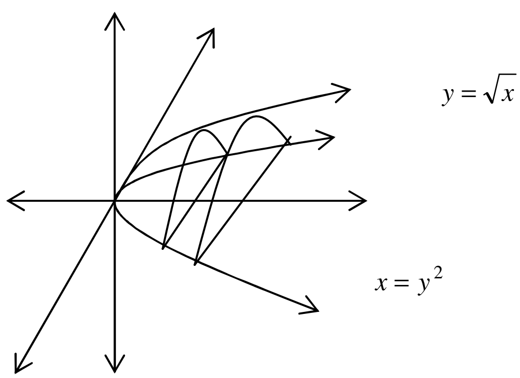


- Can approximate the volume by a parallel cross-section.

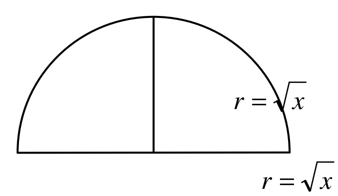
$$\text{Volume} \approx \sum_{i=1}^{n} \left(\text{volume of parallel cross-sections } C\right.$$
$$= \sum_{i=1}^{n} A\left(x_i^{\,*}\right)\left(x_i - x_{i-1}\right)$$

- As width $x_i - x_{i-1} \to 0$, the upper and lower sums converge. So Volume = $\int_a^b A(x)dx$.

Example



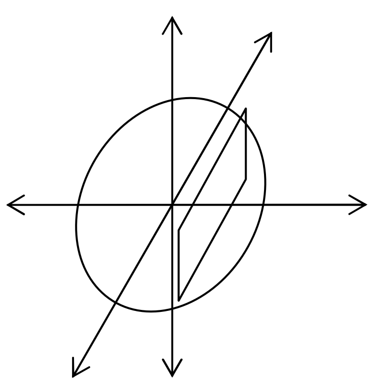
- Cross sections are half circles:



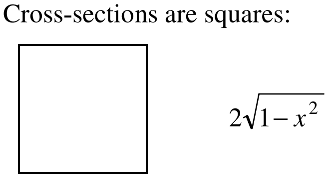
- $A(x) = \dfrac{\pi r^2}{2} = \dfrac{\pi x}{2}$.
- $V = \displaystyle\int_0^2 \frac{\pi}{2} x dx = \frac{\pi}{2}\int_0^2 x dx = \frac{\pi}{2}\left(\frac{x^2}{2}\right)\Bigg|_0^2 = \pi$.

Example

Find the volume where the base of the volume is given by the area within $x^2 + y^2 = 1$ and cross-sections are <u>squares</u>.



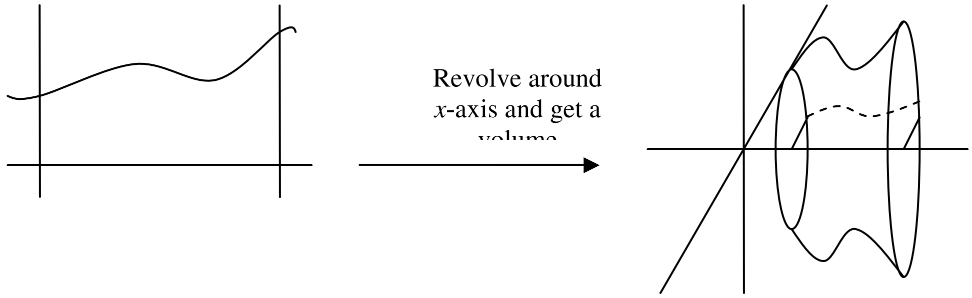
- Cross-sections are squares:



- $A(x) = \left(2\sqrt{1-x^2}\right)^2 = 4\left(1-x^2\right)$.
- $V = \displaystyle\int_{-1}^{1} 4\left(1-x^2\right)dx = 2\int_0^1 4\left(1-x^2\right)dx = 8\left(x - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{16}{3}$.

PART 1 (A): SPECIAL CASE "DISC METHOD"

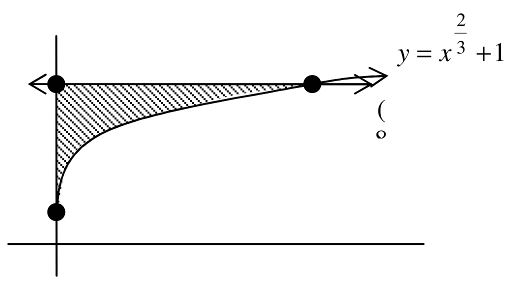
- Suppose *f* is a non-negative continuous function on [*a*, *b*].

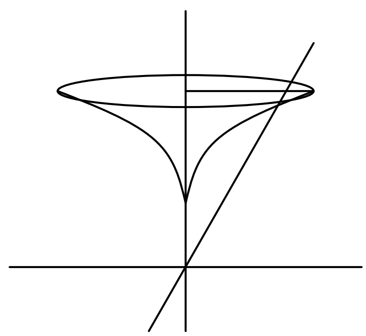


- Cross-sections are circles! So, at $x \in [a, b]$, the area of cross-sections is $A(x) = \pi(f(x))^2$.
- Therefore, the volume of solid of revolution is $\int_a^b \pi(f(x))^2\,dx$.

Example

Rotate the region bounded by $\begin{cases} y = x^{\frac{2}{3}} + 1 \\ x = 0 \\ y = 5 \end{cases}$ around the *y*-axis. Find the volume.

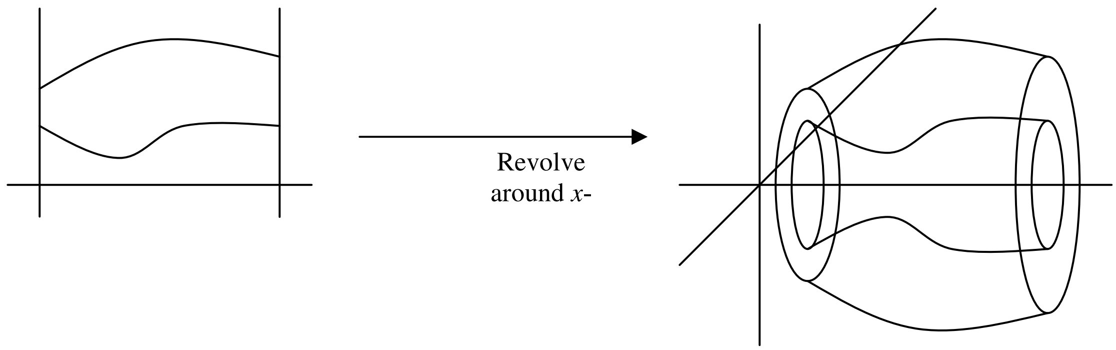




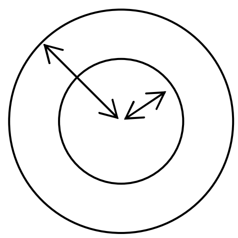
- Here, *a* = 1, *b* = 5, variable is *y*.
- Cross-section at *y*: $y - 1 = x^{\frac{2}{3}} \Rightarrow x = (y-1)^{\frac{3}{2}}$, so $A(x) = \pi(y-1)^3$.
- $V = \int_1^5 \pi(y-1)^3\,dy = \pi\int_1^5 (y-1)^3\,dy$. Let $u = y - 1$, $du = dy$. So $V = \pi\int_0^4 u^3\,du = \pi\left(\frac{u^4}{4}\right)\Bigg|_0^4 = 64\pi$.

PART 1 (B): "WASHER METHOD"

- Suppose $f(x)$, $g(x)$ are non-negative continuous functions on [*a*, *b*], and $g(x) \le f(x)$ on on [*a*, *b*]. Revolve around *x*-axis and find volume.



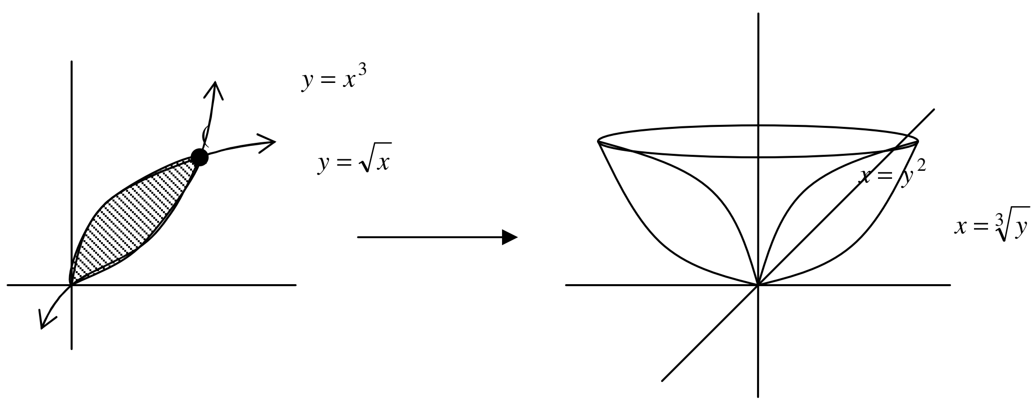
- Cross-section: $A(x)=\pi(f(x))^2-\pi(g(x))^2=\pi\left((f(x))^2-(g(x))^2\right)$.



- So $V=\int_a^b \pi\left(f(x)^2-g(x)^2\right)dx$.

Example

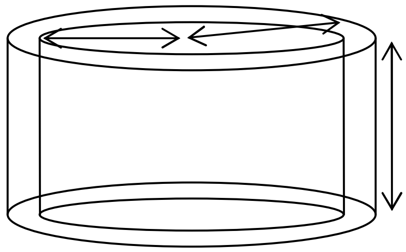
Revolve the region bounded by the curves $\begin{cases} y=\sqrt{x} \\ y=x^3 \end{cases}$ around y-axis. Find the volume.



- $V=\int_0^1 \pi\left(\left(\sqrt[3]{y}\right)^2-\left(y^2\right)^2\right)dy=\pi\int_0^1\left(y^{\frac{2}{3}}-y^4\right)dy=\pi\left(\frac{2}{5}y^{\frac{5}{2}}-\frac{1}{5}y^5\right)\Bigg|_0^1=\frac{2\pi}{5}$.

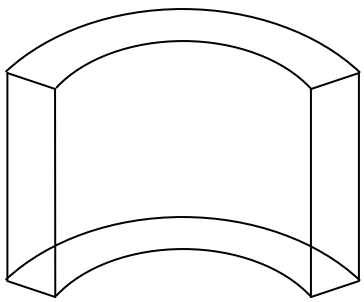
PART 2: VOLUME BY THE "SHELL METHOD"

A Shell

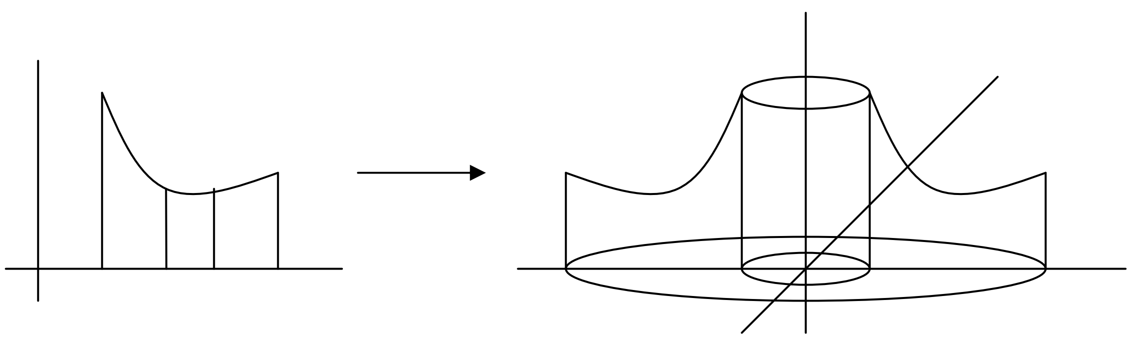


- Volume = $\pi R^2 h - \pi r^2 h = \pi h(R+r)(R-r)$.

- In the situation where $R \approx r$, $R-r$ very small, volume $\approx 2\pi Rh(R-r)$.



- Let f be a non-negative continuous function on $[a, b]$, $0 \notin [a,b]$.

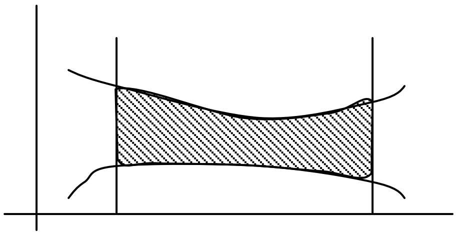


$$\begin{aligned}\text{Volume of solid} &\approx \sum_{i=1}^{n} 2\pi(x_i) f\left(x_i^*\right)(x_i - x_{i-1}) \\ &= \int_a^b 2\pi x f(x)\,dx\end{aligned}$$

Lecture #32 – Tuesday, January 27, 2004

Generalization

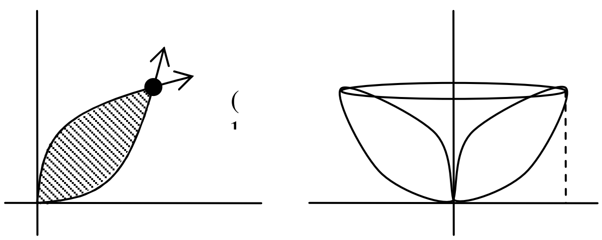
- By revolving around the y-axis, volume of solid = $\int_a^b 2\pi x\left(f(x)-g(x)\right)dx$.



- Warning/Advice: There will be volumes that can be calculated using different strategies. Choose what you prefer or what works best.

Example

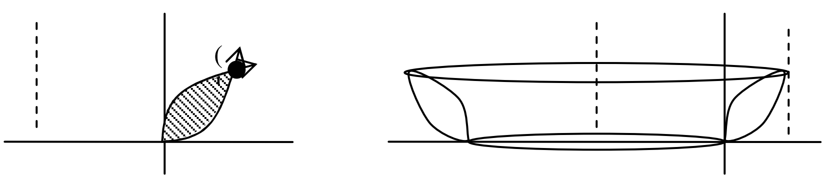
Consider the region bounded by $y=\sqrt{x}$ and $y=x^2$ rotated around the y-axis. Find the volume.



- Washer method:
  $\int_0^1 \pi\left(\left(\sqrt{y}\right)^2-\left(y^2\right)^2\right)dy=\frac{3\pi}{10}$.
- Washer method: $\int_0^1 2\pi x\left(\sqrt{x}-x^2\right)dx=\frac{3\pi}{10}$.

Example

Consider the region bounded by $y=\sqrt{x}$ and $y=x^2$ rotated around $x=-2$. Find the volume.



- By the shell method, $V=\int_0^1 2\pi(2+x)\left(\sqrt{x}+x^2\right)dx$.

Transcendental Functions

- First, some preliminaries on inverse functions.

Examples

1) $y = f(x) = \sin x$. Note that $f(x) = \sin x$ is a function so that <u>different</u> values of *x* have the <u>same</u> value of $\sin x$, so "solve $\sin x = 0$" has many possible answers!

2) $y = f(x) = x^3$. This function satisfies the property that for every *y*, there is exactly <u>one</u> *x* so that $f(x) = x^3 = y$ ("passes the <u>horizontal line test</u>").

Definition

A function *f* is <u>one-to-one</u> if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (or the contra-positive $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$).

Definition

Let *f* be one-to-one. The <u>inverse of *f*</u>, denoted f^{1}, is the (unique) function that satisfies $f\left(f^{-1}(x)\right) = x$ for all *x* in range(*f*).

- <u>Note</u>: The $\text{domain}\left(f^{-1}\right) = \text{range}(f)$, and $\text{range}\left(f^{-1}\right) = \text{domain}(f)$.

Example

Consider $f(x) = x^2$ on [0, 2].

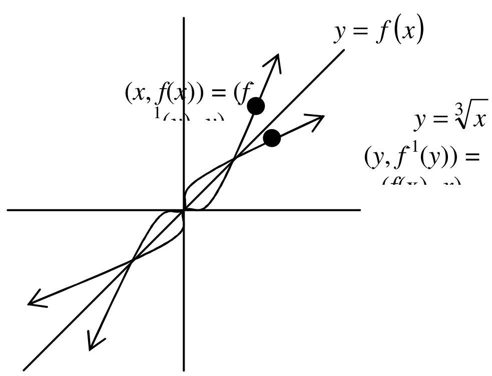
- $\text{domain}(f) = [0,2] = \text{range}\left(f^{-1}\right)$.
- $\text{range}(f) = [0,4] = \text{domain}\left(f^{-1}\right)$.

- <u>Note</u>: $f^{-1}(f(x)) = x, \forall x \in \text{domain}(f)$.
- Proof: Take $x \in \text{domain}(f)$. Let $y = f(x) \in \text{range}(f)$. So $f\left(f^{-1}(y)\right) = y \Rightarrow f\left(f^{-1}(f(x))\right) = f(x)$. Because *f* is one-to-one, $f^{-1}(f(x)) = x$.

GRAPHS OF THE INVERSE F^{-1} AND F

Example

Graph $y = f(x) = x^3$ and its inverse.



x	*y*
-1	-1
1	1
1/2	1/8
2	8
3	27

x	$y = \sqrt[3]{x}$
-1	-1
1	1
1/8	1/2
8	2
27	3

- Important: The graph of f^1 is graph of f reflected across the line $y = x$ ("switch roles of x and y").

Theorem

If f is continuous on interval I, then f^1 is also continuous on $f(I)$.

Corollary

Let f be one-to-one, differentiable on I. Suppose $f(a) = b$. If $f'(a) \neq 0$, then f^1 is differentiable at b, and $\left(f^{-1}\right)'(b) = \dfrac{1}{f'(a)}$.

Lecture #33 – Thursday, January 29, 2004

- Let $B > 0, B \neq 1$. "$\log_B x$" is the inverse function to B^x – i.e. $\log_B x = y \Leftrightarrow B^y = x$.

Recall

- $B^{a+b} = B^a \cdot B^b$
- $B^{-a} = \dfrac{1}{B^a}$
- $\left(B^a\right)^b$
- $\log_B(xy) = \log_B x + \log_B y$
- $\log_B\left(\dfrac{1}{x}\right) = -\log_B x$
- $\log_B x^r = r \log_B x$

- Our approach: Use $\log_B(xy) = \log_B x + \log_B y$ as starting point and build the function from scratch. It will agree with what you know already, and it will be defined for all real numbers (so we can use calculus).

Definition

A logarithm function is a non-constant differentiable function on $(0, \infty)$ so that $\forall\ a, b > 0$, $f(ab) = f(a) + f(b)$.

- From $f(ab) = f(a) + f(b)$:
  - $f(1) = f(1 \cdot 1) = f(1) + f(1) \Rightarrow f(1) = 0$.
  - $f\left(\dfrac{a}{b}\right) = f(a) - f(b)$.

What about the derivative?

- Fix $x > 0$.
- $$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right) - f(1)}{\frac{h}{x}}\cdot\frac{1}{x} = \lim_{\frac{h}{x}\to 0}\frac{f\left(1+\frac{h}{x}\right) - f(1)}{\frac{h}{x}}\cdot\frac{1}{x} = \frac{1}{x}f'(x).$$

  So $f'(x) = \dfrac{1}{x} f'(1)$.
- $f'(1) \neq 0$. If $f'(1) = 0$, then $f'(x) = 0 \forall x$. So assume $f'(1) \neq 0$.
- Choose: $f'(1) = 1$ (this corresponds to choice of base).

- Then $f'(x) = \frac{1}{x} \forall x$, $f(1) = 0$.
- By the Fundamental Theorem of Calculus, $f(x) = \int_1^x \frac{1}{t}\,dt$.

Definition

The function $L(x) = \int_1^x \frac{dt}{t}$ is called the <u>natural logarithm function</u>.

Theorem

$L(ab) = L(a) + L(b)$.

Proof

- $\frac{d}{dx}[L(x)] = \frac{1}{x}$. Then $\frac{d}{dx}[L(ax)] = \frac{1}{ax} \cdot a = \frac{1}{x} \Rightarrow L(ax) = L(x) + C$.
- What's C? Plug in $x = 1$. $L(a) = L(1) + C \Rightarrow C = L(a) \Rightarrow L(ax) = L(a) + L(x)$.
- Plug in $x = b$.

Theorem

For $a > 0$, $\frac{p}{q}$ rational. $L\left(a^{\frac{p}{q}}\right) = \frac{p}{q} L(a)$.

Proof

- $\frac{d}{dx}\left[L\left(x^{\frac{p}{q}}\right)\right] = \frac{1}{x^{\frac{p}{q}}} \cdot \frac{p}{q} x^{\frac{p}{q}-1} = \frac{p}{q} \cdot \frac{1}{x} = \frac{d}{dx}\left[\frac{p}{q} L(x)\right] \Rightarrow L\left(x^{\frac{p}{q}}\right) = \frac{p}{q} L(x) + C$.
- Plug in $x = 1 \Rightarrow C = 0$. So $L\left(x^{\frac{p}{q}}\right) = \frac{p}{q} L(x)$.
- Plug in $x = a$.

Theorem

The <u>range</u> of L is $(-\infty, \infty)$.

Proof

- Want: For any $M > 0$, can find x so that $L(x) > M$.
- Consider $L(2) > 0$. So for a large enough n, $L(x) > M$.
- So pick $x = 2^n$, and $L(x) = L(2^n) > M$.
- Similarly, can prove for $L(x) < M$.

THE NATURAL NUMBER: *E*

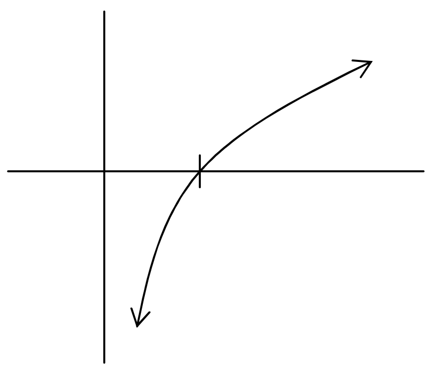
- <u>Recall</u>: $L(x)=\int_1^x \frac{dt}{t}$.
- There is a special number x where this area is 1.

Definition

e is the number such that $\int_1^e \frac{dt}{t}=1$ or $L(e)=1$.

- <u>Know</u>: $L\left(e^{\frac{p}{q}}\right)=\frac{p}{q}L(e)=\frac{p}{q}$.
- <u>Notation</u>: $L(x)=\log_e x=\ln(x)$ is called the natural logarithm.

Graph of ln(*x*)



- $\ln(1)=0$.
- $\frac{d}{dx}[\ln x]=\frac{1}{x}>0$ increasing on $(0, \infty)$.
- $\frac{d^2}{dx^2}[\ln x]=-\frac{1}{x^2}<0$ concave down on $(0, \infty)$.

MORE ON DERIVATIVES AND INTEGRATION

Derivatives

- Because $\frac{d}{dx}(\ln x)=\frac{1}{x}, x>0$:
  - $\frac{d}{dx}(\ln u)=\frac{1}{u}\frac{du}{dx}$.
  - $\frac{d}{dx}[\ln|x|]=\frac{1}{x}, x\neq 0$. Proof: If $x>0$, $|x|=x\Rightarrow\frac{d}{dx}[\ln(x)]=\frac{1}{x}$. If $x<0$, $|x|=-x\Rightarrow\frac{d}{dx}[\ln(-x)]=\frac{1}{-x}(-1)=\frac{1}{x}$.
  - $\frac{d}{dx}(\ln|u|)=\frac{1}{u}\frac{du}{dx}, u\neq 0$.

Example

Find the domain and derivative of $f(x)=\ln(\ln x)$.

- Domain: (1, ∞).
- $\dfrac{df}{dx} = \dfrac{1}{\ln x} \cdot \dfrac{1}{x}$.

Integration

- $\int \dfrac{dx}{x} = \ln|x| + C$.
- $\int \dfrac{g'(x)}{g(x)} dx = \ln|g(x)| + C$.

Examples

1) $\int \tan x dx = \int \dfrac{\sin x}{\cos x} dx$. Let $u = \cos x$, $du = -\sin x dx$. So $\int \tan x dx = \int \dfrac{-du}{u} = -\ln|\cos x| + C$.
2) $\int \dfrac{dx}{x(\ln x)^2}$. Let $u = \ln x$, $du = \dfrac{1}{x} dx$. So $\int \dfrac{du}{u^2} = -\dfrac{1}{u} + C = -\dfrac{1}{\ln x} + C$.

Trick: Logarithmic Differentiation

Suppose $g(x) = g_1(x) g_2(x) \cdots g_n(x)$. Task: Calculate $g'(x)$.

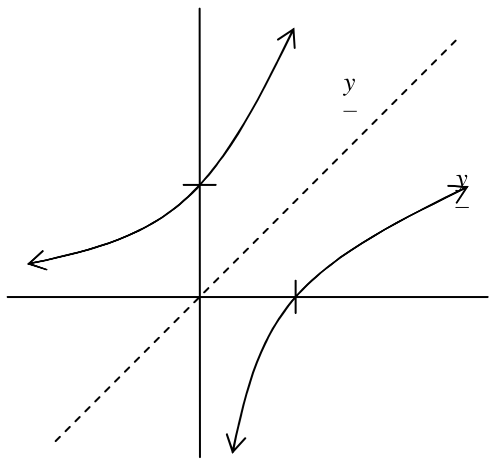
1) Take ln of both sides: $\ln|g(x)| = \ln|g_1(x) g_2(x) \cdots g_n(x)| = \ln(g_1(x)) + \ln(g_2(x)) + \cdots + \ln(g_n(x))$.
2) Take derivate: $\dfrac{g'(x)}{g(x)} = \dfrac{g_1'(x)}{g_1(x)} + \dfrac{g_2'(x)}{g_2(x)} + \cdots + \dfrac{g_n'(x)}{g_n(x)}$.
3) Multiply by $g(x)$: $g'(x) = g(x)\left[\dfrac{g_1'(x)}{g_1(x)} + \dfrac{g_2'(x)}{g_2(x)} + \cdots + \dfrac{g_n'(x)}{g_n(x)}\right]$.

EXPONENTIAL FUNCTIONS

- We <u>define</u> the exponential function to be the inverse of $\ln(x)$.

Definition

We <u>define</u> e^x so that $\ln(e^x) = x$ defined for ∀ $x \in$ (-∞, ∞).
The <u>range</u> of e^x is (0, ∞).



- $e^0 = 1$.
- $\ln\left(e^x\right) = x$.
- $e^{\ln x} = x$.
- $e^x > 0 \forall x$.

Theorem

$e^{a+b} = e^a \cdot e^b$.

Proof

- $\ln\left(e^a \cdot b^a\right) = \ln\left(e^a\right) + \ln\left(e^b\right) = a + b = \ln\left(e^{a+b}\right)$.
- Because $\ln(x)$ is one-to-one, $e^{a+b} = e^a \cdot e^b$.

Theorem

$\frac{d}{dx}\left[e^x\right] = e^x$. Mantra: "Exponential is its own derivative".

Proof

- We know that $\ln(x)$ is differentiable and the derivate is never 0, so e^x is differentiable everywhere.
- We also know that $\ln\left(e^x\right) = x$ ..
- Differentiate both sides: $\frac{1}{e^x} \cdot \frac{d}{dx}\left(e^x\right) = 1 \Rightarrow \frac{d}{dx}\left(e^x\right) = e^x$.

Integral Versions

- $\int e^x dx = e^x + C$.
- $\int e^{g(x)} g'(x) dx = e^{g(x)} + C$.

Examples

Differentiate:

1) $\frac{d}{dx}\left[e^x \ln x\right] = e^x \ln x + e^x \frac{1}{x}$.
2) $\frac{d}{dx}\left[\left(e^{x^2} + 1\right)^2\right] = 2\left(e^{x^2} + 1\right)\left(e^{x^2}\right)(2x)$.

Integrate:

1) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^{\sqrt{x}} + C$.

- Let $u = \sqrt{x} = x^{\frac{1}{2}}$, $2du = \frac{1}{\sqrt{x}}\,dx$.

2) $\int \frac{xe^{ax^2}}{e^{ax^2}+1}\,dx = \frac{1}{2a}\int \frac{du}{u} = \frac{1}{2a}\left(\ln e^{ax^2}+1\right) + C$.

- Let $u = e^{ax^2}+1$, $du = \left(e^{ax^2}\right)(2ax)dx \Rightarrow \frac{du}{2a} = xe^{ax^2}\,dx$.

Lecture #34 – Tuesday, February 3, 2004

OTHER POWERS, OTHER BASES (NOT NECESSARILY e^x AND $\ln x$)

Arbitrary Powers

- Note: For rational powers ($x > 0$), $x^r = x^{\frac{p}{q}} = \left(e^{\ln x}\right)^{\frac{p}{q}} = e^{\frac{p}{q}\ln x}$.

Definition

For any real r, $x > 0$, $x^r = e^{r\ln x}$.

Theorem

$\frac{d}{dx}\left[x^r\right] = rx^{r-1}$ $(x > 0)$.

Proof

$\frac{d}{dx}\left[x^r\right] = \frac{d}{dx}\left[e^{r\ln x}\right] = e^{r\ln x}\cdot\frac{r}{x} = x^r\cdot\frac{r}{x} = rx^{r-1}$.

- Also, $\frac{d}{dx}\left[u^r\right] = ru^{r-1}\frac{du}{dx}$.

Arbitrary Bases

- We already understand $y = e^x$. What about $y = p^x$ $(p > 0, p \neq 1)$?

Definition

$p^x = \left(e^{\ln p}\right)^x = e^{x\ln p}$.

Theorem

$\frac{d}{dx}\left[p^x\right] = p^x\cdot\ln p$.

Proof

$$\frac{d}{dx}\left[p^x\right]=\frac{d}{dx}\left[e^{x\ln p}\right]=e^{x\ln p}\cdot\ln p=p^x\cdot\ln p\ .$$

- $\frac{d}{dx}\left[p^u\right]=p^u\cdot\ln p\cdot\frac{du}{dx}$.

Example

Differentiate 2^{3x^2} .

- $\frac{d}{dx}\left[2^{3x^2}\right]=\left(2^{3x^2}\right)(\ln 2)(6x)$.

Integral Form

$$\int p^x dx=\frac{p^x}{\ln p}+C\ .$$

Example

$$\int 2^{-e^x}\cdot e^x dx=-\int 2^u\,du=-\left(\frac{2^{-e^x}}{\ln 2}\right)+C\ .$$

  - Let $u=-e^x$, $du=-e^x dx$.

Arbitrary Logarithms

- Again, we already understand $y=\ln x=\log_e x$. Want to understand $y=\log_p x$ $(p>0)$.
- Notice: $\ln\left(p^x\right)=x\cdot\ln p\Rightarrow\frac{\ln\left(p^x\right)}{\ln p}=x$. This means $f(x)=\frac{\ln(x)}{\ln(p)}$ is an inverse function to p^x. So $\log_p x=\frac{\ln(x)}{\ln p}$ $(x>0, p>0, p\neq 1)$.

Theorem

$$\frac{d}{dx}\left[\log_p(x)\right]=\frac{1}{\ln p}\cdot\frac{1}{x}\ .$$

Proof

$$\frac{d}{dx}\left[\log_p(x)\right]=\frac{d}{dx}\left[\frac{\ln(x)}{\ln(p)}\right]=\frac{1}{\ln p}\cdot\frac{1}{x}\ .$$

Example

1) $\log_2 32=\frac{\ln 32}{\ln 2}=\frac{\ln\left(2^5\right)}{\ln 2}=\frac{5\ln 2}{\ln 2}5$.

2) Find the derivative of $f(x) = (\log_3 x)^{\frac{1}{3}}$.

- $f'(x)\frac{1}{3}(\log_3 x)^{-\frac{2}{3}}\left(\frac{1}{\ln 3}\right)\left(\frac{1}{x}\right)$.

3) Find the integral $\int \frac{\log_5\left(x^{10}\right)}{x}dx$.

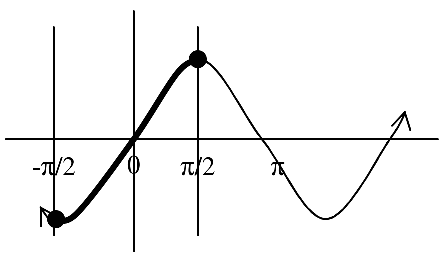
- $\int \frac{\log_5\left(x^{10}\right)}{x}dx = 10\int \frac{\log_5 x}{x}dx = 10\int \frac{\ln x}{x\ln 5}dx = \frac{10}{\ln 5}\int \frac{\ln x}{x}dx$. Let $u = \ln x$, $du = \frac{1}{x}dx$. So $\int \frac{\log_5\left(x^{10}\right)}{x}dx = \frac{10}{\ln 5}\int u\,du = \frac{5}{\ln 5}(\ln x)^2 + C$.

Lecture #35 – Thursday, February 5, 2004

More Techniques of Integration: Inverse Trigonometric Functions

- Warning: The trigonometric functions aren't one-to-one on their "usual" domains.

Example



- Part of the definition of the inverse trigonometric functions is their domain and range, and you have to memorize these (they're just by convention).

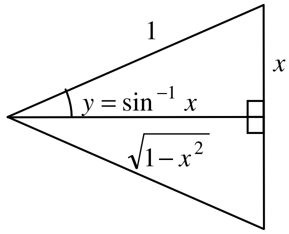
Definition

The function $\sin^{-1}(x)$ has domain [-1, 1] and range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and is the inverse of $y = \sin(x)$.

Examples

1) $\sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$.
2) $\sin^{-1}\left(\sin\frac{7\pi}{3}\right) = \frac{\pi}{3}$.
3) $\sin\left(\sin^{-1}(1)\right) = 1$.
4) $\sin\left(\sin^{-1}(2)\right)$ is undefined.

Now Some Geometry



Observe:

- $\sin\left(\sin^{-1}(x)\right) = x$.
- $\cos\left(\sin^{-1}(x)\right) = \sqrt{1-x^2}$.
- $\tan\left(\sin^{-1}(x)\right) = \dfrac{x}{\sqrt{1-x^2}}$.

Derivatives

$$y = \sin^{-1}(x) \Rightarrow x = \sin y \Rightarrow \cos y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

So $\dfrac{d}{dx}\left[\sin^{-1}(x)\right] = \dfrac{1}{\sqrt{1-x^2}}$, and $\dfrac{d}{dx}\left[\sin^{-1}(u)\right] = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$.

Example

$$\frac{d}{dx}\left[\sin^{-1}(2x)\cdot x\right] = \frac{2}{\sqrt{1-4x^2}}\cdot x + \sin^{-1}(2x).$$

Integral Version

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C, (a > 0).$$

Proof

- u-substitution: $u = \dfrac{x}{a}$ d $du = \dfrac{dx}{a}$.
- $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{adu}{\sqrt{a^2-a^2u^2}} = \int \frac{adu}{a\sqrt{1-u^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

Example

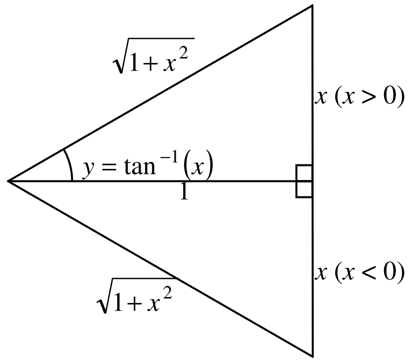
$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \int_0^1 \frac{dx}{\sqrt{2^2-x^2}} = \sin^{-1}\left(\frac{x}{2}\right)\Bigg|_0^1 = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) = \frac{\pi}{6}.$$

INVERSE TANGENT

Definition

$y = \tan^{-1}(x)$ is the <u>inverse</u> of $\tan(x)$ with domain $(-\infty, \infty)$ and range $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Picture



Observe:

- $\tan\left(\tan^{-1}(x)\right)=x$.
- $\cos\left(\tan^{-1}(x)\right)=\dfrac{1}{\sqrt{1+x^2}}$.
- $\sec\left(\tan^{-1}(x)\right)=\sqrt{1+x^2}$.

Derivative

$$y=\tan^{-1}(x)\Rightarrow \tan y=x\Rightarrow \sec^2 y\frac{dy}{dx}=1\Rightarrow \frac{dy}{dx}=\frac{1}{1+x^2}.$$

So, $\dfrac{d}{dx}\left(\tan^{-1}(x)\right)=\dfrac{1}{1+x^2}$, and $\dfrac{d}{dx}\left(\tan^{-1}(u)\right)=\dfrac{1}{1+u^2}\dfrac{du}{dx}$.

Example

$$\frac{d}{dx}\left[\tan^{-1}\left(2x^2+3x\right)\right]=\frac{1}{1+\left(2x^2+3x\right)}(4x+3).$$

Integral Version

$$\int\frac{dx}{1+x^2}=\tan^{-1}(x)+C\text{, and }\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C, a>0.$$

Proof

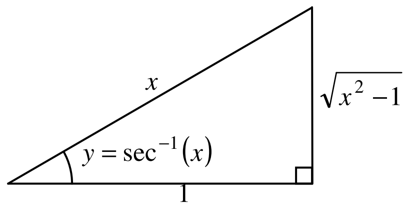
Let $u=\dfrac{x}{a}$, $du=\dfrac{1}{a}$. $\displaystyle\int\frac{dx}{a^2+x^2}=\int\frac{adx}{a^2+a^2x^2}=\frac{1}{a}\int\frac{du}{1+u^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C, a>0$.

INVERSE SECANT

Definition

The <u>inverse secant</u> $\sec^{-1}(x)$ is the inverse of $\sec(x)$ with <u>domain</u> $(-\infty,-1]\cup[1,\infty)$ and <u>range</u> $\left[0,\dfrac{\pi}{2}\right)\cup\left(\dfrac{\pi}{2},\pi\right]$.

Picture



Observe:

- $\sec\left(\sec^{-1}(x)\right) = x$.
- $\sin\left(\sec^{-1}(x)\right) = \dfrac{\sqrt{x^2-1}}{x}$.
- $\tan\left(\sec^{-1}(x)\right) = \sqrt{x^2-1}$.

Derivative

$y = \sec^{-1} x \Rightarrow \sec y = x \Rightarrow \sec y \tan y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$. Have to be a little careful. So far, we were assuming $x > 1$, but on the interval $\left(0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right)$, $\sec y \tan y$ is always > 0.

So, $\dfrac{d}{dx}\left[\sec^{-1}(x)\right] = \dfrac{1}{|x|\sqrt{x^2-1}}$ and $\dfrac{d}{dx}\left[\sec^{-1}(u)\right] = \dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}$.

Integral Version

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}\left(|x|\right) + C.$$

Proof

- Case 1: $x > 0$. Then $|x| = x$, $\dfrac{d}{dx}\left[\sec^{-1}|x|\right] = \dfrac{d}{dx}\left[\sec^{-1}(x)\right] = \dfrac{1}{x\sqrt{x^2-1}}$.
- Case 2: $x < 0$. Then $|x| = -x$, $\dfrac{d}{dx}\left[\sec^{-1}|x|\right] = \dfrac{d}{dx}\left[\sec^{-1}(-x)\right] = \dfrac{1}{-x\sqrt{x^2-1}}(-1) = \dfrac{1}{x\sqrt{x^2-1}}$.

Example

$\displaystyle\int_4^8 \frac{dx}{x\sqrt{x^2-16}}$. Let $u = \dfrac{x}{4}$, $du = \dfrac{1}{4}dx$.

So $\displaystyle\int_4^8 \frac{dx}{x\sqrt{x^2-16}} = \int_1^2 \frac{4dx}{4u\sqrt{16u^2-16}} = \frac{1}{4}\int_1^2 \frac{dx}{u\sqrt{u^2-1}} = \frac{1}{4}\sec^{-1}\left(|u|\right)\Big|_1^2 = \frac{1}{4}\left[\frac{\pi}{3} - 0\right] = \frac{\pi}{12}$.

Integration By Parts

- Mantra: "The integral form of the product rule".
- Recall: $\left(f(x)\cdot g(x)\right)' = f'(x)g(x) + f(x)g'(x)$.

- Integrate both sides:
  $$\int \left(f(x)\cdot g(x)\right)' dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$
  $$f(x)\cdot g(x) + C = \int f'(x)g(x)dx + \int f(x)g'(x)dx.$$
  $$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

- In practice, it's usually written: $\int udv = uv - \int vdu$, $\begin{matrix} u = f(x) & du = f'(x)dx \\ v = g(x) & dv = g'(x)dx \end{matrix}$.

- <u>The point</u>: Choose u, v so that $\int vdu$ is an easier integral than $\int udv$.

Example

$\int xe^x dx$. Let $\begin{matrix} u = x & du = dx \\ v = e^x & dv = e^x dx \end{matrix}$. So $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$.

Lecture #36 – Tuesday, February 10, 2004

Examples

1) $\int \sqrt{x} \ln x dx$. Let $\begin{matrix} u = \ln x & du = \frac{1}{x}dx \\ v = \frac{2}{3}x^{\frac{3}{2}} & dv = x^{\frac{1}{2}}dx \end{matrix}$. So $\int \sqrt{x}\ln x dx = \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{9}x^{\frac{3}{2}} + C$.

<u>Warning</u>: Sometimes, you have to apply integration by parts several times.

2) $\int e^x \cos x dx$. Let $\begin{matrix} u = \cos x & du = -\sin x dx \\ v = e^x & dv = e^x dx \end{matrix}$. $\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx$. Let $\begin{matrix} u = \sin x & du = \cos x dx \\ v = e^x & dv = e^x dx \end{matrix}$.

$\int e^x \cos x dx = e^x \cos x + e^x \sin x - \int e^x \cos x dx \Rightarrow \int e^x \cos x dx = \frac{1}{2}e^x(\cos x + \sin x) + C$.

3) $\int \ln x dx$. Let $\begin{matrix} u = \ln x & du = \frac{1}{x}dx \\ v = x & dv = dx \end{matrix}$. So $\int \ln x dx = x\ln x - \int dx = x\ln x - x + C$.

4) $\int (\ln x)^2 dx$. Let $\begin{matrix} u = \ln x & du = \frac{1}{x}dx \\ v = x\ln x - x & dv = \ln x dx \end{matrix}$.

$\int (\ln x)^2 dx = (x\ln x - x)(\ln x) - \int (\ln x - 1)dx = x(\ln x)^2 - x\ln x - x\ln x + x + x = x(\ln x)^2 - 2x\ln x + 2x + C$.

Definite Integrals

$$\int_a^b f(x)g'(x) = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x).$$

Example

$\int_1^{e^2} x\ln\left(\sqrt{x}\right)dx$. Let $u = \ln\sqrt{x} \quad du = \left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{2x}$, $v = \frac{x^2}{2} \quad dv = xdx$.

$$\int_1^{e^2} x\ln\left(\sqrt{x}\right)dx = \frac{x^2}{2}\ln\sqrt{x}\Big|_1^{e^2} - \int_1^{e^2}\frac{1}{4}xdx = \frac{e^4}{2} - \left(\frac{x^2}{8}\right)\Big|_1^{e^2} = \frac{3}{8}e^4 + \frac{1}{8}.$$

Powers and Products of Trigonometric Functions

- Mantra: "Use the trig identities".

Case 1

$\int (\sin x)^m (\cos x)^n\,dx$, at least one of m or n is odd.

Examples

1) $\int (\sin x)^6 \cos x dx = \frac{1}{2}\sin^7 x + C$.

2) $\int \sin^6 x\cos^3 x = \int \sin^6 x\cos^2 x \cdot \cos x dx = \int \sin^6 x\left(1-\sin^2 x\right)\cos x dx = \int \sin^6 x\cos x dx - \int \sin^8 x\cos x dx$.

Algorithm

As long as n is odd, $n = 2k+1$, $\int (\sin x)^m (\cos x)^n\,dx = \int (\sin x)^m (\cos x)^{2k}\cos x dx$. Use now $u = \sin x$, $du = \cos x dx$.

If m is odd, do the same thing, but substitute $\sin^2 x = 1 - \cos^2 x$.

Example

$$\int \sin^2 x\cos^3 x dx = \int \sin^2 x\cos^2 x\cos x dx = \int \sin^2 x\left(1-\sin^2 x\right)\cos x dx = \int \left(\sin^2 x\cos x - \sin^4 x\cos x\right)dx$$

$$\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

Lecture #37 – Thursday, February 12, 2004

Case 2

What if both m, n even? $\int (\sin x)^m (\cos x)^n \, dx$.

Use double angle formulas:

- $\sin 2x = 2\sin x \cos x \Rightarrow \sin x \cos x = \frac{1}{2}\sin(2x)$.
- $\cos 2x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.
- $\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$.

Idea: Reduce everything using the formulas, repeat as necessary.

Example

$$\int \sin^4 x \cos^2 x dx = \int \sin^2 x (\sin x \cos x)^2 \, dx = \frac{1}{8}\int (1 - \cos 2x)(\sin 2x)^2 \, dx = \frac{1}{8}\int \left(\sin^2 2x - \sin^2 2x \cos 2x\right) dx$$

$$= \frac{1}{8}\left[\frac{1}{2}\int (1 - \cos 4x) dx - \frac{1}{6}(\sin 2x)^3\right] = \frac{1}{8}\left[\frac{1}{2}\left(x - \frac{\sin 4x}{4} - \frac{1}{6}\sin^3 2x\right)\right]$$

Case 3

$\int \sin(mx)\cos(nx) dx$, $\int \sin(mx)\sin(nx) dx$, $\int \cos(mx)\cos(nx) dx$.

If $m = n$, we're back to Case 1 and 2. For $m \neq n$, use angle addition formulas:

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$, $\sin(A - B) = \sin A \cos B - \cos A \sin B$.
- Adding the two equations, we get
  $\sin(A + B) + \sin(A - B) = 2\sin A \cos B \Rightarrow \sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$.

Similarly, using the angle addition formulas for $\cos x$, we get:

- $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$.
- $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$.

Example

$$\int \sin(5x)\cos(3x) dx = \frac{1}{2}\int (\sin(5x + 3x) + \sin(5x - 3x)) dx = \frac{1}{2}\int (\sin(8x) + \sin(2x)) dx = \frac{1}{2}\left[-\frac{\cos 8x}{8} - \frac{\cos 2x}{2}\right] + C$$

Note: About "reduction formulas". Suppose you have something like $\int \left(\sin^n x\right) dx$. The idea is to reduce the power of the exponent, using integration by parts.

Example

$\int \sin^n x dx$. Let $\begin{matrix} u = \sin^{n-1} x & du = (n-1)\sin^{n-2} x \cos x dx \\ v = -\cos x & dv = \sin x dx \end{matrix}$. So

$\int \sin^n x dx = -\cos x \sin^{n-1} x + \int (n-1)\cos^2 x \sin^{n-2} x dx = -\cos x \sin^{n-1} x + \int (n-1)(1-\sin^2 x)\sin^{n-2} x dx$

$= -\cos x \sin^{n-1} x + \int (n-1)\sin^{n-2} x dx - \int (n-1)\sin^n x dx$

$\Rightarrow n\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx \Rightarrow \int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx$

This is the "reduction formula for $\int \sin^n x dx$.

TRIGONOMETRIC SUBSTITUTION

Idea: Use the inverse trigonometric functions and their derivatives.

Recall: $\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}$, $\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{|x|\sqrt{x^2-1}}$.

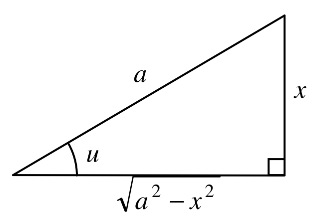
General Strategy

If you see a…	…try a u-substitution of:
$\sqrt{a^2 - x^2}$	$x = a \sin u$
$\sqrt{a^2 + x^2}$	$x = a \tan u$
$\sqrt{x^2 - a^2}$	$x = a \sec u$

Note: Choose a so that $a > 0$.

Example

$\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}} = \int \frac{dx}{\left(\sqrt{a^2 - x^2}\right)^3}$. Let $x = a \sin u \Rightarrow \sin u = \frac{x}{a}$,



$dx = a \cos u du$. So

$\int \frac{dx}{\left(\sqrt{a^2 - x^2}\right)^3} = \int \frac{a \cos u du}{\left(\sqrt{a^2 - a^2 \sin^2 u}\right)^3} = \int \frac{a \cos u du}{\left(a\sqrt{1 - \sin^2 u}\right)^3}$

$= \int \frac{a \cos u du}{a^3 \cos^3 u} = \frac{1}{a^2}\int \frac{du}{\cos^2 u} = \frac{1}{a^2}\int \sec^2 u du = \frac{1}{a^2}\tan u + C$

$= \frac{x}{a^2\sqrt{a^2 - x^2}} + C$

Example

$\int \frac{dx}{x^2\sqrt{x^2 - 4}}$. Let $x = 2\sec u \Rightarrow \sec u = \frac{x}{2} \Rightarrow \cos u = \frac{2}{x}$, $dx = 2\sec u \tan u du$.

So $\int \frac{dx}{x^2\sqrt{x^2-4}} = \int \frac{2\sec u \tan u du}{4\sec^2 u\sqrt{4\sec^2 u-4}} = \int \frac{\sec u \tan u du}{4\sec^2 u \tan u} = \frac{1}{4}\int \cos u du = \frac{1}{4}\sin u + C = \frac{\sqrt{x^2-4}}{4x} + C$.

Example

$\int \frac{dx}{\sqrt{x^2-2x-3}} = \int \frac{dx}{\sqrt{(x^2-2x+1)-1-3}} = \int \frac{dx}{\sqrt{(x-1)^2-2^2}}$. Let $(x-1) = 2\sec u$, $dx = 2\sec u \tan u du$.

So, $\int \frac{2\sec u \tan u du}{\sqrt{4\sec^2 u-4}} = \int \frac{\sec u \tan u du}{\tan u} = \int \sec u du = \ln|\sec u + \tan u| + C = \ln\left|\frac{x-1}{2} + \frac{\sqrt{(x+1)^2-4}}{2}\right| + C$.

PARTIAL FUNCTIONS

- Now, deal with rational functions.
- Idea: Write $\frac{p(x)}{q(x)}$ as a sum of "simpler" rational functions.
- Caveat: We'll only deal with degree of $p(x)$ less than degree of $q(x)$. Otherwise, do long division.

Example

$\frac{p(x)}{q(x)} = \frac{3x+1}{x^2+2x-3}$.

1) Factor $q(x)$. $x^2+2x-3 = (x+3)(x-1)$.
2) $\frac{3x+1}{x^2+2x-3} = \frac{A}{x+3} + \frac{B}{x-1}$.
3) Find A and B: $3x+1 = (x-1)A + (x+3)B = (A+B)x + (-A+3B) \Rightarrow \begin{cases} A+B=3 \\ -A+3B=1 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=1 \end{cases}$.

So, $\frac{3x+1}{x^2+2x-3} = \frac{2}{x+3} + \frac{1}{x-1}$.

What is the use? Now, can calculate $\int \frac{3x+1}{x^2+2x-3}dx = \int \frac{2}{x+3}dx + \int \frac{1}{x-1}dx = 2\ln|x+3| + \ln|x-1| + C$.

Rule 1

Each linear factor $(x-\alpha)$ of $q(x)$ gives rise to a single term in the sum of the of form $\frac{A}{x-\alpha}$.

Lecture #38 – Tuesday, February 24, 2004

Example (Repeated Factor)

- $\dfrac{2x^2+3}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.
- Find A, B, C: $2x^2+3 = A(x-1)^2 + Bx(x-1) + Cx = (A+B)x^2 + (-2A-B+C)x + A \Rightarrow \begin{cases} A=3 \\ B=-1 \\ C=5 \end{cases}$.
- So, $\dfrac{2x^2+3}{x(x-1)^2} = \dfrac{3}{x} - \dfrac{1}{x-1} + \dfrac{5}{(x-1)^2}$.

Rule 2

In general, each factor of $q(x)$ of the form $(x-\alpha)^k$ gives rise to terms $\dfrac{A_1}{x-\alpha} + \dfrac{A_2}{(x-\alpha)^2} + \cdots + \dfrac{A_k}{(x-\alpha)^k}$.

Rule 3

In general, each irreducible quadratic factor $x^2 + \beta x + \gamma$ gives rise to a term $\dfrac{Ax+B}{x^2+\beta x+\gamma}$.

Rule 4

In general, each factor of $q(x)$ of the form $\left(x^2+\beta x+\gamma\right)^k$ gives rise to a sum of terms $\dfrac{A_1x+B_1}{x^2+\beta x+\gamma} + \dfrac{A_2x+B_2}{\left(x^2+\beta x+\gamma\right)^2} + \cdots + \dfrac{A_kx+B_k}{\left(x^2+\beta x+\gamma\right)^k}$.

Examples

1) $\displaystyle\int \frac{2x^2+3}{x(x-1)^2} = \int\left(\frac{3}{x} - \frac{1}{x-1} + \frac{5}{(x-1)^2}\right)dx = 3\ln|x| - \ln|x-1| - \frac{5}{x-1} + C$.

2) $$\int \frac{x^2+5x+2}{(x+1)(x^2+1)}dx = \int\left(\frac{-1}{x+1} + \frac{2x+3}{x^2+1}\right)dx = -\ln|x+1| + \int\left(\frac{2x+3}{x^2+1}\right)dx = -\ln|x+1| + \int\frac{2x}{x^2+1}dx + 3\int\frac{1}{x^2+1}dx.$$
$$= -\ln|x+1| + \ln\left(x^2+1\right) + 3\tan^{-1}(x) + C$$

3) $$\int \frac{dx}{x\left(x^2+x+1\right)} = \int\left(\frac{1}{x} - \frac{x+1}{x^2+x+1}\right)dx = \ln|x| - \int\left(\frac{\frac{1}{2}(2x+1)}{x^2+x+1} + \frac{\frac{1}{2}}{x^2+x+1}\right)dx$$
$$= \ln|x| - \frac{1}{2}\ln\left|x^2+x+1\right| - \frac{1}{2}\int\frac{dx}{x^2+x+1} = \ln|x| - \frac{1}{2}\ln\left|x^2+x+1\right| - \frac{1}{2}\int\frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}.$$
$$= \ln|x| - \frac{1}{2}\ln\left|x^2+x+1\right| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C$$

APPENDIX TO INTEGRATION

Idea

When you see something like $\sqrt[n]{f(x)}$, try a substitution $u = \sqrt[n]{f(x)}$ and see what happens.

Example

$\int \frac{dx}{1-\sqrt{x}}$. Let $\begin{matrix} u = \sqrt{x} \Rightarrow u^2 = x \\ 2udu = dx \end{matrix}$. So, $\int \frac{dx}{1-\sqrt{x}} = \int \frac{2udu}{1-u} = \int \left(-2 + \frac{2}{1-u}\right)du = -2\sqrt{x} - 2\ln\left|1-\sqrt{x}\right| + C$.

Lecture #39 – Thursday, February 26, 2004

IMPROPER INTEGRALS

So far, we've dealt with integrals of the form $\int_a^b f(x)dx$, where a, b are fixed (finite) numbers, and $f(x)$ is a bounded (usually even continuous) function on $[a,b]$.

Idea: Loosen these requirements.

1. Integrals On An Unbounded Domain

- Suppose $f(x)$ is continuous on $[a,\infty)$. If $\lim_{b\to\infty}\int_a^b f(x)dx$ exists and is equal to L, then $\int_a^\infty f(x)dx = L$.
- We say "the improper integral $\int_a^\infty f(x)dx = L$ converges and is equal to L". Otherwise, if the limit doesn't exist, we say "the improper integral $\int_a^\infty f(x)dx = L$ diverges.
- Similarly for $\int_{-\infty}^b f(x)dx = L$.

Examples

1) $\int_0^\infty e^{-2x}dx = \lim_{b\to\infty}\int_0^b e^{-2x}dx = \lim_{b\to 0}\left[\frac{e^{-2x}}{-2}\right]\Bigg|_0^b = \lim_{b\to 0}\left[-\frac{1}{2}\left(e^{-2b}-1\right)\right] = \frac{1}{2}$.

2) $\int_1^\infty \cos(\pi x)dx = \lim_{b\to\infty}\int_1^b \cos(\pi x)dx = \lim_{b\to\infty}\left[\frac{\sin(\pi x)}{\pi}\Bigg|_1^b\right] = \lim_{b\to\infty}\left[\frac{\sin(\pi b)}{\pi} - \frac{\sin(\pi)}{\pi}\right] = \lim_{b\to\infty}\frac{1}{\pi}\sin(\pi b)$. Since this oscillates, so there is no limit. So $\int_1^\infty \cos(\pi x)dx$ diverges.

2. Integral of Unbounded Functions

- Suppose f is continuous on $[a,b)$ and unbounded at $x = b$. Then, for any c, $a \le c < b$, can calculate $\int_a^b f(x)dx$ by computing $\lim_{c\to b^-}\int_a^c f(x)dx$.

- If $\lim_{c\to b^-}\int_a^c f(x)dx$ exists and is equal to *L*, we say "the improper integral $\int_a^b f(x)dx$ <u>converges</u> to *L*".
  Otherwise, we say "the improper integral $\int_a^b f(x)dx$ <u>diverges</u>".
- Similarly, for *f* continuous on $(a,b]$, take $\lim_{c\to a^+}\int_c^b f(x)dx$.

Examples

1) $\int_0^1 \frac{1}{x}dx = \lim_{c\to 0^+}\int_c^1 \frac{1}{x}dx = \lim_{c\to 0^+}(\ln x)\Big|_c^1 = \lim_{c\to 0^+}[\ln(1)-\ln(c)]$. Since the limit is unbounded above, $\int_0^1 \frac{1}{x}dx$ diverges.

2) $\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{c\to 0^+}\int_c^1 \frac{1}{\sqrt{x}}dx = \lim_{c\to 0^+}\left[2\sqrt{x}\right]_c^1 = \lim_{c\to 0^+}\left(2-2\sqrt{c}\right)=2$. So, $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges.

Comparison Test

Suppose *f*, *g* continuous on $[a,\infty)$, and suppose $0\le f(x)\le g(x)$, then:

1) If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ also converges.
2) If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ also diverges.