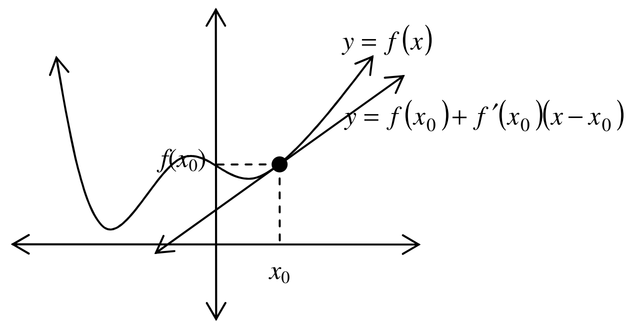
Lecture #39 – Thursday, February 26, 2004

TAYLOR POLYNOMIALS

Recall: Throughout the year, so far, we've thought of the tangent line to the curve $y = f(x)$ at $(x_0, f(x_0))$ as the "best linear approximation" to the curve a that point $(x_0, f(x_0))$.

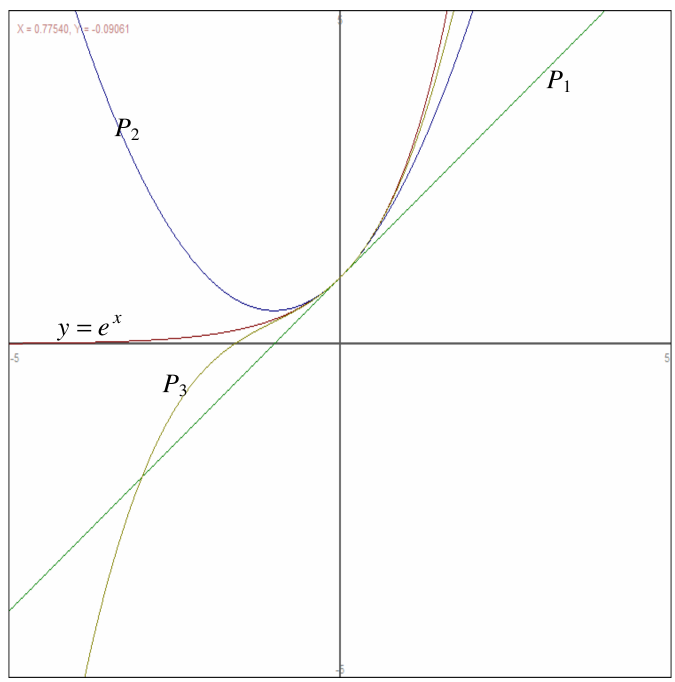


- Idea: What if, instead of "just" using straight lines to approximate $f(x)$, we want to use (higher degree) polynomials? "Taylor polynomials", using higher derivatives of $f(x)$.
- The Point: Polynomials are easy to deal with! Can take derivates, integrals, etc…

Example

$f(x) = e^x$. Pick $x_0 = 0$.

- First linear approximation: $y = 1 + x$.
- Second quadratic approximation: $y = 1 + x + \dfrac{x^2}{2}$.
- Second quadratic approximation: $y = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$.



How to get the coefficients?

1) Idea: Since we already know that $\left(e^x\right)' = e^x$, we want to mimic that in the Taylor polynomial. So, for example, $(\text{third approximation})' = \text{second approximation}$, etc. So in general,

$P_n\left(\text{for } e^x\right) = 1 + x + \dfrac{x^2}{2} + \cdots + \dfrac{x^n}{n!}$.

2) Idea: We want the derivatives of the approximations to agree with the derivative of the original function.

Above: $\begin{cases} P_2(0) = 1 + x + \dfrac{x^2}{2} = 1 = f(0) \\ P_2'(0) = 1 + x = 1 = f'(0) \\ P_2''(0) = 1 = f''(0) \end{cases}$.

Theorem: Taylor's Theorem

If f has $n+1$ derivatives on an open interval I containing 0, then for any $x \in I$,
$f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2}x^2 + \cdots + \dfrac{f^{(n)}(x)}{n!} + R_n(x)$, where $R_n(x) = \dfrac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt$ the "n^{th} (Taylor) remainder term".

Proof:

- Fix $x \in I$.
- $f(x) - f(0) = \int_0^x f'(t)dt \Rightarrow f(x) = f(0) + \int_0^x f'(t)dt$.
- Let $\begin{matrix} u = f'(t) & du = f''(t)dt \\ v = -(x-t) & dv = dt \end{matrix}$. Integration by parts gives

  $f(x) = f(0) + \left(-f'(t)(x-t)\right)\Big|_0^x + \int_0^x f''(t)(x-t)dt = f(0) + f'(0)x + \int_0^x f''(t)(x-t)dt$.
- Let $\begin{matrix} u = f''(t) & du = f'''(t)dt \\ v = -\dfrac{(x-t)^2}{2} & dv = (x-t)dt \end{matrix}$. Integration by parts gives

  $$f(x) = f(0) + f'(0)x + \left(-\frac{f''(t)(x-t)^2}{2}\right)\Bigg|_0^x + \int_0^x \frac{f'''(t)(x-t)^2}{2}dt$$
  $$= f(0) + f'(0)x + \frac{f'''(0)}{2}x^2 + \frac{1}{2}\int_0^x f'''(t)(x-t)^2\,dt$$.
- Keep going, out to $f(x) = f(0) + f'(0)x + \dfrac{f'''(0)}{2}x^2 + \cdots + \dfrac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt$.

Lecture #40 – Tuesday, March 2, 2004

How Big Is The Error?

- Fact: A generalization of the Mean Value Theorem (which says for some $c \in (a,b)$, $f'(c)(b-a) = f(b) - f(a)$) is that if f, g continuous on $[a,b]$, $g \geq 0$, then there is some $c \in (a,b)$ so that $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$.

- Use this to get "Lagrange formula for the remainder term": For some $c \in (0,x)$, $R(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$.

- This gives us (sometimes) an expression bounding the remainder: Suppose $\left|f^{(n+1)}(x)\right| \le M$ on $[0,x]$, then $|R(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \le \frac{M\left|x^{n+1}\right|}{(n+1)!}$.
- Note: $\lim_{n\to\infty} \frac{\left|x^{n+1}\right|}{(x+1)!} \to 0$, so our error $R_n(x)$ can be made arbitrarily small, so Taylor's polynomials give better and better approximations.

Example

$f(x) = e^x$.

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{f^{(n+1)}(x)}{(n+1)!}x^{n+1}$.
- Recall: $f^{(n+1)}(x) = e^x$. So on $[0,x]$, $\left|f^{(n+1)}(x)\right| \le e^x$.
- Pick $x = 1$. In that case, the upper bound is $e^1 = e$. So $|R(x)| \le \frac{e}{(n+1)!}$.

Example

Compute the first 5 Taylor polynomial for $f(x) = \sin x$.

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{(4)}(x) = \sin x$ – repeats	Repeats…

- $P_1(x) = x = P_2(x)$.
- $P_3(x) = x - \frac{x^3}{3!} = P_4(x)$.
- $P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} = P_6(x)$.

Example

Use Taylor Polynomials to approximate $e^{0.8}$ within 0.01.

- $e^{0.8} = f(0.8) = 1 + (0.8) + \frac{(0.8)^2}{2!} + \cdots + \frac{(0.8)^n}{n!} + R_n(0.8)$.
- $|R_n(0.8)| = \frac{\left|f^{(n+1)}(c)\right|}{(n+1)!}(0.8)^{n+1} = \frac{e^c}{(n+1)!}(0.8)^{n+1}, c \in (0, 0.8) \le \frac{e(0.8)^{n+1}}{(n+1)!}$.
- Want $\frac{e(0.8)^{n+1}}{(n+1)!} < 0.01$. Take $n = 4$.

- So $e^{0.8} \approx 1 + 0.8 + \frac{(0.8)^2}{2!} + \frac{(0.8)^3}{3!} + \frac{(0.8)^4}{4!} = 2.2224$.

Lecture #31 – Thursday, March 4, 2004

Dealing With Taylor Series of Several Functions: "Little-o" Notation

Definition

Let $g(x)$ be a function defined on an open interval *I* around 0. Then "$g(x)$ is $o(x)$"(g is "little-o of x") for $n \in \mathbf{Z}$ if $\lim_{x\to 0} \frac{g(x)}{x^n} = 0$.

Example

$g(x) = x^3 = o(x^2)$ because $\lim_{x\to 0} \frac{x^3}{x^2} = 0$.

- Idea: $f(x) \approx 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} \approx 1 + x + \frac{x^2}{2} + o(x^2)$.

Facts

1) "$o(x^n) + o(x^n) = o(x^n)$" means if $g(x)$ is $o(x^n)$ and $h(x)$ is $o(x^n)$, then $g + h$ is also $o(x^n)$. Proof: $\lim_{x\to 0} \frac{g(x) + h(x)}{x^n} = \lim_{x\to 0} \frac{g(x)}{x^n} + \frac{h(x)}{x^n} = 0 + 0 = 0$.
2) If c is a constant, $c \cdot o(x^n) = o(x^n)$.
3) $o(x^m) + o(x^n) = o(x^m)$, $m < n$.
4) $o(x^m) \cdot o(x^n) = o(x^{m+n})$.
5) Suppose $q_k(x)$ is a polynomial with smallest non-zero degree k (ex: If $g(x) = x^{10} + 5x^5 + 5x^2$, $k = 2$), then $q_k(x) \cdot o(x^n) = o(x^{n+k})$.
6) $\frac{o(x^n)}{q_k(x)} = o(x^{n-k}), n > k$.

- Note: $\frac{o(x^n)}{o(x^m)} \neq o(x^{n-m}), n > m$.

Example

Compute $\lim_{x\to 0} \frac{1 - \frac{x^6}{2} - \cos(x^3)}{x^6 (x - \sin x)}$.

- Recall: $\sin(x) = x - \frac{x^3}{3!} + o(x^3)$ and $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4) \Rightarrow \cos(x^3) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} + o(x^{12})$.
- $$\lim_{x\to 0} \frac{1 - \frac{x^6}{2} - \cos(x^3)}{x^6(x - \sin x)} = \lim_{x\to 0} \frac{1 - \frac{x^6}{2} - \left(1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} + o(x^{12})\right)}{x^6\left[x - \left(x - \frac{x^3}{3!} + o(x^3)\right)\right]} = \lim_{x\to 0} \frac{-\frac{x^{12}}{2} - o(x^{12})}{x^6\left(\frac{x^3}{3!} - o(x^3)\right)} = \lim_{x\to 0} \frac{-\frac{x^{12}}{2} - o(x^{12})}{\frac{x^9}{3!} - o(x^9)}$$
$$= \lim_{x\to 0} \frac{-\frac{x^3}{2} - o(x^3)}{\frac{1}{3!} - \frac{o(x^9)}{x^9}} = 0$$

Sequences of Real Numbers

Definition

A <u>sequence of real numbers</u> is a real-value function defined on the set of positive integers $\{1,2,3,\cdots\}$. So we have $\{a(1), a(2), a(3), \cdots\}, a(n) \in \mathbf{R}$.

- <u>Convention</u>: $a(n) = a_n$, the "n^{th} term in the sequence".
- With sequences, we can do the same basic operations as we did for functions. For example, for two sequences $\{a_n\}$ and $\{b_n\}$, we can:
  - "sum": $a_n + b_n$
  - "product": $a_n \cdot b_n$
  - "ratios": If $b_n \neq 0 \forall n$, $\frac{a_n}{b_n}$.

Examples

1) $a_n = \frac{1}{n}$. $\left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$.
2) $a_n = (-1)^n$. $\{-1,1,-1,1,\cdots\}$.
3) $a_n = n$. $\{1,2,3,\cdots\}$.
4) $a_n = \frac{n-1}{n}$. $\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\right\}$.

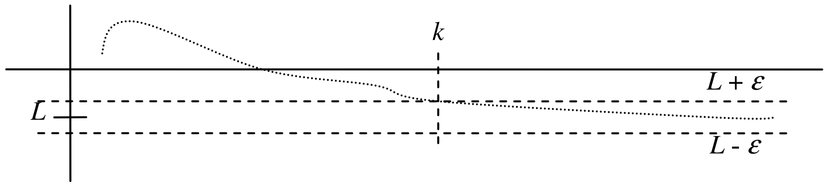
Definition

A sequence $\{a_n\}$ is:

- increasing if $a_n > a_{n-1}$
- non-decreasing if $a_n \geq a_{n-1}$
- decreasing if $a_n < a_{n-1}$
- non-increasing if $a_n \leq a_{n-1}$.

Definition

We say $\lim_{n\to\infty} a_n = L$ if $\forall \varepsilon > 0$, $\exists k \in \mathbf{Z}$ such that $\forall n \geq k$, $|a_n - L| < \varepsilon$.



Example

Prove $\lim_{n\to\infty}\left(a_n = \frac{n-1}{n}\right) = 1$.

- Want: $\left|\frac{n-1}{n} - 1\right| < \varepsilon \Rightarrow \left|\frac{n-1-n}{n}\right| < \varepsilon \Rightarrow \left|\frac{1}{n}\right| < \varepsilon$.
- Pick $k \in \mathbf{Z} > 0$ such that $k > \frac{1}{\varepsilon}$. Then $|a_k - 1| = \left|\frac{1}{k}\right| < \varepsilon$, so for $n \geq k$, $\left|\frac{1}{n}\right| < \varepsilon$.
- Note: As with other limits, if a limit does not exist, we say $\{a_n\}$ diverges.

Theorem: Uniqueness of Limits

If $a_n \to L$ and $a_n \to L'$, then $L = L'$.

Theorem

Every convergent sequence is bounded, i.e. $\exists M > 0$ such that $|a_n| < M$.

Proof:
- Let $\{a_n\}$ be a convergent sequence with limit L.
- Pick $\varepsilon = 1$. Then there exists $k \in \mathbf{Z}$ so that $\forall n \geq k$, $|a_n - L| < 1 \Rightarrow |a_n| < |L| + 1$.
- Let $M = \max\{|L| + 1, |a_1|, \cdots, |a_k|\}$. Then, by construction, $|a_i| \leq M \forall i \geq 1$.

- Warning: The converse is not true. It is not true that every bounded sequence is convergent (ex: $a_n = (-1)^n$.
- But: If a sequence is bounded and monotonic (either non-decreasing or non-increasing), then it is convergent. Actually, we can say more!

Theorem

A bounded non-decreasing sequence converges to the least upper bound.
A bounded non-increasing sequence converges to the greatest lower bound.

Proof: Non-decreasing case
- Let $L = \text{lub}(\{a_n\})$. Then $a_n \leq L$ and $\forall \varepsilon > 0$, $\exists a_k$ such that $L - \varepsilon < a_k \leq L$.
- Because $\{a_n\}$ is non-decreasing, this means $\forall n \geq k$,
  $L - \varepsilon < a_k \leq a_n \leq L \Rightarrow \forall n \geq k, |a_n - L| < \varepsilon$.

Theorem

Let $\{a_n\}$, $\{b_n\}$ be sequences, $a_n \to L$ and $b_n \to M$. Then:

- $a_n + b_n \to L + M$.
- $\alpha a_n \to \alpha L$, α constant.
- $a_n \cdot b_n \to L \cdot M$.
- If $b_n \neq 0$ and $M \neq 0$, then $\dfrac{1}{b_n} \to \dfrac{1}{M}$ and $\dfrac{a_n}{b_n} \to \dfrac{L}{M}$.

Examples

Find the limit if it exists:

1) $$a_n = \frac{3n^3 + 2n^2 + n}{n^5 - 7n^2} = \frac{\frac{3}{n^2} + \frac{2}{n^3} + \frac{1}{n^4}}{1 - \frac{7}{n^3}} = 0.$$

2) $$a_n = \frac{n^6 + 3n^7}{10n^7 + 2n^2 + 5} = \frac{\frac{1}{n} + 3}{10 + \frac{2}{n^5} + \frac{5}{n^7}} = \frac{3}{10}.$$

Theorem

$a_n \to L$ iff $a_n - L \to 0$ iff $|a_n - L| \to 0$.

This allows you to get rid of the minus signs.

Lecture #32 – Tuesday, March 9, 2004

Theorem: Pinching Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences. Suppose for all $n \geq k$, $a_n \leq b_n \leq c_n$. If $a_n \to L$ and $c_n \to L$ as $n \to \infty$, then $b_n \to L$.

Corollary

Suppose for all $n \geq k$, $|b_n| \leq c_n$ ($-c_n \leq b_n \leq c_n$). If $c_n \to 0$, then $b_n \to 0$.

Example

What is the limit of $a_n = \dfrac{\cos(n)}{n}$ as $n \to \infty$?

- $|a_n| = \left|\dfrac{\cos(n)}{n}\right| \leq \left|\dfrac{1}{n}\right| = \dfrac{1}{n} =: c_n$. We know $c_n \to 0$, so by the corollary, $a_n \to 0$.

Example

What is the limit of $b_n = \left(1+\frac{1}{n}\right)^n$ as $n \to \infty$?

- We'll use the following estimate: $\left(1+\frac{1}{n}\right)^n \le e \le \left(1+\frac{1}{n}\right)^{n+1}$.
- We'll use the pinching theorem. We already know $\left(1+\frac{1}{n}\right)^n \le e =: c_n$. Also, we know $e \le \left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right) \Rightarrow a_n =: \frac{e}{1+\frac{1}{n}} \le \left(1+\frac{1}{n}\right)^n$.
- So, $a_n = \frac{e}{1+\frac{1}{n}} \le b_n = \left(1+\frac{1}{n}\right)^n \le c_n = e$. By the Pinching Theorem, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \to e$.

Theorem

Suppose that $c_n \to c$, and all c_n are in domain of *f*, and *f* is continuous at *c*. Then $f(c_n) \to f(c)$.

Proof (sketch of proof):

- We know *f* is continuous at *c*. So if $|x-c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.
- Since $c_n \to c$, it means $\exists\ k$ such that $\forall\ n \ge k$, $|c_n - c| < \delta$.
- But it means, by continuity of *f*, that $|f(c_n) - f(c)| < \varepsilon$. But since ε is arbitrary, $f(c_n) \to f(c)$.

Some Important Examples

These examples show up a lot, especially in Taylor approximations.

1) Prove if $x > 0$, then $x^{\frac{1}{n}} \to 1$ as $n \to \infty$.
   - Mantra: "Take the natural log".
   - Define $b_n = \ln\left(a_n =: x^{\frac{1}{n}}\right) = \frac{1}{n}\ln(x) \to 0$.
   - $a_n = e^{b_n} = f(b_n)$ where $f(u) = e^u$.
   - Since $f(u)$ is continuous at $u = 0$, then by theorem above, $a_n = f(b_n) \to f(0) = e^0 = 1$.

2) Prove if $|x| < 1$, then $x^n \to 0$ as $n \to \infty$.
   - For $x = 0$, it is clear. So assume $x \ne 0$ from now on.
   - Let $|a_n| = |x^n| = |x||x^{n-1}| < |a_{n-1}|$. So $|a_n|$ is a decreasing sequence.
   - Now let $\varepsilon > 0$. By 1) above, $\varepsilon^{\frac{1}{n}} \to 1$ as $n \to \infty$.
   - Since $|x| < 1$, at some point *k*, $|x| < \varepsilon^{\frac{1}{k}}$. So $|x|^k < \left(\varepsilon^{\frac{1}{k}}\right)^k = \varepsilon$.

- But since $\left|x^n\right|$ is decreasing, $\forall \quad n > k$, $\left|a_n\right| < \varepsilon$. So $0 \le \left|a_n\right| \le \varepsilon$.
- Since ε is arbitrary, $\left|a_n\right| \to 0 \Rightarrow a_n \to 0$.

Lecture #33 – Thursday, March 11, 2004

3) Prove for any $\alpha > 0$, $\dfrac{1}{n^{\alpha}} \to 0$ as $n \to \infty$.

   - There is a positive k so that $0 < \dfrac{1}{k} < \alpha$. Notice $f(x) = x^{\frac{1}{k}}$ is continuous at 0.
   - $0 \le \left(\dfrac{1}{n}\right)^{\alpha} \le \left(\dfrac{1}{n}\right)^{\frac{1}{k}}$. But because $\dfrac{1}{n} \to 0$ as $n \to \infty$, apply f and $f\left(\dfrac{1}{n}\right) = \left(\dfrac{1}{n}\right)^{\frac{1}{k}} \to 0^{\frac{1}{k}} = 0$. So by Pinching Theorem, $\dfrac{1}{n^{\alpha}} \to 0$ also.

4) Prove $a_n = n^{\frac{1}{n}} \to 1$ as $n \to \infty$.

   - "Take natural log first". Consider $b_n = \ln\left(n^{\frac{1}{n}}\right) = \dfrac{\ln n}{n} \to 0$ by l'Hôpital.
   - $a_n = e^{(b_n)}$, so $a_n \to e^0 = 1$.

5) Prove for each real x, $\left(1 + \dfrac{x}{n}\right)^n \to e^x$ as $n \to \infty$.

   - "Take natural log" and recognize a derivative of ln.

6) For each real x, $\dfrac{x^n}{n!} \to 0$ as $n \to \infty$. (Recall this is the n^{th} term in the Taylor expansion of e^x.)

   - Fix k a positive integer such that $|x| < k$. We'll show that $\left|\dfrac{x^n}{n!}\right| = \dfrac{\left|x^n\right|}{n!} \to 0$.
   - By choice of k, for $n > k + 1$,
   $\dfrac{\left|x^n\right|}{n!} < \dfrac{\left|k^k\right|}{k!} = \dfrac{k \times k \times \cdots \times k \times k}{n \times (n-1) \times \cdots \times k \times \cdots \times 2 \times 1} = \dfrac{k}{n}\left(\dfrac{k}{n-1} \times \dfrac{k}{n-2} \times \cdots \times \dfrac{k}{k+1}\right)\left(\dfrac{k^k}{k!}\right) < \dfrac{k}{n} \times \dfrac{k^k}{k!}$. Since $\dfrac{k}{n} \times \dfrac{k^k}{k!} \to 0$, by Pinching Theorem, $\dfrac{|x|^n}{n!} \to 0$.

Series

Instead of just dealing with individual terms in the sequence $\{a_1, \cdots, a_n\}$, we'll think about $a_1 + \cdots + a_n$.

Example

Taylor's series: $a_n = \dfrac{f^{(n)}(0)x^n}{n!}$.

INFINITE SERIES

1) Starting ingredient: Infinite sequence $\{a_0, a_1, a_2, \cdots\}$.
2) Then, form the partial sums:
   - $s_0 = a_0$.
   - $s_1 = a_0 + a_1$.
   - $s_2 = a_0 + a_1 + a_2$.
   - $s_n = a_0 + \cdots a_n = \sum_{k=0}^{n} a_k$.
3) We get another sequence, the sequence of partial sums $\{s_n\}$. The limit (if it exists), is called the limit of the infinite series $\sum_{k=0}^{\infty} a_k$.

Definition

If $\{s_n\}$ converges to a finite limit *L*, then we say the infinite series $\sum_{k=0}^{\infty} a_k$ converges and we write

$\sum_{k=0}^{\infty} a_k = L$ ($\lim_{n\to\infty} s_n$).

If it diverges, then we say the infinite series $\sum_{k=0}^{\infty} a_k$ diverges.

GEOMETRIC SERIES

Definition

Let $a_n = x^n$. Then $s_n = a_0 + \cdots + a_n = 1 + x + x^2 + \cdots + x^n$.

Important Trick

- Notice: $(1 + x + x^2 + \cdots + x^n)(1 - x) = 1 + x + x^2 + \cdots + x^n - x - x^2 - \cdots - x^n - x^{n+1} = 1 - x^{n+1}$. So $s_n = 1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$.
- If $|x| < 1$, $\sum_{k=o}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=o}^{n} x^k = \lim_{n\to\infty} \dfrac{1 - x^{n+1}}{1 - x} = \dfrac{1}{1 - x}$.
- If $|x| > 1$, $\sum_{k=o}^{\infty} x^k$ diverges..

Example: Achilles

$$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k}=\lim_{n\to\infty}s_n=\lim_{n\to\infty}\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\left(\frac{1}{2}\right)}=2.$$

Example

If $x=-1$, what is $\sum_{k=0}^{\infty}(x)^{k}$?

- $s_n=1+(-1)+1+(-1)+\cdots+(-1)^n \Rightarrow s_n=\begin{cases}1, \text{if } n \text{ even}\\ \text{-}1, \text{if } n \text{ odd}\end{cases}$.
- So the limit, although it's bounded, doesn't exists. So $\sum_{k=0}^{\infty}(-1)^{k}$ diverges.

Example (Another Trick: Partial Fractions)

Show that $\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)}=\sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=1$.

- $$\begin{aligned}s_n&=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}\\&=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=1-\frac{1}{n-2}\end{aligned}$$. This is called "telescoping sum".
- So $\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(1-\frac{1}{n+2}\right)=1$.

General Techniques

Are there general techniques? Yes and no…We'll start with the fist question to be answered: Does the infinite series converge?

Theorem

1) If $\sum_{k=0}^{\infty}a_k$ converges to *L*, and if $\sum_{k=0}^{\infty}b_k$ converges to *M*, then the sum $\sum_{k=0}^{\infty}(a_k+b_k)$ converges to $L+M$.
2) If $\sum_{k=0}^{\infty}a_k$ converges to *L*, and α is any real number, then $\sum_{k=0}^{\infty}\alpha a_k$ converges to αL.

Theorem

If $\sum_{k=0}^{\infty}a_k$ converges, then $a_k\to 0$ as $k\to\infty$.

Sketch of proof:

- For all $\varepsilon > 0$, there exists k such that for all $n > k$, $|s_n - L| < \varepsilon$ and $|s_{n+1} - L| < \varepsilon$.
- Because $s_{n+1} = s_n + a_{n+1}$, this means $|a_{n+1}| < 2\varepsilon$.
- Because ε is arbitrary, $a_n \to 0$ as $n \to \infty$.

Theorem

If $\{a_n\}$ is a sequence and $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{k=0}^{\infty} a_k$ diverges.

Example

Does $\sum_{k=0}^{\infty} \frac{k}{k+1}$ converge or diverge?

- Since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, so $\sum_{k=0}^{\infty} \frac{k}{k+1}$ diverges.

Lecture #34 – Tuesday, March 16, 2004

THE INTEGRAL TEST

- Preliminary note: If $\sum a_k$ is a series and each $a_k \geq 0$ (non-negative), then the sequence of partial sums
$$\begin{cases} s_1 = a_1 \\ s_2 = a_1 + a_2 \\ \quad\vdots \\ s_n = a_1 + a_2 + \cdots + a_n \end{cases}$$
is non-decreasing.

Theorem

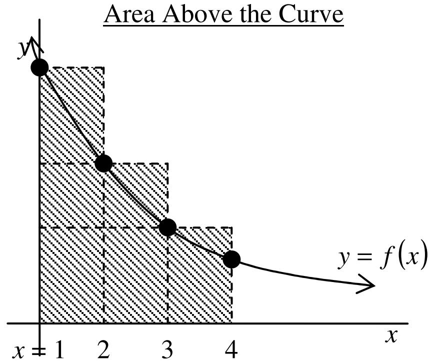
A series with non-decreasing terms converges if and only if the sequence of partial sums is bounded.

Proof: Suppose $a_k \geq 0$. Then $\{s_n\}$ is non-decreasing. So if it is bounded, it also converges. On the other hand, if it converges, it has a limit, and it's bounded.

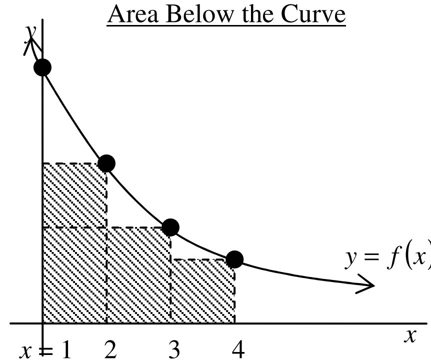
Picture: "Sequences that come from functions"

Define $a_k := f(x)$.

Area Above the Curve



Area Below the Curve



- Note: $\sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} a_k$ = area above the curve, and $\sum_{k=2}^{n} f(k) = \sum_{k=2}^{n} a_k$ = area below the curve.

Theorem: Integral Test

If f is continuous, decreasing, and positive, then $\sum_{k=1}^{\infty} f(k)$ converges iff $\int_1^{\infty} f(x)dx$ converges.

Proof:

- From the picture, we have two inequalities:
  - $f(2) + f(3) + \cdots + f(n) \le \int_1^n f(x)dx$. (1)
  - $\int_1^n f(x)dx \le f(1) + f(2) + \cdots + f(n)$. (2)
- If the integral converges, then by (1), s_n is bounded, so the series converges.
- If the integral diverges, then by (2), s_n is unbounded, so the series diverges.

Example: Harmonic Series

Does $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ converge?

- The function we'll use is $f(x) = \frac{1}{x}, x \in [1, \infty)$.
- By the Integral Test, we compute $\int_1^{\infty} \frac{1}{x} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x} dx = \lim_{b \to \infty} (\ln |b|) = \infty$, so the series diverges.

Example

Show that $\sum_{k=1}^{\infty} \frac{1}{k \ln(k+1)}$ diverges.

- The function we'll use is $f(x) = \frac{1}{x \ln(x+1)}, x \in [1, \infty)$.
- We want to show, by Integral Test, $\int_1^{\infty} \frac{1}{x \ln(x+1)} dx$ diverges.
- Look instead at $\int_1^{\infty} \frac{1}{(x+1)\ln(x+1)} dx < \int_1^{\infty} \frac{1}{x \ln(x+1)} dx$ (estimate).

- $\int_1^\infty \frac{1}{(x+1)\ln(x+1)}dx = \lim_{b\to\infty}\int_1^b \frac{1}{(x+1)\ln(x+1)}dx = \lim_{b\to\infty}\ln(\ln(x+1))\Big|_1^b = \lim_{b\to\infty}\ln(\ln(b+1)) - \ln(\ln(2))$, which diverges. So $\int_1^\infty \frac{1}{x\ln(x+1)}dx$ also diverges.
- So, since $\int_1^\infty \frac{1}{x\ln(x+1)}dx$ diverges, $\sum_{k=1}^{\infty}\frac{1}{k\ln(k+1)}$ also diverges.
- <u>Moral</u>: Don't get "stuck" on a specific integral…estimate and get integrals you can do.

Remark

Notice that $\sum_{k=1}^{\infty} a_k$ converges iff $\sum_{k=N}^{\infty} a_k$ converges for any N positive integer. So, series can be written as $\sum a_k$.

Lecture #35 – Thursday, March 18, 2004

THE COMPARISON TEST

Theorem: Comparison Test

Let $\sum a_k$ be a series with nonnegative terms.

1) $\sum a_k$ converges if there exists convergent series $\sum c_k$ with nonnegative terms and $a_k \le c_k$ for all sufficiently large k.
2) $\sum a_k$ diverges if there exists convergent series $\sum c_k$ with nonnegative terms and $a_k \ge c_k$ for all sufficiently large k.

Proof:

- Since $\sum c_k$ converges, the partial sums for c_k are bounded. This means that the partial sums for a_k are also bounded. Because the partials sums s_k are also non-decreasing, so they converge.
- Since $\sum c_k$ diverges, the partial sums for c_k are unbounded. But then because $a_k \ge c_k$ for k large enough, the partial sums for a_k are also bounded, so $\sum a_k$ diverges.

Examples

1) Show that $\sum \frac{1}{3k+1}$ diverges.
   - Compare: $\frac{1}{3k+1} > \frac{1}{3k+3}$. Since $\sum \frac{1}{3k+3} = \frac{1}{3}\sum\frac{1}{k+1}$ is the harmonic series with shifted index, it diverges. So by Comparison Test, $\sum \frac{1}{3k+1}$ diverges.

2) Show that $\sum \frac{k^2}{2k^5+10k+7}$ converges.

- Compare: $\frac{k^2}{2k^5+10k+7} < \frac{k^2}{2k^5} = \frac{1}{2k^3}$. Since $\sum \frac{1}{2k^3} = \frac{1}{2}\sum \frac{1}{k^3}$ converges by Integral Test, so by Comparison Test, $\sum \frac{k^2}{2k^5+10k+7}$ converges.

Non-Example

There are limits to the use of the Comparison Test. Sometimes, there is an "obvious" candidate to compare with, but the inequality doesn't work!

Consider $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$. The "obvious" comparison is with $\sum \frac{1}{k^2}$, but $\frac{1}{k^2-1} > \frac{1}{k^2}$.

THE LIMIT COMPARISON TEST

A slightly more sophisticated test.

Theorem: Limit Comparison Test

Let $\sum a_k$, $\sum b_k$ be series with positive terms. If $\frac{a_k}{b_k} \to L, L > 0$, then $\sum a_k$ converges iff $\sum b_k$ converges.

Proof:

- Since $\frac{a_k}{b_k} \to L$, for any $\varepsilon > 0$, $\exists\, N$ such that $\forall\ n \geq N$, $\left|\frac{a_k}{b_k} - L\right| < \varepsilon$.
- This means $L - \varepsilon < \frac{a_n}{b_n} < L + \varepsilon$ for $n \geq N$. So $(L-\varepsilon)b_n < a_n < (L+\varepsilon)b_n$ for $n \geq N$.
- So by Comparison Test, if $\sum a_k$ converges, then $\sum b_k$ converges, and vice versa.

Examples

1) Show that $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ converges.

   - Let $a_k = \frac{1}{k^2-1}$, $b_k = \frac{1}{k^2}$. Compute $\lim_{k\to\infty} \frac{\frac{1}{k^2-1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1 > 0$.
   - Since $\sum \frac{1}{k^2}$ converges, by Limit Comparison Test, $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ converges.

2) Show that $\sum \frac{k^{\frac{3}{2}}}{k^{\frac{5}{2}}+2k+10}$ diverges.

- Compute $\lim_{k\to\infty}\frac{k^{\frac{3}{2}}}{k^{\frac{5}{2}}+2k+10}\cdot\frac{k}{1}=\lim_{k\to\infty}\frac{k^{\frac{5}{2}}}{k^{\frac{5}{2}}+2k+10}=1$.
- Since $\sum\frac{1}{k}$ diverges (harmonic series), $\sum\frac{k^{\frac{3}{2}}}{k^{\frac{5}{2}}+2k+10}$ diverges.

TESTS USING THE GEOMETRIC SERIES

Reminder

- If $|\mu|<1$, then $\sum_{k=0}^{\infty}\mu^k=\frac{1}{1-\mu}$.
- If $|\mu|>1$, then $\sum_{k=0}^{\infty}\mu^k$ diverges.

Theorem: Root Test

Let $\sum a_k$ be a series with nonnegative terms. Suppose $a_k^{\frac{1}{k}}\to\rho$.

1) If $\rho<1$, then $\sum a_k$ converges.
2) If $\rho>1$, then $\sum a_k$ diverges.
3) If $\rho=1$, then the test is inconclusive.

Proof:

- Case 1: Choose μ so that $\rho<\mu<1$. Since $a_k^{\frac{1}{k}}\to\rho$, so for k large enough, $a_k^{\frac{1}{k}}<\mu\Rightarrow a_k{}^k<\mu^k$ for k large enough. Since $\sum\mu^k$ converges (geometric series with $\mu<1$), by Comparison Test, $\sum a_k$ converges.
- Case 2: Similar to Case 1.
- Case 3: Consider $\sum\frac{1}{k}$, which diverges. $\lim_{k\to\infty}\left(\frac{1}{k}\right)^{\frac{1}{k}}=\lim_{k\to\infty}\frac{1}{k^{\frac{1}{k}}}=1$. Now consider $\sum\frac{1}{k^2}$, which converges. $\lim_{k\to\infty}\left(\frac{1}{k^2}\right)^{\frac{1}{k}}=\lim_{k\to\infty}\left(\frac{1}{k^{\frac{1}{k}}}\right)^2=1$. So if $\rho=1$, $\sum a_k$ could either converge or diverge, and thus the test is inconclusive.

Examples

1) Show that $\sum k\left(\frac{2}{3}\right)^k$ converges.

- Compute $\lim_{k\to\infty}\left(k\left(\frac{2}{3}\right)^k\right)^{\frac{1}{k}} = \lim_{k\to\infty} k^{\frac{1}{k}}\left(\frac{2}{3}\right) = \frac{2}{3} < 1$. So by Root Test, $\sum k\left(\frac{2}{3}\right)^k$ converges.

2) Does $\sum \frac{\ln k}{k^2}$ converge or diverge?

- Compute $\lim_{k\to\infty}\left(\frac{\ln k}{k^2}\right)^{\frac{1}{k}} = \lim_{k\to\infty}\frac{(\ln k)^{\frac{1}{k}}}{\left(k^{\frac{1}{k}}\right)^2} = 1$. The Root Test is inconclusive.

Theorem: Ratio Test

Let $\sum a_k$ be a series with positive terms. Suppose $\frac{a_{k+1}}{a_k} \to \lambda$.

1) If $\lambda < 1$, then $\sum a_k$ converges.
2) If $\lambda > 1$, then $\sum a_k$ diverges.
3) If $\lambda = 1$, then the test is inconclusive.

Proof (Case 1):

- Choose μ so that $\lambda < \mu < 1$.
- Since $\frac{a_{k+1}}{a_k} \to \lambda$, for $k \ge N$, $\frac{a_{k+1}}{a_k} < \mu \Rightarrow a_{k+1} < \mu a_k$.
- Because $a_{n+1} < \mu a_n$, $a_{n+2} < \mu a_{n+1} < \mu^2 a_n$, etc., so $\sum_{l=0}^{\infty} a_{N+l} < \sum_{l=0}^{\infty} a_N \mu^l$.
- By Comparison Test, $\sum_{l=0}^{\infty} a_{N+l}$, so $\sum_{k=0}^{\infty} a_k$ converges.

Examples

1) Show that $\sum \frac{1}{k!}$ converges.

- Let $a_k = \frac{1}{k!}$. Compute $\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{k!}{(k+1)!} = \lim_{k\to\infty}\frac{1}{k+1} = 0 < 1$. By Ratio Test, $\sum \frac{1}{k!}$ converges.

2) Show that $\sum \frac{k^k}{k!}$ diverges.

- Compute $\lim_{k\to\infty}\frac{(k+1)^{(k+1)}}{(k+1)!}\times\frac{k!}{k^k} = \lim_{k\to\infty}\frac{(k+1)(k+1)^k}{(k+1)k^k} = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e > 1$. By Ratio Test, $\sum \frac{k^k}{k!}$ diverges.

Lecture #36 – Tuesday, March 23, 2004

ABSOLUTE VS. CONDITIONAL CONVERGENCE

- Consider $\sum a_k$ a series with possibly both positive and negative terms.

Theorem

If $\sum |a_k|$ converges, then $\sum a_k$ converges.

Proof: $-|a_k| \le a_k \le |a_k| \Rightarrow 0 \le a_k + |a_k| \le 2|a_k|$. Because $\sum |a_k|$ converges, $\sum 2|a_k|$ converges. By comparison, $\sum (a_k + |a_k|)$ converges, so $\sum a_k$ converges.

Definition

A series $\sum a_k$ is called <u>absolutely convergent</u> if $\sum |a_k|$ converges.

<u>Warning</u>: Convergent series are not necessarily absolutely convergent!

Example

Consider $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \cdots$.

- This series $\displaystyle\sum \left| \frac{(-1)^k}{k} \right| = \sum \frac{1}{k}$ is the harmonic series and diverges. So it is not absolutely convergent.
- This series is convergent because the positive and negative terms are "cancelling" nicely.

ALTERNATING SERIES

Definition

A series in which consecutive terms always have opposite signs is called an <u>alternating series</u>. (This might seem contrived, but examples abound: Taylor series for $\cos(x)$, $\sin(x)$, $\ln(1+x)$…all are alternating)

Examples and Non-Example

- $\displaystyle\sum \frac{(-1)^k}{\sqrt{k}}$ is an alternating series.
- $\displaystyle 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \cdots$ is not an alternating series.

<u>Notice</u>: Any alternating series is of the form $\sum (-1)^k a_k$ or $\sum (-1)^{k+1} a_k$, where a_k are positive.

<u>Focus</u>: Series that look like $\sum (-1)^k a_k$.

Theorem

Let $\{a_k\}$ be a decreasing sequence of positive numbers. If $a_k \to 0$ as $k \to \infty$, then $\sum (-1)^k a_k$ converges.

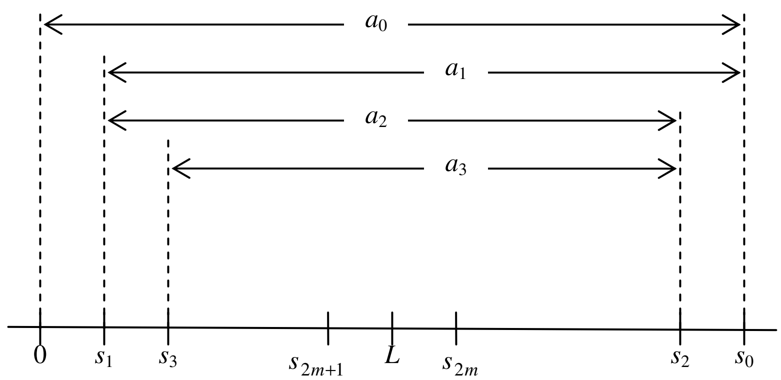
Proof:

- Look at even partial sums: $s_{2m} = (a_0 - a_1) + (a_2 - a_3) + \cdots + (a_{2m-2} + a_{2m-1}) + a_{2m}$.
- Because a_k are decreasing, $a_0 - a_1 > 0$ etc., so $s_{2m} > 0$.
- $s_{2m+2} = s_{2m} - a_{2m+1} + a_{2m+2} = s_{2m} - (a_{2m+1} - a_{2m+2})$. Since $a_{2m+1} > a_{2m+2} \Rightarrow s_{2m+2} < s_{2m}$.
- So the even partial sums are bounded below and monotonic, so they converge, so does the odd partial sums (increasing) by similar argument.
- $s_{2m+1} = s_{2m} - a_{2m+1}$. But $a_{2m+1} \to 0$, so the limit of the odd and even partial sums are equal.

Notice/Warning: This is very different from the case where all the terms in the series are positive.

ESTIMATES USING ALTERNATING SERIES

- In the proof of the above theorem, we've already seen that we can give good estimates for the limit L.



- The sum L of the convergent series $\sum (-1)^k a_k$ lies between two consecutive partial sums. Thus s_n approximates L to within a_{n+1}, i.e. $|s_n - L| < |s_n - s_{n+1}| = a_{n+1}$.
- Can estimate for alternating series.

Example

Estimate $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!}$.

- First, this does converge by theorem because $a_k = \dfrac{1}{(2k+1)!} \to 0$ as $k \to \infty$.
- Suppose you want to estimate the sum to within 0.0005. We need to find a_n so that $a_n < 0.0005$:
  $a_0 = 1$, $a_1 = \dfrac{1}{6}$, $a_2 = \dfrac{1}{120}$, $a_3 = \dfrac{1}{5040} < 0.0005$.
- So an estimate is $s_2 = \dfrac{101}{120}$.

Lecture #37 – Thursday, March 25, 2004

POWER SERIES

- So far, we've been discussing how to tell whether a given series $\sum_{k=0}^{\infty} b_k$ converges.
- What if the b_k's are functions not just of k, but also some other variable x? For example $\sum_{k=0}^{\infty} b_k(x)$.
- Then it makes sense to ask: For what values of x does this series converge?
- Good News: We'll focus on very special functions of x.
- Recall: Taylor series " $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$ ". Notice that the Taylor series is always of the form $\sum_{k=0}^{\infty} a_k x^k$.

Definition

A power series (in x) is a series of the form $\sum_{k=0}^{\infty} a_k x^k$.

Definition

A power series $\sum a_k x^k$ is said to converge:

- at c if $\sum a_k c^k$ converges
- on the set S if $\sum a_k x^k$ converges for any x in S.

Theorem

1) If $\sum a_k x^k$ converges absolutely at $c \neq 0$, then it converges for all x such that $|x| < |c|$.
2) If $\sum a_k x^k$ diverges at $d \neq 0$, then it diverges for all x such that $|x| > |d|$.

Proof (1):

- Since $\sum a_k c^k$, so $a_k c^k \to 0$ as $k \to \infty$.
- For k large enough, $|a_k c^k| < 1$. So $0 < |a_k x^k| = |a_k c^k| \cdot \left|\frac{x^k}{c^k}\right| < \left|\frac{x}{c}\right|^k$ for k large.
- For $|x| < |c|$, $\left|\frac{x}{c}\right| < 1$. So for some μ such that $\left|\frac{x}{c}\right| < \mu < 1$, $|a_k x^k| < \mu^k$ for k large.
- By comparison with geometric series, $\sum a_k x^k$ converges for $|x| < |c|$.

From the Theorem: "Fundamental" Observation about Power Series

There are exactly 3 possibilities for a power series:

1) It converges only at $x = 0$, and nowhere else.
2) It converges absolutely for all x.

3) There is a positive number r such that the series converges absolutely for $|x| < r$, and diverges for $|x| > r$.

Examples

1) $\sum k^k x^k$ converges only at $x = 0$.
2) $\sum \frac{x^k}{k!}$ converges everywhere.
3) $\sum x^k$ converges for $|x| < 1$, diverges for $|x| > 1$.

Definition

The r above is called the radius of convergence for the power series.

Warning: In general, behaviour at $|x| = r$ is unpredictable.

Example

1) $\sum x^k$ converges on $(-1,1)$.
2) $\sum \frac{(-1)^k}{k} x^k$ converges on $(-1,1]$.

Example

Check that $\sum_{k=1}^{\infty} \frac{1}{k^2} x^k$ have interval of convergence $[-1,1]$.

- Since we already know that the series absolutely on some interval , so let's look at $a_k = \sum_{k=1}^{\infty} \left| \frac{1}{k^2} x^k \right| = \sum_{k=1}^{\infty} \frac{1}{k^2} |x|^k$. Where does this converge?
- Apply the Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{1}{(k+1)^2} |x|^{k+1} \times \frac{k^2}{|x|^k} = \left(\frac{k}{k+1} \right)^2 |x| \to |x|$ as $k \to \infty$. By Ratio Test, the radius of convergence is 1.
- Check Endpoints:
  - For $x = 1$, $\sum \frac{1}{k^2}$ converges by Integral Test.
  - For $x = -1$, $\sum \frac{(-1)^k}{k^2}$. This is an alternating series with terms → 0, so it converges.
- Therefore, the interval of converges is $[-1,1]$.

Example

Check that $\sum_{k=1}^{\infty} \frac{1}{k} x^{k-1}$ has interval of converges $[-1,1)$.

- Ratio Test: $\frac{|x|^k}{k+1} \times \frac{k}{|x|^{k-1}} = \left(\frac{k}{k+1} \right) |x| \to |x|$ as $k \to \infty$. So the radius of convergence is 1.

- Endpoints:
  - For $x=1$, we have $\sum \frac{1}{k}$ the harmonic series, so it diverges.
  - For $x=-1$, $\sum \frac{(-1)^{k-1}}{k}$, we have an alternating series with terms → 0, so it converges.
- Therefore, the interval of convergence is $[-1,1)$.

Example

Find the interval of convergence for $\sum \frac{k}{6^k} x^k$.

- Again, look at $\sum \left|\frac{k}{6^k} x^k\right| = \sum \frac{k}{6^k} |x|^k$.
- Root Test: Compute the limit $\lim_{k\to\infty} \sqrt[k]{\frac{k}{6^k}|x|^k} = \lim_{k\to\infty}\left(\frac{k}{6^k}|x|^k\right)^{\frac{1}{k}} = \lim_{k\to\infty}\frac{k^{\frac{1}{k}}}{6}|x| = \frac{|x|}{6}$. So the radius of convergence is 6.
- Endpoints:
  - For $x=6$, we have $\sum k$, which diverges.
  - For $x=-6$, we have $\sum \frac{k}{6^k}(-6)^k = \sum (-1)^k k$, which diverges.
- Therefore, the interval of converge is $(-6,6)$.

Examples: Taylor Series

Important Taylor Series:

1) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ converges for all *x*.
2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}\cdots = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$ converges for all *x*.
3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\cdots = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}$ converges for all *x*.
4) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}\cdots = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k}x^k$ converges on $(-1,1]$.

Warnings

When we write "$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$ for $x \in S$", we are saying <u>several</u> things:

1) The RHS converges for $x \in S$.
2) The RHS has a limit (by (1)), and is equal to LHS.

Lecture #38 – Tuesday, March 30, 2004

POWER SERIES AND CALCULUS

- Mix our knowledge of power series with calculus (differentiation, integration, etc.).
- Motto: Dealing with power series is the "next best thing" to dealing with polynomials.
- Remember: A polynomial looks like $p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$. A power series looks like $f(x)=a_0+a_1x+a_2x^2+\cdots=\sum_{k=0}^{\infty}a_kx^k$.
- Recall: When we differentiate polynomials, we do it "term-by-term" $p'(x)=\frac{d}{dx}\left[a_0+a_1x+a_2x^2+\cdots+a_nx^n\right]=a_1+2a_2x+\cdots+na_nx^{n-1}$. Same thing when we integrate $\int p(x)dx\int=a_0+a_1x+a_2x^2+\cdots+a_nx^n dx=C+a_0x+\frac{a_1}{2}x^2+\frac{a_2}{3}x^3+\cdots+\frac{a_n}{n+1}x^{n+1}$.
- Our Goal: "Basically", we can do the same for power series: "$\frac{d^n}{dx^n}\left[\sum_{k=0}^{\infty}a_kx^k\right]=\sum_{k=0}^{\infty}\frac{d^n}{dx^n}\left[a_kx^k\right]$" and "$\int\left(\sum_{k=0}^{\infty}a_kx^k\right)=C+\sum_{k=0}^{\infty}\frac{a_k}{k+1}x^{k+1}$".
- Warning: don't take this at face value yet – there's still a lot to check.
- Advertisement: This will be immensely powerful for calculus. If we know $f(x)=\sum_{k=0}^{\infty}a_kx^k$, the we automatically know a lot about all the derivatives $f^{(n)}(x)$ and all the integrals.

DERIVATIVES OF A POWER SERIES

First, let's address the convergence issue.

Theorem

Suppose $\sum_{k=0}^{\infty}a_kx^k=a_0+a_1x+a_2x^2+\cdots$ converges on an interval $(-c,c)$. Then $\sum_{k=0}^{\infty}\frac{d}{dx}\left[a_kx^k\right]=\sum_{k=0}^{\infty}ka_kx^{k-1}=a_1+2a_2x+3a_3x^2+\cdots$ also converges on $(-c,c)$.

- Idea: $\sum_{k=0}^{\infty}ka_kx^{k-1}$ is our "candidate" for $\frac{d}{dx}\left[\sum_{k=0}^{\infty}a_kx^k\right]$.

- Warning: The theorem <u>does not</u> say that $\sum_{k=0}^{\infty}\frac{d}{dx}\left[a_k x^k\right]$ converges if and only if $\sum_{k=0}^{\infty} a_k x^k$ converges. The intervals of convergence might not be the same.

Examples

1) $\sum_{k=1}^{\infty}\frac{1}{k^2}x^k$ has interval of convergence $[-1,1]$, but $\sum_{k=1}^{\infty}\frac{1}{k}x^{k-1}=\sum_{k=1}^{\infty}\frac{d}{dx}\left[\frac{1}{k^2}x^k\right]$ has interval of convergence $[-1,1)$.
2) $\sum_{k=0}^{\infty}x^k$ converges on $(-1,1)$, and $\sum_{k=1}^{\infty}kx^{k-1}$, $\sum_{k=2}^{\infty}k(k-1)x^{k-2}$, $\sum_{k=3}^{\infty}k(k-1)(k-2)x^{k-3}$, etc., all converge on $(-1,1)$.

Going back to our goal, if $\sum_{k=0}^{\infty}a_k x^k$ converges on $(-c,c)$, we can define $f(x)=\sum_{k=0}^{\infty}a_k x^k$ on $(-c,c)$.

Since $\sum_{k=0}^{\infty}\frac{d}{dx}\left[a_k x^k\right]$ also converges on $(-c,c)$, we can also define $g(x)=\sum_{k=0}^{\infty}\frac{d}{dx}\left[a_k x^k\right]$ on $(-c,c)$.

<u>The point</u>: $g(x)$ is our "candidate" for $f'(x)$.
<u>Good news</u>: It's true!

Theorem

If $f(x)=\sum_{k=0}^{\infty}a_k x^k$ for all $x\in(-c,c)$, then $f(x)$ is differentiable, and $f'(x)=\sum_{k=0}^{\infty}\frac{d}{dx}\left[a_k x^k\right]=g(x)$ for all $x\in(-c,c)$.

- <u>Mantra</u>: "Can take derivatives term-by-term".
- <u>Note</u>: We have something actually better. The derivative of a power series is again a power series converging again on $(-c,c)$, so we can keep going. By repeating, we get $f''(x)=\sum_{k=0}^{\infty}\frac{d^2}{dx^2}\left[a_k x^k\right]=\sum_{k=2}^{\infty}k(k-1)x^{k-2}$, …, $f^{(n)}(x)=\sum_{k=0}^{\infty}\frac{d^n}{dx^n}\left[a_k x^k\right]$. So a power series is infinitely differentiable on the interior of its interval of convergence.

Lecture #39 – Thursday, April 1, 2004

Examples

We can now see explicit and rigorous calculations that we used to motivate Taylor Series:

1) $\frac{d}{dx}\left[e^x\right]=\frac{d}{dx}\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right]=\frac{d}{dx}[1]+\frac{d}{dx}[x]+\cdots+\frac{d}{dx}\left[x^2\right]+\cdots=0+1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=e^x$.

2) $\frac{d}{dx}[\sin x]=\frac{d}{dx}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right]=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots=\cos x$.

3) Here's something we didn't already know. We know that $\sum_{k=1}^{\infty} \frac{x^k}{k!}$ converges on $(-1,1)$. Define $g(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$ on $(-1,1)$. Then $g'(x) = \sum_{k=1}^{\infty} x^{k-1} = \sum_{l=0}^{\infty} x^l = \frac{1}{1-x}$. This means $g(x) = -\ln(1-x) + C = \ln\left(\frac{1}{1-x}\right) + C = \ln\left(\frac{1}{1-x}\right)$. We've reason backwards from our usual strategy! So $\ln\left(\frac{1}{1-x}\right) = \sum_{k=1}^{\infty} \frac{x^k}{k}$ on $(-1,1)$.

RELATIONSHIP BETWEEN POWER SERIES AND TAYLOR SERIES

Motto: On its interval of convergence, a power series is the Taylor Series of its sum.

How do we know that $\sum_{k=1}^{\infty} \frac{x^k}{k}$ is the Taylor series expansion of $\ln\left(\frac{1}{1-x}\right)$? Here's why:

- Suppose $\sum_{k=0}^{\infty} a_k x^k$ converges on $(-c,c)$. Define $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on $(-c,c)$.
- What's the Taylor series for $f(x)$?
- Compute $\frac{f^{(n)}(0)}{n!}$: $f^{(n)}(x) = \sum_{k=0}^{\infty} \frac{d^n}{dx^n}\left[a_k x^k\right] = \frac{d^n}{dx^n}[a_0] + \frac{d^n}{dx^n}[a_1 x] + \cdots$. So $\frac{f^{(n)}(0)}{n!} = \frac{a_n (n)(n-1)\cdots(2)(1)}{n!} = \frac{a_n n!}{n!} = a_n$.
- So the Taylor series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} a_n x^n$. So the power series is the Taylor series.
- So in particular, $\ln\left(\frac{1}{1-x}\right) = \sum_{k=1}^{\infty} \frac{x^k}{k}$ is the Taylor series.

INTEGRATION

Again, the idea is to follow what we know from polynomials: $\int \left(a_0 + a_1 x + \cdots + a_n x^n\right) dx = \int a_0 + \int a_1 x + \cdots + \int a_n x^n = C + a_0 x + \frac{a_1}{2} x^2 + \cdots + \frac{a_n}{n+1} x^{n+1}$. So can integrate term-by-term!

Theorem: "Term-by-term integration of power series"

If $f(x) = \sum_{k+0}^{\infty} a_k x^k$ converges on $(-c,c)$, then $g(x) = \sum_{k+0}^{\infty} \frac{a_k}{k+1} x^{k+1}$ also converges on $(-c,c)$ and $\int f(x) dx = g(x) + C$.

Sketch of proof:

- If $\sum_{k+0}^{\infty} a_k x^k$ converges on $(-c,c)$, then we want to show $\sum_{k+0}^{\infty} \frac{a_k}{k+1} x^{k+1}$ also converges on $(-c,c)$.
- If $f(x)=\sum_{k+0}^{\infty} a_k x^k$ and $g(x)=\sum_{k+0}^{\infty} \frac{a_k}{k+1} x^{k+1}$, term-by-term differentiation gives $g'(x)=f(x)$.
- By FTC, $g(x)=\int f(x)dx+C$.

Notation

Shorthand notation: $\int\left(\sum_{k=0}^{\infty} a_k x^k\right)dx=\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}+C$.

Warning: Endpoints have to be checked separately.

Example

Yet another way to get Taylor series. We already know $\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{k=0}^{\infty}(-x)^k=\sum_{k=0}^{\infty}(-1)^k(x)^k, |x|<1$.

Integrating both sides, we obtain $\ln(1+x)=\int\left(\sum_{k=0}^{\infty}(-1)^k x^k\right)dx=C+\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}x^{k+1}$. Since $\ln(1)=0$, $C=0$. So $\ln(1+x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}x^{k+1}, |x|<1$ is the Taylor series expansion of $\ln(1+x)$ on $(-1,1)$.

Example

Something we didn't already know. Consider $\frac{1}{1+x^2}=\sum_{k=0}^{\infty}(-1)^k x^{2k}, |x|<1$. Integrating both sides, we obtain $\int\frac{1}{1+x^2}=\int\sum_{k=0}^{\infty}(-1)^k x^{2k}\Rightarrow \arctan x=C+\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1}=\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1}$. So $\arctan x=x-\frac{1}{3}x^3+\frac{1}{5}x^5$.

Note on Definite Integrals

- $\int_c^d\left(\sum_{k=0}^{\infty}a_k x^k\right)dx=\sum_{k=0}^{\infty}\int_c^d a_k x^k\,dx=\sum_{k=0}^{\infty}\frac{a_k}{k+1}\left(d^{k+1}-c^{k+1}\right)$.
- Be careful: c, d should be in the interval of convergence.

Estimates of Integrals

Example

Estimate $\int_0^1 e^{x^2}dx$.

- Recall: $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$. We have an estimate for $R_n(x) \le \max\left(\left|f^{(n+1)}(x)\right| \cdot \frac{|x|^{n+1}}{(n+1)!}\right)$.
- For us, $f(x) = e^x$, so $f^{(n+!)}(t) = e^t$. So $\max\left(\left|e^t\right|\right) = e^x \le e^1 < 3$, and $|x|^{n+1} \le 1$. So $R_n(x) \le \frac{3}{(n+1)!}$.
- So $0 \le e^x - \left[1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right] \le \frac{3}{(n+1)!}$.
- We want e^{x^2}. But if $0 \le x \le 1$, then $0 \le x^2 \le 1$. So $0 \le e^{x^2} - \left[1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{2n}}{n!}\right] \le \frac{3}{(n+1)!}$, so $0 \le \int_0^1 e^{x^2}\,dx - \int_0^1 \left(1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{2n}}{n!}\right)dx \le \int_0^1 \frac{3}{(n+1)!}\,dx$.
- At $n = 6$, $\frac{3}{7!} = \frac{1}{1680} < 0.0006$, so $\int_0^1 e^{x^2}\,dx \approx 1 + \frac{1}{3} + \frac{1}{5 \cdot 2!} + \cdots + \frac{1}{13 \cdot 6!}$ up to within 0.0006. This means that $1.4626 \le \int_0^1 e^{x^2}\,dx \le 1.4632$.

Lecture #40 – Tuesday, April 6, 2004

Binomial Series

First, recall the usual Binomial Theorem:

$$(1+x)^n = (1+x)\cdots(1+x) = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3 \cdot 2}x^2 + \cdots + nx^{n-1} + x^n.$$

In general, the coefficient in front of x^k is $\binom{n}{k} = \frac{n(n-1)\cdots(n-(k-1))}{k(k-1)\cdots(2)(1)} = \frac{n!}{(n-k)!\,k!}$. Reminder: $0! = 1$.

Example

$$(1+x)^0 = 1$$
$$(1+x)^1 = 1 + x$$
$$(1+x)^2 = 1 + 2x + x^2$$
$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$
$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

In general, $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ the binomial/Taylor expansion for $(1+x)^n$.

Theorem

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Back to calculus: We know that we can define $(1+x)^{\alpha} := e^{\alpha\ln(1+x)}, 1+x>0$. Can we derive a Taylor expansion like the binomial expansion?

Recall: For $f_n(x)=(1+x)^n$, $\frac{f_n^{\,k}(0)}{k!}=\binom{n}{k}=\frac{n(n-1)\cdots(n-(k-1))}{k(k-1)\cdots(2)(1)}$.

What we'll find: Define $f_{\alpha}(x):=(1+x)^{\alpha}$, then $\frac{f_{\alpha}(x)}{k!}=\frac{\alpha(\alpha-1)\cdots(2)(1)}{k(k-1)\cdots(2)(1)}$.

Warning: This is no longer an integer.

Example

$$\binom{\frac{3}{2}}{3}=\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{3\cdot 2\cdot 1}=-\frac{1}{16}.$$

Theorem

$$(1+x)^{\alpha}=\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k, \forall x\in(-1,1).$$

Sketch of Proof:

- First of all, we want to show that RHS converges. This is just a ratio test.
- Secondly, let $\phi(x)=\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$. A calculation shows that $\phi(x)$ has the property $\frac{\phi'(x)}{\phi(x)}=\frac{\alpha}{1+x}$. Integrating both sides, we obtain $\ln\phi(x)=\alpha\ln(1+x)+C \Leftrightarrow \phi(x)=e^{\alpha\ln(1+x)}\cdot e^{C}=e^{\alpha\ln(1+x)}$.

Examples

1) Expand $\sqrt[4]{1-x}$ in powers of x, up to x^4.
   - $$\begin{aligned}\sqrt[4]{1-x}&=(1+(-x))^{\frac{1}{4}}=1+\frac{1}{4}(-x)+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2\cdot 1}(-x)^2+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3\cdot 2\cdot 1}(-x)^3\\&+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\left(\frac{1}{4}-3\right)}{4\cdot 3\cdot 2\cdot 1}(-x)^4+R_4(x)\end{aligned}.$$

2) Estimate $\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{1-x^3}}$ to within 0.001.
   - $\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{1-x^3}}=\int_0^{\frac{1}{2}}\left(1+\left(-x^3\right)\right)^{\frac{1}{2}}dx$. Because the limits of integration are in the interval of convergence, can expand:
     $$\int_0^{\frac{1}{2}}\left(1+\left(-x^3\right)\right)^{\frac{1}{2}}dx=\int_0^{\frac{1}{2}}\left(1-\frac{1}{2}\left(-x^3\right)+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2\cdot 1}x^6+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3\cdot 2\cdot 1}(-x)^9+R_3(x)\right)dx.$$