Lecture #1 – Tuesday, January 6, 2004

1.1 SOLUTIONS AND ELEMENTARY OPERATIONS

Example

Create a diet from fish and meal that contains 193g of proteins and 83g of carbohydrate. We know that fish contains 70% protein and 10% carbohydrate, and meal contains 30% protein and 60% carbohydrate.

- Assume that the diet contains xg of fish and yg of meal, we obtain $\begin{cases} 0.7x + 0.3y = 193 \\ 0.1x + 0.6y = 83 \end{cases}$.

Definition

A <u>linear equation</u> is an equation of the form $a_1 x_1 + ... + a_n x_n = b$ where:

- $x_1, \ldots, x_n$ are variables;
- $a_1, \ldots, a_n$ are real numbers called coefficients;
- b is the constant term.

Examples

1) $ax + by = c$.
2) $3x_1 + 2x_2 - x_3 = 0$.
3) $2{x_1}^2 + x_2 = 1$ – not a linear equation.

Definition

A finite collection of linear equations in the variables $x_1, \ldots, x_n$ is called a <u>system of linear equations</u>.

Examples

1) $\begin{cases} x_1 = 1 \\ x_2 = 3 \end{cases}$.
2) $\begin{cases} 2x_1 + 4x_2 = 14 \\ 3x_1 - x_2 = 0 \end{cases}$

Definition

Given a linear equation $a_1 x_1 + ... + a_n x_n = b$, a sequence of n real numbers $s_1, \ldots, s_n$ is called a solution to the linear equation if $a_1 s_1 + ... + a_n s_n = b$. Similarly, this also applies to a given system of linear equations.

Example

Given $(S_1)\begin{cases} x_1 = 1 \\ x_2 = 3 \end{cases}$ and $(S_2)\begin{cases} 2x_1 + 4x_2 = 14 \\ 3x_1 - x_2 = 0 \end{cases}$, what is the solution to (S_1) and (S_2)?

- (1, 3) is a solution of (S_1).
- (1, 3) is also a solution of (S_2) because $2(1) + 4(3) = 14$ and $3(1) - (3) = 0$.

Example

- $(S_3)\begin{cases} x+y=1 \\ x+y=2 \end{cases}$ has no solution.

Example

Prove that for any s, t in **R**, $\begin{cases} s_1 = \frac{3}{2}+s-t \\ s_2 = s \\ s_3 = t \\ s_4 = \frac{1}{2}-2s \end{cases}$ is a solution to the system $\begin{cases} x_1 - 3x_2 + x_3 - x_4 = 1 \\ x_1 + x_2 + x_3 + x_4 = 2 \end{cases}$.

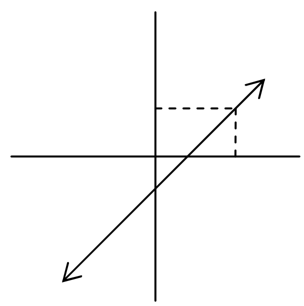
- $s_1 - 3s_2 + s_3 - s_4 = \frac{3}{2}+s-t-3s+t-\left(\frac{1}{2}-2s\right) = \frac{3}{2}-\frac{1}{2}+s-3s+s-t+t = 1$.
- $s_1 + s_2 + s_3 + s_4 = \frac{3}{2}+s-t+s+t+\frac{1}{2}-2s = \frac{3}{2}+\frac{1}{2}+s+s-2s-t+t = 2$.

Definition

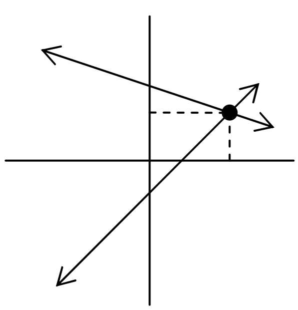
s, t are called <u>parameters</u>. $s_1,\ldots, s_4$ described this way is said to be given in <u>parametric form</u> and is called the <u>general solution</u> of the system.

Remarks

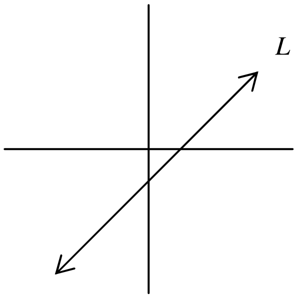
- When only 2 variables are involved, solution to systems of linear equations can be described geometrically because a linear equation $L: ax+by=c$ is a straight line if a, b are not both 0. $P(s_1, s_2)$ is in L if it is a solution of $ax+by=c$.



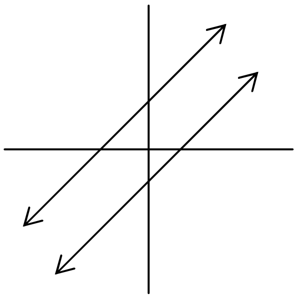
- If there are two linear equations, $L_1: ax+by=c$ and $L_2: dx+ey=f$, then the solution to the system $(S)\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$ is the intersection of L_1 and L_2.



- The solution of (S) is (s_1, s_2) where $P(s_1, s_2) = L_1 \cap L_2$.



- The solution of (S) are given by the (s_1, s_2) such that $as_1 + bs_2 = c$ (and this implies that $ds_1 + es_2 = f$).



- (S) has no solution.

Definition

The <u>elementary operations</u> are:

1) Interchange 2 equations.
2) Multiply one equation by a <u>non-zero</u> number.
3) Add a multiple of one equation to a different equation.

Definition

Two systems of linear equations are said to be equivalent iff the solutions of the systems are the same.

Theorem

Suppose that an elementary operation is performed on a system of linear equations, then the resulting system is equivalent to the original one.

Example

Solve: $\begin{cases} x + 2y = 1 \\ 3x - y = 4 \end{cases}$.

- $$\begin{cases} x+2y=1 & R_1 \\ 3x-y=4 & R_2 \end{cases} \Leftrightarrow \begin{cases} x+2y=1 & R_1 \\ 6x-2y=8 & R_2' = 2R_2 \end{cases} \Leftrightarrow \begin{cases} x+2y=1 & R_1 \\ 7x=9 & R_2'' = R_1 + R_2' \end{cases} \Leftrightarrow \begin{cases} 7x=9 & R_1' = R_2'' \\ x+2y=1 & R_2''' = R_1 \end{cases}$$
$$\Leftrightarrow \begin{cases} x = \dfrac{9}{7} \\ 2y = 1 - \dfrac{9}{7} = -\dfrac{2}{7} \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{9}{7} \\ y = -\dfrac{1}{7} \end{cases}$$
- The solution of the system is $\left(\dfrac{9}{7}, -\dfrac{1}{7}\right)$.

Definition

Consider the specific system $\begin{cases} 3x_1+3x_2-x_3-x_4=2 \\ x_1-x_2+3x_3=4 \\ x_1+2x_2+x_3+4x_4=5 \end{cases}$ of 3 linear equations in 4 variables. We define the array of numbers associated to this system as $\left[\begin{array}{cccc|c} 3 & 3 & -1 & -1 & 2 \\ 1 & -1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 4 & 5 \end{array}\right]$. This is called the augmented matrix associated to the system.

Example

Write the augmented matrix of the system $\begin{cases} x+2y=1 \\ 3x-y=4 \end{cases}$.

- The corresponding augmented matrix is $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & -1 & 4 \end{array}\right]$.

Example

Solve $\begin{cases} x+2y=1 \\ 3x-y=4 \end{cases}$ using augmented matrix.

- $$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & -1 & 4 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \end{array} \Leftrightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 6 & -2 & 8 \end{array}\right]\begin{array}{l} R_1 \\ R_2'=2R_2 \end{array} \Leftrightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 7 & 0 & 9 \end{array}\right]\begin{array}{l} R_1 \\ R_2''=R_1+R_2'' \end{array} \Leftrightarrow \left[\begin{array}{cc|c} 7 & 0 & 9 \\ 1 & 2 & 1 \end{array}\right]$$
  $$\Leftrightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{9}{7} \\ 1 & 2 & 1 \end{array}\right]\begin{array}{l} R_1'=\frac{1}{7}R_1 \\ R_2'' \end{array} \Leftrightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{9}{7} \\ 0 & 2 & -\frac{2}{7} \end{array}\right]\begin{array}{l} R_1' \\ R_2'''=R_2''-R_1' \end{array} \Leftrightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{9}{7} \\ 0 & 1 & -\frac{1}{7} \end{array}\right]\begin{array}{l} R_1' \\ R_2''''=\frac{1}{2}R_2''' \end{array}$$
- So $x=\frac{9}{7}$, $y=-\frac{1}{7}$.

Definition

The elementary operations for the augmented matrix are:
1) Interchange 2 rows.
2) Multiply one row by a non-zero number.
3) Add a multiple of a row to a different row.

Example

Solve $\begin{cases} 2x_1+x_2-x_3=0 \\ x_1+x_2+2x_3=1 \end{cases}$ using augmented matrix.

- $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 1 & 2 & 1 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 2 & 2 & 4 & 2 \end{array}\right]\times 2 \Leftrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & 5 & 2 \end{array}\right]$.
- $\begin{cases} 2x_1+x_2-x_3=0 \\ x_2+5x_3=2 \end{cases}$.
- $x_3=s$, $s\in \mathbf{R}$.

- $x_2 = 2 - 5s$.
- $2x_1 = -x_2 + x_3 = -(2 - 5s) + s = -2 + 6s \Rightarrow x_1 = -1 + 3s$.
- So, $\begin{cases} x_1 = 3x - 1 \\ x_2 = 2 - 5s \\ x_3 = s \end{cases}, s \in \mathbf{R}$.

1.2 Gaussian Elimination

Definition

A matrix is said to be in <u>row echelon form</u> (and will be a <u>row echelon matrix</u>) if it satisfies the three conditions:

1) All zero rows are at the bottom.
2) The first non-zero entry from the left is 1 and is called a "leading 1" for that row.
3) Each "leading 1" is to the right of all the "leading 1's" in the row above.

Moreover, a row echelon matrix is said to be in <u>reduced row echelon form</u> if it also satisfy the last condition:

4) Each "leading 1" is the only non-zero entry of its column.

Examples

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- This matrix is in row echelon form.

$$\begin{bmatrix} 0 & 1 & 0 & * & 0 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- This matrix is in reduced row echelon form.

Theorem

Every matrix can be brought to row (reduced) echelon form by a series of elementary row operations.

Examples

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Row echelon matrix.
- Reduced row echelon matrix.

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

- Not a row echelon matrix.

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Row echelon matrix.
- Not a reduced row echelon matrix.

Lecture #2 – Thursday, January 8, 2004

Definition

Let (S) be a system of linear equations and M be the corresponding augmented matrix when M has been brought to the reduced row echelon form, variables corresponding to the column containing a leading 1 are called leading variables.

- Remark: The non-leading variables end up as parameters in the final solution.

Example

Solve $\begin{cases} x+2y-z=2 \\ 2x+5y+2z=-1 \\ x+3y+3z=-3 \end{cases}$.

- $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 5 & 2 & -1 \\ 1 & 3 & 3 & -3 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 4 & -5 \\ 0 & 1 & 4 & -5 \end{array}\right)\begin{array}{l} R_1 \\ R_2-2R_1 \\ R_3-R_1 \end{array} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_2-R_3 \end{array}$.
- $\begin{cases} x+2y-z=2 \\ y+4z=-5 \\ 0=0 \end{cases} \Leftrightarrow \begin{cases} x=12+9s \\ y=-5-4s, s\in \mathbf{R} \\ z=s \end{cases}$.

Example

Solve $\begin{cases} x+2y-z=2 \\ 2x+5y+2z=-1 \\ x+3y+3z=2 \end{cases}$.

- $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 5 & 2 & -1 \\ 1 & 3 & 3 & 2 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 4 & -5 \\ 0 & 1 & 4 & 0 \end{array}\right)\begin{array}{l} R_1 \\ R_2-2R_1 \\ R_3-R_1 \end{array} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 5 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_2-R_3 \end{array} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 5 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ \frac{1}{5}R_2 \end{array}$
- $\begin{cases} x+2y-z=2 \\ y+4z=-5 \\ 0=1 \end{cases}$. The system has no solution

Gaussian Algorithm

Let M be an augmented matrix of a system.

1) If $M=0$ (i.e. all entries are 0), stop.
2) Otherwise, find the first column from the left containing a non-zero entry (call it a), and move the row containing that entry to the top position.
3) Now multiply that row by $\frac{1}{a}\neq 0$ to create a leading 1.
4) By subtracting multiples of that row from rows below, make each entry below the leading 1 zero.
5) Repeat steps 1 to 4 on the matrix consisting of the remaining rows.

Example

Bring the following matrix to row echelon form using the Gaussian Algorithm:

$$\begin{pmatrix}0&0&3&1&4\\0&2&-4&2&0\\0&1&-2&1&0\\0&4&-9&0&3\end{pmatrix}\to\begin{pmatrix}0&2&-4&2&0\\0&0&3&1&4\\0&1&-2&1&0\\0&4&-9&0&3\end{pmatrix}\begin{matrix}R_2\\R_1\\ \\ \\\end{matrix}\to\begin{pmatrix}0&1&-2&1&0\\0&0&3&1&4\\0&1&-2&1&0\\0&4&-9&0&3\end{pmatrix}\begin{matrix}\frac{1}{2}R_1\\ \\ \\ \\\end{matrix}\to\begin{pmatrix}0&1&-2&1&0\\ \hline 0&0&3&1&4\\0&0&0&0&0\\0&0&-1&-4&3\end{pmatrix}\begin{matrix}\frac{1}{2}R_1\\ \\R_1-R_3\\R_4-R_1\end{matrix}$$

$$\to\begin{pmatrix}0&1&-2&1&0\\ \hline 0&0&1&\frac{1}{3}&\frac{4}{3}\\0&0&-1&-4&3\\0&0&0&0&0\end{pmatrix}\begin{matrix}\\ \frac{1}{3}R_2\\R_4\\R_3\end{matrix}\to\begin{pmatrix}0&1&-2&1&0\\ \hline 0&0&1&\frac{1}{3}&\frac{4}{3}\\ \hline 0&0&0&-\frac{11}{3}&\frac{13}{3}\\0&0&0&0&0\end{pmatrix}\begin{matrix}\\ \\R_2+R_3\\ \\\end{matrix}$$

$$\to\begin{pmatrix}0&1&-2&1&0\\0&0&1&\frac{1}{3}&\frac{4}{3}\\0&0&0&1&-\frac{13}{11}\\0&0&0&0&0\end{pmatrix}\begin{matrix}\\ \\-\frac{3}{11}R_3\\ \\\end{matrix}$$

Definition

A system of linear equations that has no solution is called <u>inconsistent</u>. A system that has solution is <u>consistent</u>.

Example

Consider $(S)\begin{cases}x+2y+z=a\\2x-y+3z=b\\5y-z=c\end{cases}$. Find condition on (a, b, c) so that (S) is consistent.

- $\left(\begin{array}{ccc|c}1&2&1&a\\2&-1&3&b\\0&5&-1&c\end{array}\right)\Leftrightarrow\left(\begin{array}{ccc|c}1&2&1&a\\0&-5&1&b-2a\\0&5&-1&c\end{array}\right)\Leftrightarrow\left(\begin{array}{ccc|c}1&2&1&a\\0&1&-\frac{1}{5}&\frac{2a-b}{5}\\0&1&-\frac{1}{5}&\frac{c}{5}\end{array}\right)\Leftrightarrow\left(\begin{array}{ccc|c}1&2&1&a\\0&1&-\frac{1}{5}&\frac{2a-b}{5}\\0&0&0&\frac{2a-b-c}{5}\end{array}\right)$.
- $\begin{cases}x+2y+z=a\\ y-\dfrac{1}{5}z=\dfrac{2a-b}{5}\\ 0=\dfrac{2a-b-c}{5}\end{cases}$.

Definition

It can be proven the reduced row echelon form of matrix A is uniquely determined by A. The number r of leading 1 is called the <u>rank</u> of A ($r=\operatorname{rank} A$).

Example

Determine to rank of the matrix: $\begin{pmatrix}1&2&3&4\\2&1&-1&0\\3&3&2&4\end{pmatrix}$.

- Applying the Gaussian Algorithm, we obtain this row echelon matrix: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \frac{7}{3} & \frac{8}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$. So $\text{rank } A = 2$.

- Remark: The number of leading 1's is the same for the reduced row echelon matrix and the row echelon matrix.

Theorem

Suppose a system of m equations and n variables has a solution. If the rank of the corresponding row echelon matrix is r, then the set of solution involves exactly $n - r$ parameters.

1.3 System of Homogeneous Equations

Definition

A system of linear equations is homogeneous if all the constant terms are 0 ($a_1x_1 + a_2x_2 + ... + a_nx_n = 0$), (0, 0, …, 0) is a solution of homogenous system. It is called the trivial solution. A non-trivial solution is any other solution.

Theorem

If a homogenous system of linear equations has more variables than equations, then it has a non-trivial solution.

Lecture #3 – Tuesday, January 13, 2004

2.1 Matrix Addition, Scalar Multiplication, and Transposition

Definition

A rectangular array of number is called a matrix, and the numbers are the entries. A $m \times n$ matrix is a matrix with m rows and n columns.
A $1 \times n$ matrix is a row matrix.
A $m \times 1$ matrix is a column matrix.

Examples

1) $A = \begin{pmatrix} 1 & 2 & 4 \\ 5 & -1 & 0 \end{pmatrix}$ is a 2×3 matrix.
2) $\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$ is a 2×2 matrix.
3) $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ is a column matrix with size 3×1.

Definition

The (i, j)-entry of a matrix is the number lying simultaneously in row i and column j.

Notation

Let A be a $m\times n$ matrix. We denote a_{ij} the (i, j)-entry of A so that $A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$, $A=(i, j)$.

Example

If $A=\begin{pmatrix} 1 & 4 & 5 & 0 \\ 0 & 2 & -1 & 2 \end{pmatrix}$, the (1, 2)-entry of A is 4, and the (2, 4)-entry of A is -2.

Definition

A matrix $n\times n$ is called a square matrix.
Let $A=(a_{ij})$ be a $n\times n$ matrix. The entries $a_{11}, a_{22},\dots, a_{nn}$ are said to lie on the main diagonal of A.

Definition

Let A, B be two matrices $A=(a_{ij})$, $b=(b_{ij})$. $A=B$ iff:

- A and B have the same size.
- Their corresponding entries are equal: $a_{ij}=b_{ij}$.

Example

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B=\begin{pmatrix} 1 & 4 & 5 \\ 0 & -1 & 2 \end{pmatrix}$, $C=\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$. Discuss the possibility that $A=B$, $A=C$.

- $A=B$ is impossible: A is a 2×2 matrix, but B is a 2×3 matrix.
- $A=C$ iff $a=1$, $b=2$, $c=-1$, $d=-2$.

Definition

Let $A=(a_{ij})$ and $b=(b_{ij})$, two $m\times n$ matrices. We define $C=A+B=(a_{ij}+b_{ij})$ and $D=A-B=(a_{ij}-b_{ij})$, where C, D are $m\times n$ matrices.

- Remark: Addition is not defined for matrices of different sizes.

Example

If $A=\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, $B=\begin{pmatrix} 1 & -1 \\ 4 & -2 \end{pmatrix}$, then $A+B=\begin{pmatrix} 1+1 & 2-1 \\ 0+4 & 3-2 \end{pmatrix}=\begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix}$.

Example

Solve $\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}+X=\begin{pmatrix} -1 & 3 \\ -2 & -4 \end{pmatrix}$

- X must be a 2×2 matrix.

- Write $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We want $\begin{pmatrix} -1 & 3 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+1 & b+2 \\ c-3 & d+1 \end{pmatrix}$
- So $\begin{cases} -1 = a+1 \\ 3 = 2+b \\ -2 = -3+c \\ -4 = 1+d \end{cases} \Leftrightarrow \begin{cases} a = -2 \\ b = 1 \\ c = 1 \\ d = -5 \end{cases} \rightarrow X = \begin{pmatrix} -2 & 1 \\ 1 & -5 \end{pmatrix}$.
- Similarly, we could have written $X = \begin{pmatrix} -1 & 3 \\ -2 & -4 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -5 \end{pmatrix}$.

Definition

Let k be a number and $A = (a_{ij})$ a matrix. Then kA is the matrix $B = (ka_{ij})$.

Example

Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$. Compute $\frac{3}{2}A$.

- $\frac{3}{2}A = \begin{pmatrix} \frac{3}{2}(2) & \frac{3}{2}(-1) \\ \frac{3}{2}(3) & \frac{3}{2}(1) \end{pmatrix} = \begin{pmatrix} 3 & -\frac{3}{2} \\ \frac{9}{2} & \frac{3}{2} \end{pmatrix}$.

Example

If $kA = 0$, show that either $k = 0$ or $A = (0) = 0$.

- Suppose $A = (a_{ij})$, then $kA = (ka_{ij})$. So then $kA = 0 \Leftrightarrow ka_{ij} = 0$ for all i, j.
- If $k = 0$, there is nothing to do.
- If $k \neq 0$, then $ka_{ij} = 0 \Rightarrow a_{ij} = 0$ for all i, j. So $A = (a_{ij}) = 0$.

Theorem

Let A, B, C be $m\times n$ matrices. Let $k, p \in \mathbf{R}$. Then:

1) $A + B = B + A$ – commutative.
2) $A + (B + C) = (A + B) + C$ – associative.
3) $0 + A = A$.
4) $A + (-A) = 0$.
5) $k(A + B) = kA + kB$.
6) $(k + p)A = kA + pA$.
7) $(kp)A = k(pA)$.
8) $1 \times A = A$.

Example

Let A, B, C, be $m\times n$ matrices. Simplify $6(A + 3B) - 2(C - B) + 4[2(2A - B + C) + 4(B + 2C)]$.

- We calculate:
$$\begin{aligned}&6A+18B-2C+2B+4(4A-2B+2C+4B+8C)\\&=6A+20B-2C+16A+8B+40C\\&=22A+28B+38c\end{aligned}.$$

Example

Find X, Y such that $\begin{matrix}X+Y=(2 & -1 & 3)\\ X-2Y=(-1 & 2 & 3)\end{matrix}$.

- $X+Y-(X-2Y)=(2\quad -1\quad 3)-(-1\quad 2\quad 3)\Leftrightarrow 3Y=(3\quad -3\quad 0)\Leftrightarrow Y=(1\quad -1\quad 0)$.
- $X=(2\quad -1\quad 3)-Y=(2\quad -1\quad 3)-(1\quad -1\quad 0)=(1\quad 0\quad 3)$.

Defintion

Let A be a $m\times n$ matrix, $A=(a_{ij})$. The <u>transpose</u> of A, written A^T, is the $n\times m$ matrix given by $A^T=(a_{ji})$.

Example

Let $A=\begin{pmatrix}1 & -1 & 0 & 6\\ 2 & 3 & -4 & 0\\ 7 & 1 & -2 & 3\end{pmatrix}$ be a 3×4 matrix. Then $A^T=\begin{pmatrix}1 & 2 & 7\\ -1 & 3 & 1\\ 0 & -4 & -2\\ 6 & 0 & 3\end{pmatrix}$ is a 4×3 matrix.

Theorem

Let A, B be matrices of the same size. Let $k\in\mathbf{R}$. Then:

1) $(A^T)^T=A$.
2) $(kA)^T=k(A^T)$.
3) $(A+B)^T=A^T+B^T$.

Definition

A matrix A is called <u>symmetric</u> if $A=A^T$.

- Remark: A symmetric matrix is <u>necessarily square</u>.

Example

Let $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. When is A symmetric.

- $A^T=\begin{pmatrix}a & c\\ b & d\end{pmatrix}$, so $A=A^T$ iff $b=c$.

Example

Let A, B be two symmetric matrices. Show that $A+B$ is symmetric.

- $(A+B)^T=A^T+B^T=A+B$, since $A=A^T$ and $B=B^T$ (definition of symmetric).

- So, since $(A+B)^T = A+B$, $A+B$ is symmetric.

Example

Suppose that A is a $m\times n$ matrix such that $A^T = 3A$. Show that $A=0$.

- $(A^T)^T = A$, so $(3A)^T = A \Leftrightarrow 3(A^T) = A \Leftrightarrow 3(3A) = A \Leftrightarrow 9A = A \Leftrightarrow 8A = 0 \Rightarrow A = 0$.

2.2 Matrix Multiplication

Definition

Let $A = (a_{ij})$ be a $m\times n$ matrix, and $B = (b_{ij})$ be a $n\times k$ matrix. Then $C = (c_{ij}) = A \cdot B$ is a $m\times k$ matrix such that $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$.

Example

Let $A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 & -1 & 1 \\ 0 & -4 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{pmatrix}$. Compute the (1, 2)-entry and (2, 3)-entry of AB.

- Let $C = c_{ij} = AB$. C is a 2×4 matrix. We want to compute c_{12} and c_{23}.
- $c_{12} = (1)(1) + (0)(-4) + (-2)(2) = -3$.
- $c_{23} = (2)(-1) + (-1)(2) + (0)(3) = -4$.

Example

Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$. Compute AB and BA.

- $AB = \begin{pmatrix} (1)(0)+(2)(-2) & (1)(1)+(2)(3) \\ (-1)(0)+(1)(-2) & (-1)(1)+(1)(-3) \end{pmatrix} = \begin{pmatrix} -4 & 7 \\ -2 & 2 \end{pmatrix}$.
- $BA = \begin{pmatrix} -1 & 1 \\ -5 & -1 \end{pmatrix}$.
- Notice $AB \neq BA$.

Example

Let $A = \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix}$. Compute $A^2 = A \cdot A$.

- $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$ although $A \neq 0$.

Definition

Let A, B be two matrices such that they can be multiplied. If $AB = BA$, we say that A and B <u>commute</u>.

Example

A commutes with A (hence $A^2 = A \cdot A$).

Definition

For $n \in \mathbf{N}$, I_n is the identity matrix defined by $I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$, where I_n is a $n \times n$ matrix.

Examples

1) $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
2) $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Theorem

Let k be a scalar and A, B, C be matrices of sizes such that the indicated operations can be performed. Then:

1) $IA = A$, $IB = B$.
2) $A(BC) = (AB)C$.
3) $A(B + C) = AB + AC$, $(B + C)A = BA + CA$ – distributive laws.
4) $k(AB) = (kA)B = A(kB)$.
5) $(AB)^T = B^T A^T$.

Example

Let A, B, C be matrices such that A and B commute with C. Show that AB commutes with C.

- Since A and B commute with C, $AC = CA$ and $BC = CB$.
- $(AB)C = A(BC) = A(CB) = (AC)B = (CA)B = C(AB)$. Since $(AB)C = C(AB)$, AB commutes with C.

Lecture #4 – Thursday, January 15, 2004

Matrices and Linear Systems

Example

Consider the system $\begin{cases} 3x_1 - x_2 + 2x_3 = b_1 \\ x_1 + 2x_2 - 4x_3 = b_2 \end{cases}$.

- It can be written as $\begin{pmatrix} 3x_1 - x_2 + 2x_3 \\ x_1 + 2x_2 - 4x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ or $\begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.
- It has the form $AX = B$.

Definition

$AX = B$ is called the matrix form of the system.

- A is the coefficient matrix.
- B is the constant matrix.

X_1 is a solution if $AX_1 = B$.

Definition

Given a system $AX = B$, the system $AX = 0$ is called the associated homogeneous system.

Theorem

Let X_1 be a particular solution of $AX = B$. Then every solution X_2 to $AX = B$ has the form $X_2 = X_1 + X_0$ for some solution X_0 to $AX = 0$.

Proof

- X_1 is a solution of $AX = B$. It means that $AX_1 = B$.
- Let's consider X_2 any solution of $AX = B$. So $AX_2 = B$.
- Define $X_0 = X_2 - X_1$. So $AX_0 = A(X_2 - X_1) = AX_2 - AX_1 = B - B = 0$.
- So $AX_0 = 0 \Rightarrow X_2 = X_1 + X_0$.

Example

Consider the system $\begin{cases} x - y + z = 1 \\ x + y + 2z = 0 \\ x - 3y = 2 \end{cases}$. Gaussian Elimination gives the row-echelon augmented matrix:

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$ so that $\begin{cases} x = \frac{1}{2} - \frac{3}{2}s \\ y = -\frac{1}{2} - \frac{1}{2}s, s \in \mathbf{R} \\ z = s \end{cases}$.

- $x_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a solution iff $x_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} + s\begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = x_1 + x_0$.
- <u>Remark</u>: This method gives $x_1 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$ a particular solution, $x_0 = \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$ the solution to the associated homogeneous system.

Example

Solve $AX = 0$ where $A = \begin{pmatrix} 1 & -2 & 1 & -4 \\ 0 & 1 & -3 & 2 \\ 1 & -1 & -2 & -2 \end{pmatrix}$.

- The augmented matrix is $\left[\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 0 \\ 0 & 1 & -3 & 2 & 0 \\ 1 & -1 & -2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 0 \\ 0 & 1 & -3 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$.
- So $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, the solutions are $X = \begin{pmatrix} 5s \\ 3s-2t \\ s \\ t \end{pmatrix}, s, t \in \mathbf{R}$. So $X = s\begin{pmatrix} 5 \\ 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} = sX_1 + tX_2$

Definition

X_1 and X_2 are called the basic solutions of $AX = 0$. $X = sX_1 + tX_2$ is a <u>linear combination</u> of the basic solutions X_1 and X_2.

Theorem

Consider the homogeneous system $AX = 0$ in n variables, where $\operatorname{rank} A = r$. Then:

1) The system has exactly $n - r$ basic solutions.
2) Every solution is a linear combination of the basic solution.

Block Multiplication

Theorem

Let $A = (c_1, c_2, \ldots, c_n)$ be a $m\times n$ matrix with columns $c_1, c_2, \ldots, c_n$. If $X = (x_1, x_2, \ldots, x_n)^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is any

$$AX = (c_1, c_2, \ldots, c_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1c_1 + x_2c_2 + \ldots + x_nc_n .$$

Example

Let $A = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 3 & -2 & 1 & 0 \\ -2 & 4 & 1 & 3 & -1 \end{array}\right) = \left(\begin{array}{c|c} I_2 & 0_{2\times 3} \\ \hline P & Q \end{array}\right)$, P is a 2×2 matrix and Q is a 2×3 matrix. Let

$B = \left(\begin{array}{cc} 1 & -1 \\ 2 & 4 \\ \hline 1 & 3 \\ -2 & 2 \\ 1 & -5 \end{array}\right) = \left(\begin{array}{c} X \\ \hline Y \end{array}\right)$, X is a 2×2 matrix and Y is a 3×2 matrix.

- We can calculate $AB = \begin{pmatrix} I & 0 \\ P & Q \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} IX + 0Y \\ PX + QY \end{pmatrix} = \begin{pmatrix} X \\ PX + QY \end{pmatrix}$.

Theorem

Suppose that $A = \begin{pmatrix} B & X \\ 0 & C \end{pmatrix}$, $A_1 = \begin{pmatrix} B_1 & X_1 \\ 0_1 & C_1 \end{pmatrix}$ are $n\times n$ matrices where B and B_1 are $p\times p$ matrices and C and C_1 are $q\times q$ matrices ($n = p + q$). Then we can conclude that $AA_1 = \begin{pmatrix} BB_1 & B_1X_1 + XC_1 \\ 0 & CC_1 \end{pmatrix}$.

Lecture #5 – Tuesday, January 20, 2004

2.3 MATRIX INVERSES

Definition

If A is a square matrix, a matrix B is called an inverse of A iff $AB = I = BA$. A then is called an <u>invertible matrix</u>.

Example

Show that $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is an inverse of $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

- Compute $AB = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = I$.
- Similarly, $BA = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = I$.

Example

Show that $A = \begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix}$ has no inverse.

- Let $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary matrix. We want to solve $AB=I$ and $BA=I$.
- $AB=\begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow \begin{cases} c=1 \\ d=0 \\ 4c=0 \\ 4d=1 \end{cases} \Rightarrow \begin{cases} c=1 \\ c=0 \\ d=0 \\ d=4 \end{cases}$, therefore, there is no solution.

Theorem

If B and C are both inverses of A, then $B = C$.

Proof

- "B and C are both inverses of A" means $AB=I=BA$, $AC=I=CA$.
- Hence, $C=CI=C(AB)=(CA)B=IB=B$.

Definition

If A is invertible, we denote A^{-1} as the <u>unique inverse</u> of A.

Example

Let $A=\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Show that $A^3=I$ and deduce A^{-1}.

- $A^2=A\times A=\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. $A^3=A^2\times A=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=I$.
- So $A^3=A^2\times A=I=A\times A^2 \Rightarrow A^{-1}=A^2=\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

Example

If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad-bc\neq 0$, show that $A^{-1}=\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

- We have to prove that $AA^{-1}=I$ and $A^{-1}A=I$.
- $AA^{-1}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\times\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}=\dfrac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}=I$.
- $A^{-1}A=\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\times\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\dfrac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}=I$.

Inverses and Linear Systems

Theorem

Suppose a system of n equations and m variables is written in matrix form: $AX=B$, A is of size $m\times m$. If A is invertible, the system has the unique solution $X=A^{-1}B$.

Example

Let $A=\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}$. We know that $A^{-1}=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$. Use it to solve the system $\begin{cases} x-y=2\\ -x+y=4\end{cases}$.

- Let $X=\begin{pmatrix}x\\ y\end{pmatrix}$, $B=\begin{pmatrix}2\\ 4\end{pmatrix}$. The system can be written $AX=B$.
- So $X=A^{-1}B=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}2\\ 4\end{pmatrix}=\begin{pmatrix}8\\ 6\end{pmatrix}$.
- Finally, we have $\begin{cases} x=8\\ y=6\end{cases}$.

MATRIX INVERSION ALGORITHM

If A is an invertible matrix, there exists a sequence of elementary row operations that carry A to the identity matrix of the same size, written $A\to I$. Then the same series of row operations carries $I\to A^{-1}$. We write $\begin{bmatrix}A & I\end{bmatrix}\to\begin{bmatrix}I & A^{-1}\end{bmatrix}$ and perform the row operations on A and I simultaneously.

- Remark: If A cannot be brought to I, then A is not invertible.

Example

Use the Inversion Algorithm to find the inverse of $A=\begin{pmatrix}1 & 0 & 4\\ 2 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}$.

- We write $\begin{bmatrix}A & I\end{bmatrix}=\left[\begin{array}{ccc|ccc}1 & 0 & 4 & 1 & 0 & 0\\ 2 & 1 & 0 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & 0 & 1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{1}{5} & \frac{4}{5} & -\frac{4}{5}\\ 0 & 1 & 0 & -\frac{2}{5} & -\frac{3}{5} & \frac{8}{5}\\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5}\end{array}\right]=\begin{bmatrix}I & A\end{bmatrix}$.
- So $A^{-1}=\frac{1}{5}\begin{pmatrix}1 & 4 & -4\\ -2 & -3 & 8\\ 1 & -1 & 1\end{pmatrix}$.

Theorem

Let A be a $n\times n$ matrix. There are 2 possibilities:

1) We can reduce A to I by elementary row operations and obtain A^{-1} by the Inversion Algorithm.
2) We cannot reduce A to I, and then A is not invertible.

Example

If A is an invertible matrix, show that A^{-1} is also invertible and $\left(A^T\right)^{-1}=\left(A^{-1}\right)^T$.

- We have $AA^{-1}=I$, so $\left(AA^{-1}\right)^T=I^T=I$. Finally, $\left(A^{-1}\right)^T(A)^T=I$.
- Similarly, $AA^{-1}=I$ gives $A^T\left(A^{-1}\right)^T=I\Rightarrow\left(A^T\right)^{-1}=\left(A^{-1}\right)^T$.

Example

Let A, B, be invertible matrices. Show that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

- Let us calculate $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$.
- Similarly, $(B^{-1}A^{-1})(AB) = I$.

Theorem

All the following matrices are square matrices of the same size:

1) I is invertible and $I^{-1} = I$.
2) If A is invertible, A^{-1} is also invertible and $(A^{-1})^{-1} = A$.
3) If A and B are invertible, so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.
4) If $A_1, A_2, \ldots, A_k$ are invertible, then $(A_1, A_2, \ldots, A_k)^{-1} = A_k^{-1}, \ldots, A_1^{-1}$.
5) If A is invertible, so is $A^k, k \geq 1$, and $(A^k)^{-1} = (A^{-1})^k$.
6) If A is invertible and $a \neq 0$, then aA is invertible. $(aA)^{-1} = \frac{1}{a}A^{-1}$.
7) If A is invertible, so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

Corollary

A square matrix A is invertible iff A^T is invertible.

Example

Find A if $(A^{-1} - 3I)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$.

- $A^{-1} - 3I = \left((A^{-1} - 3I)^{-1}\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. So $A^T = 3I + \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 1 & 4 \end{pmatrix}$, and $A = \begin{pmatrix} 5 & 1 \\ 0 & 4 \end{pmatrix}$.

Theorem

Let A be a $n \times n$ matrix. Then the following conditions are equivalent:

1) A is invertible.
2) The homogeneous system $AX = 0$ has only the trivial solution $X = 0$.
3) A can be carried to I by elementary row operations.
4) For every column B, the system $AX = B$ has at least one solution.
5) There exists a $n \times n$ matrix C such that $AC = I_n$.

Corollary

If $AC = I_n$, then $CA = I_n$ for A, C $n \times n$ matrices. In particular, A and C are invertible and $A^{-1} = C$, $C^{-1} = A$.

Example

Show that $A=\begin{pmatrix}4 & -2\\ -10 & 5\end{pmatrix}$ has no inverse.

- Remark that $A\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, so there exist a non-trivial solution. Therefore, A has no inverse.

2.4 Elementary Matrices

Definition

A $n\times n$ matrix is an elementary matrix if it is obtained from the $n\times n$ identity by an elementary row operation.

Examples

$E_1=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$, $E_2=\begin{pmatrix}1 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 1\end{pmatrix}$, $E_3=\begin{pmatrix}1 & 3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ are elementary matrices.

Theorem

Let A be a $m\times n$ matrix. Let E be a $m\times m$ elementary matrix corresponding to some elementary row operation. If the same elementary row operation is performed on A, the resulting matrix is EA.

Example

Let $A=\begin{pmatrix}3 & 1 & 4 & -2\\ 0 & 3 & -1 & 1\\ 4 & -2 & 3 & 5\end{pmatrix}$, $E=\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$. Then $EA=\begin{pmatrix}3 & 1 & 4 & -2\\ 6 & 5 & 7 & -3\\ 4 & -2 & 3 & 5\end{pmatrix}$ is obtained also by adding two times R_2 to R_1.

Theorem

An elementary matrix E is invertible. E^{-1} is the elementary matrix corresponding to the elementary row operation that transformed the matrix EA into A.

Example

Write the inverses of $E_1=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$, $E_2=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$, $E_3=\begin{pmatrix}1 & 0 & 5\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$.

- $E_1^{-1}=E_1=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$, $E_2^{-1}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{pmatrix}$, $E_3^{-1}=\begin{pmatrix}1 & 0 & -5\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$.

Lecture #6 – Thursday, January 22, 2004

Theorem

Let A be a $m\times n$ matrix. Assume that A can be carried to a matrix B by elementary row operations. Then:

1) $B = UA$, where U is invertible, U is $m\times m$ matrix.
2) $U = E_R \times E_{R-1} \times \cdots \times E_1$ where E_j are elementary matrices.
3) We have $\begin{bmatrix} A & I_m \end{bmatrix} \to \begin{bmatrix} B & U \end{bmatrix}$.

Example

$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & -2 \end{pmatrix}$.

- $\left[\begin{array}{ccc|cc} 2 & 1 & 3 & 1 & 0 \\ 1 & 4 & -2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 4 & -2 & 0 & 1 \\ 0 & 1 & -1 & -\frac{1}{7} & \frac{2}{7} \end{array}\right]$.
- So $B = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & -1 \end{pmatrix} = UA$, $U = \begin{pmatrix} 0 & 1 \\ -\frac{1}{7} & \frac{2}{7} \end{pmatrix}$.

Corollary

A square matrix A is invertible iff A is the product of elementary matrices.

Example

Let $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$. Write A as a product of elementary matrices.

- Since $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$.
- $E_3 E_2 E_1 A = I \Rightarrow A = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$, and $E_1^{-1} = E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$,
  $E_3^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$.
- $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$.
- <u>Remark</u>: $A^{-1} = E_3 E_2 E_1$.

<u>Operation</u>	<u>Inverse Operaton</u>
• Interchange row p and row q.	• Exchange row p and row q.
• Multiple row p by $c \neq 0$.	• Multiple row p by $\frac{1}{c}$.
• Add k times row p to row q.	• Subtract k times row p to row q.

Lecture #7 – Tuesday, January 27, 2004

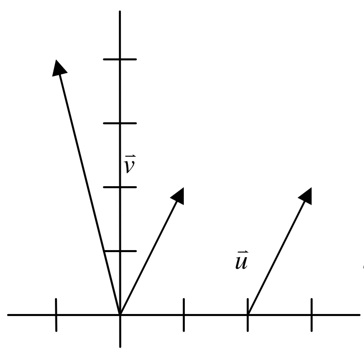
VECTOR SPACE IN $\mathbf{R}^N$: VECTORS AND LINES

Definition

A vector $\mathbf{v}$ in $\mathbf{R}^n$ is an ordered sequence of n real numbers called n-tuples. We will consider $\mathbf{v}$ as an $n\times 1$ matrix (or $1\times n$ matrix).

Example

In $\mathbf{R}^n$, $\mathbf{u}=\begin{pmatrix}1\\2\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}-1\\4\end{pmatrix}$.



- O = initial point of $\mathbf{u}$, $\mathbf{v}$
- A = terminal point of $\mathbf{u}$
- B = terminal point of $\mathbf{v}$

Definition

The magnitude (length) of a vector is the distance between the initial point and the terminal point. We denote it $\|\mathbf{v}\|$.

Example

In $\mathbf{R}^n$, $\mathbf{u}=\begin{pmatrix}1\\2\end{pmatrix}$. So $\|\mathbf{u}\|=\left(1^2+2^2\right)^{\frac{1}{2}}=\sqrt{5}$.

- Remark: $\mathbf{0}=\begin{pmatrix}0 & 0\end{pmatrix}$ is the only vector such that $\|\mathbf{0}\|=0$.

Definition

Addition and multiplication by a scalar are defined as for the matrices.

Theorem 1

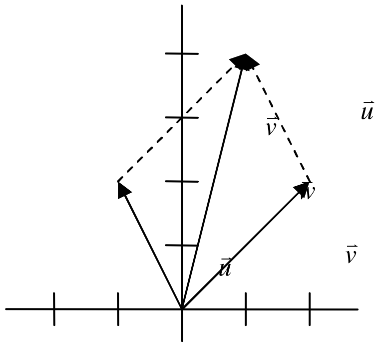
Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbf{R}^n$. Then:

1) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
2) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{v}+\mathbf{u})+\mathbf{w}$.
3) $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
4) $1\mathbf{u}=\mathbf{u}$.
5) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6) $a(b\mathbf{u})=(ab)\mathbf{u}, a,b\in\mathbf{R}$.

7) $(a+b)(\mathbf{u}) = a\mathbf{u} + b\mathbf{u}$.
8) $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

Example

In $\mathbf{R}^2$, $\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. $\mathbf{w} = \mathbf{u} + \mathbf{v} = \begin{pmatrix} -1+2 \\ 2+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.



Proposition

Let $\mathbf{u} = \mathbf{R}^n$ and $a \in \mathbf{R}$. Then $\|a\mathbf{u}\| = |a|\|\mathbf{u}\|$.

Definition

$\mathbf{v}$ is a unit-vector if $\|\mathbf{v}\| = 1$.

Example

If $\mathbf{v} \neq \mathbf{0}$, then $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector.

Proposition

Let $\mathbf{u}$, $\mathbf{v}$ be non-zero vectors in $\mathbf{R}^n$. Then $\mathbf{u}$, $\mathbf{v}$ are parallel iff $\exists a \in \mathbf{R}$ such that $\mathbf{u} = a\mathbf{v}$.

VECTOR SPACE IN $\mathbf{R}^N$: SUBSPACES OF $\mathbf{R}^N$

Definition

A set U is a subspace of $\mathbf{R}^n$ iff:
1) $\mathbf{0} \in U$.
2) If $\mathbf{u}$, $\mathbf{v}$ are in U, then $\mathbf{u} + \mathbf{v} \in U$. U is closed under the addition.
3) If $\mathbf{u} \in U$, $a \in \mathbf{R}$, then $a\mathbf{u} \in U$. U is closed under the multiplication of scalar.

Example

1) $U = \{\mathbf{0}\}$ is a subspace.
2) $U = \mathbf{R}^n$ is a subspace.

Definition

Any subspaces different from $\{\mathbf{0}\}$ and $\mathbf{R}^n$ are called proper subspaces.

Example

Let A be a $m\times n$ matrix. Define:

- null $A = \{X \in \mathbf{R}^n : AX = 0\}$, the null space of A.
- im $A = \{Y \in \mathbf{R}^n : Y = AX, X \in \mathbf{R}^n\}$, the image of A.

Show that null$A$ is a subspace of $\mathbf{R}^n$.

1) $X = 0$ satisfy $AX = 0$, so $\mathbf{0} \in \text{null } A$.
2) Let X_1 and X_2 be in null$A$. Then $AX_1 = 0$ and $AX_2 = 0$. Let $X = X_1 + X_2$. So $AX = A(X_1 + X_2) = AX_1 + AX_2 = 0 + 0 = 0$. So $X = X_1 + X_2 \in \text{null } A$.
3) Let $X_1 \in$ null$A$, $a \in$ R. So $AX_1 = 0$. $A(aX_1) = a(AX_1) = a0 = 0$. So $aX_1 \in$ null$A$.

Counter-Example

Let $U = \{(x, y), x \geq 0\}$. Prove $U \subset \mathbf{R}^n$.

- Let $U_1 = (1,2) \in U$.
- $-1 \cdot U_1 = (-1,-2) \notin U$. So U is not a subspace.

Vector Space In R^N: Spanning Sets

Definition

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be vectors in $\mathbf{R}^n$. Let $\mathbf{x} = r_1\mathbf{x}_1 + r_2\mathbf{x}_2 + \cdots + r_k\mathbf{x}_k, r_i \in \mathbf{R}$ be a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. rj is called the coefficient of $\mathbf{x}_j$ in the linear combination. The set of all linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is called the span of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$.

$\text{span}(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k) = \{\mathbf{x} = r_1\mathbf{x}_1 + r_2\mathbf{x}_2 + \cdots + r_k\mathbf{x}_k\}$.

Theorem

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be vectors in $\mathbf{R}^n$. Then:

1) $\text{span}(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k)$ is a subspace of $\mathbf{R}^n$.
2) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are in W, a subspace of $\mathbf{R}^n$, then $\text{span}(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k) \subseteq W$.

Corollary

$\text{span}(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k)$ is the smallest subspace that contains $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$.

Example

Let $\mathbf{X}$ and $\mathbf{Y}$ be vectors in $\mathbf{R}^n$. Show that $\text{span}\{\mathbf{X}, \mathbf{Y}\} = \text{span}\{\mathbf{X}+\mathbf{Y}, \mathbf{X}-\mathbf{Y}\}$.

- $\mathbf{X}+\mathbf{Y} \in \text{span}\{\mathbf{X}, \mathbf{Y}\}$ and $\mathbf{X}-\mathbf{Y} \in \text{span}\{\mathbf{X}, \mathbf{Y}\}$, so $\text{span}\{\mathbf{X}+\mathbf{Y}, \mathbf{X}-\mathbf{Y}\} \subseteq \text{span}\{\mathbf{X}, \mathbf{Y}\}$.

- Since $\mathbf{X}=\frac{1}{2}((\mathbf{X}+\mathbf{Y})+(\mathbf{X}-\mathbf{Y}))=\frac{1}{2}(\mathbf{X}+\mathbf{Y})+\frac{1}{2}(\mathbf{X}-\mathbf{Y})$, so $\mathbf{X}\in\operatorname{span}\{\mathbf{X}+\mathbf{Y},\mathbf{X}-\mathbf{Y}\}$. Similarly, $\mathbf{Y}=\frac{1}{2}((\mathbf{X}+\mathbf{Y})-(\mathbf{X}-\mathbf{Y}))=\frac{1}{2}(\mathbf{X}+\mathbf{Y})-\frac{1}{2}(\mathbf{X}-\mathbf{Y})$, so $\mathbf{Y}\in\operatorname{span}\{\mathbf{X}+\mathbf{Y},\mathbf{X}-\mathbf{Y}\}$. Thus $\operatorname{span}\{\mathbf{X},\mathbf{Y}\}\subseteq\operatorname{span}\{\mathbf{X}+\mathbf{Y},\mathbf{X}-\mathbf{Y}\}$.
- Since $\operatorname{span}\{\mathbf{X}+\mathbf{Y},\mathbf{X}-\mathbf{Y}\}\subseteq\operatorname{span}\{\mathbf{X},\mathbf{Y}\}$ and $\operatorname{span}\{\mathbf{X},\mathbf{Y}\}\subseteq\operatorname{span}\{\mathbf{X}+\mathbf{Y},\mathbf{X}-\mathbf{Y}\}$, $\operatorname{span}\{\mathbf{X},\mathbf{Y}\}=\operatorname{span}\{\mathbf{X}+\mathbf{Y},\mathbf{X}-\mathbf{Y}\}$.

Definition

If U is a subspace, $U=\operatorname{span}\{\mathbf{X}_1,\mathbf{X}_2,\cdots,\mathbf{X}_k\}, \mathbf{X}_j\in\mathbf{R}^n$. We say that $\mathbf{X}_1, \mathbf{X}_2,\ldots, \mathbf{X}_k$ is the spanning set of U. U is spanned by $\mathbf{X}_1, \mathbf{X}_2,\ldots, \mathbf{X}_k$.

Example

Show that $\operatorname{im} A=\{Y\in\mathbf{R}^n : Y=AX, X\in\mathbf{R}^n\}$ is a subspace of $\mathbf{R}^n$. Also prove that $\operatorname{im} A=\operatorname{span}\{C_1,C_2,\cdots,C_n\}$.

- Let $C_1, C_2,\ldots, C_n$ be the columns of A. So $A=(C_1,C_2,\cdots,C_n)$. Then $\operatorname{im} A=\operatorname{span}(C_1,C_2,\cdots,C_n)$

1) $\mathbf{0}\in\operatorname{im}A$. $\mathbf{0}=A\mathbf{0}$.
2) Let $\mathbf{Y}_1, \mathbf{Y}_2\in\operatorname{im}A$. There is $\mathbf{X}_1, \mathbf{X}_2\in\mathbf{R}^n$ such that $\mathbf{Y}_1=A\mathbf{X}_1$, $\mathbf{Y}_2=A\mathbf{X}_2$.
   $\mathbf{Y}_1+\mathbf{Y}_2=A\mathbf{X}_1+A\mathbf{X}_2=A(\mathbf{X}_1+\mathbf{X}_2)$. So $\mathbf{Y}_1+\mathbf{Y}_2=A\mathbf{X}$ with $\mathbf{X}=\mathbf{X}_1+\mathbf{X}_2$. So $\mathbf{Y}_1+\mathbf{Y}_2\in\operatorname{im}A$.
3) Let $\mathbf{Y}\in\operatorname{im}A$ and $a\in\mathbf{R}$. By definition, $\mathbf{Y}=A\mathbf{X}, \mathbf{X}\in\mathbf{R}^n$. So $a\mathbf{Y}=aA\mathbf{X}=A(a\mathbf{X})=A\mathbf{X}'$ with $\mathbf{X}'=a\mathbf{X}\in\mathbf{R}^n$. Thus, $a\mathbf{Y}\in\operatorname{im}A$.

- $A\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}=(C_1,C_2,\cdots,C_n)\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}=C_1$, so $C_1\in\operatorname{im}A$. $A\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}=(C_1,C_2,\cdots,C_n)\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}=C_2$, so $C_2\in\operatorname{im}A$.

  Therefore, $C_1, C_2,\ldots, C_n\in\operatorname{im}A$. Since $\operatorname{im}A$ is a subspace, $\operatorname{span}(C_1,C_2,\cdots,C_n)\subseteq\operatorname{im}A$.

- Let $\mathbf{Y}\in\operatorname{im}A$. It means that $\exists\,\mathbf{X}\in\mathbf{R}$ such that $A\mathbf{X}=\mathbf{Y}$. Let $\mathbf{X}=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}, x_j\in\mathbf{R}$.

  $\mathbf{Y}=A\mathbf{X}=(C_1,C_2,\cdots,C_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=x_1C_1+x_2C_2+\cdots+x_2C_2$. Since $\mathbf{Y}\in\operatorname{span}(C_1,C_2,\cdots,C_n)$, so $\operatorname{im}A\subseteq\operatorname{span}(C_1,C_2,\cdots,C_n)$.

VECTOR SPACE IN R^N: INDEPENDENCE

Definition

A set $\{\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_k\}$ of vectors in $\mathbf{R}^n$ is <u>independent</u> if it satisfies:
$t_1\mathbf{X}_1 + t_2\mathbf{X}_2 + \cdots + t_k\mathbf{X}_k = 0 \Rightarrow t_1 = t_2 = \cdots = t_k = 0$.

Example

In $\mathbf{R}^4$, $\mathbf{X}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \\ -5 \end{pmatrix}$, $\mathbf{X}_2 = \begin{pmatrix} 0 \\ 1 \\ -4 \\ 2 \end{pmatrix}$, $\mathbf{X}_3 = \begin{pmatrix} 3 \\ 4 \\ 1 \\ -2 \end{pmatrix}$. Prove $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ is independent.

- $t_1\mathbf{X}_1 + t_2\mathbf{X}_2 + t_3\mathbf{X}_3 \Leftrightarrow t_1\begin{pmatrix} 1 \\ 0 \\ 3 \\ -5 \end{pmatrix} + t_2\begin{pmatrix} 0 \\ 1 \\ -4 \\ 2 \end{pmatrix} + t_3\begin{pmatrix} 3 \\ 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.
- So $\begin{cases} t_1 + 3t_3 = 0 \\ t_2 + 4t_3 = 0 \\ 3t_1 - 4t_2 + t_3 = 0 \\ -5t_1 - 2t_2 - 2t_3 = 0 \end{cases} \Leftrightarrow \begin{cases} t_1 + 3t_3 = 0 \\ t_2 + 4t_3 = 0 \\ -4t_2 - 8t_3 = 0 \\ 2t_2 + 13t_3 = 0 \end{cases} \Leftrightarrow \begin{cases} t_1 + 3t_3 = 0 \\ t_2 + 4t_3 = 0 \\ 8t_3 = 0 \\ 7t_3 = 0 \end{cases} \Leftrightarrow \begin{cases} t_1 + 3t_3 = 0 \\ t_2 + 4t_3 = 0 \\ 8t_3 = 0 \\ 7t_3 = 0 \end{cases} \Leftrightarrow \begin{cases} t_1 t = 0 \\ t_2 = 0 \\ t_3 = 0 \end{cases}$.
- Since $t_1\mathbf{X}_1 + t_2\mathbf{X}_2 + t_3\mathbf{X}_3 = 0 \Rightarrow t_1 = t_2 = t_3 = 0$, so $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ is independent.

Theorem

If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ are independent, then $\mathbf{X} \in \operatorname{span}\{\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_k\}$ has a unique representation as a linear combination of $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$. So if $\mathbf{X} = r_1\mathbf{X}_1 + r_2\mathbf{X}_2 + \cdots + r_k\mathbf{X}_k$, then $r_1, r_1, \ldots, r_k$ are unique.

Example

Let $\mathbf{X}$ and $\mathbf{Y}$ be independent vectors of $\mathbf{R}^n$. Prove $\mathbf{X} - \mathbf{Y}$ and $\mathbf{X} + 3\mathbf{Y}$ are also independent.

- Let $t_1(\mathbf{X} - \mathbf{Y}) + t_2(\mathbf{X} + 3\mathbf{Y}) = 0 \Leftrightarrow t_1\mathbf{X} - t_1\mathbf{Y} + t_2\mathbf{X} + 3t_2\mathbf{Y} = 0 \Leftrightarrow (t_1 + t_2)\mathbf{X} + (-t_1 + 3t_2)\mathbf{Y} = 0$.
- Because $\mathbf{X}$ and $\mathbf{Y}$ are independent, $\begin{cases} t_1 + t_2 = 0 \\ -t_1 + 3t_2 = 0 \end{cases} \Leftrightarrow \begin{cases} t_1 + t_2 = 0 \\ 4t_2 = 0 \end{cases} \Leftrightarrow \begin{cases} t_1 = 0 \\ t_2 = 0 \end{cases}$.
- Therefore, $\mathbf{X} - \mathbf{Y}$ and $\mathbf{X} + 3\mathbf{Y}$ are independent.

Theorem

Let A be an $n\times n$ matrix. Then the following conditions are equivalent:

1) A is invertible.
2) The columns of A are independent.
3) The rows of A are independent.
4) The columns of A span $\mathbf{R}^n$.
5) The rows of A span $\mathbf{R}^n$.

Example

Consider $\mathbf{X}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{X}_2 = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}$, $\mathbf{X}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. Prove that $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ are independent.

- Prove $A = \begin{pmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{pmatrix}$ is invertible.

VECTOR SPACE IN R^N: BASIS AND DIMENSION

Definition

Let U be a subspace of $\mathbf{R}^n$. A set $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k\}$ of vectors is a basis of U iff:

1) $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k\}$ is independent.
2) $U = \operatorname{span}\{\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_k\}$.

- Remark: If $\mathbf{X} \in U$, then $\mathbf{X} = r_1\mathbf{X}_1 + r_2\mathbf{X}_2 + \cdots + r_k\mathbf{X}_k$.

Theorem

Let U be a subspace of $\mathbf{R}^n$. Let $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k\}$ and $\{\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_k\}$ be two basis of U. Then $k = m$.

Definition

The dimension of a subspace U is the number of vectors in a basis. It is denoted $\dim U$.

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Definition

Let $E_1, E_2, \ldots, E_n$ be the columns of I_n. Then $\{\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n\}$ is the standard basis of $\mathbf{R}^n$. Then $\dim \mathbf{R}^n = n$.

Example

Let $U = \{(x \quad y \quad x) : x, y \in \mathbf{R}\}$. Prove U is a subspace of $\mathbf{R}^3$. Find a basis of U.

- Let $\mathbf{u} \in U$, so $\exists\, x, y \in \mathbf{R}$ such that $\mathbf{u} = (x \quad y \quad x) = x(1 \quad 0 \quad 1) + y(0 \quad 1 \quad 0) = x\mathbf{u}_1 + y\mathbf{u}_2$. Since $\mathbf{u} = \operatorname{span}(\mathbf{u}_1, \mathbf{u}_2)$, so U is a subspace.
- Are $\mathbf{u}_1$, $\mathbf{u}_2$ independent? Let $t_1\mathbf{u}_1 + t_2\mathbf{u}_2 = 0 \Rightarrow (t_1 \quad t_2 \quad t_1) = (0 \quad 0 \quad 0) \Rightarrow t_1 = 0 = t_2$.
- So $\mathbf{u}_1$, $\mathbf{u}_2$ are basis of U, and $\dim U = 2$.

Theorem

Let U, W be subspaces of $\mathbf{R}^n$. Then:

1) U has a subspace and $\dim U \le n$.
2) If $U \subseteq W$, then $\dim U \le \dim W$.
3) If $U \subseteq W$ and $\dim U = \dim W$, then $U = W$.

Theorem

Let U be a subspace of $\mathbf{R}^n$, $\dim U = m$. Let B be a set of m vectors. Then B is independent iff B spans U. B is then a basis of U.

Example

Let $\mathbf{X}_1 = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 4 \end{pmatrix}$, $\mathbf{X}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{X}_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 0 \end{pmatrix}$ be vectors in $\mathbf{R}^4$. Find a basis of $\mathbf{R}^4$ with $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$.

- Let $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, $\mathbf{E}_4$ be the standard basis of $\mathbf{R}^4$.
- Is $\mathbf{E}_1$ independent with $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$? Let $t_1\mathbf{X}_1 + s_1\mathbf{E}_1 + s_2\mathbf{E}_2 + s_3\mathbf{E}_3 = 0$. So

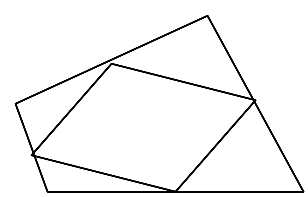
$$\begin{pmatrix} t_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -s_1 \\ 3s_1 \\ 0 \\ 4s_1 \end{pmatrix} + \begin{pmatrix} 2s_2 \\ 0 \\ -s_2 \\ 3s_2 \end{pmatrix} + \begin{pmatrix} s_3 \\ 3s_3 \\ 2s_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} t_1 - s_1 - 2s_2 + s_3 = 0 \\ 3s_1 + 3s_3 = 0 \\ -s_2 + 2s_3 = 0 \\ 4s_1 + 3s_2 = 0 \end{cases} \Leftrightarrow \begin{cases} t_1 - s_1 - 2s_2 + s_3 = 0 \\ s_1 = -s_3 \\ s_2 = 2s_3 \\ 2s_3 = 0 \end{cases} \Leftrightarrow \begin{cases} t_1 = 0 \\ s_1 = 0 \\ s_2 = 0 \\ s_3 = 0 \end{cases}.$$

- So $\{\mathbf{E}_1, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ is a basis of $\mathbf{R}^4$.

Dimension 2 and 3: Geometric Applications

Example

Consider $ABCD$ a quadrilateral. Show that the quadrilateral from by the midpoints E, F, G, H is a parallelogram.

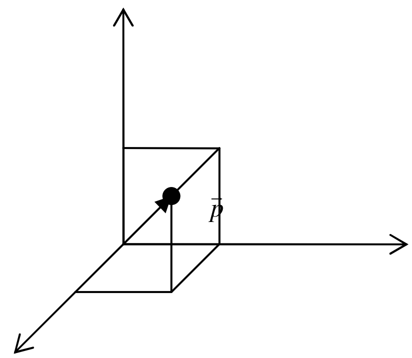


- It is enough to prove $\overrightarrow{EH} = \overrightarrow{FG}$.
- $\overrightarrow{EH} = \overrightarrow{EA} + \overrightarrow{AH} = \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AD} = \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{AD}\right) = \frac{1}{2}\overrightarrow{BD}$.
- $\overrightarrow{FG} = \overrightarrow{FC} + \overrightarrow{CG} = \frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{CD} = \frac{1}{2}\left(\overrightarrow{BC} + \overrightarrow{CD}\right) = \frac{1}{2}\overrightarrow{BD}$.
- So $\overrightarrow{EH} = \overrightarrow{FG}$.

Coordinates in $\mathbf{R}^3$

Definition

A point $P(x, y, z)$, x, y, z are unique.



Defintion

Given a point in $\mathbf{R}^3$, the position vector **P** is $\vec{p} = \overrightarrow{OP}$.
$P(x, y, z)$ is then $\vec{p} = (x \quad y \quad z)$.

Theorem

Given P_1, P_2 points in $\mathbf{R}^3$, $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, then $\overrightarrow{P_1P_2} = (x_2 - x_1 \quad y_2 - y_1 \quad z_2 - z_1)$.

- Remark: So if $P_1 = 0(0,0,0)$, $\overrightarrow{OP_2} = (x_2 \quad y_2 \quad z_2)$.

Example

Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$. Let I be the midpoint of P_1, P_2. What are the coordinates of I?

- $\overrightarrow{P_1I} = \overrightarrow{IP_2}$. Suppose $I = (x, y, z)$. So $\overrightarrow{P_1I} = \overrightarrow{IP_2} \Rightarrow (x - x_1 \quad y - y_1 \quad z - z_1) = (x_2 - x \quad y_2 - y \quad z_2 - z)$. So

$$\begin{cases} x - x_1 = x_2 - x \\ y - y_1 = y_2 - y \\ z - z_1 = z_2 - z \end{cases} \Leftrightarrow \begin{cases} 2x = x_1 + x_2 \\ 2y = y_1 + y_2 \\ 2z = z_1 + z_2 \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{x_1 + x_2}{2} \\ y = \dfrac{y_1 + y_2}{2} \\ z = \dfrac{z_1 + z_2}{2} \end{cases}.$$

Proposition

The line L is parallel to the vector $\vec{v}(v_1 \quad v_2 \quad v_3)$ passing through the point $P_0(x_0, y_0, z_0)$ is

$L = \{P_0 + t\vec{v}, t \in \mathbf{R}\}$. The parametric equation of L is $L = \left\{ P(x, y, z) : \begin{matrix} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{matrix}, t \in \mathbf{R} \right\}$.

RANK OF MATRICES

Definition

Let A be a $m\times n$ matrix. Let $R_1, R_2,\dots,R_n$ be the rows of A. So $A=\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$. The <u>row space</u> of A is

$\text{row } A=\text{span}(R_1,R_2,\cdots,R_n)\subset \mathbf{R}^n$. Similarly, if $A=(C_1,C_2,\cdots,C_n)$, C_j is the columns of A, then $\text{col } A=\text{span}(C_1,C_2,\cdots,C_n)\subset \mathbf{R}^m$ is the <u>column space</u> of A.

Theorem

Let A, B, C be matrices of sizes $m\times n$, $p\times m$, $n\times q$ respectively. Then:

1) $\text{col}(AC)\subset\text{col}(A)$, with equality if C is invertible.
2) $\text{row}(BA)\subset\text{row}(A)$, with equality if B is invertible.

Theorem: Rank Theorem

Let A be a $m\times n$ matrix. Then $\dim(A)=\dim(\text{row } A)$.

Let R be the row-echelon form of A. Then $r=\text{rank } A$ is the number of leading 1s in R, and $\dim(\text{col } A)=\dim(\text{row } A)=r$.

Moreover, the r non-zero rows of R are a basis of row A.

Let $C_1, C_2,\dots,C_r$ be the columns with a leading 1 in R. Then the columns $C_1, C_2,\dots,C_r$ of A is a basis of col A.

Example

Let $A=\begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 1 & 3 \\ -1 & -2 & 5 & 1 \end{pmatrix}$. Find the basis of col A and row A.

- $A\to R=\begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\text{rank } A=3$. The basis of row A is $\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \\ 4 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{5}{3} \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$. The basis of col A is $\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix},\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}\right\}$.

Corollary

1) $\text{rank } A=\text{rank } A^T$.
2) If A is a $m\times n$ matrix, then $\text{rank } A\le m$ and $\text{rank } A\le n$.
3) $\text{rank } A=\text{rank } AU=\text{rank } VA$ if U, V are invertible.
4) If A is a $n\times n$ matrix, then A is invertible iff $\text{rank } A=n$.

Example

Let $U = \operatorname{span}\left(\begin{pmatrix}1\\-2\\1\end{pmatrix},\begin{pmatrix}0\\-1\\3\end{pmatrix},\begin{pmatrix}1\\0\\-5\end{pmatrix}\right)$. Find a basis of U.

- Consider $A = \begin{pmatrix}1 & -2 & 1\\0 & -1 & 3\\1 & 0 & -5\end{pmatrix}$. Now, $U = \operatorname{row} A$. $A \to R = \begin{pmatrix}1 & 2 & 1\\0 & 1 & -3\\0 & 0 & 0\end{pmatrix}$. So a basis of U is $\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix},\begin{pmatrix}0\\1\\-3\end{pmatrix}\right\}$.

Example

If $A = \begin{pmatrix}1 & 1 & 1 & -2\\2 & -1 & 1 & 0\\1 & -2 & 0 & 2\end{pmatrix}$, find a basis of $\operatorname{null} A = \{X : AX = 0\} \subset \mathbf{R}^4$.

- The augmented matrix is $[A \quad 0] = \left[\begin{array}{cccc|c}1 & 1 & 1 & -2 & 0\\0 & 1 & \frac{1}{3} & -\frac{4}{3} & 0\\0 & 0 & 0 & 0 & 0\end{array}\right]$.
- Let $X = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$. So $\begin{cases}x_1 + x_2 + x_3 - 2x_4 = 0\\ \quad 3x_2 + x_3 - 4x_4 = 0\end{cases} \Leftrightarrow \begin{cases}x_4 = -\frac{2}{3}s + \frac{2}{3}t\\ x_2 = -\frac{1}{3}s + \frac{4}{3}t\\ x_3 = s\\ x_4 = t\end{cases}, s, t \in \mathbf{R}$.
- So $X = s\begin{pmatrix}-\frac{2}{3}\\-\frac{1}{3}\\1\\0\end{pmatrix} + t\begin{pmatrix}\frac{2}{3}\\\frac{4}{3}\\0\\1\end{pmatrix} = sX_1 + tX_2$.
- So $\operatorname{null} A \subset \operatorname{span}(X_1, X_2)$. Moreover, $X_1 \in \operatorname{null} A$ and $X_2 \in \operatorname{null} A$, so $\operatorname{span}(X_1, X_2) = \operatorname{null} A$ a subspace.
- X_1, X_2 are independent: $t_1 X_1 + t_2 X_2 = 0 \Rightarrow t_1\begin{pmatrix}-\frac{2}{3}\\-\frac{1}{3}\\1\\0\end{pmatrix} + t_2\begin{pmatrix}\frac{2}{3}\\\frac{4}{3}\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow t_1 = t_2 = 0$.

Theorem

Let A be a $m\times n$ matrix, $r = \operatorname{rank} A$.

1) If $X_1, X_2, \ldots, X_{n-r}$ are basic solutions of $AX = 0$, then it is a basis of null A.
2) Since $\operatorname{im} A = \operatorname{col} A$, in particular…???

Theorem

Let A be a $m\times n$ matrix. Then $\operatorname{rank} A = r$ iff there exists invertible matrices such that $UAV = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

Example

Let $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 4 \\ 2 & -1 & 4 & 8 \end{pmatrix}$. Find rank A.

- <u>Method</u>: $\begin{pmatrix} A & I_3 \end{pmatrix} \to \begin{pmatrix} R & U \end{pmatrix}$. Then, $\begin{pmatrix} R^T & I_4 \end{pmatrix} \to \begin{pmatrix} \begin{matrix} I_r & 0 \\ 0 & 0 \end{matrix} & V^T \end{pmatrix}$.
- $\begin{pmatrix} A & I_3 \end{pmatrix} = \left(\begin{array}{cccc|ccc} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 4 & 0 & 1 & 0 \\ 2 & -1 & 4 & 8 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|ccc} 1 & 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) = \begin{pmatrix} R & U \end{pmatrix}$.
- $\begin{pmatrix} R^T & I_4 \end{pmatrix} = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 1 & 0 \end{array}\right) = \begin{pmatrix} \begin{matrix} I_r \\ 0 \end{matrix} & V^T \end{pmatrix}$.
- So, $U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -2 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. $UAV = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} I_3 & 0 \end{pmatrix}$. So $\operatorname{rank} A = 3$.

Theorem

Let A be a $m\times n$ matrix. Then the following conditions are equivalent:

1) $AX = 0$ has only the trivial solution.
2) The columns of A are independent ($\dim(\operatorname{col} A) = n$).
3) $\operatorname{rank} A = n \le m$.
4) $A^T A$ is invertible.

Theorem

Let A be a $m\times n$ matrix. Then the following conditions are equivalent:

1) $AX = B$ has a solution for any $B \in \mathbf{R}^m$.
2) The columns of A span $\mathbf{R}^m$ ($\operatorname{col} A = \mathbf{R}^m$).
3) $\operatorname{rank} A = m$.
4) AA^T is invertible.

Example

Let $A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}$. What is rank A?

- $AA^T = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 0 \\ -1 & 2 \end{pmatrix} = 5I_2$. Since $(5I_2)^{-1} = \frac{1}{5}(I_2)^{-1} = \frac{1}{5}I_2$, so AA^T is invertible, and $\operatorname{rank} A = 2$.

Lecture #9 – Tuesday, February 3, 2004

Complex Numbers

Definition

A complex number $z \in \mathbf{C}$ is given by $z = a + bi$, $a, b \in \mathbf{R}$, i is a root of $x^2 + 1 = 0 \Rightarrow i^2 = -1$.

- $a = \operatorname{Re} z$ is the real part of z.
- $b = \operatorname{Im} z$ is the imaginary part of z.

- Remark: If $a \in \mathbf{R}$, then $a = a + 0i \in \mathbf{C}$.
- Remark: $0 + bi$ is a pure imaginary number. i is the imaginary unit.

Definition

- $a + ib = a' + ib'$ iff $a = a'$ an $b = b'$.
- $(a + ib) + (a' + ib') = (a + a') + i(b + b')$.
- $(a + ib)(a' + ib') = aa' - bb' + i(ab' + a'b)$.

Example

Let $z = 2 + i$, $w = 3 - 2i$.

1) $z - w = -1 + 3i$.
2) $zw = 6 + 2 - 4i + 3i = 8 - i$.
3) $\frac{1}{4}z = \frac{1}{4}(2 + i) = \frac{1}{2} + \frac{i}{4}$.

Example

Find all complex number z such that $z^2 = -i$.

- Let $z = a + ib$. So $z^2 = a^2 + 2abi - b^2 = -i$.
- So $\begin{cases} a^2 - b^2 = 0 \\ 2ab = -1 \end{cases} \Leftrightarrow \begin{cases} a = -b, \because ab < 0 \\ 2a^2 = 1 \end{cases} \Leftrightarrow \begin{cases} b = -a \\ a = \pm \frac{1}{\sqrt{2}} \end{cases}$. So $z_1 = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$, $z_2 = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$.

Definition

Let $z = a + bi$ be a complex number. The conjugate of z is $\bar{z} = a - bi$. The modulus of z is $|z| = \sqrt{a^2 + b^2}$.

- Remark: $|z|^2 = a^2 + b^2 = z\bar{z}$.

- Remark: $|z|^2 = 0 \Leftrightarrow z = 0$.
- If $z \neq 0$, $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$.
- If $z = a + ib$, $\frac{1}{z} = \frac{a - ib}{a^2 + b^2}$.

Example

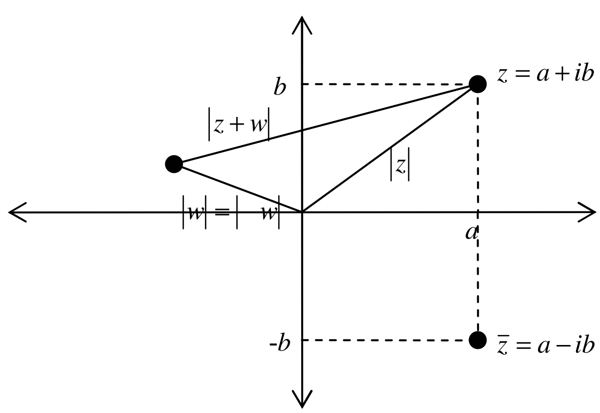
Write $\frac{3+i}{1-i}$ in the form $a + bi$.

- $\frac{1}{1-i} = \frac{1+i}{2}$. So $\frac{3+i}{1-i} = (3+i)\left(\frac{1+i}{2}\right) = 1 + 2i$.

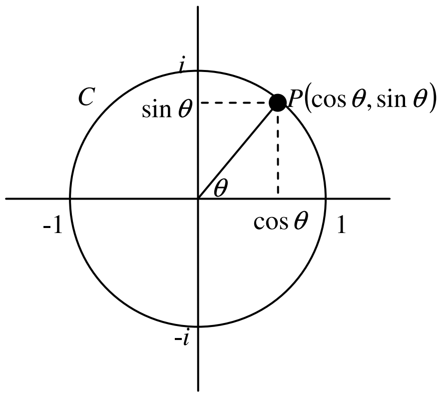
Proposition

1) $\overline{z \pm w} = \bar{z} \pm \bar{w}$.
2) $\overline{zw} = \bar{z} \cdot \bar{w}$.
3) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$.
4) $\overline{(\bar{z})} = z$.
5) $\bar{z} = z \Rightarrow z \in \mathbf{R}$.
6) $|zw| = |z||w|$.
7) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.
8) $|z + w| \leq |z| + |w|$ ("triangle inequality").

The Complex Plane



POLAR FORM OF COMPLEX NUMBERS



- $-\pi < \theta \le \pi$.
- $P(\cos\theta, \sin\theta) = \cos\theta + i\sin\theta$.
- $|P| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$.

Examples

1) $\theta = 0$: $\cos\theta + i\sin\theta = 1$.
2) $\theta = \frac{\pi}{2}$: $\cos\theta + i\sin\theta = i$.
3) $\theta = \pi$: $\cos\theta + i\sin\theta = -1$.
4) $\theta = \frac{3\pi}{2}$: $\cos\theta + i\sin\theta = -i$.

Definition

Let $z = a + ib \neq 0$. Then $P = \frac{z}{|z|} = \frac{a+ib}{\sqrt{a^2+b^2}}$, and $|P| = \left|\frac{z}{|z|}\right| = \frac{|z|}{|z|} = 1$. Since $P \in \mathbf{C}$, so there exists a θ such that $P = \cos\theta + i\sin\theta$. θ is the principle argument of z, denoted $\arg z$.

- Remark: If θ is the principle argument of z, $\theta + 2k\pi, k \in \mathbf{Z}$ is an argument of z.

Lecture #10 – Thursday, February 5, 2004

Definition

We write $\cos\theta + i\sin\theta = e^{i\theta}$. If $|z| = r$ and $\arg z = \theta$, then the polar form is $z = re^{i\theta}$.

Example

Write $z_1 = 3 - 3i$ and $z_2 = 2i$ ins polar form.

- Let $r_1 = |z_1| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$. So $z_1 = r_1\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = 3\sqrt{2}e^{-i\frac{\pi}{4}}$.
- Let $r_2 = |z_2| = \sqrt{0^2 + 2^2} = 2$. So $z_1 = r_2(0 - i) = 2e^{i\frac{\pi}{2}}$.

Theorem

If $z_1 = r_1e^{i\theta_1}$, $z_2 = r_2e^{i\theta_2}$, then:

1) $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$.
2) $|z_1z_2| = |z_1||z_2|$.
3) $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$.

Example

Multiply $(1-i)(1+\sqrt{3}i)$.

- $|1-i|=\sqrt{2}$, so $1-i=\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)=\sqrt{2}e^{-i\frac{\pi}{4}}$. $|1+\sqrt{3}i|=2$, so $1+\sqrt{3}i=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2e^{i\frac{\pi}{3}}$. So, $(1-i)(1+\sqrt{3}i)=\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)\left(2e^{i\frac{\pi}{3}}\right)=2\sqrt{2}e^{i\frac{\pi}{12}}=2\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$.
- Since $(1-i)(1+\sqrt{3}i)=(\sqrt{3}+1)+(\sqrt{3}-1)i$, we know $\cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}$ and $\sin\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}$.

Remark: If $z=re^{i\theta}$ is given in polar form, then $z^2=r^2e^{2i\theta}$, $z^3=r^3e^{3i\theta}$, and $\frac{1}{z}=\frac{1}{r}e^{-i\theta}$.

Theorem: De Moivre's Theorem

$\left(e^{i\theta}\right)^n=e^{in\theta}$ for $n\in \mathbf{Z}$. So $(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$.

Example

Calculate $(1-i)^4$.

- $|1-i|=\sqrt{2}$, so $1-i=\sqrt{2}e^{-i\frac{\pi}{4}}$. Hence, $(1-i)^4=\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^4=4e^{-i\pi}=-4$.

Example

Find the 4th root of unity, i.e. all $z\in\mathbf{C}$ such that z^4-1.

- We write $z=re^{i\theta}$, so $z^4=r^4e^{4i\theta}$.
- $z^4=1=e^{i0}$ gives $r^4=1\Rightarrow r=1$.
- $4\theta=0\Rightarrow 4\theta=2k\pi, k\in\mathbf{Z}\Rightarrow\theta=\frac{k\pi}{2}$.
    - When $k=0$, $\theta=0$, so $z_0=1$.
    - When $k=1$, $\theta=\frac{\pi}{2}$, so $z_1=e^{i\frac{\pi}{2}}=i$.
    - When $k=2$, $\theta=\pi$, so $z_2=e^{i\pi}=-1$.
    - When $k=3$, $\theta=\frac{3\pi}{2}$, so $z_3=e^{i\frac{3\pi}{2}}=-i$.
    - When $k=4$, $\theta=2\pi$, so $z_4=e^{i2\pi}=0=z_0$. Therefore, there are no more roots.

Theorem

If $m\geq 1$, then the m^{th} roots of unity are $z_k=e^{\frac{i2k\pi}{m}}, k=0,1,\cdots,m-1$.

Example

Find the 3rd roots of $4+4\sqrt{3}i$.

- We look for z such that $z^3 = 4+4\sqrt{3}i$.
- $\left|4+4\sqrt{3}i\right| = 8$, so $4+4\sqrt{3}i = 8e^{i\frac{\pi}{3}}$.
- If $z = re^{i\theta}$, $z^3 = r^3e^{3i\theta} = 8e^{i\frac{\pi}{3}}$. So $r^3 = 8 \Rightarrow r = 2$, and $3\theta = \frac{\pi}{3} + 2k\pi \Rightarrow \theta = \frac{\pi}{9} + \frac{2k\pi}{3}$.
- So, $z_0 = 2e^{i\frac{\pi}{9}}$, $z_1 = 2e^{i\frac{7\pi}{9}}$, $z_2 = 2e^{i\frac{13\pi}{9}}$.

Definition

A real quadratic is given by $ax^2 + bx + c, a, b, c \in \mathbf{R} \neq 0$. a root of the quadratic is a complex number such that $au^2 + bu + c = 0$. $u = \frac{-b \pm \sqrt{\Delta}}{2a}$, where $\Delta = b^2 - 4ac$ is the <u>discriminate</u>. If $\Delta \geq 0$, $u \in \mathbf{R}$. If $\Delta < 0$, $\sqrt{r} = \pm i\sqrt{|\Delta|}$ and $\begin{cases} u \in \mathbf{C} \\ \overline{u} \in \mathbf{C} \end{cases}$.

Example

Find a real quadratic such that $u = 3 + 2i$ is a root.

- $(x-u)(x-\overline{u})$ is the solution. So the quadratic is $(x-3-2i)(x-3+2i) = x^2 - 6x + 13$.

Complex Quadratic

- $ax^2 + bx + c, a, b, c \in \mathbf{C}, a \neq 0$.
- $\Delta = b^2 - 4ac \in \mathbf{C}$, and $\sqrt{\Delta} = z$.
- $u_1 = \frac{-b + z_1}{2a}$, $u_2 = \frac{-b + z_2}{2a}$.

Example

Find the roots of $x^2 + (2-i)x + 1 - i$.

- $z = \frac{-(2-i) \pm \sqrt{\Delta}}{2}$. $\Delta = (2-i)^2 - 4(1-i) = -1$, so $\sqrt{\Delta} = \pm i$. So $z = \frac{-(2-i) \pm i}{2} \Rightarrow \begin{cases} z_1 = -1 + i \\ z_2 = -1 \end{cases}$.
- <u>Remark</u>: $x^2 + (2-i)x + 1 - i = x^2 - (z_1 + z_2)x + z_1z_2 = (x - z_1)(x - z_2)$.

Theorem

Every polynomial of positive degree with complex coefficient has a complex root. If f such a polynomial, then $f(x) = a(x-u_1)(x-u_2)\cdots(x-u_n)$ for some a, $u_i \in \mathbf{C}$.

Corollary

If f has real coefficient, f can be factored as a product of linear and inducible quadratic factors.