Linear Transformations

EXAMPLES AND ELEMENTARY PROPERTIES

Definition

If V and W are two vector spaces, a function $T:V\to W$ is called a linear transformation if it satisfies the following axioms:

T$_1$: $T(v+_V v_1)=T(V)+_W T(v_1), v, v_1\in V, T(V), T(v_1)\in W$.

T$_2$: $T(r\cdot_V v)=r\cdot_W T(v), r\in \mathbf{R}, v\in V$.

Example

Define $T:P_2\to M_{22}$, and $v=a+bx+cx^2\in P_2\to T(v)=\frac{1}{2}\begin{bmatrix} a+b-c & -a+b+c \\ -a+b+c & a-b+c\end{bmatrix}$. Show that T is linear.

- T$_1$:
  - $$\begin{aligned} T(v+v_1)&=T((a+a_1)+(b+b_1)x+(c+c_1)x^2) \\ &=\frac{1}{2}\begin{bmatrix} (a+a_1)+(b+b_1)-(c+c_1) & -(a+a_1)+(b+b_1)+(c+c_1) \\ -(a+a_1)+(b+b_1)+(c+c_1) & (a+a_1)-(b+b_1)+(c+c_1)\end{bmatrix}\end{aligned}.$$
  - $$\begin{aligned} T(v)+T(v_1)&=T(a+bx+cx^2)+T(a_1+b_1x+c_1x^2) \\ &=\frac{1}{2}\begin{bmatrix} a+b-c & -a+b+c \\ -a+b+c & a-b+c\end{bmatrix}+\frac{1}{2}\begin{bmatrix} a_1+b_1-c_1 & -a_1+b_1+c_1 \\ -a_1+b_1+c_1 & a_1-b_1+c_1\end{bmatrix} \\ &=\frac{1}{2}\begin{bmatrix} (a+a_1)+(b+b_1)-(c+c_1) & -(a+a_1)+(b+b_1)+(c+c_1) \\ -(a+a_1)+(b+b_1)+(c+c_1) & (a+a_1)-(b+b_1)+(c+c_1)\end{bmatrix}\end{aligned}.$$
- T$_2$:
  - $$\begin{aligned} T(r\cdot v)&=T(ra+rbx+rcx^2)=\frac{1}{2}\begin{bmatrix} ra+rb-rc & -ra+rb+rc \\ -ra+rb+rc & ra-rb+rc\end{bmatrix} \\ &=r\frac{1}{2}\begin{bmatrix} a+b-c & -a+b+c \\ -a+b+c & a-b+c\end{bmatrix}=r\cdot T(v)\end{aligned}.$$
- Therefore, T is linear.

Example

The following are linear transformations:

- $D:P_n\to P_{n-1}$ where $D(p_n(x))=(p_n(x))'$ – ex: $D(x^2+3x)=2x+3$.
- $I;P_n\to P_{n+1}$ where $I(p_n(x))=\int_a^x p_n(y)dy$.

Theorem

Let $T:V\to W$ be a linear transformation.

1) $T(0_V)=0_W$.
2) $T(-v)=-T(v), \forall v\in V$.
3) $T\left(\sum_{i=1}^n a_i\cdot v_i\right)=\sum_{i=1}^n a_i\cdot T(v_i)$.

Theorem

Let $T:V \to W$ and $S:V \to W$ be two linear transformations. Suppose that $V = \text{span}\{v_1,\dots,v_n\}$. If $T(v_i) = S(v_i), \forall i$, then $T = S$.

Proof: Let $v = \sum_{i=1}^{n} a_i \cdot v_i \in V$. So $T(v) = T\left(\sum_{i=1}^{n} a_i \cdot v_i\right) = \sum_{i=1}^{n} a_i \cdot T(v_i)$, and $S(v) = S\left(\sum_{i=1}^{n} a_i \cdot v_i\right) = \sum_{i=1}^{n} a_i \cdot S(v_i)$. Thus, $T(v) = S(v)$.

Theorem

Let *V* and *W* be vector spaces, and $\{e_1,\dots,e_n\}$ a basis of *V*. Given any vector $w_1,\dots,w_n \in W$, there exits a unique linear transformation $T:V \to W$ satisfying $T(e_i) = w_i, \forall i$. In fact, the action of *T* is as follows: Given $v = \sum_{i=1}^{n} a_i \cdot v_i \in V$, then $T(v) = \sum_{i=1}^{n} a_i \cdot T(v_i)$.

Example

Find a linear transformation $T:P_2 \to M_{22}$ such that $T(1+x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $T(x+x^2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $T(1+x^2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

- $\{(1+x),(x+x^2),(1+x^2)\}$ is a basis of P_2.
  $a+bx+cx^2 = c_1(1+x)+c_2(x+x^2)+c_3(1+x^2)$
- $\Rightarrow (a-c_1-c_3)+(b-c_1-c_2)x+(c-c_2-c_3)x^2 = 0 \Rightarrow \begin{cases} a-c_1-c_3 = 0 \\ b-c_1-c_2 = 0 \\ c-c_2-c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = \dfrac{a+b-c}{2} \\ c_2 = \dfrac{-a+b+c}{2} \\ c_3 = \dfrac{a-b+c}{2} \end{cases}$.
- $T(v) = T\left(c_1(1+x)+c_2(x+x^2)+c_3(1+x^2)\right) = c_1 \cdot T(1+x)+c_2 \cdot T(x+x^2)+c_3 \cdot T(1+x^2)$
  $= c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{a+b-c}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+\frac{-a+b+c}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}+\frac{a-b+c}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
  $\Rightarrow T(v) = \begin{bmatrix} \dfrac{a+b-c}{2} & \dfrac{-a+b+c}{2} \\ \dfrac{-a+b+c}{2} & \dfrac{a-b+c}{2} \end{bmatrix}$

Kernel and Image of a Linear Transformation

Definition

Let $T:V \to W$ be a linear transformation. Then:

- $\ker T = \{v \in V \mid T(v) = 0\}$.

- $\operatorname{im} T = \{T(v) \mid v \in V\}$.

Theorem

If $T : V \to W$ is a linear transformation, then $\ker T$ is a subspace of *V*, and $\operatorname{im} T$ is a subspace of *W*.

Definition

$\operatorname{nullity}(T) = \dim(\ker T)$.
$\operatorname{rank}(T) = \dim(\operatorname{im} T)$.

Example

Given an *m*×*n* matrix *A*, show that $\operatorname{im} T_A = \operatorname{col} A$ (so $\operatorname{rank} T_A = \operatorname{rank} A$), where $T_A : \mathbf{R}^n \to \mathbf{R}^m \mid T(X) = AX$.

- Write $A = [C_1 \quad \cdots \quad C_n]$ where C_i are columns, and $X = [x_1 \quad \cdots \quad x_n]^T, x_i \in \mathbf{R}$.
- Then $\operatorname{im} A = AX = [C_1 \quad \cdots \quad C_n][x_1 \quad \cdots \quad x_n]^T = x_1 C_1 + \cdots x_n C_n = \operatorname{col} A$.

One-to-One and Onto Transformation

Definition

Let $T : V \to W$ be a linear transformation. Then:

- *T* is said to be <u>onto</u> if $\operatorname{im} T = W$.
- *T* is said to be <u>one-to-one</u> if $T(v) = T(v_1) \Rightarrow v = v_1$ (each vector in *W* corresponds to only one element in *V*).

Theorem

If $T : V \to W$ is a linear transformation, then *T* is one-to-one if and only if $\ker T = 0$.

Proof:

- Want: *T* is one-to-one $\Rightarrow$ $\ker T = 0$.
  - Let $v \in \ker T$.
  - $T(v) = 0 = T(0) \Rightarrow v = 0$ because *T* is one-to-one. So $\ker T = 0$.
- Want: $\ker T = 0$ $\Rightarrow$ *T* is one-to-one.
  - Let $T(v) = T(v_1) \Rightarrow T(v - v_1) = 0$.
  - But since $\ker T = 0$, $v - v_1 = 0 \Rightarrow v = v_1$. So *T* is one-to-one.

Example

Given $T : \mathbf{R}^2 \to \mathbf{R}^3 \mid T(x, y) = (x + y, x - y, x)$, show that *T* is one-to-one but not onto.

- Want: *T* is one-to-one.
  - $\ker T = \{(x, y) \mid T(x, y) = 0\}$.
  - $T(x, y) = (x + y, x - y, x) = 0 \Rightarrow \begin{cases} x + y = 0 \\ x - y = 0 \\ x = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$. So $\ker T = 0$.
  - Since $\ker T = 0$, *T* is one-to-one.

- Want: T is not onto.
  - Consider $w = (0,0,1) \in \mathbf{R}^3$.
  - $T(v) = (0,0,1)$ is not possible because $\begin{cases} x+y=0 \\ x-y=0 \\ x=1 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=1 \\ y=-1 \end{cases}$ has no solution.
  - So T is not onto.

Theorem

Let A be an $m \times n$ matrix and let $T_A : \mathbf{R}^n \to \mathbf{R}^m$ be the linear transformation induced by A, that is, $T_A(X) = AX, \forall X \in \mathbf{R}^n$. Then:

1) T_A is onto if and only if $\operatorname{rank} A = m$.
2) T_A is one-to-one if and only if $\operatorname{rank} A = n$.

THE DIMENSION THEOREM

Theorem: Dimension Theorem

Let $T : V \to W$ be a linear transformation and assume that $\ker T$ and $\operatorname{im} T$ are both finite dimensional. Then V is also finite dimensional and $\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$.

Proof:

- Let $\{T(e_1), \ldots, T(e_r)\}$ be a basis of $\operatorname{im} T$, and $\{f_1, \ldots, f_k\}$ be a basis of $\ker T$.
- Then $\dim(\operatorname{im} T) = r$ and $\dim(\ker T) = k$. It suffice to show $B = \{e_1, \ldots, e_r, f_1, \ldots, f_k\}$ is a basis of V.
- Want: B spans V.
  - If v lies in V, then $T(v)$ lies in $\operatorname{im} T$.
  - So $T(v) = \sum_{i=1}^{r} t_i \cdot T(e_i), t_i \in \mathbf{R}$.
  - This means $v - \sum_{i=1}^{r} t_i \cdot e_i$ lies in $\ker T$, because

    $T(v) - \sum_{i=1}^{r} T(t_i \cdot e_i) = 0 \Rightarrow T\left(v - \sum_{i=1}^{r} t_i \cdot e_i \right) = 0 \in \ker T$.
  - Since $v - \sum_{i=1}^{r} t_i \cdot e_i = \sum_{i=1}^{k} a_i \cdot f_i \Rightarrow v = \sum_{i=1}^{r} t_i \cdot e_i + \sum_{i=1}^{k} a_i \cdot f_i$, so B spans V.
- Want: $B = \{e_1, \ldots, e_r, f_1, \ldots, f_k\}$ is linearly independent.
  - Let $\sum_{i=1}^{r} t_i \cdot e_i + \sum_{i=1}^{k} a_i \cdot f_i = 0$.
  - Apply T:

    $T\left(\sum_{i=1}^{r} t_i \cdot e_i + \sum_{i=1}^{k} a_i \cdot f_i \right) = T(0) \Rightarrow T\left(\sum_{i=1}^{r} t_i \cdot e_i \right) + T\left(\sum_{i=1}^{k} a_i \cdot f_i \right) = 0 \Rightarrow T\left(\sum_{i=1}^{r} t_i \cdot e_i \right) = 0$

    since $\{f_1, \ldots, f_k\}$ is a basis of $\ker T$.

- So $T\left(\sum_{i=1}^{r} t_i \cdot e_i\right) = 0 \Rightarrow \sum_{i=1}^{r} t_i \cdot T(e_i) = 0 \Rightarrow t_i = 0$ since $\{T(e_1), \ldots, T(e_r)\}$ is a basis, so it is linearly independent.
- So $\sum_{i=1}^{r} t_i \cdot e_i + \sum_{i=1}^{k} a_i \cdot f_i = 0 \Rightarrow \sum_{i=1}^{k} a_i \cdot f_i = 0 \Rightarrow a_i = 0$ since $\{f_1, \ldots, f_k\}$ is a basis, so it is linearly independent.
- So $B = \{e_1, \ldots, e_r, f_1, \ldots, f_k\}$ is linearly independent.

Theorem

Let $T : V \to W$ be a linear transformation and let $\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_n\}$ be a basis of *V* such that $\{e_{r+1}, \ldots, e_n\}$ is a basis of $\ker T$. Then $\{T(e_1), \ldots, T(e_r)\}$ is a basis of $\operatorname{im} T$.

Isomorphism

Definition

A linear transformation $T : V \to W$ is an <u>isomorphism</u> if it is both one-to-one and onto. In this case, *V* is isomorphic to *W*.

Examples

1) $1_V : V \to V$ the identity transformation is an isomorphism.
2) $T : P_2 \to \mathbf{R}^3, T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.
   - *T* is a linear transformation.
   - *T* is one-to-one: Let $T(ax^2 + bx + c) = 0 \Rightarrow a = b = c = 0 \Rightarrow \ker T = 0$, so *T* is one-to-one.
   - *T* is onto: Given $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbf{R}^3$, $T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, so *T* is onto.

Remark

Consider an isomorphism $T : V \to W$. then we can identify *V* with *W* through the pairing $V \leftrightarrow T(V) = W$. In the category of vector spaces, *V* and *W* are "identical" (in the same class).

Theorem

If *V* and *W* are finite dimensional, the following conditions are equivalent for a linear transformation $T : V \to W$:

1) *T* is an isomorphism.
2) If $\{e_1, \ldots, e_n\}$ is any basis of *V*, then $\{T(e_1), \ldots, T(e_n)\}$ is a basis of *W*.
3) There exists a basis $\{e_1, \ldots, e_n\}$ of *V* such that $\{T(e_1), \ldots, T(e_n)\}$ is a basis of *W*.

Proof:

- 1⇒2:
  - Let $\{e_1, \ldots, e_n\}$ be a basis of *V*.

- Linear independence: Let $t_1 \cdot T(e_1) + \cdots + t_n \cdot T(e_n) = 0 \Rightarrow T(t_1 \cdot e_1 + \cdots + t_n \cdot e_n) = 0 \Rightarrow t_1 \cdot e_1 + \cdots + t_n \cdot e_n = 0$ since $\ker T = 0$, so $t_1 = \cdots = t_n = 0$ since $\{e_1, \ldots, e_n\}$ is a basis.
- Span: Take $w \in W$. Since T is onto, there exists $v \in V$ such that $T(v) = W$. Write $v = t_1 \cdot e_1 + \cdots + t_n \cdot e_n$. So $w \in \operatorname{span}\{T(e_1), \ldots, T(e_n)\}$.

- 2⇒3:
  - V has a basis.
- 3⇒1:
  - One-to-one: Suppose $T(v) = 0$. Write $v = t_1 \cdot e_1 + \cdots + t_n \cdot e_n 0$, so $T(v) = 0 \Rightarrow v = t_1 \cdot T(e_1) + \cdots + t_n \cdot T(e_n) = 0 \Rightarrow t_1 = \cdots = t_n = 0 \Rightarrow v = 0$. Since $\ker T = \{0\}$, so T is one-to-one.
  - Onto: Let $w \in W$. Write $w = t_1 \cdot T(e_1) + \cdots + t_n \cdot T(e_n) = T(t_1 \cdot e_1 + \cdots + t_n \cdot e_n) = T(v) \Rightarrow w = T(v)$. So T is onto.

Theorem

Two finite dimensional vector spaces V and W are isomorphic if and only if $\dim V = \dim W$.

Proof:

- Assume $T : V \to W$ is an isomorphism. By the previous theorem, there exits a basis $\{e_1, \ldots, e_n\}$ of V such that $\{T(e_1), \ldots, T(e_n)\}$ is a basis of W. So $\dim V = \dim W$.
- Assume $\dim V = \dim W = n$. Let $\{e_1, \ldots, e_n\}$ be a basis of V and $\{f_1, \ldots, f_n\}$ a basis of W. Define a linear transformation $T : V \to W$ by $T(e_i) = f_i, \forall i$. By construction, $\{T(e_1) = f_1, \ldots, T(e_n) = f_n\}$ is a basis of W. By previous theorem, T is an isomorphism.

Corollary

Every vector space of dimension n is isomorphic to $\mathbf{R}^n$.

Examples

$$\mathbf{R}^4 \cong P_3 \cong M_{22}, \text{ so } \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \cong ax^3 + bx^2 + cx + d \cong \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Theorem

If $\dim V = \dim W = n$, a linear transformation $T : V \to W$ is an isomorphism if it is either one-to-one or onto.

Proof:

- If T is one-to-one, then $\ker T = 0 \Rightarrow \dim(\operatorname{im} T) = n = \dim W \Rightarrow \operatorname{im} T = \mathrm{W}$. So T is onto.
- If T onto, then $\operatorname{im} T = W \Rightarrow \dim(\operatorname{im} T) = \dim W = n \Rightarrow \dim(\ker T) = 0$. So T is one-to-one.

COMPOSITION

Definition

Given linear transformations $V \xrightarrow{T} W \xrightarrow{S} U$, the <u>composite</u> $ST : V \to U$ of T and S is defined by $(ST)(v) = S(T(v)), \forall v \in V$.

Remark

In general, $ST \neq TS$.

Theorem

Let $V \xrightarrow{T} W \xrightarrow{S} U \xrightarrow{R} Z$ be linear transformations.

1) The composite ST is again a linear transformation.
2) $T1_V = T$ and $1_W T = T$.
3) $(RS)T = R(ST)$.

Theorem

Let V, W be finite dimensional vector spaces. The following conditions are equivalent for a linear transformation $T : V \to W$:

1) T is an isomorphism.
2) There exists a linear transformation $S : W \to V$ such that $ST = 1_V$ and $TS = 1_W$.

Moreover, in this case S is also an isomorphism and is uniquely determined by T. If $w \in W$ is written as $w = T(v)$, then $S(w) = v$.

Proof:

- Assume T is an isomorphism. If $B = \{e_1, \ldots, e_n\}$ is a basis of V, then $\{T(e_1), \ldots, T(e_n)\}$ is a basis of W. Define $S : W \to V$ by $S(T(e_i)) = e_i, \forall i$. Then $ST = 1_V$ and $TS = 1_W$.
- Assume S is a linear transformation such that $ST = 1_V$ and $TS = 1_W$. Let $T(v) = T(v_1) \Rightarrow ST(v) = ST(v_1) \Rightarrow v = v_1$, so T is one-to-one. If $w \in W$, then let $v = S(w) \Rightarrow T(v) = TS(w) = w$, so T is onto.

THE MATRIX OF A LINEAR TRANSFORMATION

- Every linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$ is induced by an $m \times n$ matrix A – $T : \mathbf{R}^n \to \mathbf{R}^m, T_A(X) = AX, X \in \mathbf{R}^n$.
- This idea can be extended to any pair of vector spaces V and W of dimensions n and m respectively.
- How to transform V into $\mathbf{R}^n$?

Definition

Let $B = \{e_1, \ldots, e_n\}$ be an ordered basis of V. We know $v = v_1 e_1 + \cdots + v_n e_n, \forall v \in V$. The <u>coordinate vector of V with respect to B</u> is defined to be $C_B(v) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$.

Theorem

If V has dimension n and B is any ordered basis of V, the coordinate transformation $C_B : V \to \mathbf{R}^n$ is an isomorphism.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "C_B"'] & W \arrow[d, "C_D"] \\ \mathbf{R}^n \arrow[r, "T_A"] & \mathbf{R}^m \end{tikzcd}$$

- Fixed bases $B = \{e_1, \ldots, e_n\}$ and $D = \{f_1, \ldots, f_n\}$ are in V and W respectively.
- $C_D T = T_A C_B \Leftrightarrow C_D T = A C_B \Leftrightarrow T = (C_D)^{-1}(A C_B)$.

Definition

The <u>matrix of T corresponding to the ordered bases B and D</u> is $M_{DB}(T) = [C_D(T(e_1)) \ \cdots \ C_D(T(e_n))]$.

Theorem

Let $T : V \to W$ be a linear transformation where $\dim V = n$ and $\dim W = m$, and let $B = \{e_1, \ldots, e_n\}$ and $D = \{f_1, \ldots, f_n\}$ be ordered bases of V and W respectively. Then the matrix $M_{DB}(T)$ is the unique $m \times n$ matrix A that satisfies $C_D T = T_A C_B$. Hence the defining property of $M_{DB}(T)$ is $C_D(T(v)) = M_{DB} C_B(v), \forall v \in V$.

Example

Define $T : P_2 \to \mathbf{R}^2$ by $T(a + bx + cx^2) = (a + c, b - a - c)$. If $B = \{1, x, x^2\}$ is an ordered basis of P_2 and $D = \{(1,0), (0,1)\}$ is an ordered basis of $\mathbf{R}^n$, find $M_{DB}(T)$.

- $M_{DB}(T) = [C_D(T(1,-1)) \quad C_D(T(0,1)) \quad C_D(T(1,-1))] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} = A$.
- So $T(v) = (C_D)^{-1}(A C_B(v))$.

Example

Given $T : P_2 \to \mathbf{R}^2$ and $M_{DB} = \begin{bmatrix} 5 & 2 & -1 \\ 3 & 0 & 4 \end{bmatrix}$ where $B = \{1, x, x^2\}$ and $D = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, express T explicitly.

- $C_D(T(v)) = M_{DB} C_B(v), \forall v \in V$.
- Let $v = a + bx + cx^2$. So $C_D(T(v)) = \begin{bmatrix} 5 & 2 & -1 \\ 3 & 0 & 4 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5a + 2b - c \\ 3a - 4c \end{pmatrix}$.
- So $T(v) = (5a + 2b - c)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (3a - 4c)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8a + 2b - 5c \\ 3a - 4c \end{pmatrix}$.

Theorem

Let $V \xrightarrow{T} W \xrightarrow{S} U$ be linear transformations and let *B*, *D*, *E* be finite ordered bases of *V*, *W*, *U* respectively. Then $M_{EB}(ST) = M_{ED}(S)M_{DB}(T)$.

Proof:

- If $v \in V$, then $M_{ED}(S)M_{DB}(T)C_B(v) = M_{ED}(S)C_D(T(v)) = C_E(ST(v)) = M_{EB}C_B(v)$.
- So $M_{EB} = M_{ED}M_{DB}$.

Theorem

Let $T : V \to W$ be a linear transformation where $\dim V = \dim w = n$. The following are equivalent:

1) *T* is an isomorphism.
2) $M_{DB}(T)$ is invertible for all ordered bases *B* and *D* of *V* and *W*.
3) $M_{DB}(T)$ is invertible for some pair of ordered basis *B* and *D* of *V* and *W*.

When this is the case, $M_{DB}(T)^{-1} = M_{BD}(T^{-1})$.

Proof:

- **1** $\Rightarrow$ **2**: We have $V \xrightarrow{T} W \xrightarrow{T^{-1}} V$. So $M_{BD}(T^{-1})M_{DB}(T) = M_B(T^{-1}T) = M_{BB}(1_v) = I_n$. So $M_{DB}(T)$ has an inverse, $M_{BD}(T^{-1})$.
- **2** $\Rightarrow$ **3**: (3) is a particular case of (2).
- **3** $\Rightarrow$ **1**: Let $M_{DB}(T)$ be invertible for some *B* and *D*. Then $C_D T = T_M C_B$. Hence $T = C_D{}^{-1} T_M C_B$ is an isomorphism.

Theorem

Let $T : V \to W$ be a linear transformation where $\dim V = n$ and $\dim W = m$. If *B* and *D* are ordered bases of *V* and *W*, then $\operatorname{rank}(T) = \operatorname{rank}(M_{DB}(T))$.

Proof:

- Remember $\operatorname{rank} T = \dim(\operatorname{im} T)$.
- The column space of *M* is $U = \{MX \mid X \in \mathbf{R}^n\} \Rightarrow \operatorname{rank} M = \dim U$. So it suffice to find an isomorphism $S : \operatorname{im} T \to U$.
- Define $S(T(v)) = C_D(T(v)), \forall T(v) \in \operatorname{im} T$.
- C_D is linear and one-to-one, so *S* is also linear and one-to-one. It remains to show that C_D is onto.
- Let $MX \in U$. Then $X = C_B(v)$ for some $v \in V$. Hence, $MX = MC_B(v) = C_D T(v)$. This means C_D is onto since C_B is onto. So *S* is onto, and thus is an isomorphism.

CHANGE OF BASIS

- How to go from $C_B(v)$ to $C_D(v)$ where *B*, *D* are two bases of *V* and $C_B : V \to \mathbf{R}^n$, $C_D : V \to \mathbf{R}^n$?

Theorem

Let $B = \{b_1, \dots, b_n\}$ and $D = \{d_1, \dots, d_n\}$ be two ordered bases of a vector space *V*. Define the <u>change matrix</u> $P_{D \leftarrow B}$ as $P_{D \leftarrow B} = [C_D(b_1) \ \cdots \ C_D(b_n)]$. Then $C_D(v) = P_{D \leftarrow B} C_B(v)$ holds for every vector $v \in V$.

Example

Let the vector space $V = P_2$. Find $P_{D\leftarrow B}$ if $B = \{1, x, x^2\}$ and $D = \{1, 1-x, (1-x)^2\}$.

- $P_{D\leftarrow B} = \begin{bmatrix} C_D(1) & C_D(x) & C_D(x^2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

Theorem

Let B, D, E be three ordered bases of an n-dimensional vector space V.

1) $P_{B\leftarrow B} = I$.
2) $P_{D\leftarrow B}$ is invertible and $P_{D\leftarrow B}{}^{-1} = P_{B\leftarrow D}$.
3) $P_{E\leftarrow D} P_{D\leftarrow B} = P_{E\leftarrow B}$.

- Given a linear operator $T : V \to V$, how simple can the matrix $M_{BB}(T)$ be made by an appropriate chose of basis B?
- Denote $M_{BB}(T) = M_B(T)$.

Theorem

Let B_0 and B be two ordered basis of a finite dimensional vector space V. If $T : V \to V$ is any linear transformation, the matrices $M_B(T)$ and $M_{B_0}(T)$ of T with respect to these bases are similar.

$M_B(T) = P^{-1} M_{B_0}(T) P$ where $P = P_{B_0\leftarrow B}$ is the change matrix from B to B_0.

Proof:

- We know $C_{B_0}(v) = P_{B_0\leftarrow B} C_B(v)$ and $C_B(T(v)) = M_B(T) C_B(v)$.
- $P_{B_0\leftarrow B} M_B(T) C_B(v) = P_{B_0\leftarrow B} C_B(T(v)) = C_{B_0}(T(v)) = M_{B_0}(T) C_{B_0}(v) = M_{B_0}(T) P_{B_0\leftarrow B} C_B(v)$, so $P_{B_0\leftarrow B} M_B(T) = M_{B_0}(T) P_{B_0\leftarrow B} \Rightarrow M_B(T) = P_{B_0\leftarrow B}{}^{-1} M_{B_0}(T) P_{B_0\leftarrow B}$.

Theorem

Let A be an $n\times n$ square matrix. Let B_0 be the standard basis of $\mathbf{R}^n$. Let $T_A : \mathbf{R}^n \to \mathbf{R}^n, T_A(X) = AX$. Then:

1) $M_{B_0}(T_A) = A$.
2) If $A' = P^{-1}AP$, let B be the ordered basis of $\mathbf{R}^n$ consisting of the columns of P in order, then $M_B(T_A) = A' = P^{-1}AP$.
3) If B is any ordered basis of $\mathbf{R}^n$, let P be the invertible matrix whose columns are the vectors of B in order, then $M_B(T_A) = P^{-1}AP$.

Example

Given a matrix A, $P = \begin{bmatrix} 2 & -1 \\ -3 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, find a basis of $\mathbf{R}^2$ such that $M_B(T_A) = D$.

- Let $B = \operatorname{col} B = \left\{ \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$.

Definitions

1) A property of $n \times n$ matrices is called a <u>similarity invariant</u> whenever a given matrix A has that property and every other similar matrices has that property also (ex: determinants).
2) If $T : V \to V$ is a linear operator, the <u>determinant of T</u>, $\det T$, is defined by $\det T = \det M_B(T)$ where B is any basis of V.
3) The <u>characteristic polynomial</u> of a linear operator T is defined by $c_T(x) = c_A(x)$ where $A = M_B(T)$ and B any basis of V.

Note that $\det T$ and $c_T(x)$ are both independent of the choice of B.

INVARIANT SUBSPACES

Definition

If U is a subspace of V, write $T(u) = \{T(u) \mid u \in U\}$. If $T : V \to V$ is a linear operator, a subspace U is said to be <u>T-invariant</u> if $T(u)$ lies in U for every vector $u \in U$.

Theorem

Let $T : V \to V$ be a linear operator where V has dimension n. Suppose U is any T-invariant subspace of V. Let $B_1 = \{e_1, \ldots, e_k\}$ be any basis if U and extend it to a basis $B = \{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$ of V in any way. then $M_B(T)$ has the block triangular form $M_B(T) = \begin{bmatrix} M_{B_1}(T) & Y \\ 0 & Z \end{bmatrix}$ where Y is $k \times (n-k)$, Z is $(n-k) \times (n-k)$, and $M_{B_1}(T)$ is the matrix of the restriction of T to U.

Proof:

- The matrix of the restriction of $T : U \to U$ with respect to the basis B_1 is the $k \times k$ matrix $M_{B_1}(T)$.
- Compare the first column $C_{B_1}(T(e_1))$ with the first column of $C_B(T(e_1))$ of $M_B(T)$. Since $T(e_1)$ lies in U, $T(e_1) = t_1 e_1 + \cdots + t_n e_k + 0 e_{k+1} + \cdots + 0 e_n$.
- So $M_B(T) = \begin{bmatrix} M_{B_1}(T) & Y \\ 0 & Z \end{bmatrix}$.

Theorem

Let A be a block upper triangle matrix, say $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{nn} \end{bmatrix}$, where the diagonal blocks are square. Then:

1) $\det A = \det A_{11} \times \cdots \times \det A_{nn}$.
2) $c_A(x) = c_{A_{11}}(x) \times \cdots \times c_{A_{nn}}(x)$.

Example

Let $T : P_2 \to P_2, T(a + bx + cx^2) = (-2a - b + 2c) + (a + b)x + (-6a - 2b + 5c)x^2$. Show that $U = \operatorname{span}\{x, 1 + 2x^2\}$ is T-invariant and use it to find a block upper triangle representation of T.

- $T(x) = -1 + x - 2x^2 = (1)x + (-1)(1 + 2x^2) \Rightarrow T(x) \in U$, and $T(1 + 2x^2) = 2 + x + 4x^2 = (1)x + (2)(1 + 2x^2) \Rightarrow T(1 + 2x^2) \in U$. So U is a T-invariant subspace.
- Extend $B_1 = \{x, 1 + 2x^2\}$ to a basis $B = \{x, 1 + 2x^2, x^2\}$ of P_2.
  $M_B(T) = [C_B(T(x)) \quad C_B(T(1 + 2x^2)) \quad C_B(T(x^2))]$
- $$= [C_B(-1 + x - 2x^2) \quad C_B(2 + x + 4x^2) \quad C_B(2 + 5x^2)] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} M_{B_1}(T) & Y \\ 0 & Z \end{bmatrix}.$$

EIGENVALUES

Definitions

1) A real number λ is called an <u>eigenvalue</u> of an operator $T : V \to V$ if $T(v) = \lambda v$ holds for some non-zero vector $v \in V$.
2) The subspace $E_\lambda(T) = \{v \in V \mid T(v) = \lambda \cdot v\}$ is called the <u>eigenspace</u> of T corresponding to λ.

Theorem

Let $T : V \to V$ be a linear operator where $\dim V = n$, let B be any ordered basis of V, and let $C_B : V \to \mathbf{R}^n$ denote the coordinate isomorphism. Then:

1) The eigenvalues λ of T are precisely the eigenvalues of the matrix $M_B(T)$, and thus, are the roots of the characteristic polynomial $c_T(x)$.
2) In this case, the eigenspace $E_\lambda(T)$ and $E_\lambda(M_B(T))$ are isomorphic via $C_B : E_\lambda(T) \to E_\lambda(M_B(T))$.

Proof of 1:

- Write $A = M_B(T)$.
- If $T(v) = \lambda v$, then applying C_B gives $\lambda C_B(v) = C_B(T(v)) = AC_B(v) \Rightarrow \lambda C_B(v) = AC_B(v)$.
- So λ is an eigenvalue of A with eigenvector $C_B(v)$.

Proof of 2:

- $C_B(v)$ lies in $E_\lambda(A)$, so we have a function $C_B : E_\lambda(T) \to E_\lambda(A)$. We know C_B is linear and one-to-one. Claim C_B is isomorphic.
- $C_B(T(v)) = AC_B(v) = \lambda C_B(v) = C_B(\lambda v) \Leftrightarrow T(v) = \lambda v$.

Theorem

Each eigenspace of a linear operator $T : V \to V$ is a T-invariant subspace of V.

Proof:

- If v lies in the eigenspace $E_\lambda(T)$, then $T(v) = \lambda v$.
- $T(T(v)) = T(\lambda v) = \lambda T(v) \Rightarrow T(T(v)) \in E_\lambda(T)$. So $T(v)$ lies in the eigenspace $E_\lambda(T)$.

DIRECT SUMS

Definitions

1) $U+W=\{u+w \mid u\in U, w\in W\}$ and $U\cap W=\{v \mid v\in U \text{ and } v\in W\}$.
2) A vector space V is said to be the direct sum of subspaces U and W if $U+W=V$ and $U\cap W=0$. In this case, $V=U\oplus W$.
3) Given a subspace U, any subspace W such that $V=U\oplus W$ is called a complement of U in V.

Theorem

Let U and W be subspaces of a finite dimensional vector space V. The following conditions are equivalent.

1) $V=U\oplus W$.
2) Each vector $v\in V$ can be written uniquely in the form $v=u+w, u\in U, w\in W$.
3) If $\{u_1,\dots,u_k\}$ and $\{w_1,\dots,w_n\}$ are basis of U and W respectively, then $B=\{u_1,\dots,u_k,w_1,\dots,w_n\}$ is a basis of V.

Theorem

If a finite dimensional vector space $V=U\oplus W$ is a direct sum of subspaces U and W, then $\dim V=\dim U+\dim W$.

- Note: If U_1 is a T-invariant subspace, then $M_B(T)=\begin{bmatrix} M_{B_1}(T) & Y \\ 0 & Z \end{bmatrix}$ where B_1 is a basis of U_1 and $B=\{B_1,e_{k+1},\dots,e_n\}$ is a basis of V. Now, if we can find $\{e_{k+1},\dots,e_n\}$ a basis of another T-invariant subspace U_2 such that $V=U_1\oplus U_2$, then $Y=0$.

Theorem

Let $T:V\to V$ be a linear operator where $\dim V=n$. Suppose $V=U_1\oplus U_2$ where both U_1 and U_2 are T-invariant. If $B_1=\{e_1,\dots,e_k\}$ and $B_2=\{e_{k+1},\dots,e_n\}$ are bases of U_1 and U_2 respectively, then $B=\{B_1,B_2\}$ is a basis of V and $M_B(T)$ has the block diagonal form $M_B(T)=\begin{bmatrix} M_{B_1}(T) & 0 \\ 0 & M_{B_2}(T) \end{bmatrix}$ where $M_{B_1}(T)$ and $M_{B_2}(T)$ are the matrices of the restriction of T to U_1 and U_2 respectively.

Definition

T is said to be reducible if non-zero T-invariant subspaces U_1 and U_2 can be found such that $V=U_1\oplus U_2$.

Example

Consider $T:\mathbf{R}^2\to\mathbf{R}^2, T\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} a+b \\ b \end{pmatrix}$. Show that $U_1=\left\{u \mid u=a\begin{pmatrix} 1 \\ 0 \end{pmatrix}, a\in\mathbf{R}\right\}$ is T-invariant but it has no T-invariant complement in $\mathbf{R}^2$.

- $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}\in U_1$, so U_1 is T-invariant.

- Assume U_1 has a T-invariant complement U_2, so $U_1 \oplus U_2 = \mathbf{R}^2$ and $T(u_2) \in U_2, u_2 \in U_2$. Now, $\dim U_2 = 1$, so $U_2 = \left\{ u_2 \mid u_2 = a\begin{pmatrix} p \\ q \end{pmatrix}, a, p, q \in \mathbf{R} \right\}$ and $\begin{pmatrix} p \\ q \end{pmatrix} \notin U_1 \Rightarrow q \neq 0$. Let $T\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p+q \\ q \end{pmatrix} = a\begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow \begin{cases} p+q = ap \\ q = aq \end{cases} \Rightarrow a = 1 \Rightarrow p+q = p \Rightarrow q = 0$. Contradiction!

BLOCK TRIANGULAR FORM

Theorem: Block Triangular Theorem

Let A be an $n \times n$ matrix with real eigenvalues and let $c_A(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots(x-\lambda_k)^{m_k}$ where the λ_i are distinct. Then an invertible matrix P exists such that $P^{-1}AP = \begin{bmatrix} U_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & U_k \end{bmatrix}$ where the

$U_i = \begin{bmatrix} \lambda_i & \cdots & Y \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_i \end{bmatrix}$ is an $m_i \times m_i$ upper triangular matrix with every entry on the main diagonal equal to λ_i.

Definition

The <u>generalized eigenspace</u> $G_{\lambda_i}(A)$ is defined by $G_{\lambda_i}(A) = \operatorname{null}\left[(\lambda_i I - A)^{m_i}\right]$.

- Notice that $E_{\lambda_i}(A) = \operatorname{null}(\lambda_i I - A) = \{v \in V \mid Av = \lambda_i v\}$ is a subspace of $G_{\lambda_i}(A)$.

Lemma

$\dim(G_{\lambda_i}(A)) = m_i$.

Proof:

- Write $A_i = (\lambda_i I - A)^{m_i}$. Let P be defined as in the Block Triangulation Theorem.
- Then the spaces $G_{\lambda_i}(A) = \operatorname{null} A_i$ and $\operatorname{null}(P^{-1}AP)$ are isomorphic via $f : X \to P^{-1}X$.
- To see that f is one-to-one, let $P^{-1}X_1 = P^{-1}X_2 \Leftrightarrow P^{-1}(X_1 - X_2) = 0 \Leftrightarrow X_1 - X_2 = 0 \Leftrightarrow X_1 = X_2$. Now, let $X \in \operatorname{null} A_i$, so $(\lambda_i I - A)^{m_i} X = A_i X = 0 \Rightarrow P^{-1}A_i P(P^{-1}X) = 0 \Rightarrow (P^{-1}A_i P)X = 0$ and $X \in \operatorname{null}(P^{-1}A_i P)$. So f is one-to-one.
- So $\dim(\operatorname{null}(P^{-1}A_i P)) = m_i$ because $P^{-1}A_i P = \begin{bmatrix} \lambda_i I - U_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_i I - U_k \end{bmatrix}^{m_i} = \begin{bmatrix} (\lambda_i I - U_1)^{m_i} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\lambda_i I - U_k)^{m_i} \end{bmatrix}$. The matrix $(\lambda_i I - U_j)^{m_i}$ is invertible if $i \neq j$ and zero if $i = j$.
- It follows that $\dim(G_{\lambda_i}(A)) = m_i$ as required.

Lemma

If P is as in the Block Triangulation Theorem, denote the columns of P as follows: $\{P_{11},\ldots,P_{1m_1};P_{21},\ldots,P_{2m_2};\cdots;P_{k1},\ldots,P_{km_k}\}$. Then $\{P_{i1},\ldots,P_{im_1}\}$ is a basis of $G_{\lambda_i}(A)$..

Lemma

If B_i is any basis of $G_{\lambda_i}(A)$, then $B = B_1 \cup B_2 \cup \cdots \cup B_k$ is a basis of $\mathbf{R}^n$.

Proof: $E_{\lambda_1}(A) \subseteq G_{\lambda_i}(A) \Rightarrow E_{\lambda_1}(A) = \operatorname{null}(\lambda_i I - A) \subseteq \operatorname{null}\left((\lambda_i I - A)^2\right) \subseteq \cdots \subseteq \operatorname{null}\left((\lambda_i I - A)^{m_i}\right) = G_{\lambda_i}(A)$.

Triangulation Algorithm

Suppose A has characteristic polynomial $c_A(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots(x-\lambda_k)^{m_k}$.

1) Choose a basis of $\operatorname{null}(\lambda_i I - A)$, enlarge it by adding vectors to a basis of $\operatorname{null}\left((\lambda_i I - A)^2\right)$, and so on. Continue to obtain an ordered basis $\{P_{i1}, P_{i2},\ldots,P_{im_1}\}$ of $G_{\lambda_i}(A)$ for each i.
2) Let $P = \left[P_{11} \quad \cdots \quad P_{1m_1} \quad \cdots \quad P_{k1} \quad \cdots \quad P_{km_k}\right]$ be the matrix with these basis vectors (in order) as columns. Then $P^{-1}AP = \operatorname{diag}(U_1,\ldots,U_k)$ as in the Block Triangulation Theorem.