Inner Product Spaces

INNER PRODUCTS

Definition
An inner product on a vector space $V$ is a function that assigns a number $\langle v, w \rangle$ to every pair $v, w$ of vector space $V$ in such a way that the following axioms holds:

$P_1$: $\langle v, w \rangle$ is a real number.
$P_2$: $\langle v, w \rangle = \langle w, v \rangle$.
$P_3$: $\langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle$.
$P_4$: $\langle rv, w \rangle = r \langle v, w \rangle$.
$P_5$: $\langle v, v \rangle \geq 0, \forall v \in V$.

Definition
A vector space $V$ with an inner product is called an inner product space.

Note
- $\langle \cdot \rangle : V \times V \to \mathbb{R}$.
- $(V, \mathbb{R}, +, \cdot)$ is a vector space.
- $(V, \mathbb{R}, +, \cdot, \langle \cdot \rangle)$ is an inner product space.

Examples
The following are inner product spaces.
1) $(\mathbb{R}^n, \mathbb{R}, +, \cdot, \langle \cdot \rangle)$, define $\langle X, Y \rangle = X \cdot Y$ the dot product.
2) $(C[a, b], \mathbb{R}, +, \cdot, \langle \cdot \rangle)$, define $\langle f, g \rangle = \int_a^b f(x)g(x)dx$.
3) $(M_{max}, \mathbb{R}, +, \cdot, \langle \cdot \rangle)$, define $\langle A, B \rangle = \text{tr}(AB^T)$.

Theorem
Let $\langle \cdot \rangle$ be an inner product on a space $V$. Let $u, v, w$ denote vectors in $V$, $r$ a real number. Then:
1) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$.
2) $\langle v, rw \rangle = r \langle v, w \rangle$.
3) $\langle v, 0 \rangle = 0 = \langle 0, v \rangle$.
4) $\langle v, v \rangle = 0$ if and only if $v = 0$.

Theorem
If $A$ is any $n \times n$ positive definite matrix, then $\langle X, Y \rangle = X^TAY, \forall X, Y \in \mathbb{R}^n$ defines an inner product on $\mathbb{R}^n$, and every inner product on $\mathbb{R}^n$, and every inner product on $\mathbb{R}^n$ arises in this way.

Proof:
\[ \langle X, Y \rangle = X^T A Y \] is an inner product.

Any \( \langle X, Y \rangle \) on \( \mathbb{R}^n \) can be expressed as \( X^T A Y \):

- Let \( E = \{E_1, \ldots, E_n\} \) be the standard basis of \( \mathbb{R}^n \). Then \( X = \sum_{i=1}^{n} x_i E_i \) and \( Y = \sum_{j=1}^{n} y_j E_j \).

\[ \langle X, Y \rangle = \left( \sum_{i=1}^{n} x_i E_i, \sum_{j=1}^{n} y_j E_j \right) = \sum_{i,j=1}^{n} x_i y_j \langle E_i, E_j \rangle \]

- \[ \begin{bmatrix} \langle E_1, E_1 \rangle & \cdots & \langle E_1, E_n \rangle \\ \vdots & \ddots & \vdots \\ \langle E_n, E_1 \rangle & \cdots & \langle E_n, E_n \rangle \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = X^T A Y \]

- Moreover, \( A = A^T \).

**NORMS AND DISTANCE**

**Definition**

The norm of \( v \) in \( V \) is defined as \( \|v\| = \sqrt{\langle v, v \rangle} \) (length).

**Definition**

The distance between vectors \( v, w \) in an inner product space is \( d(v, w) = \|v - w\| \).

**Theorem**

If \( v \neq 0 \) is any vector in an inner product space, then \( \hat{v} = \frac{v}{\|v\|} \) is the unique unit vector that is a positive multiple of \( V \).

**Theorem: Schwarz Inequality**

If \( v \) and \( w \) are two vectors in an inner product space \( V \), then \( \langle v, w \rangle^2 \leq \|v\|^2 \|w\|^2 \). Moreover, equality occurs if and only if one of \( v \) or \( w \) is a scalar multiple of the other.

**Proof:**

- Assume \( \|v\| = a > 0 \) and \( \|w\| = b > 0 \).
- \( \|v - aw\|^2 = \langle v - aw, v - aw \rangle = 2ab - \langle v, w \rangle \geq 0 \Rightarrow \langle v, w \rangle \leq ab \), and \( \|v + aw\|^2 = \langle v + aw, v + aw \rangle = 2ab + \langle v, w \rangle \geq 0 \Rightarrow \langle v, w \rangle \geq -ab \).
- So \( -ab \leq \langle v, w \rangle \leq ab \Rightarrow 0 \leq \langle v, w \rangle^2 \leq a^2 b^2 = \|v\|^2 \|w\|^2 \).

- Note: \( \frac{\langle v, w \rangle^2}{\|v\|^2 \|w\|^2} \leq 1 \) or \( -1 \leq \frac{\langle v, w \rangle}{\|v\| \|w\|} \leq 1 \).
Example
Consider the vector space \( C[a, b] \) of all continuous functions on \( [a, b] \). Define \( \langle f, g \rangle = \int_a^b f(x)g(x)dx \).
Then \( \left( \int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b (f(x))^2 \cdot \int_a^b (g(x))^2 dx \).

**Theorem: Properties of Norms**
1) \( \|v\| \geq 0 \).
2) \( \|v\| = 0 \) if and only if \( v = 0 \).
3) \( \|rv\| = |r| \|v\| \).
4) \( \|v + w\| \leq \|v\| + \|w\| \) (triangle inequality).

**Theorem: Properties of Distance**
1) \( d(v, w) \geq 0 \).
2) \( d(v, w) = 0 \) if and only if \( v = w \).
3) \( d(v, w) = d(w, v) \).
4) \( d(v, w) \leq d(v, u) + d(u, w), \forall v, u, w \in V \)

**ORTHOGONAL SETS OF VECTORS**

**Definition**
Two vectors \( v, w \) in an inner product space \( V \) are said to be **orthogonal** if \( \langle v, w \rangle = 0 \).

**Definition**
A set of vectors \( \{e_1, \ldots, e_n\} \) is called an **orthogonal set** if each \( e_i \neq 0 \) and \( \langle e_i, e_j \rangle = 0, \forall i \neq j \). If, in addition, \( \|e_i\| = 1, \forall i \), then the set is called an **orthonormal set**.

**Example**
Consider \( \{\sin x, \cos x\} \) in \( C[-\pi, \pi] \) with \( \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx \). Then \( \langle \sin x, \cos x \rangle = 0 \), so \( \{\sin x, \cos x\} \) is an orthogonal set.

**Theorem: Pythagorean Theorem**
If \( \{e_1, \ldots, e_n\} \) is an orthogonal set of vectors, then \( \|e_1 + \cdots + e_n\|^2 = \|e_1\|^2 + \cdots + \|e_n\|^2 \).

**Theorem**
Let \( \{e_1, \ldots, e_n\} \) be an orthogonal set of vectors. Then:
1) \( r_1 e_1, \ldots, r_n e_n \) is also orthogonal for all \( r_i \neq 0 \) in \( \mathbb{R} \).
2) \[ \left\{ \frac{e_1}{\|e_1\|}, \ldots, \frac{e_n}{\|e_n\|} \right\} \] is an orthonormal set.

3) \( \{e_1, \ldots, e_n\} \) is linearly independent.

**Theorem: Expansion Theorem**

Let \( \{e_1, \ldots, e_n\} \) be an orthogonal basis of an inner product space \( V \). If \( v \) is any vector in \( V \), then

\[ v = \sum_{i=1}^{n} \frac{\langle v, e_i \rangle}{\|e_i\|^2} e_i \]

is the expansion of \( v \) as a linear combination of the basis vectors. The coefficients are called Fourier coefficients of \( v \) with respect to the orthogonal basis \( \{e_1, \ldots, e_n\} \).

**Lemma: Orthogonal Lemma**

Let \( \{e_1, \ldots, e_m\} \) be an orthogonal set of vectors in an inner product space \( V \), and let \( v \) be any vector not in \( \text{span}\{e_1, \ldots, e_m\} \). Define \( e_{m+1} = v - \sum_{i=1}^{m} \frac{\langle v, e_i \rangle}{\|e_i\|^2} e_i \), then \( \{e_1, \ldots, e_m, e_{m+1}\} \) is an orthogonal set of vectors.

**Gram-Schmidt Orthogonalization Algorithm**

Let \( V \) be an inner product space and \( \{v_1, \ldots, v_n\} \) be any basis of \( V \). Define vectors \( \{e_1, \ldots, e_n\} \) in \( V \) successively as follows:

- \( e_1 = v_1 \).
- \( e_2 = v_2 - \frac{\langle v_2, e_1 \rangle}{\|e_1\|^2} e_1 \).
- \( e_3 = v_3 - \frac{\langle v_3, e_1 \rangle}{\|e_1\|^2} e_1 - \frac{\langle v_3, e_2 \rangle}{\|e_2\|^2} e_2 \).
- \( \ldots \)
- \( e_n = v_n - \sum_{i=1}^{n-1} \frac{\langle v_n, e_i \rangle}{\|e_i\|^2} e_i \).

Then \( \{e_1, \ldots, e_n\} \) is orthogonal and \( \text{span}\{v_1, \ldots, v_n\} = \text{span}\{e_1, \ldots, e_n\} \).

**Definition**

The orthogonal complement \( U^\perp \) of \( U \) in \( V \) is defined by \( U^\perp = \{v \in V | \langle v, u \rangle = 0, \forall u \in U \} \).

**Theorem**

Let \( U \) be a finite dimensional subspace of an inner product space \( V \). Then:

1) \( U^\perp \) is a subspace of \( V \) and \( V = U \oplus U^\perp \).
2) If \( \dim V = n \), then \( \dim U + \dim U^\perp = n \).
3) If \( \dim V = n \), then \( \dim U^{\perp\perp} = U \).

Proof of 1:

- \( U^\perp \) is a subspace of \( V \) because:
  - \( 0 \in U^\perp \).
• $a_1u_1^\perp + a_2u_2^\perp \in U^\perp$.

• $V = U \oplus U^\perp$ because:
  • Let $x \in U \cap U^\perp$, then $x \in U^\perp \Rightarrow \langle x, u \rangle = 0, \forall u \in U$. But $\langle x, x \rangle = 0$ since $x \in U$, so $x = 0$. So $U \cap U^\perp = \{0\}$.
  • Take a basis in $U \{b_1, \ldots, b_m\}$ and a basis in $U^\perp \{b_{m+1}, \ldots, b_k\}$. Assume $V \neq \text{span}\{b_1, \ldots, b_m, b_{m+1}, \ldots, b_k\}$. Define $v^\perp = v - \sum_{i=1}^{k} a_i b_i = v - \sum_{i=1}^{m} a_i b_i - \sum_{i=m+1}^{k} a_i b_i$. So $\{b_1, \ldots, b_m, b_{m+1}, \ldots, b_k, v^\perp\}$ is an orthogonal set in $V$, so $\langle v^\perp, b_i \rangle = 0$ for $i = 1, \ldots, m$.
  This means $v^\perp \in U^\perp$. Contradiction! So $V = \text{span}\{b_1, \ldots, b_m, b_{m+1}, \ldots, b_k\}$ and $V = U + U^\perp$.

Proof of 2: Since $V = U \oplus U^\perp$, so $\dim V = n = \dim U + \dim U^\perp$.

Proof of 3: $U^{\perp\perp} = \{v \in V \mid \langle v, u \perp \rangle = 0, \forall u \perp \in U^\perp\}$. It is clear that $U^{\perp\perp} = U$.

Definition

$\text{proj}_U : V \rightarrow V, \text{proj}_U(v) = u$ where $v = u + w$ for $u \in U$, $w \in W$, $V = U \oplus W$ is called the projection on $U$ with kernel $W$.

Theorem: Projection Theorem

Let $U$ be a finite dimensional subspace of an inner product space $V$ and let $v \in V$. Then:
1) $\text{proj}_U : V \rightarrow V$ is a linear operator with image $U$ and kernel $U^\perp$.
2) $\text{proj}_U(v) \in U$ and $v - \text{proj}_U(v) \in U^\perp$.
3) If $\{e_1, \ldots, e_m\}$ is any orthogonal basis of $U$, then $\text{proj}_U(v) = \sum_{i=1}^{m} \langle v, e_i \rangle e_i$.

Proof of 1:
• $\text{proj}_U : V \rightarrow V$ is a linear operator because:
  • Let $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$.
  • $\text{proj}_U(v_1 + v_2) = \text{proj}_U(u_1 + u_2 + w_1 + w_2) = u_1 + u_2 = \text{proj}_U(v_1) + \text{proj}_U(v_2)$.
  • $\text{proj}_U(a \cdot v) = \text{proj}_U(a \cdot u + a \cdot w) = au = a \cdot \text{proj}_U(v)$.
  • $\text{im}(\text{proj}_U) = \{\text{proj}_U(v) \mid v \in V\}$. Take $v = u, u \in U$. Then $\text{proj}_U(v) = \text{proj}_U(u) = u$. So $\text{im}(\text{proj}_U) = U$.
  • $\text{ker}(\text{proj}_U) = \{v \in V \mid \text{proj}_U(v) = 0\}$. $\text{proj}_U(v) = 0 \Rightarrow \text{proj}_U(u + u^\perp) = 0 \Rightarrow u = 0$. So $\text{ker}(\text{proj}_U) = U^\perp$.

Proof of 2:
• $\text{proj}_U(v) \in U$ follows from definition.
• $v - \text{proj}_U(v) = (u + u^\perp) - u = u^\perp \in U^\perp$.

Proof of 3:
If \( \{e_1, \ldots, e_m\} \) is an orthogonal basis of \( U \), and \( \{e_{m+1}, \ldots, e_n\} \) is an orthogonal basis of \( U^\perp \), then \( \{e_1, \ldots, e_{m+1}, e_{m+1}, \ldots, e_n\} \) is an orthogonal basis of \( V \).

Since \( v = \sum_{i=1}^{m} \frac{\langle v, e_i \rangle}{\| e_i \|^2} e_i + \sum_{i=m+1}^{n} \frac{\langle v, e_i \rangle}{\| e_i \|^2} e_i = u + u^\perp \),

\[
\text{proj}_U (v) = \text{proj}_U \left( \sum_{i=1}^{m} \frac{\langle v, e_i \rangle}{\| e_i \|^2} e_i + \sum_{i=m+1}^{n} \frac{\langle v, e_i \rangle}{\| e_i \|^2} e_i \right) = \sum_{i=1}^{m} \frac{\langle v, e_i \rangle}{\| e_i \|^2} e_i + 0 = \sum_{i=1}^{m} \frac{\langle v, e_i \rangle}{\| e_i \|^2} e_i .
\]

**Theorem: Approximation Theorem**

Let \( U \) be a finite dimensional subspace of an inner product space \( V \). If \( v \in V \), then \( \text{proj}_U (v) \) is the vector in \( U \) that is closest to \( v \). Closest means that \( \| v - \text{proj}_U (v) \| < \| v - u \| \) \( \forall u \in U, u \neq \text{proj}_U (v) \).

**Example**

Find the polynomial in \( P_2 \) that best approximates the function \( f(x) = |x| \). Assume \( V = C[-1,1] \) and \( \langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx \).

- \( B = \{1, x, 3x^2 - 1\} \) is an orthogonal basis of \( P_2 \).
- \( \text{proj}_{P_2} (|x|) = \left( \frac{\langle |x|, 1 \rangle}{\| 1 \|^2} \cdot 1 + \frac{\langle |x|, x \rangle}{\| x \|^2} \cdot x + \frac{\langle |x|, 3x^2 - 1 \rangle}{\| 3x^2 - 1 \|^2} \cdot (3x^2 - 1) \right) = 3 \frac{1}{16} \left( 5x^2 + 1 \right) \).

**ORTHOGONAL DIAGONALIZATION**

**Theorem**

Let \( T : V \to V \) be a linear operator on \( V \). Then the following conditions are equivalent:

1) \( V \) has a basis of eigenvectors of \( T \).
2) There exists a basis \( B \) of \( V \) such that \( M_B (T) \) is diagonal.

**Proof:** Take \( B = \{e_1, \ldots, e_n\} \) a basis of \( V \). Then \( T(e_i) = \lambda_i e_i \iff M_B (T) = [C_B (T(e_i))] = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \).

**Theorem**

Let \( T \) be a linear operator on an inner product space \( V \). If \( \{e_1, \ldots, e_n\} \) is an orthogonal basis of \( V \), then

\[
M_B (T) = \begin{bmatrix} \langle e_1, T(e_1) \rangle & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \langle e_n, T(e_n) \rangle \end{bmatrix}. 
\]

**Proof:**

- Write \( M_B (T) = \begin{bmatrix} a_{ij} \end{bmatrix} \).
\[ T(e_j) = a_{ij}e_1 + \cdots + a_{nj}e_n \iff C_B(T(e_j)) = \begin{pmatrix} a_{ij} \\ \vdots \\ a_{nj} \end{pmatrix}. \]

- Since \( v = \sum_{i=1}^{n} \frac{\langle v, e_i \rangle}{\|e_i\|^2} e_i \) for all \( v \in V \), \( T(e_j) = \sum_{i=1}^{n} \frac{\langle T(e_j), e_i \rangle}{\|e_i\|^2} e_i \). So
  \[
  \sum_{i=1}^{n} \frac{\langle e_i, T(e_j) \rangle}{\|e_i\|^2} e_i = \sum_{i=1}^{n} a_{ij} e_i \Rightarrow a_{ij} = \frac{\langle e_i, T(e_j) \rangle}{\|e_i\|^2}.
  \]

**Definition**

A linear operator is called *symmetric* if \( \langle v, T(w) \rangle = \langle T(v), w \rangle \) holds for all \( v, w \in V \).

**Theorem**

Let \( V \) be a finite dimensional inner product space. The following conditions are equivalent for a linear operator \( T : V \to V \).

1) \( \langle v, T(w) \rangle = \langle T(v), w \rangle \) for all \( v, w \in V \).
2) The matrix of \( T \) is symmetric with respect to every orthonormal basis of \( V \).
3) The matrix of \( T \) is symmetric with respect to some orthonormal basis of \( V \).
4) There is an orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( V \) such that \( \langle e_i, T(e_j) \rangle = \langle T(e_i), e_j \rangle \) for all \( i, j \).

**Theorem**

Let \( T : V \to V \) be a symmetric linear operator on an inner product space \( V \), and let \( U \) be a \( T \)-invariant subspace of \( V \). Then:

1) The restriction of \( T \) to \( U \) is a symmetric linear operator on \( U \).
2) \( U^\perp \) is also \( T \)-invariant.

Proof:

- \( U \) is itself an inner product space using the same inner product as \( V \). Thus if \( \langle T(v), w \rangle = \langle v, T(w) \rangle \), \( \forall v, w \in V \), then, in particular, it holds for \( v, w \in U \).
- If \( v \in U^\perp \) and \( u \in U \), then \( \langle T(v), u \rangle = \langle v, T(u) \rangle = \langle v, u^\prime \rangle, u^\prime \in U \). So \( \langle v, u^\prime \rangle = 0 \). Thus \( \langle T(v), u \rangle = 0 \Rightarrow T(v) \in U^\perp \).

**Theorem: Principle Axis Theorem**

The following conditions are equivalent for a linear operator \( T \) on a finite dimensional inner product space \( V \).

1) \( T \) is symmetric.
2) \( V \) has an orthogonal basis consisting of eigenvectors of \( T \).

**Example**

Let \( T : P_2 \to P_2 \) be given by \( T(a + bx + cx^2) = (8a - 2b + 2c)x + (2a + 4b + 5c)x^2 \). Define \( \langle a + bx + cx^2, a' + b'x + c'x^2 \rangle = aa' + bb' + cc' \). Show that \( T \) is symmetric and find an orthonormal basis of \( P_2 \) consisting of eigenvectors.

- Want: \( T \) is symmetric.
• Take an orthonormal basis of $P_2$, $B_0 = \{1, x, x^2\}$.

• Then $M_{B_0}(T) = \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}$. So $T$ is symmetric.

• Want: Orthonormal basis consisting of eigenvectors.

  • We know the eigenvalues of $M_{B_0}(T)$ and thus eigenvectors of $M_{B_0}(T)$ to be

    \[ \{f_1, f_2, f_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\} \in \mathbb{R}^3. \]

  \[ f_1 = C_{B_0}(e_1), \quad f_2 = C_{B_0}(e_2), \quad f_3 = C_{B_0}(e_3) \]

• We are looking for $\{e_1, e_2, e_3\} \in P_2$ such that

  \[ \begin{bmatrix} f_1 = C_{B_0}(e_1) \\ f_2 = C_{B_0}(e_2) \\ f_3 = C_{B_0}(e_3) \end{bmatrix}. \]

• If $M_B(T) = P^{-1}M_{B_0}(T)P$ is diagonal, then

  \[ P = P_{B_0 \rightarrow B} = \begin{bmatrix} C_{B_0}(e_1) & C_{B_0}(e_2) & C_{B_0}(e_1) \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}. \]

  So

  \[ \begin{bmatrix} f_1 = C_{B_0}(e_1) \Rightarrow e_1 = \frac{1}{3} - \frac{1}{3} x + \frac{2}{3} x^2 \\ f_2 = C_{B_0}(e_2) \Rightarrow e_2 = \frac{2}{3} - \frac{1}{3} x + \frac{2}{3} x^2 \\ f_3 = C_{B_0}(e_3) \Rightarrow e_3 = \frac{2}{3} - \frac{1}{3} x + \frac{2}{3} x^2 \].

### ISOMETRIES

**Theorem**

Let $T : V \to V$ be a linear operator on a finite dimensional inner product space $V$. Then the following conditions are equivalent:

1) $\|T(v)\| = \|v\| \quad \forall v \in V$ (T preserves norm).

2) $\|T(v) - T(v_1)\| = \|v - v_1\| \quad \forall v, v_1 \in V$ (T preserves distance).

3) $\langle T(v), T(v) \rangle = \langle v, v \rangle, \quad \forall v \in V$ (T preserves inner product).

4) If $\{e_1, \ldots, e_n\}$ is any orthonormal basis in $V$, then $\{T(e_1), \ldots, T(e_n)\}$ is also an orthonormal basis (T preserves basis).

**Definition**

A linear operator is called an isometry if it satisfies one of the conditions in the previous theorem.

**Corollary**

1) Every isometry is an isomorphism.

2) The composite of two isometries is an isometry.

**Example**

Consider $T : M_{nn \to nn}$ and define $\langle A, B \rangle = \text{tr}(AB^T)$. Then $T(A) = A^T$ is an isometry.
Theorem
Let $T : V \to V$ be an operator where $V$ is a finite dimensional inner product space. Then the following conditions are equivalent:
1) $T$ is an isometry.
2) $M_B(T)$ is an orthogonal matrix for every orthonormal basis $B$.
3) $M_B(T)$ is an orthogonal matrix for some orthonormal basis $B$.

Proof:
- 1$\Rightarrow$2: Let $B = \{e_1, \ldots, e_n\}$ be an orthonormal basis. Then the $j^{th}$ column of $M_B(T)$ is $C_B(T(e_j))$. Now $\langle C_B(T(e_j)), C_B(T(e_j)) \rangle = \langle T(e_j), T(e_j) \rangle$ since $C_B : V \to \mathbb{R}^n$ is an isometry, and $\langle T(e_i), T(e_j) \rangle = \langle e_i, e_j \rangle$ since $T : V \to V$ is an isometry. $\langle e_i, e_j \rangle = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$, so the columns of $M_B(T)$ are orthogonal.
- 3$\Rightarrow$1: Let $B = \{e_1, \ldots, e_n\}$ be the orthonormal basis. Then $\langle T(e_i), T(e_j) \rangle = \langle C_B(T(e_i)), C_B(T(e_j)) \rangle = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$ because $M_B(T)$ is orthogonal. So $\{T(e_1)\ldots, T(e_n)\}$ is an orthonormal basis of $V$. So $T$ is an isometry.

Corollary
If $T : V \to V$ is an isometry where $V$ is a finite dimensional inner product space, then $\det T = \pm 1$.

Theorem
Let $T : V \to V$ be an isometry on a two dimensional inner product space $V$. Then there are two possibilities.
Either:
1) There is an orthonormal basis $B$ of $V$ such that $M_B(T) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ $0 \leq \theta < 2\pi$ (rotation).
   Or:
2) There is an orthonormal basis $B$ of $V$ such that $M_B(T) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $0 \leq \theta < 2\pi$ (reflection).

Lemma
Let $T : V \to V$ be an isometry on a finite dimensional inner product space $V$. Then:
1) If $U$ is $T$-invariant, then $U^\perp$ is also $T$-invariant.
2) If $\lambda$ is a complex eigenvalues of $T$, then $|\lambda| = 1$.
3) If $T$ has a non-real eigenvalues, then $V$ has a 2-dimensional $T$-invariant subspace.