Inner Product Spaces

INNER PRODUCTS

Definition

An inner product on a vector space V is a function that assigns a number $\langle v, w\rangle$ to every pair v, w of vector space V in such a way that the following axioms holds:

P_1: $\langle v, w\rangle$ is a real number.

P_2: $\langle v, w\rangle = \langle w, v\rangle$.

P_3: $\langle v+w, u\rangle = \langle v, u\rangle + \langle w, u\rangle$.

P_4: $\langle rv, w\rangle = r\langle v, w\rangle$.

P_5: $\langle v, v\rangle \geq 0, \forall v \in V$.

Definition

A vector space V with an inner product is called an inner product space.

Note

- $\langle , \rangle : V \times V \to \mathbf{R}$.
- $(V, \mathbf{R}, +, \cdot)$ is a vector space.
- $(V, \mathbf{R}, +, \cdot, \langle , \rangle)$ is an inner product space.

Examples

The following are inner product spaces.

1) $(\mathbf{R}^n, \mathbf{R}, +, \cdot, \langle , \rangle)$, define $\langle X, Y\rangle = X \cdot Y$ the dot product.
2) $(C[a,b], \mathbf{R}, +, \cdot, \langle , \rangle)$, define $\langle f, g\rangle = \int_a^b f(x)g(x)dx$.
3) $(M_{mn\times}, \mathbf{R}, +, \cdot, \langle , \rangle)$, define $\langle A, B\rangle = \mathrm{tr}(AB^T)$.

Theorem

Let $\langle , \rangle$ be an inner product on a space V. Let u, v, w denote vectors in V, r a real number. Then:

1) $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$.
2) $\langle v, rw\rangle = r\langle v, w\rangle$.
3) $\langle v, 0\rangle = 0 = \langle 0, v\rangle$.
4) $\langle v, v\rangle = 0$ if and only if $v = 0$.

Theorem

If A is any $n\times n$ positive definite matrix, then $\langle X, Y\rangle = X^T AY, \forall X, Y \in \mathbf{R}^n$ defines an inner product on $\mathbf{R}^n$, and every inner product on $\mathbf{R}^n$, and every inner product on $\mathbf{R}^n$ arises in this way.

Proof:

- $\langle X, Y\rangle = X^T AY$ is an inner product.
- Any $\langle,\rangle$ on $\mathbf{R}^n$ can be expressed as $X^T AY$:
  - Let $E = \{E_1, \ldots, E_n\}$ be the standard basis of $\mathbf{R}^n$. Then $X = \sum_{i=1}^{n} x_i E_i$ and $Y = \sum_{j=1}^{n} y_j E_j$.

    $\langle X, Y\rangle = \left\langle \sum_{i=1}^{n} x_i E_i, \sum_{j=1}^{j} y_j E_j \right\rangle = \sum_{i,j=1}^{n} x_i y_j \langle E_i, E_j\rangle$

  - $= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \langle E_1, E_1\rangle & \cdots & \langle E_1, E_n\rangle \\ \vdots & \ddots & \vdots \\ \langle E_n, E_1\rangle & \cdots & \langle E_n, E_n\rangle \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = X^T AY$
- Moreover, $A = A^T$.

Norms and Distance

Definition

The <u>norm</u> of v in V is defined as $\text{norm}(v) = \|v\| = \sqrt{\langle v, v\rangle}$ (length).

Definition

The <u>distance</u> between vectors v, w in an inner product space is $\mathrm{d}(v, w) = \|v - w\|$.

Theorem

If $v \neq 0$ is any vector in an inner product space, then $\hat{v} = \frac{v}{\|v\|}$ is the unique unit vector that is a positive multiple of V.

Theorem: Schwarz Inequality

If v and w are two vectors in an inner product space V, then $\langle v, w\rangle^2 \leq \|v\|^2 \|w\|^2$. Moreover, equality occurs if and only if one of v or w is a scalar multiple of the other.

Proof:

- Assume $\|v\| = a > 0$ and $\|w\| = b > 0$.
- $\|bv - aw\|^2 = \langle bv - aw, bv - aw\rangle = 2ab(ab - \langle v, w\rangle) \geq 0 \Rightarrow \langle v, w\rangle \leq ab$, and

  $\|bv + aw\|^2 = \langle bv + aw, bv + aw\rangle = 2ab(ab + \langle v, w\rangle) \geq 0 \Rightarrow \langle v, w\rangle \geq -ab$.
- So $-ab \leq \langle v, w\rangle \leq ab \Rightarrow 0 \leq \langle v, w\rangle^2 \leq a^2 b^b = \|v\|^2 \|w\|^2$.

- Note: $\frac{\langle v, w\rangle^2}{\|v\|^2 \|w\|^2} \leq 1$ or $-1 \leq \frac{\langle v, w\rangle}{\|v\| \|w\|} \leq 1$.

Example

Consider the vector space $C[a,b]$ of all continuous functions on $[a,b]$. Define $\langle f,g\rangle = \int_a^b f(x)g(x)dx$.

Then $\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b (f(x))^2\,dx \cdot \int_a^b (g(x))^2\,dx$.

Theorem: Properties of Norms

1) $\|v\| \ge 0$.
2) $\|v\| = 0$ if an only if $v = 0$.
3) $\|rv\| = |r|\|v\|$.
4) $\|v+w\| \le \|v\| + \|w\|$ (triangle inequality).

Theorem: Properties of Distance

1) $\mathrm{d}(v,w) \ge 0$.
2) $\mathrm{d}(v,w) = 0$ if and only if $v = w$.
3) $\mathrm{d}(v,w) = \mathrm{d}(w,v)$.
4) $\mathrm{d}(v,w) \le \mathrm{d}(v,u) + \mathrm{d}(u,w), \forall v,u,w \in V$

ORTHOGONAL SETS OF VECTORS

Definition

Two vectors v, w in an inner product space V are said to be orthogonal if $\langle v,w\rangle = 0$.

Definition

A set of vectors $\{e_1,\dots,e_n\}$ is called an orthogonal set if each $e_i \ne 0$ and $\langle e_i, e_j\rangle = 0, \forall i \ne j$. If, in addition, $\|e_i\| = 1, \forall i$, then the set is called an orthonormal set.

Example

Consider $\{\sin x, \cos x\}$ in $C[-\pi,\pi]$ with $\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$. Then $\langle \sin x, \cos x\rangle = 0$, so $\{\sin x, \cos x\}$ is an orthogonal set.

Theorem: Pythagorean Theorem

If $\{e_1,\dots,e_n\}$ is an orthogonal set of vectors, then $\|e_1+\cdots+e_n\|^2 = \|e_1\|^2 + \cdots + \|e_n\|^2$.

Theorem

Let $\{e_1,\dots,e_n\}$ be an orthogonal set of vectors. Then:

1) $\{r_1e_1,\dots,r_ne_n\}$ is also orthogonal for all $r_i \ne 0$ in $\mathbf{R}$.

2) $\left\{\frac{e_1}{\|e_1\|},\dots,\frac{e_n}{\|e_n\|}\right\}$ is an orthonormal set.
3) $\{e_1,\dots,e_n\}$ is linearly independent.

Theorem: Expansion Theorem

Let $\{e_1,\dots,e_n\}$ be an orthogonal basis of an inner product space *V*. If *v* is any vector in *V*, then $v=\sum_{i=1}^{n}\frac{\langle v,e_i\rangle}{\|e_i\|^2}e_i$ is the expansion of *v* as a linear combination of the basis vectors. The coefficients are called Fourier coefficients of *v* with respect to the orthogonal basis $\{e_1,\dots,e_n\}$.

Lemma: Orthogonal Lemma

Let $\{e_1,\dots,e_m\}$ be an orthogonal set of vectors in an inner product space *V*, and let *v* be any vector not in $\operatorname{span}\{e_1,\dots,e_m\}$. Define $e_{m+1}=v-\sum_{i=1}^{m}\frac{\langle v,e_i\rangle}{\|e_i\|^2}e_i$, then $\{e_1,\dots,e_m,e_{m+1}\}$ is an orthogonal set of vectors.

Gram-Schmidt Orthogonalization Algorithm

Let *V* be an inner product space and $\{v_1,\dots,v_n\}$ be any basis of *V*. Define vectors $\{e_1,\dots,e_n\}$ in *V* successively as follows:

- $e_1=v_1$.
- $e_2=v_2-\frac{\langle v_2,e_1\rangle}{\|e_1\|^2}e_1$.
- $e_3=v_3-\frac{\langle v_3,e_1\rangle}{\|e_1\|^2}e_1-\frac{\langle v_3,e_2\rangle}{\|e_2\|^2}e_2$.
- …
- $e_n=v_n-\sum_{i=1}^{n-1}\frac{\langle v_n,e_i\rangle}{\|e_i\|^2}e_i$.

Then $\{e_1,\dots,e_n\}$ is orthogonal and $\operatorname{span}\{v_1,\dots,v_n\}=\operatorname{span}\{e_1,\dots,e_n\}$.

Definition

The orthogonal complement $U^{\perp}$ of *U* in *V* is defined by $U^{\perp}=\{v\in V\mid\langle v,u\rangle=0,\forall u\in U\}$.

Theorem

Let *U* be a finite dimensional subspace of an inner product space *V*. Then:

1) $U^{\perp}$ is a subspace of *V* and $V=U\oplus U^{\perp}$.
2) If $\dim V=n$, then $\dim U+\dim U^{\perp}=n$.
3) If $\dim V=n$, then $\dim U^{\perp\perp}=U$.

Proof of 1:

- $U^{\perp}$ is a subspace of *V* because:
  - $0\in U^{\perp}$.

    - $a_1u_1^{\perp} + a_2u_2^{\perp} \in U^{\perp}$.
- $V = U \oplus U^{\perp}$ because:
    - Let $x \in U \cap U^{\perp}$, then $x \in U^{\perp} \Rightarrow \langle x, u\rangle = 0, \forall u \in U$. But $\langle x, x\rangle = 0$ since $x \in U$, so $x = 0$. So $U \cap U^{\perp} = 0$.
    - Take a basis in U $\{b_1, \ldots, b_m\}$ and a basis in $U^{\perp}$ $\{b_{m+1}, \ldots, b_k\}$. Assume $V \neq \operatorname{span}\{b_1, \ldots, b_m, b_{m+1}, \ldots, b_k\}$. Define $v^* = v - \sum_{i=1}^{k} a_i b_i = v - \sum_{i=1}^{m} a_i b_i - \sum_{i=m+1}^{k} a_i b_i$. So $\{b_1, \ldots, b_m, b_{m+1}, \ldots, b_k, v^*\}$ is an orthogonal set in *V*, so $\langle v^*, b_i\rangle = 0$ for $i = 1, \ldots, m$. This means $v^* \in U^{\perp}$. Contradiction! So $V = \operatorname{span}\{b_1, \ldots, b_m, b_{m+1}, \ldots, b_k\}$ and $V = U + U^{\perp}$.

Proof of 2: Since $V = U \oplus U^{\perp}$, so $\dim V = n = \dim U + \dim U^{\perp}$.

Proof of 3: $U^{\perp\perp} = \{v \in V \mid (v, u^{\perp}) = 0, \forall u^{\perp} \in U^{\perp}\}$. It is clear that $U^{\perp\perp} = U$.

Definition

$\operatorname{proj}_U : V \to V, \operatorname{proj}_U(v) = u$ where $v = u + w$ for $u \in U$, $w \in W$, $V = U \oplus W$ is called the <u>projection on *U* with kernel *W*</u>.

Theorem: Projection Theorem

Let *U* be a finite dimensional subspace of an inner product space *V* and let $v \in V$. Then:

1) $\operatorname{proj}_U : V \to V$ is a linear operator with image *U* and kernel $U^{\perp}$.
2) $\operatorname{proj}_U(v) \in U$ and $v - \operatorname{proj}_U(v) \in U^{\perp}$.
3) If $\{e_1, \ldots, e_m\}$ is any orthogonal basis of *U*, then $\operatorname{proj}_U(v) = \sum_{i=1}^{m} \frac{\langle v, e_i\rangle}{\|e_i\|^2} e_i$.

Proof of 1:

- $\operatorname{proj}_U : V \to V$ is a linear operator because:
    - Let $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$.
    - $\operatorname{proj}_U(v_1 + v_2) = \operatorname{proj}_U(u_1 + u_2 + w_1 + w_2) = u_1 + u_2 = \operatorname{proj}_U(v_1) + \operatorname{proj}_U(v_2)$.
    - $\operatorname{proj}_U(a \cdot v) = \operatorname{proj}_U(a \cdot u + a \cdot w) = au = a \cdot \operatorname{proj}_U(v)$.
- $\operatorname{im}(\operatorname{proj}_U) = \{\operatorname{proj}_U(v) \mid v \in V\}$. Take $v = u, u \in U$. Then $\operatorname{proj}_U(v) = \operatorname{proj}_U(u) = u$. So $\operatorname{im}(\operatorname{proj}_U) = U$.
- $\ker(\operatorname{proj}_U) = \{v \in V \mid \operatorname{proj}_U(v) = 0\}$. $\operatorname{proj}_U(v) = 0 \Rightarrow \operatorname{proj}_U(u + u^{\perp}) = 0 \Rightarrow u = 0$. So $\ker(\operatorname{proj}_U) = U^{\perp}$.

Proof of 2:

- $\operatorname{proj}_U(v) \in U$ follows from definition.
- $v - \operatorname{proj}_U(v) = (u + u^{\perp}) - u = u^{\perp} \in U^{\perp}$.

Proof of 3:

- If $\{e_1, \ldots, e_m\}$ is an orthogonal basis of *U*, and $\{e_{m+1}, \ldots, e_n\}$ is an orthogonal basis of $U^{\perp}$, then $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ is an orthogonal basis of *V*.
- Since $v = \sum_{i=1}^{n} \frac{\langle v, e_i \rangle}{\|e_i\|^2} e_i = \sum_{i=1}^{m} \frac{\langle v, e_i \rangle}{\|e_i\|^2} e_i + \sum_{i=m+1}^{n} \frac{\langle v, e_i \rangle}{\|e_i\|^2} e_i = u + u^{\perp}$,

$$\operatorname{proj}_U(v) = \operatorname{proj}_U\left(\sum_{i=1}^{m} \frac{\langle v, e_i \rangle}{\|e_i\|^2} e_i + \sum_{i=m+1}^{n} \frac{\langle v, e_i \rangle}{\|e_i\|^2} e_i \right) = \sum_{i=1}^{m} \frac{\langle v, e_i \rangle}{\|e_i\|^2} e_i + 0 = \sum_{i=1}^{m} \frac{\langle v, e_i \rangle}{\|e_i\|^2} e_i \ .$$

Theorem: Approximation Theorem

Let *U* be a finite dimensional subspace of an inner product space *V*. If $v \in V$, then $\operatorname{proj}_U(v)$ is the vector in *U* that is closest to *v*. Closest means that $\|v - \operatorname{proj}_U(v)\| < \|v - u\|, \forall u \in U, u \neq \operatorname{proj}_U(v)$.

Example

Find the polynomial in P_2 that best approximates the function $f(x) = |x|$. Assume $V = C[-1,1]$ and $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$.

- $B = \{1, x, 3x^2 - 1\}$ is an orthogonal basis of P_2.
- $\operatorname{proj}_{P_2}(|x|) = \frac{\langle |x|, 1 \rangle}{\|1\|^2} \cdot 1 + \frac{\langle |x|, x \rangle}{\|x\|^2} \cdot x + \frac{\langle |x|, 3x^2 - 1 \rangle}{\|3x^2 - 1\|^2} \cdot (3x^2 - 1) = \frac{3}{16}(5x^2 + 1)$.

Orthogonal Diagonalization

Theorem

Let $T : V \to V$ be a linear operator on *V*. Then the following conditions are equivalent:

1) *V* has a basis of eigenvectors of *T*.
2) There exists a basis *B* of *V* such that $M_B(T)$ is diagonal.

Proof: Take $B = \{e_1, \ldots, e_n\}$ a basis of *V*. Then $T(e_i) = \lambda_i e_i \Leftrightarrow M_B(T) = [C_B(T(e_i))] = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$.

Theorem

Let *T* be a linear operator on an inner product space *V*. If $\{e_1, \ldots, e_n\}$ is an orthogonal basis of *V*, then $M_B(T) = \left[\frac{\langle e_i, T(e_j) \rangle}{\|e_i\|^2} \right]$.

Proof:

- Write $M_B(T) = [a_{ij}]$.

- $T\left(e_j\right)=a_{1j}e_1+\cdots+a_{nj}e_n \Leftrightarrow C_B\left(T\left(e_j\right)\right)=\begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}$.
- Since $v=\sum_{i=1}^{n}\frac{\langle v,e_i\rangle}{\|e_i\|^2}e_i$ for all $v\in V$, $T\left(e_j\right)=\sum_{i=1}^{n}\frac{\langle T\left(e_j\right),e_i\rangle}{\|e_i\|^2}e_i$. So $\sum_{i=1}^{n}\frac{\langle e_iT\left(e_j\right)\rangle}{\|e_i\|^2}e_i=\sum_{i=1}^{n}a_{ij}e_i \Rightarrow a_{ij}=\frac{\langle e_iT\left(e_j\right)\rangle}{\|e_i\|^2}$.

Definition

A linear operator is called <u>symmetric</u> if $\langle v,T(w)\rangle=\langle T(v),w\rangle$ holds for all $v,w\in V$.

Theorem

Let V be a finite dimensional inner product space. The following conditions are equivalent for a linear operator $T:V\to V$.

1) $\langle v,T(w)\rangle=\langle T(v),w\rangle$ for all $v,w\in V$.
2) The matrix of T is symmetric with respect to every orthonormal basis of V.
3) The matrix of T is symmetric with respect to some orthonormal basis of V.
4) The is an orthonormal basis $\{e_1,\ldots,e_n\}$ of V such that $\langle e_i,T\left(e_j\right)\rangle=\langle T\left(e_i\right),e_j\rangle,\forall i,j$.

Theorem

Let $T:V\to V$ be a symmetric linear operator on an inner product space V, and let U be a T-invariant subspace of V. Then:

1) The restriction of T to U is a symmetric linear operator on U.
2) $U^{\perp}$ is also T-invariant.

Proof:

- U is itself an inner product space using the same inner product as V. Thus if $\langle T(v),w\rangle=\langle v,T(w)\rangle,\forall v,w\in V$, then, in particular, it holds for $v,w\in U$.
- If $v\in U^{\perp}$ and $u\in U$, then $\langle T(v),u\rangle=\langle v,T(u)\rangle=\langle v,u'\rangle,u'\in U$. So $\langle v,u'\rangle=0$. Thus $\langle T(v),u\rangle=0\Rightarrow T(v)\in U^{\perp}$.

Theorem: Principle Axis Theorem

The following conditions are equivalent for a linear operator T on a finite dimensional inner product space V.

1) T is symmetric.
2) V has an orthogonal basis consisting of eigenvector of T.

Example

Let $T:P_2\to P_2$ be given by $T\left(a+bx+cx^2\right)=(8a-2b+2c)+(-2a+5b+4c)x+(2a+4b+5c)x^2$. Define $\langle a+bx+cx^2,a'+b'x+c'x^2\rangle=aa'+bb'+cc'$. Show that T symmetric and find an orthonormal basis of P_2 consisting of eigenvectors.

- Want: T is symmetric.

- Take an orthonormal basis of P_2 $B_0=\{1,x,x^2\}$.
- Then $M_{B_0}(T)=\begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}$. So T is symmetric.

- Want: Orthonormal basis consisting of eigenvectors.
  - We know the eigenvalues of $M_{B_0}(T)$ and thus eigenvectors of $M_{B_0}(T)$ to be $\{f_1,f_2,f_3\}=\left\{\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix},\frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},\frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}\right\}\in \mathbf{R}^3$.
  - We are looking for $\{e_1,e_2,e_3\}\in P_2$ such that $\begin{cases} f_1=C_{B_0}(e_1) \\ f_2=C_{B_0}(e_2) \\ f_3=C_{B_0}(e_3) \end{cases}$.
  - If $M_B(T)=P^{-1}M_{B_0}(T)P$ is diagonal, then $P=P_{B_0\leftarrow B}=\left[C_{B_0}(e_1)\quad C_{B_0}(e_1)\quad C_{B_0}(e_1)\right]=\left[f_1\quad f_2\quad f_3\right]$. So

$$\begin{cases} f_1=C_{B_0}(e_1)\Rightarrow e_1=\frac{1}{3}\cdot 1+\frac{2}{3}\cdot x-\frac{2}{3}\cdot x^2 \\ f_2=C_{B_0}(e_2)\Rightarrow e_2=\frac{2}{3}\cdot 1+\frac{1}{3}\cdot x+\frac{2}{3}\cdot x^2 \\ f_3=C_{B_0}(e_3)\Rightarrow e_3=-\frac{2}{3}\cdot 1+\frac{2}{3}\cdot x+\frac{1}{3}\cdot x^2 \end{cases}.$$

ISOMETRIES

Theorem

Let $T:V\to V$ be a linear operator on a finite dimensional inner product space *V*. Then the following conditions are equivalent:

1) $\|T(v)\|=\|v\|,\forall v\in V$ (*T* preserves norm).
2) $\|T(v)-T(v_1)\|=\|v-v_1\|,\forall v,v_1\in V$ (*T* preserves distance).
3) $\langle T(v),T(v)\rangle=\langle v,v\rangle,\forall v\in V$ (*T* preserves inner product).
4) If $\{e_1\ldots,e_n\}$ is any orthonormal basis in *V*, then $\{T(e_1)\ldots,T(e_n)\}$ is also an orthonormal basis (*T* preserves basis).

Definition

A linear operator is called an isometry if it satisfies one of the conditions in the previous theorem.

Corollary

1) Every isometry is an isomorphism.
2) The composite of two isometries is an isometry.

Example

Consider $T:M_{n\times n}\to M_{n\times n}$ and define $\langle A,B\rangle=\mathrm{tr}(AB^T)$. Then $T(A)=A^T$ is an isometry.

Theorem

Let $T: V \to V$ be an operator where V is a finite dimensional inner product space. Then the following conditions are equivalent:

1) T is an isometry.
2) $M_B(T)$ is an orthogonal matrix for every orthonormal basis B.
3) $M_B(T)$ is an orthogonal matrix for some orthonormal basis B.

Proof:

- 1⇒2: Let $B = \{e_1, \ldots, e_n\}$ be an orthonormal basis. Then the j^{th} column of $M_B(T)$ is $C_B(T(e_j))$. Now $\langle C_B(T(e_i)), C_B(T(e_j))\rangle = \langle T(e_i), T(e_j)\rangle$ since $C_B: V \to \mathbf{R}^n$ is an isometry, and $\langle T(e_i), T(e_j)\rangle = \langle e_i, e_j\rangle$ since $T: V \to V$ is an isometry. $\langle e_i, e_j\rangle = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$, so the columns of $M_B(T)$ are orthogonal.
- 3⇒1: Let $B = \{e_1, \ldots, e_n\}$ be the orthonormal basis. Then $\langle T(e_i), T(e_j)\rangle = \langle C_B(T(e_i)), C_B(T(e_j))\rangle = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$ because $M_B(T)$ is orthogonal. So $\{T(e_1), \ldots, T(e_n)\}$ is an orthonormal basis of V. So T is an isometry.

Corollary

If $T: V \to V$ is an isometry where V is a finite dimensional inner product space, then $\det T = \pm 1$.

Theorem

Let $T: V \to V$ be an isometry on a two dimensional inner product space V. Then there are two possibilities. Either:

1) There is an orthonormal basis B of V such that $M_B(T) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, 0 \leq \theta < 2\pi$ (rotation).

Or:

2) There is an orthonormal basis B of V such that $M_B(T) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, 0 \leq \theta < 2\pi$ (reflection).

Lemma

Let $T: V \to V$ be an isometry on a finite dimensional inner product space V. Then:

1) If U is T-invariant, then $U^{\perp}$ is also T-invariant.
2) If λ is a complex eigenvalues of T, then $|\lambda| = 1$.
3) If T has a non-real eigenvalues, then V has a 2-dimensional T-invariant subspace.