# The Geometry of Euclidean Space

## VECTORS IN $\mathbf{R}^N$

- A vector in $\mathbf{R}^n$ can be written as $\mathbf{u} = (u_1, \ldots, u_n)$.

### Definitions

- If $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$, then $\mathbf{u} = \mathbf{v} \Leftrightarrow \begin{cases} u_1 = v_1 \\ \vdots \\ u_n = v_n \end{cases}$.
- $\alpha\mathbf{u} = (\alpha u_1, \ldots \alpha u_n), \alpha \in \mathbf{R}$.

### Some Properties

1) $\left.\begin{array}{l} \mathbf{u} \in \mathbf{R}^n \\ \mathbf{v} \in \mathbf{R}^n \end{array}\right\} \Rightarrow (\mathbf{u} + \mathbf{v}) \in \mathbf{R}^n$.
2) $\left.\begin{array}{l} \alpha \in \mathbf{R} \\ \mathbf{u} \in \mathbf{R}^n \end{array}\right\} \Rightarrow \alpha\mathbf{u} \in \mathbf{R}^n$.
3) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
4) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{u}) + \mathbf{w}$.
5) $-\mathbf{u} = (-u_1, \ldots, -u_n)$ and $\mathbf{u} + (-\mathbf{u}) = \mathbf{0} = (0, \ldots, 0) = (-\mathbf{u}) + \mathbf{u}$.
6) $\mathbf{u} + \mathbf{0} = \mathbf{u} = \mathbf{0} + \mathbf{u}$.
7) $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.
8) $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
9) $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$.
10) $1\mathbf{u} = \mathbf{u}$.

## NORM

### Definition

Let $\mathbf{u} = (u_1, \ldots, u_n)$. The norm is $\|\mathbf{u}\| = \sqrt{{u_1}^2 + \cdots + {u_n}^2}$.

### Properties

1) $\|\mathbf{u}\| \geq 0$.
2) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.
3) $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$.

# DOT PRODUCT

Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$, then $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$.

### Properties

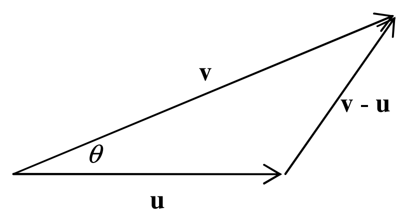
1) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
2) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.
3) $\alpha(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (\alpha\mathbf{v})$.
4) $\mathbf{u} \cdot \mathbf{u} \geq 0$, moreover, $\mathbf{u} \cdot \mathbf{u} = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$

### More Properties

1) $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.
2) $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ (Cauchy-Schwartz).
3) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$ and $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$.
4) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$.

### The Cosine Law

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{v}\|\|\mathbf{u}\| \cos\theta$$
$$\Rightarrow (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{v}\|\|\mathbf{u}\| \cos\theta$$
$$\Rightarrow \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{v}\|\|\mathbf{u}\| \cos\theta$$
$$\Rightarrow \mathbf{u} \cdot \mathbf{v} = \|\mathbf{v}\|\|\mathbf{u}\| \cos\theta$$




### Theorem

$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{v}\|\|\mathbf{u}\| \cos\theta, \mathbf{u}, \mathbf{v} \neq \mathbf{0}$.

### Some Basic Consequences

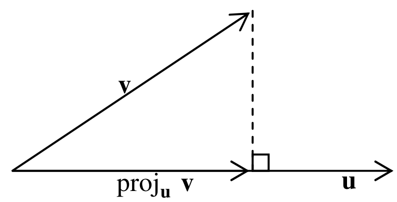
1) $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$.
2) $\mathbf{u} \cdot \mathbf{v} > 0$ means $0 \leq \theta < \frac{\pi}{2}$.

   $\mathbf{u} \cdot \mathbf{v} = 0$ means $\theta = \frac{\pi}{2}$.

   $\mathbf{u} \cdot \mathbf{v} < 0$ means .

# PROJECTION

### Geometry

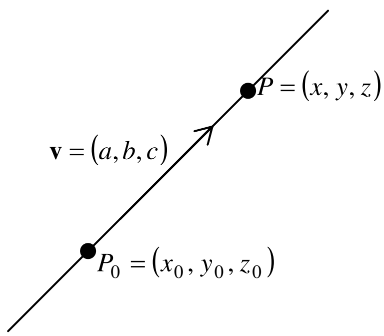



### Theorem

1) $\left\|\text{proj}_{\mathbf{u}}\ \mathbf{v}\right\| = \dfrac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|}$.

2) $\text{proj}_{\mathbf{u}}\ \mathbf{v} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$.

## LINES IN R$^3$




Some ways to characterize the line that passes through a given point $P_0 = (x_0, y_0, z_0)$ and follows the direction of a non-zero vector $\mathbf{v} = (a, b, c)$. If $P = (x, y, z)$ denotes any point on the line, then:

- $\overrightarrow{P_0P} = \lambda\mathbf{v}$ (vector equation).
- $x = x_0 + \lambda a$
  $y = y_0 + \lambda b$
  $z = z_0 + \lambda c$ (parametric equations).
- If $a, b, c \neq 0$, $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b`} = \dfrac{z - z_0}{c}$ (symmetric equations).

## THE CROSS PRODUCT

### Definition

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then $\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$ is the <u>cross product</u> of **u** and **v**.

Note: $\mathbf{u}\times\mathbf{v}=\det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{pmatrix}=\mathbf{i}\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}-\mathbf{j}\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}+\mathbf{k}\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}$.

### Some Basis Properties

1) $\mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})$.
2) $\mathbf{u}\times\mathbf{u}=\mathbf{0}$. More general, if $\mathbf{u}\parallel\mathbf{v}$, then $\mathbf{u}\times\mathbf{v}=\mathbf{0}$.
3) $(\alpha\mathbf{u})\times\mathbf{v}=\alpha(\mathbf{u}\times\mathbf{v})=\mathbf{u}\times(\alpha\mathbf{v})$.
4) $\mathbf{u}\times(\mathbf{v}+\mathbf{w})=\mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$.

Note: The cross product is not associative.

### The Geometry of the Cross Product

1) If $\mathbf{u}\neq\mathbf{0}$ and $\mathbf{v}\neq\mathbf{0}$, then $\|\mathbf{u}\times\mathbf{v}\|=\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$.
2) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\det\begin{pmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{pmatrix}$.

### Theorem

$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})=\mathbf{0}$ and $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})=\mathbf{0}$. Geometrically, $(\mathbf{u}\times\mathbf{v})\perp\mathbf{u}$, $(\mathbf{u}\times\mathbf{v})\perp\mathbf{v}$.

### Equations of Planes in $R^3$

- An equation for the plane that passes through $P_0=(x_0,y_0,z_0)$ and is perpendicular (normal) to $\mathbf{n}=(a,b,c)$ is $\overrightarrow{P_0P}\cdot\mathbf{n}=0$.
- In terms of coordinates, $(x-x_0,y-y_0,z-z_0)\cdot(a,b,c)=0\Leftrightarrow a(x-x_0)+b(y-y_0)+c(z-z_0)=0$, or $ax+by+cz=d$ where $d=ax_0+by_0+cz_0$.
- Another way to describe a plane is $\begin{cases}x=x_0+su_1+tv_1\\ y=y_0+su_2+tv_2\\ z=z_0+su_3+tv_3\end{cases}$. This is the parametric equation through $(x_0,y_0,z_0)$ and parallel to **u** and **v**.

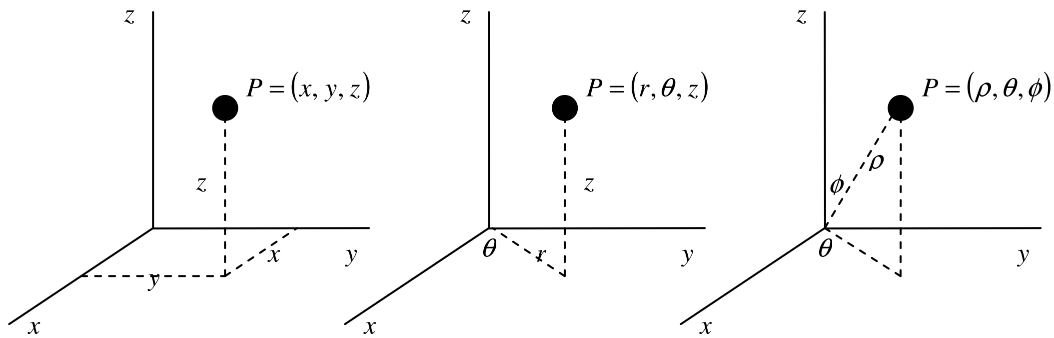
### Theorem

Let $\mathbf{u}\neq\mathbf{0}$, $\mathbf{v}\neq\mathbf{0}$, $\mathbf{w}\neq\mathbf{0}$.

1) The area of the parallelogram generated by **u** and **v** is $\|\mathbf{u}\times\mathbf{v}\|$.
2) The volume of the parallelepiped generated by the vectors **u**, **v**, **w** is $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$.

## COORDINATES IN $R^3$

Rectangular Coordinates | Cylindrical Coordinates | Spherical Coordinates




**Example**

| Rectangular $(x,y,z)$ | Cylindrical $(r,\theta,z)$ | Spherical $(\rho,\theta,\phi)$ |
|---|---|---|
| $(-2,0,2)$ | $(2,\pi,2)$ | $\left(2\sqrt{2},\pi,\frac{\pi}{4}\right)$ |
| $\left(\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2},0\right)$ | $\left(3,\frac{\pi}{4},0\right)$ | $\left(3,\frac{\pi}{4},\frac{\pi}{2}\right)$ |

**Equations of Transformations**

- Rectangular and Cylindrical: $\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array}$ and $\begin{array}{l} r = x^2 + y^2 \\ \theta = \arctan\left(\frac{y}{x}\right) \\ z = z \end{array}$.
- Rectangular and Spherical: $\begin{array}{l} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{array}$.

## VECTORS IN $\mathbf{R}^N$

**Definition**

Let $\mathbf{u}=(u_1,\ldots,u_n)\in\mathbf{R}^n$. Then the norm of $\mathbf{u}$ is $\|\mathbf{u}\|=\sqrt{u_1{}^2+\cdots+u_n{}^2}$.

**Definition**

Let $P=(p_1,\ldots,p_n)$ and $Q=(q_1,\ldots,q_n)$ be two points in $\mathbf{R}^n$. Then the distance between $P$ and $Q$ is $\sqrt{(p_1-q_1)^2+\cdots+(p_n-q_n)^2}$.

**Definition**

If $P_0 = \left(p_1^0, \ldots, p_n^0\right)$ and $\mathbf{v} = (a_1, \ldots, a_n)$, then the <u>line</u> that passes through the point $P_0$ and follows the direction of the vector $\mathbf{v}$ is the set of all points $(x_1, \ldots x_n)$ such that $\begin{cases} x_1 = x_1^0 + ta_1 \\ \vdots \\ x_n = x_n^0 + ta_n \end{cases}, t \in \mathbf{R}$.

### Problem

Suppose that the real numbers *a*, *b*, *c*, *d* satisfy the condition $a^2 + (b-2)^2 + (c+1)^2 + (d-3)^2 = 16$. Find the values for *a*, *b*, *c*, *d* for which $f(a,b,c,d) = (a-7)^2 + (b-8)^2 + (c+10)^2 + (d-12)^2$ take its maximum and minimum values.

- In this case, we can view $a^2 + (b-2)^2 + (c+1)^2 + (d-3)^2 = 16$ as a "sphere" centered at $(0,2,-1,3)$ with radius 4, and $f(a,b,c,d)$ as the distance from $(a,b,c,d)$ to $(7,8,-10,12)$.
- Now, the line that joins the center of the "sphere" $(0,2,-1,3)$ with the point $(7,8,-10,12)$ is given by
$$\begin{cases} x_1 = 0 + 7t \\ x_2 = 2 + 6t \\ x_3 = -1 - 9t \\ x_4 = 3 + 9t \end{cases}.$$
- At the intersection with the "sphere", $(7t)^2 + (6t)^2 + (-9t)^2 + (9t)^2 = 16 \Rightarrow t = \pm \dfrac{4}{247}$. Now one of these two points (which we can solve easily) will give the maximum value for $f(a,b,c,d)$, and the other one will give the minimum.

# Limits

### Definition: Limit of One Variable

When we say "the function $f(x)$ approaches the number *L* as *x* approaches *a*" (write $\lim_{x \to a} f(x) = L$), what we mean is:

- Geometrically: For any open interval *B* that contains *L*, we can always find an open interval *A* that contains *a* such that for all $x \in A, x \neq a$, $f(x) \in B$.
- Algebraically: For every $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

### Definition: Limit of Two Variables

Let $f : \mathbf{R}^2 \to \mathbf{R}$. When we say "the function $f(x, y)$ approaches the number *L* as $(x, y)$ approaches $(a, b)$" (write $\lim_{(x,y) \to (a,b)} f(x, y) = L$), what we mean is:

- Geometrically: For any open interval *B* that contains *L*, we can always find an "open disk" *D* centered at $(a,b)$ such that for all $(x, y) \in D, (x, y) \neq (a,b)$, $f(x, y) \in B$.
- Algebraically: For every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$0 < \|(x-a, y-b)\| = \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x, y) - L| < \varepsilon.$$

## OPEN AND CLOSED SETS

### Some "Geometric" Definitions

1) The set consisting of all $\mathbf{x} \in \mathbf{R}^n$ such that $\|\mathbf{x} - \mathbf{x}_0\| < r, r > 0$ is called "the disk centered at $\mathbf{x}_0$ with radius $r$" and is denoted $D_r(\mathbf{x}_0)$.
2) A set $A \subseteq \mathbf{R}^n$ is said to be open if for every $\mathbf{x}_0 \in A$, there exists $r > 0$ such that $D_r(\mathbf{x}_0) \subseteq A$.
3) Let $A \subseteq \mathbf{R}^n$. $\mathbf{x} \in \mathbf{R}^n$ is said to be a boundary point of $A$ if for every $r > 0$, the disk $D_r(\mathbf{x})$ contains at least one point of $A$ and contains at least one point of $A^c$.

### Examples

Are the following sets open?

1) $\{(x, y) \in \mathbf{R}^2 \mid y \geq x\}$ is not open.
2) $\{(x, y) \in \mathbf{R}^2 \mid |x| < 4\}$ is open.
3) $\mathbf{R}^2$, $\phi$ are both open.

### Theorem

If $A$, $B$ are open sets, then $A \cap B$ is open also.

Proof: Let $\mathbf{x} \in A \cap B$, then $\mathbf{x} \in A$ and $\mathbf{x} \in B$. Since $A$ is open, there exists $r_A > 0$ such that $D_{r_A}(\mathbf{x}) \subseteq A$. Since $B$ is open, there exists $r_B > 0$ such that $D_{r_B}(\mathbf{x}) \subseteq B$. Now, by taking $r = \min(r_A, r_B)$,

$$\left.\begin{matrix} D_r(\mathbf{x}) \subseteq D_{r_A}(\mathbf{x}) \subseteq A \\ D_r(\mathbf{x}) \subseteq D_{r_B}(\mathbf{x}) \subseteq B \end{matrix}\right\} \Rightarrow D_r(\mathbf{x}) \subseteq A \cap B.$$

### Theorem

Let $A \subseteq \mathbf{R}^n$. If the set $A$ contains at least one of its boundary points, then $A$ is not open.

### Definition

A set $A \subseteq \mathbf{R}^n$ is said to be closed if it contains all of its boundary points.

### Note

$\mathbf{R}^2$, $\phi$ are both open and closed.

## LIMITS IN $\mathbf{R}^N$

### Definition: The General Case

Let $f : A \subseteq \mathbf{R}^m \to \mathbf{R}^n$, where $A$ is open. Let $\mathbf{a} \in A$ or a boundary point of $A$. Then $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = \mathbf{L}$ means for each open set $N \subseteq \mathbf{R}^n$ that contains $\mathbf{L}$, there exists an open set $M \subseteq \mathbf{R}^m$ that contains $\mathbf{a}$ such that $\mathbf{x} \in M \cap A, \mathbf{x} \neq \mathbf{a} \Rightarrow f(\mathbf{x}) \in N$.

### Example

Let $f(x,y)=\begin{cases}(x,y,2y),(x,y)\neq(1,1)\\ (5,7,9),(x,y)=(1,1)\end{cases}$. Then $\lim_{(x,y)\to(1,1)} f(x,y)=(1,1,2)$.

### The Algebra of Limits

1) $\lim_{\mathbf{x}\to\mathbf{a}}[f(\mathbf{x})\pm g(\mathbf{x})]=\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\pm\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$.
2) $\lim_{\mathbf{x}\to\mathbf{a}}[f(\mathbf{x})\cdot g(\mathbf{x})]=\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\cdot\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$.
3) If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})=\mathbf{L}$ and $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})=\mathbf{M}$, then $\mathbf{L}=\mathbf{M}$.
4) Let $f:A\subseteq\mathbf{R}^m\to\mathbf{R}^n$, $g:B\subseteq\mathbf{R}^n\to\mathbf{R}^p$, where $A$, $B$, $f(A)\cap B$ are open. If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})=\mathbf{L}\in B$ and $\lim_{\mathbf{y}\to\mathbf{L}} g(\mathbf{y})=\mathbf{M}$, then $\lim_{\mathbf{x}\to\mathbf{a}}(g\circ f)(\mathbf{x})=\mathbf{M}$.

### Exercise

Evaluate the limit if it exists.

1) $\lim_{(x,y)\to(2,3)}\dfrac{x^2+y^2}{x^2-y}=7$.
2) $\lim_{(x,y)\to(0,0)}\dfrac{x^2+y^2}{\sin(x^2+y^2)}$. Let $f(x,y)=x^2+y^2$ and $g(t)=\dfrac{t}{\sin t}$. Then
$\lim_{(x,y)\to(0,0)}\dfrac{x^2+y^2}{\sin(x^2+y^2)}=\lim_{(x,y)\to(0,0)}(g\circ f)(x,y)=\lim_{t\to0}\dfrac{t}{\sin t}=1$.
3) $\lim_{(x,y)\to(0,0)}\dfrac{xy}{x^2+y^2}$. Approaching $(0,0)$ along $(t,t)$, $\lim_{(x,y)\to(0,0)}\dfrac{xy}{x^2+y^2}=\lim_{t\to0}\dfrac{t^2}{2t^2}=\dfrac{1}{2}$. Approaching $(0,0)$ along $(t,2t)$, $\lim_{(x,y)\to(0,0)}\dfrac{xy}{x^2+y^2}=\lim_{t\to0}\dfrac{2t^2}{5t^2}=\dfrac{2}{5}$. So the limit doesn't exists.
4) $\lim_{(x,y)\to(0,0)}\dfrac{x^6}{x^2+y^4}$. Using polar coordinates,
$\lim_{(x,y)\to(0,0)}\dfrac{x^6}{x^2+y^4}=\lim_{r\to0}\dfrac{r^6\cos^6\theta}{r^2\cos^2\theta+r^4\sin^2\theta}=\lim_{r\to0}\dfrac{r^4\cos^6\theta}{\cos^2\theta+r^2\sin^2\theta}=0$. Alternately, we know $0\le\dfrac{x^6}{x^2+y^4}\le\dfrac{x^6}{x^2}=x^4$. As $(x,y)\to(0,0)$, $x^4\to0$, so $\lim_{(x,y)\to(0,0)}\dfrac{x^6}{x^2+y^4}=0$.

## CONTINUITY

### Defintion

Let $f:A\subseteq\mathbf{R}^m\to\mathbf{R}^n$. *f* is said to be <u>continuous</u> at $\mathbf{a}\in A$ if:

1) $f(\mathbf{a})$ is defined.
2) $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exists.
3) $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})=f(\mathbf{a})$.

**Theorem**

If $P(\mathbf{x})$ and $Q(\mathbf{x})$ are polynomials, then $\lim_{\mathbf{x}\to\mathbf{a}} \frac{P(\mathbf{x})}{Q(\mathbf{x})} = \frac{P(\mathbf{a})}{Q(\mathbf{a})}, Q(\mathbf{a}) \neq 0$.

**Example**

$f(x, y) = \begin{cases} \frac{x^2 + y^2}{\sqrt{x^2 + y^2}}, (x, y) \neq 0 \\ 0, (x, y) = 0 \end{cases}$ is continuous everywhere.

# Differentiation

## Derivative

**Definition: Derivative of a Single-Variable Function**

Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$, *I* open, $a \in I$. We say *f* is differentiable at $x = a$ if $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ exists. This limit, if it exists, is usually denoted $f'(a)$ and is called the derivative of $f(x)$ at $x = a$.

If *f* is differentiable for all $x \in I$, then the function $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ is usually called the derivative of the function *f*.

**Definition: The Partial Derivative of a Two-Variable Function**

Let $f : A \subseteq \mathbf{R}^2 \to \mathbf{R}$, *A* open, $(a,b) \in A$. If $\lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$ exists, then it is called the partial derivative of the function *f* with respect to *x* at $(a,b)$ and is denoted $\frac{\partial f}{\partial x}(a,b)$.

**Geometrically**

- $\frac{\partial f}{\partial x}(a,b)$ is just the slope of the curve of intersection of the surface $z = f(x, y)$ with respect to the plane $y = b$ at the point $(a, b, f(a,b))$.
- In general, if $z = f(x, y)$ is a surface for which a "tangent plane" exists at the point $(a, b, f(a,b))$, then the equation for the tangent plane is $\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-a) - (z - f(f,b)) = 0$.

**Application**

If $z = f(x, y)$ has a tangent plane at the point $(a, b, f(a,b))$ and *h* and *k* are "small" numbers, then $f(a+h, b+k) \approx f(a,b) + h\frac{\partial f}{\partial x}(a,b) + k\frac{\partial f}{\partial y}(a,b)$.

### Definition: Partial Derivative in General

Let $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ . Then
$\frac{\partial f}{\partial x_i}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - (a_1, \ldots, a_n)}{h}$ . If the limit exists, then it is the partial derivative of $f$ with respect to $x_i$.

### Some Properties of the Derivative

1) If $f : \mathbf{R}^m \to \mathbf{R}^n$ , $g : \mathbf{R}^m \to \mathbf{R}^n$ , $\alpha$ and $\beta$ are constants, then $\mathbf{D}(\alpha f \pm \beta g) = \alpha \mathbf{D} f \pm \beta \mathbf{D} g$ .
2) If $f : \mathbf{R}^m \to \mathbf{R}^n$ , $g : \mathbf{R}^n \to \mathbf{R}^p$ , then $\mathbf{D}(f \circ g) = (\mathbf{D}g)(f) \pm \mathbf{D}f$ .
3) If $f : \mathbf{R}^n \to \mathbf{R}$ , $g : \mathbf{R}^n \to \mathbf{R}$ , then $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$ .

## DIFFERENTIABILITY

### Definition: Differentiability of Two Variable Functions

Let $f : A \subseteq \mathbf{R}^2 \to \mathbf{R}$ . We say $f$ is differentiable at $(a,b) \in A$ if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,b) - \frac{\partial f}{\partial x}(a,b)(x-a) - \frac{\partial f}{\partial y}(a,b)(y-b)}{\|(x,y)-(a,b)\|} = 0 .$$

### Definition: Differentiability in General

Let $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ . We say $f$ is differentiable at $\mathbf{a} \in A$ if all the partial derivatives associated with $f$ at $\mathbf{a}$ exist and $\lim_{\mathbf{x} \to \mathbf{a}} \frac{\|f(\mathbf{x}) - f(\mathbf{a}) - T(\mathbf{x} - \mathbf{a})\|}{\|\mathbf{x} - \mathbf{a}\|} = 0$ , where $T$ is the matrix of partial derivatives associated with $f$ at $\mathbf{a}$.

## PATHS IN R$^N$

### Definition

If $I$ denotes an interval in $\mathbf{R}$, then the function $f : I \subseteq \mathbf{R} \to \mathbf{R}^n$ is called a path in $\mathbf{R}^n$.

### Example

$f : [0, 4\pi] \to \mathbf{R}^2, t \mapsto (\cos t, \sin t)$ is a circular path traveled twice.

### Definition

If $f : A \subseteq \mathbf{R} \to \mathbf{R}^3, t \mapsto (x(t), y(t), z(t))$ is a path, and $a \in A$ is such that $x'(a)$ , $y'(a)$ , $z'(a)$ , then the vector $f'(a) = (x'(a), y'(a), z'(a))$ is called the velocity vector of the path at $x = a$ .

### The Algebra of Path Velocities

Let $f$, $g$ be paths, $\alpha$ and $\beta$ constants, $h:\mathbf{R}\to\mathbf{R}$ an ordinary function. Then:

1) $(f \pm g)' = f' \pm g'$.
2) $(\alpha f \pm \beta g)' = \alpha f' \pm \beta g'$.
3) $(f \cdot g)' = f' \cdot g + f \cdot g'$.
4) $(f \times g)' = f' \times g + f \times g'$.
5) $(hg)' = h'g + hg'$.
6) $(g \circ f)'(a) = [g'(f(a))][f'(a)] = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{bmatrix} \begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt}$. Here $\begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{bmatrix}$ is called the gradient.

## THE CHAIN RULE IN GENERAL

Let $f : A \subseteq \mathbf{R}^m \to \mathbf{R}^n$, $g : B \subseteq \mathbf{R}^n \to \mathbf{R}^p$, $\mathbf{a} \in A$ such that $f(\mathbf{a}) \in B$. If $f$ is differentiable at $\mathbf{x} = \mathbf{a}$ and $g$ is differentiable at $\mathbf{y} = f(\mathbf{a})$, then $[\mathbf{D}(g \circ f)](a) = [[\mathbf{D}g](f(\mathbf{a}))][[\mathbf{D}f](\mathbf{a})]$.

### Useful Property

If $F(x_1, \ldots, x_n) = 0$ implicitly defines each of the variables $x_i$ as a function of the remaining variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, then $\frac{\partial x_i}{\partial x_j} = -\frac{\partial F / \partial x_j}{\partial F / \partial x_i}, i \neq j$.

## DIRECTIONAL DERIVATIVE

### Definition

Let $f : A \subseteq \mathbf{R}^2 \to \mathbf{R}$, $\mathbf{a} = (a, b) \in A$, and $\mathbf{u} = (u_1, u_2)$ unit vector. If $\lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h}$ exists, then it is called the directional vector of the function $f$ at the point $(a, b)$ in the direction of the vector $\mathbf{u}$. It is denoted $\mathbf{D_u} f(\mathbf{a})$.

### Theorem

$\mathbf{D_u} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}$, where $\nabla f = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$ the gradient.

### Example

Consider a domed roof with the shape of the surface $z = 15 - x^2 - 2y^2$. A marble is placed at $(3,1,4)$. In what direction will the marble fall?

- We want the minimum of $\mathbf{D_u} f(3,1) = \nabla f(3,1) \cdot (u_1, u_2)$.
- $\nabla f(3,1) = \left( \frac{\partial f}{\partial x}(3,1), \frac{\partial f}{\partial y}(3,1) \right) = (-6,-4)$.
- Now, $\nabla f(3,1) \cdot (u_1, u_2) = \|\nabla f(3,1)\| \|\mathbf{u}\| \cos(\nabla f, \mathbf{u}) = \sqrt{52} \cos(\nabla f, \mathbf{u})$. If $\cos(\nabla f, \mathbf{u}) = -1$, then we have the minimum. So take $\mathbf{u} = -\nabla f(3,1) = (6,4) = (3,2)$.
- So the marble will fall in the direction $(3,2)$.

**Theorem**

$-\|\nabla f(\mathbf{x})\| \leq \mathbf{D_u} f(\mathbf{x}) \leq \|\nabla f(\mathbf{x})\|$.

**Theorem**

1) $\nabla f(\mathbf{x})$ points in the direction of maximum increase of the function *f* at the point **x**.
2) $-\nabla f(\mathbf{x})$ points in the direction of maximum decrease of the function *f* at the point **x**.

**Theorem**

If *S* is the level surface given by the equation $F(x, y, z) = k$ and *S* has a tangent plane at $(x_0, y_0, z_0)$, then $\nabla F(x_0, y_0, z_0)$ is a normal vector to the tangent plane to *S* at the point $(x_0, y_0, z_0)$.

Proof: We just show that if $\mathbf{r}(t)$ denotes any path on the surface *S* that passes through $(x_0, y_0, z_0)$, the our gradient $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector of $\mathbf{r}(t)$ at $(x_0, y_0, z_0)$. So let $\mathbf{r}(t)$ be a curve on *S*, so $F(r(t)) = k, \forall t \in I$. Now $\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial t} = 0 \Rightarrow \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) \cdot \left( \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t} \right) = 0$ $\Rightarrow \nabla F \cdot \mathbf{r}'(t) = 0$.

# Higher-Order Derivatives

## The Taylor Theorem

### First Order Taylor Expansion

$f(x+h) = f(x) + hf'(x) + R_1(x,h)$ where $\lim_{h \to 0} \frac{R_1(x,h)}{h} = 0$.

Notice that $\int_x^{x+h} f'(t)dt = f(t)\Big|_x^{x+h}$, so $f(x+h) = f(x) + hf'(x) + \left( \int_x^{x+h} f'(t)dt - hf'(x) \right) = f(x) + hf'(x) + R_1(x,h)$.

**Theorem**

$$f(x+h) = f(x) + hf'(x) + \int_x^{x+h} (x+h-t) f''(t)\,dt .$$

### Second Order Taylor Expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{1}{2}\int_x^{x+h} (x+h-t)^2 f'''(t)\,dt .$$

### The General Taylor Expansion

If $f$ is $n+1$ times continuously differentiable at $x$, then

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + R_n(x,h)$, where

$$R_n(x,h) = \frac{1}{n!}\int_x^{x+h} (x+h-t)^n f^{(n+!)}(t)\,dt .$$

### Note

Notice that $f^{(n+1)}$ being continuous is bounded on $[x, x+h]$. Let $M$ denote the maximum of $f^{(n+1)}$ over $[x, x+h]$. Then $\left|R_n(x,h)\right| \le \frac{M}{n!}\int_x^{x+h} (x+h-t)^n\,dt$.

Also notice that when $x \le t \le x+h$, $\left|x+h-t\right| \le |h|$. So $\left|R_n(x,h)\right| \le \frac{M}{n!}\int_x^{x+h} |h|^n\,dt = \frac{M}{n!}\frac{|h|^{n+1}}{n+1}$.

Therefore $\left|R_n(x,h)\right| \le \frac{M|h|^{n+1}}{(n+1)!}$.

### First Order Taylor Expansion for $\mathbf{R}^n$

For $f : \mathbf{R}^n \to \mathbf{R}$, $f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{h} + R_1(\mathbf{x},\mathbf{h})$, where

$$R_1(\mathbf{x},\mathbf{h}) = \int_0^1 (1-t) \sum_{i=1}^n \sum_{j=1}^n h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}+t\mathbf{h})\,dt .$$

### Theorem: Second Order Taylor Expansion for a Function of Several Variables

If $f : \mathbf{R}^n \to \mathbf{R}$ is at least thrice differentiable at $\mathbf{x}$, then

$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}(\mathbf{x}) + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) + R_n(\mathbf{x},\mathbf{h})$, where

$$R_n(\mathbf{x},\mathbf{h}) = \frac{1}{2}\int_0^1 (1-t)^2 \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n h_i h_j h_k \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(\mathbf{x}+t\mathbf{h})\,dt .$$

## Local Extrema

### Definitions: Local Minima

1) The function $y = f(x)$ has a "local minimum at $x = x_0$" if there exists an open interval $I$ such that $x_0 \in I$ and for all $x \in I$, $f(x) \geq f(x_0)$.
2) Let $f : A \subseteq \mathbf{R}^2 \to \mathbf{R}$. The function $z = f(x, y)$ has a "local minimum at $x = x_0$" if there exists an open interval disc $D_r(x_0, y_0)$ such that for all $(x, y) \in D_r(x_0, y_0) \cap A$, $f(x, y) \geq f(x_0, y_0)$.

**Theorem**
If *f* is differentiable at $(x_0, y_0)$ and *f* has a local maximum/minimum at $(x_0, y_0)$, then $\nabla f(x_0, y_0) = (0,0)$, that is $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = (0,0)$.

**Definition: Critical Point**
The function $z = f(x, y)$ has a critical point at $(x_0, y_0)$ if $\nabla f(x_0, y_0) = (0,0)$, or if at least one of $\frac{\partial f}{\partial x}(x_0, y_0)$ or $\frac{\partial f}{\partial y}(x_0, y_0)$ does not exist.

**Theorem: Second Derivate Test**
If $f : A \subseteq \mathbf{R}^2 \to \mathbf{R}$ is differentiable at $(x_0, y_0)$ and $\nabla f(x_0, y_0) = (0,0)$, then:

1) If $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$ and $\det\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} > 0$, then *f* has a local minimum at $(x_0, y_0)$.
2) If $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$ and $\det\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} > 0$, then *f* has a local maximum at $(x_0, y_0)$.
3) If $\det\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} < 0$, then *f* has not a local extrema at $(x_0, y_0)$. We say that *f* has a saddle point at $(x_0, y_0)$.
4) If $\det\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = 0$, then this test is inconclusive.

## LAGRANGE MULTIPLIERS METHOD

**Theorem**
Let $f : \mathbf{R}^n \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}$ be of class $C^1$. Let $S := \{\mathbf{x} \in \mathbf{R}^n \mid g(\mathbf{x}) = 0\}$. Let $f \mid S : S \to \mathbf{R}, \mathbf{x} \in S \mapsto f(\mathbf{x})$. If $f \mid S$ has a local extremum at $\mathbf{x}_0 \in S$, then there exists $\lambda$ such that $\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$ where $\nabla g(\mathbf{x}_0) \neq 0$.

## THE IMPLICIT FUNCTION THEOREM

**Theorem: One Variable Implicit Function Theorem**

If $F:\mathbf{R}^{n+1}\to\mathbf{R}$ is a $C^1$ function. Let $\mathbf{x}_0\in\mathbf{R}^n$, $z_0\in\mathbf{R}$, and $F(\mathbf{x}_0,z_0)=0$. If $\frac{\partial F}{\partial z}(\mathbf{x}_0,z_0)\neq 0$, then there exists an open set $U\subseteq\mathbf{R}^n$, $\mathbf{x}_0\in U$, and a unique function $z=g(\mathbf{x})$ such that for all $\mathbf{x}\in U$, $F(\mathbf{x},g(\mathbf{x}))=0$.

**Theorem: The General Implicit Function Theorem**

Let $\begin{matrix} F_1=(x_1,\dots,x_n,z_1,\dots,z_m)=0 \\ \vdots \\ F_m=(x_1,\dots,x_n,z_1,\dots,z_m)=0 \end{matrix}$ where $F_1,\dots,F_m$ are of class $C^1$. Let $\mathbf{x}_0\in\mathbf{R}^n$ and $\mathbf{z}_0\in\mathbf{R}^n$ such that $F_i(\mathbf{x}_0,\mathbf{z}_0)=0,\forall i=1,\dots,m$. If the Jacobian Determinant $\det\begin{pmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial z_1} & \cdots & \frac{\partial F_m}{\partial z_m} \end{pmatrix}\neq 0$ (at $(\mathbf{x}_0,\mathbf{z}_0)$), then there exists an open set $U\subseteq\mathbf{R}^m$ that contains $\mathbf{z}_0\in\mathbf{R}^m$ and $m$ unique functions $g_1,\dots,g_m:U\to\mathbf{R}$ such that for each $\mathbf{z}_0\in U$, $F_i(\mathbf{x}_0,\mathbf{g}(\mathbf{x}_0))=0,\forall i=1,\dots,m$ where $\mathbf{g}=(g_1,\dots,g_m)$.

# Vector Valued Functions

## Arc Length

Suppose $\mathbf{r}(t)=(x(t),y(t),z(t)), a\le t\le b$, then the length of the curve on the interval $[a,b]$ is $L=\int_a^b\|\mathbf{r}'(t)\|dt$.

## Vector Fields

**Definition**

$\mathbf{F}:\mathbf{R}^n\to\mathbf{R}^n$ is a vector field.

**Definition: The Divergence of a Vector Field**

Let $\mathbf{F}=(F_1,F_2,F_3)$. Then $\operatorname{div}\mathbf{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$.

**Definition: The Curl of a Vector Field**

Let $\mathbf{F}=(F_1,F_2,F_3)$. Then $\operatorname{curl}\mathbf{F}=\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z},\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x},\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$.

**Notation**

If $\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$, then $\operatorname{div}\mathbf{F}=\nabla\cdot\mathbf{F}$ and $\operatorname{curl}\mathbf{F}=\nabla\times\mathbf{F}$.

### Laplacian Operator

The Laplacian operator is denoted $\nabla^2$. If $f:\mathbf{R}^n \to \mathbf{R}$, then $\nabla^2 f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_n^2}$. If $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$, then $\nabla^2 \mathbf{F} = \left( \frac{\partial^2 f}{\partial x_1^2}, \cdots, \frac{\partial^2 f}{\partial x_n^2} \right)$.

### Note

1) $\operatorname{curl}(\nabla f) = 0$. Proof: $\nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \ldots, \ldots \right) = 0$ if $f \in C^1$.
2) $\operatorname{div}(\operatorname{curl} F) = 0$. Proof: $\nabla \cdot (\nabla \times F) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$.

### Some Identities

1) $\nabla(fg) = (\nabla f)g + f(\nabla g)$.
2) $\operatorname{div}(fF) = f(\operatorname{div} F) + F \cdot (\nabla f)$.
3) $\operatorname{div}(F \times G) = G \cdot (\operatorname{curl} F) + F \cdot (\operatorname{curl} G)$.
4) $\operatorname{curl}(fF) = f(\operatorname{curl} F) + (\nabla f) \times F$.

### Potential Function

If $\operatorname{curl} F = 0$, then there exists $f:\mathbf{R}^3 \to \mathbf{R}$ such that $F = \operatorname{grad} f$.
$F(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$ is the gradient field, and $f = f(x,y,z)$ is the potential function.

## FLOW LINES

### Definition

A flow curve has tangent vectors that coincides with the vector field.

### Example

Show that $\mathbf{c}(t) = (t^2, 2t-1, \sqrt{t}), t > 0$ is a flow line of the velocity vector field $F(x,y,z) = \left( y+1, 2, \frac{1}{2z} \right)$.

Want: $\mathbf{c}'(t) = F(\mathbf{c}(t))$. Now, $\mathbf{c}'(t) = \left( 2t, 2, \frac{1}{2\sqrt{t}} \right)$, and $F(\mathbf{c}(t)) = \left( 2t, 2, \frac{1}{2\sqrt{z}} \right) = \mathbf{c}'(t)$.

### Arc Length Parametrization

Let $\sigma(t) = (x(t), y(t), z(t)), a \le t \le b$. Let $p(s)$ be the same curve parameterized arc length:

$s = s(t) = \int_a^t \|\sigma'(\tau)\| d\tau$.

1) Then the velocity vector is always a unit vector.
   Proof: $\sigma'(t) = \frac{dp}{ds}\frac{ds}{dt} = \frac{dp}{ds}\|\sigma'(t)\| \Rightarrow \frac{dp}{ds} = \frac{\sigma'(t)}{\|\sigma'(t)\|}$ which is unit.
2) Let $T = \frac{dp}{ds} = \mathbf{v}$, then $\kappa = \left\|\frac{dT}{ds}\right\|$ is called the curvature of $\sigma$ parameterized by $p(s)$.

# Integration

## Double Integral

#### Definition

Let $f : \mathbf{R}^2 \to \mathbf{R}$. Let *D* be a bounded region in $\mathbf{R}^2$. Let *P* be an inner partition. Take all rectangles $R_{ij} \subset D$.

Let $(x_i^*, y_j^*) \in R_{ij}$. Let $|P| = \max|d_{ij}|$, where $|d_{ij}|$ is the diameter of $R_{ij}$.

The Reimann Sum is $S_n = \sum_{i,j=1}^{n} f(x_i^*, y_j^*)(\Delta A_{ij})$ where $\Delta A_{ij}$ is the area of $R_{ij}$.

If $\lim_{n\to\infty} S_n$ exists and independent of the choice of $(x_i^*, y_j^*)$, then *f* is integrable and $\lim_{n\to\infty} S_n = \iint_D f(x, y)dA$.

#### Note

$\iint_D f(x, y)dA$ is the volume below the surface $f(x, y)$ over *D*.

#### Theorem: Fubini's Theorem

If *f* is continuous over $R = [a,b]\times[c,d]$, then $\iint_R f(x, y)dA = \int_a^b \left[\int_c^d f(x, y)dy\right]dx = \int_c^d \left[\int_a^b f(x, y)dx\right]dy$ (iterated integrals).

#### Properties

Let $\Omega = [a,b]\times[c,d]$, $f : \Omega \to \mathbf{R}$ integrable.

1) If $f(x, y) = 0$ for all $(x, y) \in \Omega$, then $\iint_\Omega f(x, y)dA = 0$.
2) If $f(x, y) = k$, then $\iint_\Omega f(x, y)dA = k(b-a)(d-c)$.

3) If $\begin{cases} a < \alpha < b \\ c < \beta < d \end{cases}$, then $\iint_{\Omega} f(x, y)dA = \iint_{\substack{a \le x \le \alpha \\ c \le y \le \beta}} f(x, y)dA + \iint_{\substack{\alpha \le x \le \alpha \\ c \le y \le \beta}} f(x, y)dA + \iint_{\substack{\alpha \le x \le \alpha \\ \beta \le y \le d}} f(x, y)dA + \iint_{\substack{\beta \le x \le b \\ \beta \le y \le d}} f(x, y)dA$.

4) If $f \ge 0$ over Ω, then $\iint_{\Omega} f(x, y)dA \ge 0$.

5) $\left| \iint_{\Omega} f(x, y)dA \right| \le \iint_{\Omega} |f(x, y)| dA$.

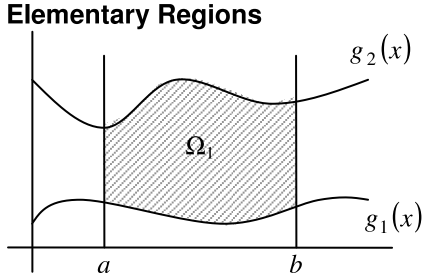
6) If $f$ is continuous over Ω, then $\iint_{\Omega} f(x, y)dA = f(x_0, y_0)(b-a)(d-c)$ for some $(x_0, y_0) \in \Omega$. This is the Mean Value Theorem for double integrals.

7) If $m \le f \le M$, then $m(b-a)(d-c) \le \iint_{\Omega} f(x, y)dA \le M(b-a)(d-c)$.

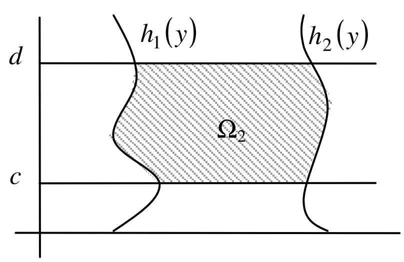
8) $\iint_{\Omega} (\alpha f(x, y) + \beta g(x, y))dA = \alpha \iint_{\Omega} f(x, y)dA + \beta \iint_{\Omega} g(x, y)dA$.

9) If $\begin{cases} a = -b, b > 0 \\ c = -d, d > 0 \end{cases}$, then $f(-x, y) = -f(x, y) \Rightarrow \iint_{\Omega} f(x, y)dA = 0$ and $f(x, -y) = -f(x, y) \Rightarrow \iint_{\Omega} f(x, y)dA = 0$.

### Elementary Regions




Type 1 Elementary Region: $\Omega_1 = \begin{cases} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{cases}$.




Type 2 Elementary Region: $\Omega_1 = \begin{cases} a \le y \le b \\ h_1(y) \le x \le h_2(y) \end{cases}$.

## CHANGE OF VARIABLES

### Definition: Jacobian Determinant

Let $T : D^* \subset \mathbf{R}^2 \to \mathbf{R}^2$ be a $C^1$ transformation given by $\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$. The Jacobian determinant is

$\dfrac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$ (the determinant of the derivative matrix $\mathbf{D}T(u, v)$).

### Theorem: Change of Variables for Double Integrals

Let $D$ and $D^*$ be elementary regions in the plane and let $T : D^* \to D$ be of class $C^1$; suppose that $T$ is one-to-one on $D^*$. Furthermore, suppose that $D = T\left(D^*\right)$. Then for any integrable function $f : D \to \mathbf{R}$ ,we have $\iint\limits_{D} f(x, y)dxdy = \iint\limits_{D^*} f(x(u,v), y(u,v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv$.

### Example: Jacobian Determinants of Popular Transformations

1) Rectangular to polar: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix}\right| = r$.
2) Rectangular to cylindrical: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $\left|\frac{\partial(x, y, z)}{\partial(r, \theta, z)}\right| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{pmatrix}\right| = r$.
3) Rectangular to spherical: $\begin{cases} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{cases}$, $\left|\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)}\right| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{pmatrix}\right| = \rho^2 \sin\phi$.

## SOME BASIC APPLICATIONS

1) If $\Omega$ is a solid region in $\mathbf{R}^3$, then is volume is $\iiint\limits_{\Omega} dV$ .
2) If $\delta(x, y, z)$ is the density at $(x, y, z)$ of the solid $\Omega$, then it's mass is $M = \iiint\limits_{\Omega} \delta(x, y, z)dV$ .
3) If $f : \Omega \subset \mathbf{R}^3 \to \mathbf{R}$, then $\dfrac{\iiint\limits_{\Omega} f(x, y, z)dV}{\iiint\limits_{\Omega} dV}$ is the average value of $f$ over $\Omega$.
4) The center of gravity is given by $\bar{x} = \dfrac{\iiint\limits_{\Omega} x\delta(x, y, z)dV}{\iiint\limits_{\Omega} \delta(x, y, z)dV}$, $\bar{y} = \dfrac{\iiint\limits_{\Omega} y\delta(x, y, z)dV}{\iiint\limits_{\Omega} \delta(x, y, z)dV}$, $\bar{z} = \dfrac{\iiint\limits_{\Omega} z\delta(x, y, z)dV}{\iiint\limits_{\Omega} \delta(x, y, z)dV}$.

## PATH INTEGRAL

### Problem

A very thin piece of wire has the shape of the arc $r(t)$. It has density $\delta(x, y, z)$ which changes continuously. What is the mass of the wire?

**Definition: The Path Integral**

$\int_a^b f(x(t), y(t), z(t))\|r'(t)\|dt$ is called the path integral of the function *f* along the path $r(t), a \le t \le b$.

Notation: $\int_c fds := \int_a^b f(r(t))\|r'(t)\|dt$.

**Property**

1) The line integral of a function $g(x, y)$ over a curve given as the graph of a function, say $y = f(x), a \le x \le b$, is $\int_c gds = \int_a^b g(r(x, f(x)))\sqrt{1+(f'(x))^2}\,dx$.
2) The line integral of a function $g(x, y)$ over a curve given as the graph of a function in polar coordinates as $r = r(\theta), \theta_1 \le \theta \le \theta_2$ is $\int_c gds = \int_{\theta_1}^{\theta_2} g(r(\theta)\cos\theta, r(\theta)\sin\theta)\sqrt{(r(\theta))^2 + (r'(\theta))^2}\,d\theta$.

## LINE INTEGRAL

**Work**

The work done is $W = \vec{F} \cdot \vec{v}$.

**Problem**

Let $\mathbf{F} = \mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$. Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a path. What is the total work done by **F** when the object moves from $r(a)$ to $r(b)$?

**Definition: Line Integral**

Let $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ be a vector field, and $c : \mathbf{r}(t) = (x(t), y(t), z(t)), a \le t \le b$ be a path. If $\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$ exists, then it is call the line integral of **F** over c.

Notation: $\int_c \mathbf{F} \cdot d\mathbf{r}$.

**Theorem**

If $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ is a gradient vector field (i.e. $\mathbf{F} = \nabla f$ for some $f : \mathbf{R}^3 \to \mathbf{R}$) and both **F** and *f* are continuous, then for smooth path $c : \mathbf{r}(t) = (x(t), y(t), z(t)), a \le t \le b$, $\int_c \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. Here, **F** is said to be path independent.

**Theorem**

If **F** is path independent and *c* is a closed curved, then $\int_c \mathbf{F} \cdot d\mathbf{r} = 0$.

### Theorem

If $\mathbf{r}(t), a \le t \le b$ and $\boldsymbol{\delta}(u), c \le u \le d$ are different parameterizations of the same curve with $\mathbf{r}(a) = \boldsymbol{\delta}(c)$ and $\mathbf{r}(b) = \boldsymbol{\delta}(d)$, and $\int_c \mathbf{F} \cdot d\mathbf{r}$ exists, then $\int_c \mathbf{F} \cdot d\boldsymbol{\delta}$ exists and $\int_c \mathbf{F} \cdot d\boldsymbol{\delta} = \int_c \mathbf{F} \cdot d\mathbf{r}$.

### Remark

The line integral and path integral are connected.

$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\|\mathbf{r}'(t)\|dt = \int_a^b \underbrace{(F \cdot T)}_{f} ds$, where $T = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \mathbf{r}'(t) \ne 0$ the unit tangent vector.

## SURFACE INTEGRAL

### Definition: Parametric Surface

Consider $\Phi : D \subseteq \mathbf{R}^2 \to \mathbf{R}^3$ where $(u,v) \mapsto (x(u,v), y(u,v), z(u,v))$. $\Phi$ is said to be a parametric surface.
Note: $\{(x(u,v), y(u,v), z(u,v)) \mid (u,v) \in D\}$ is a surface.

### Property

Let $\mathbf{T}_u := \left(\frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0)\right)$ and $\mathbf{T}_v := \left(\frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0)\right)$. If $(\mathbf{T}_u \times \mathbf{T}_v)(u_0,v_0) \ne \mathbf{0}$, then this is a vector normal to the parametric surface $\Phi(u,v)$ at the point $(u_0,v_0)$.

### Definition: Surface Integral

The surface integral of a function $f : \mathbf{R}^3 \to \mathbf{R}$ over a parameterized surface $\Phi(u,v), (u,v) \in D$ is $\iint_D f(\Phi(u,v))\|\mathbf{T}_u \times \mathbf{T}_v\| dudv$.

### Definition: Flux

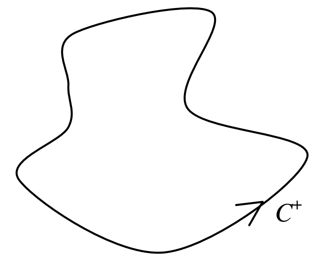
Let $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ be a vector field. Let $\Phi(u,v), (u,v) \in D$ be a parameterized surface. The integral $\iint_D \mathbf{F}(\Phi(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) dudv$ is called the flux of the vector field **F** (the surface integral of vector field **F**).

## GREEN'S THEOREM

### Orientation

If $C$ is a simple closed curve in the plane $\mathbf{R}^2$, then there are two ways to go around the curve:

1) Counterclockwise (positive): $C^+$.
2) Clockwise (negative): $C^-$.

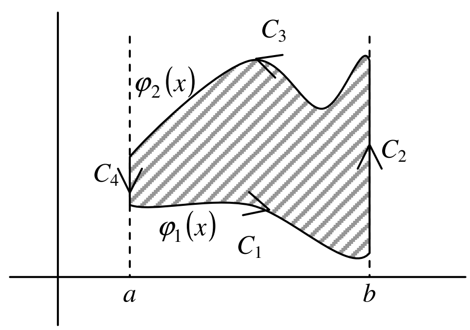



### Lemma

Suppose that $D$ is a region of type 1 and that $P : D \to \mathbf{R}$ is of class $C^1$. Then $\int_{C^+} P dx = -\iint_D \frac{\partial P}{\partial y} dxdy$, where $C$ the boundary of $D$.

Proof:

$D$ is a region of type 1 means there are continuous functions $\varphi_1, \varphi_2 : [a,b] \to \mathbf{R}$ where $\varphi_1(x) \le \varphi_2(x), \forall x \in [a,b]$ such that $D = \{(x,y) \in \mathbf{R}^2 \mid a \le x \le b, \varphi_1(x) \le y \le \varphi_2(x)\}$.




Now,

$$\iint_D \frac{\partial P}{\partial y} dxdy = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}(x,y) dydx = \int_a^b \left(P(x,\varphi_2(x)) - P(x,\varphi_1(x))\right) dx = \int_a^b P(x,\varphi_2(x)) dx - \int_a^b P(x,\varphi_1(x)) dx.$$

Also, $\int_{C^+} P dx = \int_{C_1^+} P dx + \int_{C_2^+} P dx + \int_{C_3^-} P dx + \int_{C_4^+} P dx$. Along $C_2^+$ and $C_4^+$, $x$ is a constant, so $\int_{C_2^+} P dx = \int_{C_4^+} P dx = 0$.

Since $C_1^+$ is parameterized by $x \mapsto (x, \varphi_1(x)), x \in [a,b]$, $\int_{C_1^+} P dx = \int_a^b P(x,\varphi_1(x)) dx$; since $C_3^+$ is parameterized by $x \mapsto (x, \varphi_2(x)), x \in [a,b]$, $\int_{C_3^+} P dx = \int_a^b P(x,\varphi_2(x)) dx \Rightarrow \int_{C_3^-} P dx = -\int_a^b P(x,\varphi_2(x)) dx$. So

$$\int_{C^+} P dx = \int_a^b P(x,\varphi_1(x)) dx - \int_a^b P(x,\varphi_2(x)) dx = -\iint_D \frac{\partial P}{\partial y} dxdy.$$

### Lemma

Suppose that $D$ is a region of type 2 and that $Q : D \to \mathbf{R}$ is of class $C^1$. Then $\int_{C^+} Q dy = \iint_D \frac{\partial Q}{\partial y} dxdy$, where $C$ the boundary of $D$.

### Theorem: Green's Theorem

Let $D$ be a region of type 3 and let $C$ be its boundary. Suppose that $P : D \to \mathbf{R}$ and $Q : D \to \mathbf{R}$ of class $C^1$.

Then $\int_{C^+} P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$.

### Theorem

Let $D$ be a region of type 3 with boundary $C = \partial D$. Then the area of $D$ is given by $A = \frac{1}{2}\int_{\partial D} \underbrace{-y}_{P}dx + \underbrace{x}_{Q}dy$ .

Proof: $\int_C -ydx + xdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \iint_D (1+1)dxdy = 2A$ .

## STOKE'S THEOREM

### Theorem: Stoke's Theorem

Let $S$ be the oriented surface defined by $\Phi : \Omega \subset \mathbf{R}^2 \to \mathbf{R}^3$ of class $C^1$ and $\Omega$ type 1 or 2. Let $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ of class $C^1$. Then $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$ .

### Remark

Let $\mathbf{F} = (P, Q, 0)$. Then $\operatorname{curl} \mathbf{F} = (-Q_z - P_z, Q_x - P_y)$ and $\mathbf{T}_x \times \mathbf{T}_y = (0,0,1)$, so $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$ . Now $\int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial\Omega} Pdx + Qdy$ . This is Green's Theorem!

## GAUSSES' THEOREM

### Theorem: Gausses' Theorem

Let $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ of class $C^1$. Let $\Omega$ be a solid in $\mathbf{R}^3$ with the a smooth surface $\partial\Omega$. Then $\iint_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S} = \iiint_\Omega \mathbf{F} dV$ .