The Geometry of Euclidean Space

VECTORS IN \( \mathbb{R}^n \)

- A vector in \( \mathbb{R}^n \) can be written as \( u = (u_1, \ldots, u_n) \).

**Definitions**

\[
\begin{align*}
\mathbf{u} = \mathbf{v} & \iff \begin{cases} u_1 = v_1 \\ \vdots \\ u_n = v_n \end{cases} \\
\alpha \mathbf{u} = \mathbf{c} & \iff \begin{cases} \alpha u_1 = c_1 \\ \vdots \\ \alpha u_n = c_n \end{cases}, \alpha \in \mathbb{R}.
\end{align*}
\]

- If \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_n) \), then

**Some Properties**

1) \( \mathbf{u} \in \mathbb{R}^n \), \( \mathbf{v} \in \mathbb{R}^n \) \( \implies (\mathbf{u} + \mathbf{v}) \in \mathbb{R}^n \)
2) \( \mathbf{u} \in \mathbb{R}^n \), \( \alpha \in \mathbb{R} \) \( \implies \alpha \mathbf{u} \in \mathbb{R}^n \)
3) \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \)
4) \( \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{u}) + \mathbf{w} \)
5) \( -\mathbf{u} = (-u_1, \ldots, -u_n) \) and \( \mathbf{u} + (-\mathbf{u}) = \mathbf{0} = (0, \ldots, 0) = (-\mathbf{u}) + \mathbf{u} \)
6) \( \mathbf{u} + \mathbf{0} = \mathbf{u} = \mathbf{0} + \mathbf{u} \)
7) \( \alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v} \)
8) \( (\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u} \)
9) \( (\alpha \beta)\mathbf{u} = \alpha(\beta \mathbf{u}) \)
10) \( \mathbf{0} \mathbf{u} = \mathbf{u} \).

**NORM**

**Definition**

Let \( u = (u_1, \ldots, u_n) \). The norm is \( \|\mathbf{u}\| = \sqrt{u_1^2 + \cdots + u_n^2} \).

**Properties**

1) \( \|\mathbf{u}\| \geq 0 \)
2) \( \|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| \)
3) \( \|\alpha \mathbf{u}\| = |\alpha| \|\mathbf{u}\| \).
**DOT PRODUCT**

Let \( \mathbf{u} = (u_1, \ldots, u_n) \) and \( \mathbf{v} = (v_1, \ldots, v_n) \), then \( \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n \).

**Properties**

1) \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \).
2) \( (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \).
3) \( \alpha(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (\alpha \mathbf{v}) \).
4) \( \mathbf{u} \cdot \mathbf{u} \geq 0 \), moreover, \( \mathbf{u} \cdot \mathbf{u} = 0 \iff \mathbf{u} = \mathbf{0} \)

**More Properties**

1) \( \mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 \).
2) \( \|\mathbf{u} \| \cdot \|\mathbf{v}\| \leq \mathbf{u} \cdot \mathbf{v} \) (Cauchy-Schwartz).
3) \( (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \) and \( (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \).
4) \( (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 \).

**The Cosine Law**

\[
\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{v}\|\|\mathbf{u}\| \cos \theta
\]

\( \Rightarrow (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{v}\|\|\mathbf{u}\| \cos \theta
\]

\( \Rightarrow \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{v}\|\|\mathbf{u}\| \cos \theta
\]

\( \Rightarrow \mathbf{u} \cdot \mathbf{v} = \|\mathbf{v}\|\|\mathbf{u}\| \cos \theta
\)

**Theorem**

\( \mathbf{u} \cdot \mathbf{v} = \|\mathbf{v}\|\|\mathbf{u}\| \cos \theta, \mathbf{u}, \mathbf{v} \neq \mathbf{0} \).

**Some Basic Consequences**

1) \( \|\mathbf{u} \| \cdot \|\mathbf{v}\| \leq \mathbf{u} \cdot \mathbf{v} \).
2) \( \mathbf{u} \cdot \mathbf{v} > 0 \) means \( 0 \leq \theta < \frac{\pi}{2} \).
3) \( \mathbf{u} \cdot \mathbf{v} = 0 \) means \( \theta = \frac{\pi}{2} \).
4) \( \mathbf{u} \cdot \mathbf{v} < 0 \) means \( \frac{\pi}{2} < \theta < \pi \).

**PROJECTION**
Geometry

\[ \mathbf{u} \times \mathbf{v} = \text{proj}_u \mathbf{v} \]

\[ \text{proj}_u \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \]

**Theorem**

\[ \|\text{proj}_u \mathbf{v}\| = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\|} \]

**LINES IN R^3**

Some ways to characterize the line that passes through a given point \( P_0 = (x_0, y_0, z_0) \) and follows the direction of a non-zero vector \( \mathbf{v} = (a, b, c) \). If \( P = (x, y, z) \) denotes any point on the line, then:

- \( P_0P = \lambda \mathbf{v} \) (vector equation).
  
  \[ x = x_0 + \lambda a \]
  \[ y = y_0 + \lambda b \]
  \[ z = z_0 + \lambda c \] (parametric equations).
- \( \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \) (symmetric equations).

**THE CROSS PRODUCT**

**Definition**

If \( \mathbf{u} = (u_1, u_2, u_3) \) and \( \mathbf{v} = (v_1, v_2, v_3) \), then \( \mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \) is the cross product of \( \mathbf{u} \) and \( \mathbf{v} \).
Note: $\mathbf{u} \times \mathbf{v} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$.

Some Basis Properties
1) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.
2) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$. More general, if $\mathbf{u} \parallel \mathbf{v}$, then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
3) $(\mathbf{a} \mathbf{u}) \times \mathbf{v} = \mathbf{a}(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (\mathbf{a} \mathbf{v})$.
4) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.
   Note: The cross product is not associative.

The Geometry of the Cross Product
1) If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$, then $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| ||\mathbf{v}|| \sin \theta$.
   
   $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$.

Theorem
$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$. Geometrically, $(\mathbf{u} \times \mathbf{v}) \perp \mathbf{u}$, $(\mathbf{u} \times \mathbf{v}) \perp \mathbf{v}$.

Equations of Planes in $\mathbb{R}^3$
- An equation for the plane that passes through $\mathbf{P}_0 = (x_0, y_0, z_0)$ and is perpendicular (normal) to $\mathbf{n} = (a, b, c)$ is $\mathbf{P}_0 \cdot \mathbf{n} = 0$.
- In terms of coordinates, $(x-x_0, y-y_0, z-z_0) \cdot (a, b, c) = 0 \iff a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$, or $ax + by + cz = d$ where $d = ax_0 + by_0 + cz_0$.
- Another way to describe a plane is $\begin{cases} x = x_0 + su_1 + tv_1 \\ y = y_0 + su_2 + tv_2 \\ z = z_0 + su_3 + tv_3 \end{cases}$. This is the parametric equation through $(x_0, y_0, z_0)$ and parallel to $\mathbf{u}$ and $\mathbf{v}$.

Theorem
Let $\mathbf{u} \neq \mathbf{0}$, $\mathbf{v} \neq \mathbf{0}$, $\mathbf{w} \neq \mathbf{0}$.
1) The area of the parallelogram generated by $\mathbf{u}$ and $\mathbf{v}$ is $||\mathbf{u} \times \mathbf{v}||$.
2) The volume of the parallelepiped generated by the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is $||\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})||$.

COORDINATES IN $\mathbb{R}^3$
Rectangular Coordinates

\[ P = (x, y, z) \]

Cylindrical Coordinates

\[ P = (r, \theta, z) \]

Spherical Coordinates

\[ P = (\rho, \theta, \phi) \]

**Example**

<table>
<thead>
<tr>
<th>Rectangular ((x, y, z))</th>
<th>Cylindrical ((r, \theta, z))</th>
<th>Spherical ((\rho, \theta, \phi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-2, 0, 2))</td>
<td>((2, \pi, 2))</td>
<td>(\left(2\sqrt{2}, \pi, \frac{\pi}{4}\right))</td>
</tr>
<tr>
<td>(\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 0\right))</td>
<td>(\left(3, \frac{\pi}{4}, 0\right))</td>
<td>(\left(3, \frac{\pi}{4}, \frac{\pi}{2}\right))</td>
</tr>
</tbody>
</table>

**Equations of Transformations**

- Rectangular and Cylindrical:
  \[ x = r \cos \theta \quad \text{and} \quad y = r \sin \theta \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right). \]
  \[ z = z \]

- Rectangular and Spherical:
  \[ z = \rho \cos \theta \sin \phi \]
  \[ x = \rho \cos \theta \sin \phi \]

- **Vectors in** \(\mathbb{R}^n\)**

**Definition**

Let \(\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n\). Then the **norm** of \(\mathbf{u}\) is \(\|\mathbf{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}\).

**Definition**

Let \(P = (p_1, \ldots, p_n)\) and \(Q = (q_1, \ldots, q_n)\) be two points in \(\mathbb{R}^n\). Then the **distance** between \(P\) and \(Q\) is \(\sqrt{(p_1 - q_1)^2 + \cdots + (p_n - q_n)^2}\).

**Definition**
If \( P_0 = \left( p_1^0, \ldots, p_n^0 \right) \) and \( v = (a_1, \ldots, a_n) \), then the line that passes through the point \( P_0 \) and follows the direction of the vector \( v \) is the set of all points \( (x_1, \ldots, x_n) \) such that:

\[
\begin{align*}
  x_1 &= x_1^0 + t a_1 \\
  \vdots \\
  x_n &= x_n^0 + t a_n
\end{align*}
\]

Problem
Suppose that the real numbers \( a, b, c, d \) satisfy the condition \( a^2 + (b-2)^2 + (c+1)^2 + (d-3)^2 = 16 \). Find the values for \( a, b, c, d \) for which \( f(a, b, c, d) = (a-7)^2 + (b-8)^2 + (c+10)^2 + (d-12)^2 \) take its maximum and minimum values.

- In this case, we can view \( a^2 + (b-2)^2 + (c+1)^2 + (d-3)^2 = 16 \) as a “sphere” centered at \( (0,2,-1,3) \) with radius 4, and \( f(a, b, c, d) \) as the distance from \( (a, b, c, d) \) to \( (7,8,-10,12) \).
- Now, the line that joins the center of the “sphere” \( (0,2,-1,3) \) with the point \( (7,8,-10,12) \) is given by:

\[
\begin{align*}
  x_1 &= 0 + 7t \\
  x_2 &= 2 + 6t \\
  x_3 &= -1 - 9t \\
  x_4 &= 3 + 9t
\end{align*}
\]

- At the intersection with the “sphere”, \( (7t)^2 + (6t)^2 + (-9t)^2 + (9t)^2 = 16 \Rightarrow t = \pm \frac{4}{247} \). Now one of these two points (which we can solve easily) will give the maximum value for \( f(a, b, c, d) \), and the other one will give the minimum.

Limits

Definition: Limit of One Variable
When we say “the function \( f(x) \) approaches the number \( L \) as \( x \) approaches \( a \)” (write \( \lim_{x \to a} f(x) = L \)), what we mean is:

- Geometrically: For any open interval \( B \) that contains \( L \), we can always find an open interval \( A \) that contains \( a \) such that for all \( x \in A, x \neq a \), \( f(x) \in B \).
- Algebraically: For every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon \).

Definition: Limit of Two Variables
Let \( f: \mathbb{R}^2 \to \mathbb{R} \). When we say “the function \( f(x, y) \) approaches the number \( L \) as \( (x, y) \) approaches \( (a, b) \)” (write \( \lim_{(x, y) \to (a, b)} f(x, y) = L \)), what we mean is:

- Geometrically: For any open interval \( B \) that contains \( L \), we can always find an “open disk” \( D \) centered at \( (a, b) \) such that for all \( (x, y) \in D, (x, y) \neq (a, b) \), \( f(x, y) \in B \).
- Algebraically: For every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( 0 < \| (x-a, y-b) \| = \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x, y) - L| < \epsilon \).
OPEN AND CLOSED SETS

Some “Geometric” Definitions
1) The set consisting of all \( x \in \mathbb{R}^n \) such that \( \|x - x_0\| < r, r > 0 \) is called “the disk centered at \( x_0 \) with radius \( r \)” and is denoted \( D_r(x_0) \).

2) A set \( A \subseteq \mathbb{R}^n \) is said to be open if for every \( x_0 \in A \), there exists \( r > 0 \) such that \( D_r(x_0) \subseteq A \).

3) Let \( A \subseteq \mathbb{R}^n, x \in \mathbb{R}^n \) is said to be a boundary point of \( A \) if for every \( r > 0 \), the disk \( D_r(x) \) contains at least one point of \( A \) and contains at least one point of \( A^c \).

Examples
Are the following sets open?
1) \( \{(x, y) \in \mathbb{R}^2 | y \geq x \} \) is not open.
2) \( \{(x, y) \in \mathbb{R}^2 | |x| < 4 \} \) is open.
3) \( \mathbb{R}^2, \emptyset \) are both open.

Theorem
If \( A, B \) are open sets, then \( A \cap B \) is open also.

Proof: Let \( x \in A \cap B \), then \( x \in A \) and \( x \in B \). Since \( A \) is open, there exists \( r_A > 0 \) such that \( D_{r_A}(x) \subseteq A \).
Since \( B \) is open, there exists \( r_B > 0 \) such that \( D_{r_B}(x) \subseteq B \). Now, by taking \( r = \min\{r_A, r_B\} \).
\[
D_r(x) \subseteq D_{r_A}(x) \subseteq A \\
D_r(x) \subseteq D_{r_B}(x) \subseteq B \\
\Rightarrow D_r(x) \subseteq A \cap B .
\]

Theorem
Let \( A \subseteq \mathbb{R}^n \). If the set \( A \) contains at least one of its boundary points, then \( A \) is not open.

Definition
A set \( A \subseteq \mathbb{R}^n \) is said to be closed if it contains all of its boundary points.

Note
\( \mathbb{R}^2, \emptyset \) are both open and closed.

LIMITS IN \( \mathbb{R}^n \)

Definition: The General Case
Let \( f : A \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n \), where \( A \) is open. Let \( a \in A \) or a boundary point of \( A \). Then \( \lim_{x \to a} f(x) = L \) means for each open set \( N \subseteq \mathbb{R}^n \) that contains \( L \), there exists an open set \( M \subseteq \mathbb{R}^m \) that contains \( a \) such that \( x \in M \cap A, x \neq a \Rightarrow f(x) \in N \).
Example
Let \( f(x, y) = \begin{cases} (x, y, 2y), & (x, y) \neq (1,1) \\ (5, 7, 9), & (x, y) = (1,1) \end{cases} \). Then \( \lim_{(x,y) \to (1,1)} f(x, y) = (1,1,2) \).

The Algebra of Limits
1) \( \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \).
2) \( \lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \).
3) If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \), then \( L = M \).
4) Let \( f : A \subseteq \mathbb{R}^m \to \mathbb{R}^n \), \( g : B \subseteq \mathbb{R}^n \to \mathbb{R}^p \), where \( A, B, f(A) \cap B \) are open. If \( \lim_{x \to a} f(x) = L \in B \) and \( \lim_{y \to L} g(y) = M \), then \( \lim_{x \to a} (g \circ f)(x) = M \).

Exercise
Evaluate the limit if it exists.
1) \( \lim_{(x,y) \to (2,3)} \frac{x^2 + y^2}{x^2 - y} = 7 \).
2) \( \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sin(x^2 + y^2)} \). Let \( f(x, y) = x^2 + y^2 \) and \( g(t) = \frac{t}{\sin t} \). Then
   \[ \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sin(x^2 + y^2)} = \lim_{(x,y) \to (0,0)} (g \circ f)(x, y) = \lim_{t \to 0} \frac{t}{\sin t} = 1 \).
3) \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} \). Approaching \((0,0)\) along \((t,t)\), \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} = \lim_{t \to 0} \frac{2t^2}{5t^2} = \frac{2}{5} \). So the limit doesn’t exist.
4) \( \lim_{(x,y) \to (0,0)} \frac{x^6}{x^2 + y^4} \). Using polar coordinates,
   \[ \lim_{(x,y) \to (0,0)} \frac{x^6}{x^2 + y^4} = \lim_{r \to 0} r^4 \cos^6 \theta = \lim_{r \to 0} r^4 \cos^6 \theta \cos^2 \theta + r^4 \sin^2 \theta = 0 \]. Alternately, we know
   \[ 0 \leq \frac{x^6}{x^2 + y^4} \leq \frac{x^6}{x^2} = x^4 \]. As \((x, y) \to (0,0)\), \( x^4 \to 0 \), so \( \lim_{(x,y) \to (0,0)} \frac{x^6}{x^2 + y^4} = 0 \).

CONTINUITY

Definition
Let \( f : A \subseteq \mathbb{R}^m \to \mathbb{R}^n \). \( f \) is said to be \textbf{continuous} at \( a \in A \) if:
1) \( f(a) \) is defined.
2) \( \lim_{x \to a} f(x) \) exists.
3) \( \lim_{x \to a} f(x) = f(a) \).
Theorem
If $P(x)$ and $Q(x)$ are polynomials, then
\[ \lim_{{x \to a}} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}, \quad Q(a) \neq 0. \]

Example
\[ f(x, y) = \begin{cases} \frac{x^2 + y^2}{\sqrt{x^2 + y^2}}, & (x, y) \neq 0 \\ 0, & (x, y) = 0 \end{cases} \]
is continuous everywhere.

Differentiation

**DERIVATIVE**

**Definition: Derivative of a Single-Variable Function**
Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, $I$ open, $a \in I$. We say $f$ is differentiable at $x = a$ if
\[ \lim_{{h \to 0}} \frac{f(a + h) - f(a)}{h} \]
exists. This limit, if it exists, is usually denoted $f'(a)$ and is called the derivative of $f(x)$ at $x = a$.

If $f$ is differentiable for all $x \in I$, then the function
\[ f'(x) = \lim_{{h \to 0}} \frac{f(x + h) - f(x)}{h} \]
is usually called the derivative of the function $f$.

**Definition: The Partial Derivative of a Two-Variable Function**
Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$, $A$ open, $(a, b) \in A$. If
\[ \lim_{{h \to 0}} \frac{f(a + h, b) - f(a, b)}{h} \]
exists, then it is called the partial derivative of the function $f$ with respect to $x$ at $(a, b)$ and is denoted $\frac{\partial f}{\partial x}(a, b)$.

**Geometrically**
- $\frac{\partial f}{\partial x}(a, b)$ is just the slope of the curve of intersection of the surface $z = f(x, y)$ with respect to the plane $y = b$ at the point $(a, b, f(a, b))$.
- In general, if $z = f(x, y)$ is a surface for which a “tangent plane” exists at the point $(a, b, f(a, b))$, then the equation for the tangent plane is
\[ \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - a) - (z - f(a, b)) = 0. \]

**Application**
If $z = f(x, y)$ has a tangent plane at the point $(a, b, f(a, b))$ and $h$ and $k$ are “small” numbers, then
\[ f(a + h, b + k) = f(a, b) + h \frac{\partial f}{\partial x}(a, b) + k \frac{\partial f}{\partial y}(a, b). \]
Definition: Partial Derivative in General
Let \( f: A \subseteq \mathbb{R}^n \to \mathbb{R} \). Then
\[
\frac{\partial f}{\partial x_i}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}.
\]
If the limit exists, then it is the partial derivative of \( f \) with respect to \( x_i \).

Some Properties of the Derivative
1) If \( f: \mathbb{R}^m \to \mathbb{R}^n \), \( g: \mathbb{R}^m \to \mathbb{R}^n \), \( \alpha \) and \( \beta \) are constants, then \( D(\alpha f + \beta g) = \alpha Df + \beta Dg \).
2) If \( f: \mathbb{R}^m \to \mathbb{R}^n \), \( g: \mathbb{R}^n \to \mathbb{R}^p \), then \( D(f \circ g) = (Dg)(f) \pm Df \).
3) If \( f: \mathbb{R}^n \to \mathbb{R} \), \( g: \mathbb{R}^n \to \mathbb{R} \), then
\[
\nabla \left( \frac{f}{g} \right) = \frac{g \nabla f - f \nabla g}{g^2}.
\]

DIFFERENTIABILITY

Definition: Differentiability of Two Variable Functions
Let \( f: A \subseteq \mathbb{R}^2 \to \mathbb{R} \). We say \( f \) is differentiable at \((a, b) \in A\) if
\[
\lim_{(x, y) \to (a, b)} \frac{f(x, y) - f(a, b) - \frac{\partial f}{\partial x}(a, b)(x-a) - \frac{\partial f}{\partial y}(a, b)(y-b)}{\| (x, y) - (a, b) \|} = 0.
\]

Definition: Differentiability in General
Let \( f: A \subseteq \mathbb{R}^n \to \mathbb{R} \). We say \( f \) is differentiable at \( a \in A \) if all the partial derivatives associated with \( f \) at \( a \) exist and
\[
\lim_{x \to a} \frac{f(x) - f(a) - T(x-a)}{\| x-a \|} = 0,
\]
where \( T \) is the matrix of partial derivatives associated with \( f \) at \( a \).

PATHS IN \( \mathbb{R}^n \)

Definition
If \( I \) denotes an interval in \( \mathbb{R} \), then the function \( f: I \subseteq \mathbb{R} \to \mathbb{R}^n \) is called a path in \( \mathbb{R}^n \).

Example
\( f: [0,4\pi] \to \mathbb{R}^2, t \mapsto (\cos t, \sin t) \) is a circular path traveled twice.

Definition
If \( f: A \subseteq \mathbb{R} \to \mathbb{R}^3, t \mapsto (x(t), y(t), z(t)) \) is a path, and \( a \in A \) is such that \( x'(a), y'(a), z'(a) \), then the vector \( f'(a) = (x'(a), y'(a), z'(a)) \) is called the velocity vector of the path at \( x = a \).
The Algebra of Path Velocities
Let $f, g$ be paths, $\alpha$ and $\beta$ constants, $h : \mathbb{R} \to \mathbb{R}$ an ordinary function. Then:

1) $(f \pm g)' = f' \pm g'$.

2) $(\alpha f \pm \beta g)' = \alpha f' \pm \beta g'$.

3) $(f \cdot g)' = f' \cdot g + f \cdot g'$.

4) $(f \times g)' = f' \times g + f \times g'$.

5) $(h \cdot g)' = h' \cdot g + h \cdot g'$.

6) $(g \circ f)'(a) = \left[ g' (f(a)) \right] f'(a) = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt} + \frac{\partial g}{\partial z} \frac{dz}{dt}$. Here $\left[ \frac{\partial g}{\partial x} \frac{\partial g}{\partial y} \frac{\partial g}{\partial z} \right]$ is called the gradient.

THE CHAIN RULE IN GENERAL

Let $f : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$, $g : B \subseteq \mathbb{R}^n \to \mathbb{R}^p$, $a \in A$ such that $f(a) \in B$. If $f$ is differentiable at $x = a$ and $g$ is differentiable at $y = f(a)$, then $D(g \circ f)(a) = [Dg(f(a))][Df(a)]$.

Useful Property
If $F(x_1, \ldots, x_n) = 0$ implicitly defines each of the variables $x_i$ as a function of the remaining variables $x_j$, then $\frac{\partial F}{\partial x_j} = -\frac{\partial F}{\partial x_i} \frac{dx_i}{dx_j}, i \neq j$.

DIRECTIONAL DERIVATIVE

Definition
Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$, $a = (a, b) \in A$, and $u = (u_1, u_2)$ unit vector. If $\lim_{h \to 0} \frac{f(a + hu) - f(a)}{h}$ exists, then it is called the directional vector of the function $f$ at the point $(a, b)$ in the direction of the vector $u$. It is denoted $D_u f(a)$.

Theorem
$D_u f(a) = \nabla f(a) \cdot u$, where $\nabla f = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$ the gradient.

Example
Consider a domed roof with the shape of the surface \( z = 15 - x^2 - 2y^2 \). A marble is placed at \((3,1,4)\). In what direction will the marble fall?

- We want the minimum of \( D_u f(3,1) = \nabla f(3,1) \cdot (u_1, u_2) \).
- \( \nabla f(3,1) = \left( \frac{\partial f}{\partial x}(3,1), \frac{\partial f}{\partial y}(3,1) \right) = (-6, -4) \).
- Now, \( \nabla f(3,1) \cdot (u_1, u_2) = \|\nabla f(3,1)\| \cos(\nabla f, u) = \sqrt{52} \cos(\nabla f, u) \). If \( \cos(\nabla f, u) = -1 \), then we have the minimum. So take \( u = -\nabla f(3,1) = (6, 4) = (3, 2) \).
- So the marble will fall in the direction \((3, 2)\).

Theorem

\[ -\|\nabla f(x)\| \leq D_u f(x) \leq \|\nabla f(x)\| . \]

Theorem

1) \( \nabla f(x) \) points in the direction of maximum increase of the function \( f \) at the point \( x \).

2) \( -\nabla f(x) \) points in the direction of maximum decrease of the function \( f \) at the point \( x \).

Theorem

If \( S \) is the level surface given by the equation \( F(x, y, z) = k \) and \( S \) has a tangent plane at \((x_0, y_0, z_0)\), then \( \nabla F(x_0, y_0, z_0) \) is a normal vector to the tangent plane to \( S \) at the point \((x_0, y_0, z_0)\).

Proof: We just show that if \( \mathbf{r}(t) \) denotes any path on the surface \( S \) that passes through \((x_0, y_0, z_0)\), the our gradient \( \nabla F(x_0, y_0, z_0) \) is perpendicular to the tangent vector of \( \mathbf{r}(t) \) at \((x_0, y_0, z_0)\). So let \( \mathbf{r}(t) \) be a curve on \( S \), so \( F(\mathbf{r}(t)) = k, \forall t \in I \). Now \( \frac{\partial F}{\partial t} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0 \Rightarrow \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) \cdot \left( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = 0 \Rightarrow \nabla F \cdot \mathbf{r}'(t) = 0 \).

Higher-Order Derivatives

**The Taylor Theorem**

**First Order Taylor Expansion**

\[ f(x + h) = f(x) + hf'(x) + R_1(x, h) \text{ where } \lim_{h \to 0} \frac{R_1(x, h)}{h} = 0. \]

Notice that \( \int_x^{x+h} f'(t) dt = f(t)|_x^{x+h} \), so \( f(x + h) = f(x) + hf'(x) + \left( \int_x^{x+h} f'(t) dt - hf'(x) \right) = f(x) + hf'(x) + R_1(x, h). \)

**Theorem**
Second Order Taylor Expansion

\[ f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \int_x^{x+h} (x+h-t)f'''(t)dt. \]

The General Taylor Expansion

If \( f \) is \( n+1 \) times continuously differentiable at \( x \), then

\[
 f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \ldots + \frac{h^n}{n!} f^{(n)}(x) + R_n(x,h), \text{ where}
\]

\[
 R_n(x,h) = \frac{1}{n!} \int_x^{x+h} (x+h-t)^n f^{(n+1)}(t)dt.
\]

Note

Notice that \( f^{(n+1)} \) being continuous is bounded on \([x,x+h]\). Let \( M \) denote the maximum of \( f^{(n+1)} \) over \([x,x+h]\). Then

\[
 |R_n(x,h)| \leq \frac{M}{n!} \int_x^{x+h} (x+h-t)^n dt.
\]

Also notice that when \( x \leq t \leq x+h \), \( |x+h-t| \leq |h| \). So

\[
 |R_n(x,h)| \leq \frac{M}{n!} \int_x^{x+h} |h|^n dt = \frac{M|h|^{n+1}}{n! (n+1)}.
\]

Therefore

\[
 |R_n(x,h)| \leq \frac{M|h|^{n+1}}{(n+1)!}.
\]

First Order Taylor Expansion for \( \mathbb{R}^n \)

For \( f : \mathbb{R}^n \to \mathbb{R} \), \( f(x+h) = f(x) + \nabla f(x) \cdot h + R_1(x,h) \), where

\[
 R_1(x,h) = \int_0^1 (1-t) \sum_{i=1}^n \sum_{j=1}^n h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x+th)dt.
\]

Theorem: Second Order Taylor Expansion for a Function of Several Variables

If \( f : \mathbb{R}^n \to \mathbb{R} \) is at least thrice differentiable at \( x \), then

\[
 f(x+h) = f(x) + \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + R_n(x,h), \text{ where}
\]

\[
 R_n(x,h) = \frac{1}{2} \int_0^1 (1-t)^2 \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n h_i h_j h_k \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(x+th)dt.
\]

LOCAL EXTREMA

Definitions: Local Minima
1) The function \( y = f(x) \) has a “local minimum at \( x = x_0 \)” if there exists an open interval \( I \) such that \( x_0 \in I \) and for all \( x \in I \), \( f(x) \geq f(x_0) \).

2) Let \( f : A \subseteq \mathbb{R}^2 \to \mathbb{R} \). The function \( z = f(x, y) \) has a “local minimum at \( x = x_0 \)” if there exists an open interval disc \( D_r(x_0, y_0) \) such that for all \( (x, y) \in D_r(x_0, y_0) \cap A \), \( f(x, y) \geq f(x_0, y_0) \).

**Theorem**

If \( f \) is differentiable at \( (x_0, y_0) \) and \( f \) has a local maximum/minimum at \( (x_0, y_0) \), then

\[
\nabla f(x_0, y_0) = (0, 0).
\]

**Definition: Critical Point**

The function \( f(x, y) \) has a critical point at \( (x_0, y_0) \) if

\[
\nabla f(x_0, y_0) = (0, 0),
\]

or if at least one of

\[
\frac{\partial f}{\partial x}(x_0, y_0) \quad \text{or} \quad \frac{\partial f}{\partial y}(x_0, y_0)
\]

does not exist.

**Theorem: Second Derivate Test**

If \( f : A \subseteq \mathbb{R}^2 \to \mathbb{R} \) is differentiable at \( (x_0, y_0) \) and \( \nabla f(x_0, y_0) = (0, 0) \), then:

1) If \( \frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0 \) and

\[
\begin{vmatrix}
\frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{vmatrix} > 0,
\]

then \( f \) has a local minimum at \( (x_0, y_0) \).

2) If \( \frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0 \) and

\[
\begin{vmatrix}
\frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{vmatrix} > 0,
\]

then \( f \) has a local maximum at \( (x_0, y_0) \).

3) If \( \det \left| \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 f}{\partial y^2} \right| < 0 \), then \( f \) has not a local extrema at \( (x_0, y_0) \). We say that \( f \) has a saddle point at \( (x_0, y_0) \).

4) If \( \det \left| \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 f}{\partial y^2} \right| = 0 \), then this test is inconclusive.

**LAGRANGE MULTIPLIERS METHOD**

**Theorem**

Let \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) be of class \( C^1 \). Let \( S := \{ x \in \mathbb{R}^n \mid g(x) = 0 \} \). Let \( f \mid S : S \to \mathbb{R}, x \in S \mapsto f(x) \). If \( f \mid S \) has a local extremum at \( x_0 \in S \), then there exists \( \lambda \) such that

\[
\nabla f(x_0) = \lambda \nabla g(x_0)
\]

where \( \nabla g(x_0) \neq 0 \).

**THE IMPLICIT FUNCTION THEOREM**
Theorem: One Variable Implicit Function Theorem

If \( F : \mathbb{R}^{n+1} \to \mathbb{R} \) is a \( C^1 \) function. Let \( x_0 \in \mathbb{R}^n \), \( z_0 \in \mathbb{R} \), and \( F(x_0, z_0) = 0 \). If \( \frac{\partial F}{\partial z}(x_0, z_0) \neq 0 \), then there exists an open set \( U \subseteq \mathbb{R}^n \), \( x_0 \in U \), and a unique function \( z = g(x) \) such that for all \( x \in U \), \( F(x, g(x)) = 0 \).

Theorem: The General Implicit Function Theorem

\[ F_1 = (x_1, \ldots, x_n, z_1, \ldots, z_m) = 0 \]

where \( F_1, \ldots, F_m \) are of class \( C^1 \). Let \( x_0 \in \mathbb{R}^n \) and \( z_0 \in \mathbb{R}^n \) such that \( F_i(x_0, z_0) = 0, \forall i = 1, \ldots, m \). If the Jacobian Determinant \( \det \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{vmatrix} \neq 0 \) (at \( (x_0, z_0) \)), then there exists an open set \( U \subseteq \mathbb{R}^m \) that contains \( z_0 \in \mathbb{R}^m \) and \( m \) unique functions \( g_1, \ldots, g_m : U \to \mathbb{R} \) such that for each \( z_0 \in U \), \( F_i(x_0, g(x_0)) = 0, \forall i = 1, \ldots, m \) where \( g = (g_1, \ldots, g_m) \).

Vector Valued Functions

ARC LENGTH

Suppose \( \mathbf{r}(t) = (x(t), y(t), z(t)), a \leq t \leq b \), then the length of the curve on the interval \([a, b] \) is \( L = \int_a^b \| \mathbf{r}'(t) \| \, dt \).

VECTOR FIELDS

Definition

\( \mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n \) is a vector field.

Definition: The Divergence of a Vector Field

Let \( \mathbf{F} = (F_1, F_2, F_3) \). Then \( \text{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \).

Definition: The Curl of a Vector Field

Let \( \mathbf{F} = (F_1, F_2, F_3) \). Then \( \text{curl} \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \\
\frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{vmatrix} F_1 \).

Notation

If \( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \), then \( \text{div} \mathbf{F} = \nabla \cdot \mathbf{F} \) and \( \text{curl} \mathbf{F} = \nabla \times \mathbf{F} \).
Laplacian Operator

The Laplacian operator is denoted $\nabla^2$. If $f : \mathbb{R}^n \to \mathbb{R}$, then $\nabla^2 f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$. If $F : \mathbb{R}^n \to \mathbb{R}^n$, then

$$\nabla^2 F = \left( \frac{\partial^2 F_{11}}{\partial x_1^2}, \ldots, \frac{\partial^2 F_{nn}}{\partial x_n^2} \right).$$

Note

1) $\text{curl}(\nabla f) = 0$. Proof: $\nabla \times (\nabla f) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \ldots \right) = 0$ if $f \in C^1$.

2) $\text{div}(\text{curl} F) = 0$. Proof: $\nabla \cdot (\nabla \times F) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$.

Some Identities

1) $\nabla(fg) = (\nabla f)g + f(\nabla g)$.
2) $\text{div}(fF) = f(\text{div} F) + F \cdot (\nabla f)$.
3) $\text{div}(F \times G) = G \cdot (\text{curl} F) + F \cdot (\text{curl} G)$.
4) $\text{curl}(fF) = f(\text{curl} F) + (\nabla f) \times F$.

Potential Function

If $\text{curl} F = 0$, then there exists $f : \mathbb{R}^3 \to \mathbb{R}$ such that $F = \text{grad} f$.

$F(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$ is the gradient field, and $f = f(x, y, z)$ is the potential function.

FLOW LINES

Definition

A flow curve has tangent vectors that coincides with the vector field.

Example

Show that $c(t) = \left(t^2, 2t-1, \sqrt{t}\right)$, $t > 0$ is a flow line of the velocity vector field $F(x, y, z) = \left( y + 1, 2, \frac{1}{2z} \right)$.

Want: $c'(t) = F(c(t))$. Now, $c'(t) = \left(2t, 2, \frac{1}{2\sqrt{t}}\right)$, and $F(c(t)) = \left(2t, 2, \frac{1}{2\sqrt{z}}\right) = c'(t)$.
ARC LENGTH PARAMETRIZATION

Let \( \sigma(t) = (x(t), y(t), z(t)), a \leq t \leq b \). Let \( p(s) \) be the same curve parameterized arc length:

\[
s = s(t) = \int_a^t \|\sigma'(\tau)\| d\tau.
\]

1) Then the velocity vector is always a unit vector.
   Proof: \( \sigma'(t) = \frac{dp}{ds} \),
   \[
   \sigma'(t) = \frac{dp}{ds} \Rightarrow \sigma'(t) = \frac{ds}{ds} = \|\sigma'(t)\| \text{ which is unit.}
   \]

2) Let \( T = \frac{dp}{ds} = v \), then \( \kappa = \left\| \frac{dT}{ds} \right\| \) is called the curvature of \( \sigma \) parameterized by \( p(s) \).

Integration

DOUBLE INTEGRAL

Definition
Let \( f : \mathbb{R}^2 \to \mathbb{R} \). Let \( D \) be a bounded region in \( \mathbb{R}^2 \). Let \( P \) be an inner partition. Take all rectangles \( R_{ij} \subseteq D \).

Let \( (x_i^*, y_j^*) \in R_{ij} \). Let \( |P| = \max d_{ij} \), where \( d_{ij} \) is the diameter of \( R_{ij} \).

The Riemann Sum is

\[
S_n = \sum_{i,j=1}^n f(x_i^*, y_j^*) \Delta A_{ij}
\]

where \( \Delta A_{ij} \) is the area of \( R_{ij} \).

If \( \lim_{n \to \infty} S_n \) exists and independent of the choice of \( (x_i^*, y_j^*) \), then \( f \) is integrable and

\[
\lim_{n \to \infty} S_n = \int_D f(x, y) dA.
\]

Note
\[
\int_D f(x, y) dA \text{ is the volume below the surface } f(x, y) \text{ over } D.
\]

Theorem: Fubini’s Theorem
If \( f \) is continuous over \( R = [a, b] \times [c, d] \), then

\[
\int_D f(x, y) dA = \int_a^b \int_c^d f(x, y) dx dy = \int_c^d \int_a^b f(x, y) dy dx
\]

(iterated integrals).

Properties
Let \( \Omega = [a, b] \times [c, d] \). \( f : \Omega \to \mathbb{R} \) integrable.

1) If \( f(x, y) = 0 \) for all \( (x, y) \in \Omega \), then \( \int_{\Omega} f(x, y) dA = 0 \).

2) If \( f(x, y) = k \), then \( \int_{\Omega} f(x, y) dA = k (b-a)(d-c) \).
3) If \(a < \alpha < b\) and \(c < \beta < d\), then \(\int \int_{\Omega} f(x,y) dA = \int \int_{\Omega} f(x,y) dA + \int \int_{\Omega} f(x,y) dA + \int \int_{\Omega} f(x,y) dA + \int \int_{\Omega} f(x,y) dA\).

4) If \(f \geq 0\) over \(\Omega\), then \(\int \int_{\Omega} f(x,y) dA \geq 0\).

5) \(|\int \int_{\Omega} f(x,y) dA| \leq \int \int_{\Omega}|f(x,y)| dA\).

6) If \(f\) is continuous over \(\Omega\), then \(\int \int_{\Omega} f(x,y) dA = f(x_0,y_0)(b-a)(d-c)\) for some \((x_0,y_0) \in \Omega\). This is the Mean Value Theorem for double integrals.

7) If \(m \leq f \leq M\), then \(m(b-a)(d-c) \leq \int \int_{\Omega} f(x,y) dA \leq M(b-a)(d-c)\).

8) \(\int \int_{\Omega} (af(x,y) + \beta g(x,y)) dA = a \int \int_{\Omega} f(x,y) dA + \beta \int \int_{\Omega} g(x,y) dA\).

9) If \(a = -b, b > 0\) and \(c = -d, d > 0\), then \(f(-x,y) = -f(x,y) \Rightarrow \int \int_{\Omega} f(x,y) dA = 0\) and \(f(-x,-y) = -f(x,y) \Rightarrow \int \int_{\Omega} f(x,y) dA = 0\).

**Elementary Regions**

**Type 1 Elementary Region:** \(\Omega_1 = \left\{a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\right\}\).

**Type 2 Elementary Region:** \(\Omega_1 = \left\{a \leq y \leq b, h_1(y) \leq x \leq h_2(y)\right\}\).

**CHANGE OF VARIABLES**

**Definition: Jacobian Determinant**

Let \(T : D^2 \subset \mathbb{R}^2 \to \mathbb{R}^2\) be a \(C^1\) transformation given by \(\begin{align*} x &= x(u,v) \\ y &= y(u,v) \end{align*}\). The Jacobian determinant is

\[
\frac{\partial (x,y)}{\partial (u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \] (the determinant of the derivative matrix \(DT(u,v)\)).
Theorem: Change of Variables for Double Integrals
Let \( D \) and \( D' \) be elementary regions in the plane and let \( T : D' \rightarrow D \) be of class \( C^1 \); suppose that \( T \) is one-to-one on \( D' \). Furthermore, suppose that \( D = T(D') \). Then for any integrable function \( f : D \rightarrow \mathbb{R} \), we have
\[
\iint_D f(x, y) \, dx \, dy = \iint_{D'} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv.
\]

Example: Jacobian Determinants of Popular Transformations
1) Rectangular to polar:
\[
\begin{align*}
\begin{cases}
    x = r \cos \theta \\
y = r \sin \theta
\end{cases},
\end{align*}
\]
\[
\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix}
    \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
    \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{vmatrix} = r.
\]

2) Rectangular to cylindrical:
\[
\begin{align*}
\begin{cases}
x = r \cos \theta \\
y = r \sin \theta \\
z = z
\end{cases},
\end{align*}
\]
\[
\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix}
    \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\
    \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\
    \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z}
\end{vmatrix} = r.
\]

3) Rectangular to spherical:
\[
\begin{align*}
\begin{cases}
x = \rho \cos \theta \sin \phi \\
y = \rho \sin \theta \sin \phi \\
z = \rho \cos \phi
\end{cases},
\end{align*}
\]
\[
\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix}
    \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\
    \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\
    \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}
\end{vmatrix} = \rho^2 \sin \phi.
\]

Some Basic Applications
1) If \( \Omega \) is a solid region in \( \mathbb{R}^3 \), then the volume is \( \iiint_{\Omega} dV \).

2) If \( \delta(x, y, z) \) is the density at \( (x, y, z) \) of the solid \( \Omega \), then its mass is \( M = \iiint_{\Omega} \delta(x, y, z) dV \).

3) If \( f : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R} \), then \( \iiint_{\Omega} f(x, y, z) dV \) is the average value of \( f \) over \( \Omega \).

4) The center of gravity is given by
\[
\bar{x} = \frac{\iiint_{\Omega} x \delta(x, y, z) dV}{\iiint_{\Omega} \delta(x, y, z) dV}, \quad \bar{y} = \frac{\iiint_{\Omega} y \delta(x, y, z) dV}{\iiint_{\Omega} \delta(x, y, z) dV}, \quad \bar{z} = \frac{\iiint_{\Omega} z \delta(x, y, z) dV}{\iiint_{\Omega} \delta(x, y, z) dV}.
\]

Path Integral

Problem
A very thin piece of wire has the shape of the arc \( r(t) \). It has density \( \delta(x, y, z) \) which changes continuously. What is the mass of the wire?
Definition: The Path Integral
\[ \int_a^b f(x(t), y(t), z(t)) \|r'(t)\| dt \] is called the path integral of the function \( f \) along the path \( r(t), a \leq t \leq b \).

Notation: \( \int_c fds := \int_a^b f(r(t)) \|r'(t)\| dt \).

Property
1) The line integral of a function \( g(x, y) \) over a curve given as the graph of a function, say \( y = f(x), a \leq x \leq b \), is
\[ \int_c gds = \int_a^b g(r(x, f(x)))\sqrt{1+(f'(x))^2} dx . \]
2) The line integral of a function \( g(x, y) \) over a curve given as the graph of a function in polar coordinates as \( r = r(\theta), \theta_1 \leq \theta \leq \theta_2 \) is
\[ \int_c gds = \int_{\theta_1}^{\theta_2} g(r(\theta)\cos \theta, r(\theta)\sin \theta)\sqrt{(r(\theta))^2+(r'(\theta))^2} d\theta . \]

LINE INTEGRAL

Work
The work done is \( W = \vec{F} \cdot \vec{v} \).

Problem
Let \( \vec{F} = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) \). Let \( r(t) = (x(t), y(t), z(t)) \) be a path. What is the total work done by \( \vec{F} \) when the object moves from \( r(a) \) to \( r(b) \)?

Definition: Line Integral
Let \( \vec{F} : \mathbb{R}^3 \to \mathbb{R}^3 \) be a vector field, and \( c : r(t) = (x(t), y(t), z(t)), a \leq t \leq b \) be a path. If \( \int_a^b \vec{F}(r(t)) \cdot r'(t) dt \) exists, then it is called the line integral of \( \vec{F} \) over \( c \).

Notation: \( \int_c \vec{F} \cdot dr \).

Theorem
If \( \vec{F} : \mathbb{R}^3 \to \mathbb{R}^3 \) is a gradient vector field (i.e. \( \vec{F} = \nabla f \) for some \( f : \mathbb{R}^3 \to \mathbb{R} \)) and both \( \vec{F} \) and \( f \) are continuous, then for smooth path \( c : r(t) = (x(t), y(t), z(t)), a \leq t \leq b \), \( \int_c \vec{F} \cdot dr = f(r(b)) - f(r(a)) \). Here, \( \vec{F} \) is said to be path independent.

Theorem
If \( \vec{F} \) is path independent and \( c \) is a closed curve, then \( \int_c \vec{F} \cdot dr = 0 \).
Theorem
If \( r(t), a \leq t \leq b \) and \( \delta(u), c \leq u \leq d \) are different parameterizations of the same curve with \( r(a) = \delta(c) \) and \( r(b) = \delta(d) \), and \( \int_c^d \mathbf{F} \cdot d\mathbf{r} \) exists, then \( \int_c^d \mathbf{F} \cdot d\delta \) exists and \( \int_c^d \mathbf{F} \cdot d\delta = \int_c^d \mathbf{F} \cdot d\mathbf{r} \).

Remark
The line integral and path integral are connected.

\[
\int_a^b \mathbf{F}(r(t)) \cdot r'(t) \, dt = \int_a^b \mathbf{F}(r(t)) \cdot \frac{r'(t)}{\|r'(t)\|} \, dt = \int_a^b \mathbf{F} \cdot \mathbf{T} \, ds, \text{ where } T = \frac{r'(t)}{\|r'(t)\|}, r'(t) \neq 0 \text{ the unit tangent vector.}
\]

**SURFACE INTEGRAL**

**Definition: Parametric Surface**
Consider \( \Phi : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3 \) where \( (u, v) \mapsto (x(u, v), y(u, v), z(u, v)) \). \( \Phi \) is said to be a parametric surface.
Note: \( \{(x(u, v), y(u, v), z(u, v)) \mid (u, v) \in D\} \) is a surface.

**Property**
Let \( \mathbf{T}_u := \left( \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0) \right) \) and \( \mathbf{T}_v := \left( \frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0) \right) \). If \( \langle \mathbf{T}_u \times \mathbf{T}_v \rangle(u_0, v_0) \neq 0 \), then this is a vector normal to the parametric surface \( \Phi(u, v) \) at the point \((u_0, v_0)\).

**Definition: Surface Integral**
The surface integral of a function \( f : \mathbb{R}^3 \to \mathbb{R} \) over a parameterized surface \( \Phi(u, v), (u, v) \in D \) is
\[
\iint_D f(\Phi(u, v)) \|\mathbf{T}_u \times \mathbf{T}_v\| \, dudv.
\]

**Definition: Flux**
Let \( \mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3 \) be a vector field. Let \( \Phi(u, v), (u, v) \in D \) be a parameterized surface. The integral
\[
\iint_D \mathbf{F}(\Phi(u, v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) \, dudv
\]

is called the flux of the vector field \( \mathbf{F} \) (the surface integral of vector field \( \mathbf{F} \)).

**GREEN’S THEOREM**

**Orientation**
If \( C \) is a simple closed curve in the plane \( \mathbb{R}^2 \), then there are two ways to go around the curve:

1) Counterclockwise (positive): \( C^+ \).
2) Clockwise (negative): \( C^- \).

**Lemma**

Suppose that \( D \) is a region of type 1 and that \( P : D \to \mathbb{R} \) is of class \( C^1 \). Then

\[
\int_{C^+} P \, dx = -\iint_D \frac{\partial P}{\partial y} \, dxdy,
\]

where \( C \) the boundary of \( D \).

**Proof:**

\( D \) is a region of type 1 means there are continuous functions \( \varphi_1, \varphi_2 : [a, b] \to \mathbb{R} \) where \( \varphi_1(x) \leq \varphi_2(x), \forall x \in [a, b] \) such that

\[
D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}.
\]

Now,

\[
\iint_D \frac{\partial P}{\partial y} \, dxdy = \int_a^b \varphi_2(x) \frac{\partial P}{\partial y} (x, y) \, dy \, dx = \int_a^b (P(x, \varphi_2(x)) - P(x, \varphi_1(x))) \, dx = \int_a^b P(x, \varphi_2(x)) \, dx - \int_a^b P(x, \varphi_1(x)) \, dx.
\]

Also,

\[
\int_{C^+} P \, dx = \int_{C_1^+} P \, dx + \int_{C_2^+} P \, dx + \int_{C_3^+} P \, dx + \int_{C_4^+} P \, dx.
\]

Along \( C_2^+ \) and \( C_4^+ \), \( x \) is a constant, so

\[
\int_{C_2^+} P \, dx = \int_{C_4^+} P \, dx = 0.
\]

Since \( C_1^+ \) is parameterized by \( x \mapsto (x, \varphi_1(x)), x \in [a, b] \), \( \int_{C_1^+} P \, dx = \int_a^b P(x, \varphi_1(x)) \, dx \); since \( C_3^+ \) is parameterized by \( x \mapsto (x, \varphi_2(x)), x \in [a, b] \),

\[
\int_{C_3^+} P \, dx = \int_a^b P(x, \varphi_2(x)) \, dx \Rightarrow \int_{C_3^+} P \, dx = -\int_a^b P(x, \varphi_2(x)) \, dx.
\]

So

\[
\int_{C^+} P \, dx = \int_a^b P(x, \varphi_1(x)) \, dx - \int_a^b P(x, \varphi_2(x)) \, dx = -\iint_D \frac{\partial P}{\partial y} \, dxdy.
\]

**Theorem: Green's Theorem**

Let \( D \) be a region of type 3 and let \( C \) be its boundary. Suppose that \( P : D \to \mathbb{R} \) and \( Q : D \to \mathbb{R} \) of class \( C^1 \).

Then

\[
\int_{C^+} P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy.
\]

Page 22 of 23
Theorem

Let $D$ be a region of type 3 with boundary $C = \partial D$. Then the area of $D$ is given by $A = \frac{1}{2} \oint_C -y\,dx + x\,dy$.

Proof: $\int_C -y\,dx + x\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \,dx\,dy = \iint_D (1+1)\,dx\,dy = 2A$.

STOKES’ THEOREM

Theorem: Stokes’s Theorem

Let $S$ be the oriented surface defined by $\Phi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ of class $C^1$ and $\Omega$ type 1 or 2. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ of class $C^1$. Then $\iint_S \text{curl} \cdot dS = \int_{\partial S} F \cdot ds$.

Remark

Let $F = (P, Q, 0)$. Then $\text{curl} \cdot F = \left( -Q_z - P_z, P_y - Q_y \right)$ and $T_x \times T_y = (0,0,1)$, so $\iint_S \text{curl} \cdot dS = \iint_\Omega \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$. Now $\int_{\partial \Omega} F \cdot ds = \int_{\partial \Omega} P\,dx + Q\,dy$. This is Green’s Theorem!

GAUSS’S THEOREM

Theorem: Gauss’s Theorem

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ of class $C^1$. Let $\Omega$ be a solid in $\mathbb{R}^3$ with the a smooth surface $\partial \Omega$. Then $\iint_{\partial \Omega} F \cdot dS = \iiint_{\Omega} F \,dV$.