Introduction

INTRODUCTION TO DIFFERENTIAL EQUATIONS

A differential equation is an equation involving some hypothetical function and its derivatives.

Example
$y'' + 2y' = x$ is an differential equation. As such, the differential equation is a description of some function (exists or not).

A solution to a differential equation is a function that satisfies the differential equation.

Example
$y = x^3 + 5x$ is a solution to $y'' = x + y - x^3$.

Some differential equations are famous/important:

- $y' = y$, $y = e^x$.
- $y' = ay$, $y = e^{ax}$.
- $y'' + y = 0$, $y = \cos x$.
- $y'' + ay = 0$, $y = \cos\sqrt{a}x$.

Recall that a differential equation describes a phenomenon in terms of changes. For example, if $P = mv$, then $Fdt = dm \cdot v$ or $\frac{dP}{dt} = F$.

Example
A pool contains *V* liters of water which contains *M* kg of salt. Pure water enters the pool at a constant rate of *v* liters per minute, and after mixing, exits at the same rate. Write a differential equation that describes the density of salt in the pool at an arbitrary time t.

- Let $\rho(t)$ be the density at time *t*. Then $\rho(t) = \frac{M(t)}{V}$.
- To model change in $\rho(t)$, let $\rho_1 = \rho(t_1)$ and $\rho_2 = \rho(t_2)$. Then $(\rho_2 - \rho_1)V \approx -\rho_1 v(t_2 - t_1)$, so $\frac{\rho_2 - \rho_1}{t_2 - t_1}V \approx -\rho_1 \frac{v}{V}$ or $\frac{\rho(t + \Delta t) - \rho(t)}{t + \Delta t - t}V \approx -\rho(t)\frac{v}{V}$.
- Now, as $t \to 0$, $\frac{d\rho}{dt} = -\rho(t)\frac{v}{V}$ or $\rho' = -\rho\frac{v}{V}$.
- Without solving this equation, we can predict facts about this system. As $t \to \infty$, $\rho \to 0$.
- To solve this differential equation, write $\frac{d\rho}{\rho} = -\frac{v}{V}dt$. Integrating both sides, we get $\ln|\rho(t)| = -\frac{v}{V}t + C \Rightarrow \rho(t) = e^{-\frac{v}{V}t + C} = e^C e^{-\frac{v}{V}t} = Ae^{-\frac{v}{V}t}$.

- If we add $\rho(0)=\rho$ to $\rho'=-\rho\frac{v}{V}$, then we have an IVP (initial value problem).

Issues About the Use of Differential Equations

1) How to translate a real problem to a differential equation. Keep your eyes open!
2) Some patterns of nature are ill-defined. Use different points of view and different mental differential equation models to reformulate them.
3) Differential equations have infinitely many solutions. Which one is yours? The initial value are extremely important.
4) There may be no analytic solution found.
   - Is there a solution?
   - Is this solution unique?
   - If a numeric answer is required, i.e. the value of the solution at one particular point, then use numerical approximations. It does not give any feelings for the pattern, nor does it give elbow room.
   - Use theoretical analysis if you need the behavior of the solution. This does not give any values.
   - To know the behavior locally/in a neighborhood, solve in series.
5) The data does not fit you solution. You need to repeat (as in a feedback/controlled system).

Notations With Regard to the Input/Output Systems

Example

$xy''+2y'-\sin x\cdot y=\tan x$ can be written as $L[y]=\tan x$. Solve it, and the answer is the output.

- $L[y]$ is the "black box system".
- $\tan x$ is the 'input".

For theoretical purposes, mathematicians use these equivalents: $y'''=\frac{\tan x}{x}+\frac{\sin x}{x}y-\frac{2y'}{x}$ is the same as $y'''=f(x,y,y')$ or $F(x,y,y',y'',y''')=0$.

Linear vs. Non-Linear Differential Equations

- $\tan x\cdot y''+\frac{1}{x}y'+e^{x}y=\frac{1}{1+\tan x}$ is linear.
- $y'''+y'+y^{2}=0$ is non-linear.

First Order Differential Equations

Linear Equations

First order linear equations have the form $y' + p(t)y = g(t)$.

Derivation

- Suppose I can find $\mu(t)$ so that $\mu(t)p(t) = \mu'(t)$.
- Multiply both sides of $y' + p(t)y = g(t)$ by $\mu(t)$: $\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$ which is $\mu(t)y' + \mu'(t)y = \mu(t)g(t)$, i.e. $(\mu(t)y)' = \mu(t)g(t)$.
- Integrate both sides: $\mu(t)y = \int \mu(t)g(t)dt + C$, so $y(t) = \frac{1}{\mu(t)}\int \mu(t)g(t)dt + C$.
- But what is $\mu(t)$? Since $\mu(t)p(t) = \mu' \Leftrightarrow \frac{\mu'(t)}{\mu(t)} = p(t)$, therefore $\ln|\mu(t)| = \int p(t)dt$. So $\mu(t) = e^{\int p(t)dt}$.

General Solution

To solve $y' + p(t)y = g(t)$,

1) Let $\mu(t) = e^{\int p(t)dt}$ (no constant needed).
2) The solution is $y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right]$.

Example

Solve $y' + 2ty = 2te^{-t^2}$.

- Here, $p(t) = 2t$, $g(t) = 2te^{-t^2}$.
- Let $\mu(t) = e^{\int 2tdt} = e^{t^2}$.
- The solution is $y(t) = \frac{1}{e^{t^2}}\left[\int e^{t^2} 2te^{-t^2} dt + C\right] = \frac{1}{e^{t^2}}\left[\int 2tdt + C\right] = \frac{1}{e^{t^2}}\left(t^2 + C\right) = t^2 e^{-t^2} + Ce^{-t^2}$.
- Note that as $t \to \infty$, $y \to 0$.

Importance of Analysis

One needs to have an understand of the solution before (even after) solving it, with respect to:

1) Behavior of the solution as $t \to \infty$.
2) The nature and behavior of the solution within a family (depends on y_0).

Variation of Parameter

Recall that the family of solutions of a first order linear differential equation $y' + p(t)y = g(t)$, $\mu(t) = e^{\int p(t)dt}$, $y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right] = \frac{1}{\mu(t)}\int \mu(t)g(t)dt + \frac{C}{\mu(t)}$. Notice that the family of solutions is generated by $\mu(t)$. This leads to the technique of <u>variation of parameter</u>.

Recall a differential equation $L[y] = g(x)$. If $g(x) = 0$, then we have a zero-input system, or a <u>homogeneous differential equation</u> $L[y] = 0$ which describes the solutions to a great extent.

For example, consider $y' + \frac{1}{t} y = 3\cos 2t$. First solve the corresponding homogeneous equation $y' + \frac{1}{t} y = 0$ to find $y_0(t)$. Since $\mu(t) = e^{\int \frac{1}{t} dt} = t$, $y_0(t) = \frac{1}{\mu(t)}(0 + C) = \frac{C}{t}$. Then the general solution to $y' + \frac{1}{t} y = 3\cos 2t$ looks like $y(t) = A(t)\frac{1}{t}$. Then $y' = A'(t)\frac{1}{t} - A(t)\frac{1}{t^2}$, so $A'(t)\frac{1}{t} - A(t)\frac{1}{t^2} + \frac{1}{t} A(t)\frac{1}{t} = 3\cos 2t \Rightarrow A'(t) = 3t\cos 2t \Rightarrow A(t) = \int (3t\cos 2t)dt$. Thus, $y(t) = \frac{1}{t}\left[\int (3t\cos 2t)dt + C\right]$.

ASYMPTOTIC BEHAVIOR OF SOLUTIONS

Recall from calculus that $f(x)$ and $g(x)$ are asymptotes of one another if $\lim_{x\to\infty}(f(x) - g(x)) = 0$.

Example

$2t - 5$ and $2t - 5 + ce^{-t}$ and $2t - 5 + \frac{c}{t}$ are asymptotic to each other.

SEPARATION OF VARIABLES

Idea

A differential (not necessarily linear) may appear as $\frac{dy}{dx} = \frac{f(x)}{g(y)}$. Then, $g(y)dy = f(x)dx$, and the solution is $\int g(y)dy = \int f(x)dx$.

Note

Other ways a separable differential equation can appear as: $M(y) = \frac{dy}{dx} N(x)$ or $f(x)g(y) = \frac{dy}{dx}$.

Example

$\frac{dy}{dx} = x\sqrt{y}$. So $\frac{1}{\sqrt{y}} dy = xdx \Rightarrow \int y^{-\frac{1}{2}} dy = \int xdx \Rightarrow 2y^{\frac{1}{2}} = \frac{x^2}{2} + C$.

Note

The solution about is an implicit solution. When write a solution explicitly, be careful! Pay attention to the domain and range.

Example

$\frac{dy}{dx} = x\sqrt[3]{y} \Rightarrow \int y^{-\frac{1}{3}} dy = \int x dx \Rightarrow \frac{3}{2} y^{\frac{2}{3}} = \frac{x^2}{2} + C$. Now, suppose $y(2) = 1$, then $\frac{3}{2} = 2 + C \Rightarrow C = -\frac{1}{2}$. So $\frac{3}{2} y^{\frac{2}{3}} = \frac{x^2}{2} - \frac{1}{2} \Rightarrow y^{\frac{2}{3}} = \frac{x^2}{3} - \frac{1}{3} \Rightarrow y = \left[\frac{x^2}{3} - \frac{1}{3}\right]^{\frac{3}{2}}$. We need $\frac{x^2}{3} - \frac{1}{3} > 0 \Rightarrow x^2 - 1 > 0 \Rightarrow x^2 > 1 \Rightarrow x > 1, x < -1$.

IMPLICIT VS. EXPLICIT SOLUTIONS

For separable (not linear in general), we have implicit solutions (not necessarily functions). So as the solutions to non-linear equations are implicit, the analysis of the solution is very difficult, and we need to know if such solutions have explicit forms or not (and where). We need to know an interval on which explicit solutions exists.

Example

Consider $2y\frac{dy}{dx} = 1$. The solution is $y^2 = x + c$.

- Now, if $y(1) = 1$, then $c = 0$ and the solution is $y^2 = x$. An explicit solution $y = \sqrt{x}$ exists on the interval $(0, \infty)$. This solution can't be extended to $y < 0$ because it doesn't pass the vertical line test.
- At the point $(0,0)$, $\frac{dy}{dx}$ is undefined. This indicates the possibility of a problem with defining an explicit solution to the differential equation.

Theorem: Implicit Function Theorem

If $F(x, y) = 0$ and (a,b) is such that $F(a,b) = 0$ and if $F_y(a,b) \neq 0$, then we have an explicit function $y = f(x)$ on an interval containing (a,b).

Conclusion: On any interval as long as $\frac{dy}{dx}$ is defined, we will have an explicit solution.

Example

Solve $\frac{dy}{dx} = \frac{x + 3y}{x - y}$ using separation of variables.

- Let $v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$.

- So $\frac{dy}{dx}=\frac{x+3y}{x-y}=\frac{1+3\frac{y}{x}}{1-\frac{y}{x}}$ becomes $v+x\frac{dv}{dx}=\frac{1+3v}{1-v}\Rightarrow x\frac{dv}{dx}=\frac{1+3v}{1-v}-v=\frac{(1+v)^2}{1-v}$. So we have $\int\frac{1-v}{(1+v)^2}dv=\int\frac{1}{x}dx$.
- Let $w=1+v$, $dv=dw$. So $\int\frac{2-w}{w^2}dw=\int\frac{1}{x}dx \Rightarrow \int\frac{2}{w^2}dw-\int\frac{1}{w}dw=\ln|x|+C \Rightarrow$ $-\frac{2}{w}-\ln|w|=\ln|x|+C \Rightarrow -\frac{2}{1+v}-\ln|1+v|=\ln|x|+C \Rightarrow -\frac{2}{1+\frac{y}{x}}-\ln\left|1+\frac{y}{x}\right|=\ln|x|+C \Rightarrow$ $-\frac{2x}{x+y}-\ln\left|\frac{x+y}{x}\right|=\ln|x|+C \Rightarrow -\frac{2x}{x+y}-\ln|x+y|+\ln|x|=\ln|x|+C \Rightarrow -\frac{2x}{x+y}-\ln|x+y|=C$.
- Our techniques (integration) limits us to $1+v=w\neq 0$, but are we excluding some solutions? Now $1+v=1+\frac{y}{x}=\frac{x+y}{x}=0$ means $y=-x$. Notice that if $y=-x$, $\frac{dy}{dx}=-1$ and $\frac{x+3y}{x-y}=\frac{-2x}{2x}=-1$, so we can add $y=-x$ to the family of solutions.

ISSUES ON MODELING

Example: Money Growth

- If we say that annual rate of interest is 5%, we mean that on \$100, we get \$5 in one year. So $P(1)-P(0)=0.05P(0)$.
- Equivalently, $\frac{dP(t)}{dt}=0.05P(t)$. So $P(t)=Ae^{0.05t}\Rightarrow P(t)=P(0)e^{0.05t}$. So $P(1)=P(0)e^{0.05}=P(0)\left(1+0.05+\frac{(0.05)^2}{2!}+\cdots\right)=P(0)+0.05P(0)+\frac{(0.05)^2}{2!}P(0)+\cdots$, so $P(1)-P(0)=0.05P(0)+\frac{(0.05)^2}{2!}P(0)+\cdots>0.05P(0)$.
- If we say the annual interest rate *r* is compounded semi-annually, we mean $P\left(\frac{1}{2}\right)-P(0)=\frac{r}{2}P(0)\Leftrightarrow P\left(\frac{1}{2}\right)=P(0)\left(1+\frac{r}{2}\right)$. So, $P(1)=P\left(\frac{1}{2}\right)\left(1+\frac{r}{2}\right)=P(0)\left(1+\frac{r}{2}\right)\left(1+\frac{r}{2}\right)=P(0)\left(1+\frac{r}{2}\right)^2$.
- In general, when we have an annual interest rate *r* compounding *n* times a year, $P(1)=P(0)\left(1+\frac{r}{n}\right)^n$.
- Compounding continuously means $n\to\infty$. So $P(1)=\lim_{n\to\infty}P(0)\left(1+\frac{r}{n}\right)^n=P(0)e^r$. Similarly, $P(1)=P(0)e^{rt}$.
- If we contribute continuously to the account, then $\frac{dP(t)}{dt}=rP(t)+k$ where *k* is the constant contribution.

EXISTENCE OF A UNIQUE SOLUTION

Example

- Recall that a differential equation can be built starting from the solution like $\begin{cases} y = \sqrt{x} \\ y = -\sqrt{x} \end{cases}$. They both can be "expressed" by $y^2 = x$ and $2yy' = 1$.
- Notice that the differential equation $2yy' = 1$ at $(0,0)$ is a confused initial value problem.

Note

Recall that given a first order differential equation $y' + p(t)y = g(t)$, the solution is $y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right]$ where $\mu(t) = e^{\int p(t)dt}$. So as long as $p(t)$ and $g(t)$ are integrable (continuous) functions, then we have a good solution.

Theorem

If $p(t)$ and $g(t)$ are continuous on $I = [\alpha, \beta]$, then for any value y_0 (that is already given), there is a unique solution $y = \phi(t)$ that on I it satisfies $y' + p(t)y = g(t)$, $y_0 = y(t_0)$ where $t_0 \in I$.

Theorem

Let $y' = f(y,t)$. If there is an "open window" $I \times J$ on which f and $\frac{\partial f}{\partial y}$ are continuous, then the initial value problem $y' = f(y,t)$, $y_0 = y(t_0)$ has an unique solution (for any $t_0 \in I$ and $y_0 \in J$ $\Leftrightarrow$ $(t_0, y_0) \in I \times J$).

Note

Notice that $y' = \frac{1}{2y}$ is not continuous at any neighborhood of $(0,0)$, so the theorem doesn't apply.

Example

For which initial values does $y' = (t^2 + y^2)^{\frac{3}{2}} = f(y,t)$ have a unique solution?

- f is continuous everywhere.
- $\frac{\partial f}{\partial y} = \frac{3}{2}(t^2 + y^2)^{\frac{1}{2}} 2y$ is also continuous everywhere.
- So $y' = (t^2 + y^2)^{\frac{3}{2}}$ has a unique solution for all initial values.

Example

For which initial values does $y' = \frac{\cot y}{1+y} = f(y,t)$ have a unique solution?

- f is discontinuous at $y = -1$ and $y = k\pi$.
- $\frac{\partial f}{\partial y} = \frac{\csc^2 y(1+y) - \cot y}{(1+y)^2}$ is also discontinuous at $y = -1$ and $y = k\pi$.

Bernoulli

Bernoulli solved $y' + p(t)y = y^n g(t)$ as follows: $\frac{y'}{y^n} + p(t)\frac{y}{y^n} = g(t) \Rightarrow y^{-n}y' + y^{1-n}p(t) = g(t)$. Let $v = y^{1-n}$, and $v' = \frac{dv}{dt} = (1-n)y^{1-n-1}y' = (1-n)y^{-n}y' \Rightarrow \frac{1}{1-n}v' = y^{-n}y'$. So now we get $\frac{1}{1-n}v' + p(t)v = g(t)$.

Autonomous Differential Equations

Autonomous differential equations look like $y' = f(y)$.

Examples

1) $y' = 0$; $y = c$.
2) $y' = k$; $y = kx + c$.
3) $y' = ry$ exponential growth.
4) $y' = y(1-y)$ is know as logistic growth.

Example: Logistic Growth Model

Consider the spread of a disease. If $y(t)$ is the number of infected population, then $\frac{dy}{dt} \approx y(k-y)$. So $\frac{dy}{dt} = ay(k-y) = aky\left(1-\frac{y}{k}\right) = ry\left(1-\frac{y}{k}\right)$, k is "environmental carrying capacity" or "saturation level" and r is the "intrinsic growth rate".

To solve it, $\int \frac{dy}{y\left(1-\frac{y}{k}\right)} = \int r\,dt \Rightarrow \int\left(\frac{1}{y} + \frac{1}{1-\frac{y}{k}}\right)dy = \int r\,dt \Rightarrow \ln y - \ln\left(1-\frac{y}{k}\right) = rt + C \Rightarrow$

$\ln\left(\frac{y}{1-\frac{y}{k}}\right) = rt + C \Rightarrow \frac{y}{1-\frac{y}{k}} = Ae^{rt} \Rightarrow y = Ae^{rt} - Ae^{rt}\frac{y}{k} \Rightarrow y + Ae^{rt}\frac{y}{k} = Ae^{rt} \Rightarrow y\left(1+\frac{A}{k}e^{rt}\right) = Ae^{rt} \Rightarrow$

$y(t) = \frac{Ae^{rt}}{1+\frac{A}{k}e^{rt}} = \frac{A}{e^{-rt}+\frac{A}{k}} = \frac{kA}{ke^{-rt}+A} = A\frac{k}{\frac{k}{A}e^{-rt}+1}$. Now, $y(0) = y_0$ means $\frac{y_0}{1-\frac{y_0}{k}} = A$. So

$y(t) = \cdots = \frac{ky_0}{y_0 + (k+y_0)e^{-rt}}$. As $t \to \infty$, $y \to k$.

Example

An important model: $\dfrac{dy}{dt} = ry(M - \ln y) = ry(\ln k - \ln y) = ry \ln \dfrac{k}{y}$.

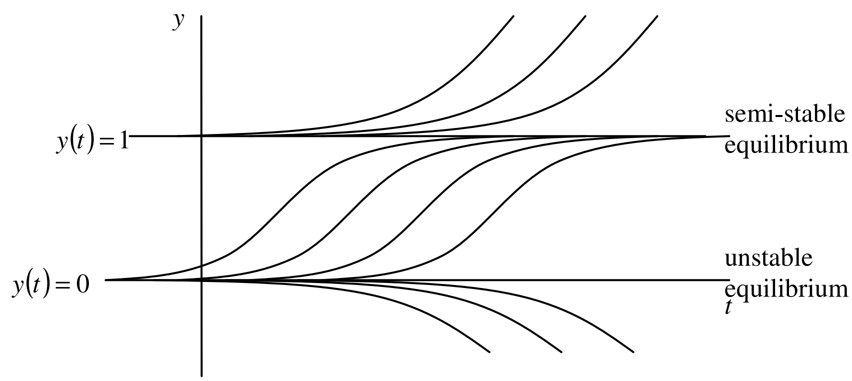
Example: Logistic Growth With Threshold

Some times we need enough of y_0 to start the epidemic. Let $\dfrac{dy}{dt} = -ry(k-y)(T-y) = f(y), r > 0$, with critical points at $y = 0, k, T$. Now we get $\dfrac{dy}{dt} = ry\left(1 - \dfrac{y}{k}\right)\left(1 - \dfrac{y}{T}\right)$.

CRITICAL POINTS OR EQUILIBRIUMS OF AN AUTONOMOUS DIFFERENTIAL EQUATION

Example

$y' = y(1-y) = 0$ gives constant solutions $y(t) = 0$ and $y(t) = 1$. These are the critical points or the equilibriums.



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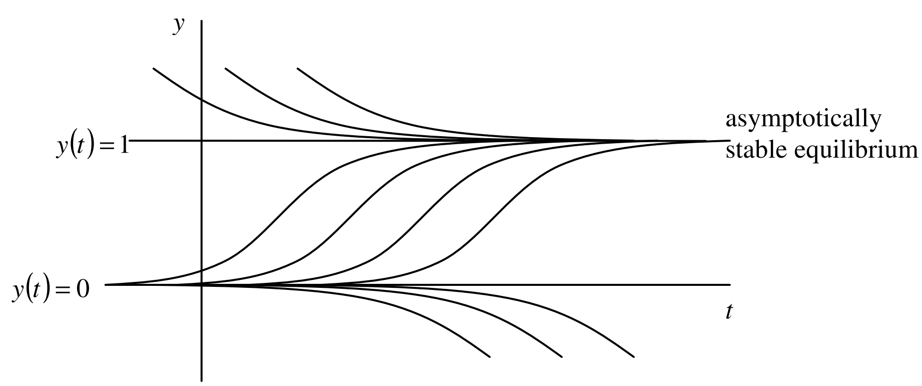


Example: Schaefer Model

Let $y(t)$ be the fish population at time *t*, which follows the logistic growth model. Then $\frac{dy}{dt} = ry\left(1 - \frac{y}{k}\right) - Ey$, where *E* the rate of harvest which is proportional to the fish population. Rewriting $\frac{dy}{dt} = y\left(r\left(1 - \frac{y}{k}\right) - E\right)$, the equilibriums are at $y(t) = 0$ and $r\left(1 - \frac{y(t)}{k}\right) - E = 0 \Rightarrow y(t) = (r - E)\frac{k}{r}$. Now, if $r > E$, the stable equilibrium is at $y = \frac{k}{r}(r - E)$. But if $r < E$, the stable equilibrium is at $y = 0$.

Parametric Differential Equations

Example

Consider $y' = ay - y^2 = y(a - y)$. The equilibriums are at $y = 0$ and $y = a$.

- When $a > 0$, we have a stable equilibrium around $y = a$.
- When $a = 0$, $y' = y^2$ and we have a semi-stable solution around $y = 0$.
- When $a < 0$, we have a unstable solution around $y = a$.

Here, $a = 0$ is called a bifurcation point.

Exact Differential Equations

Suppose $\psi(x, y) = c$. It implicitly defines a function $y(x)$.
Now $\frac{d}{dx}\psi(x, y) = 0 = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} := M(x, y) + N(x, y)\frac{dy}{dx}$. So $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ or $M(x, y)dx + N(x, y)dy = 0$. This type of differential equations are called exact equations.

Existence of Solution

If we are given $M(x, y)dx + N(x, y)dy = 0$, how would we know there is such a $\psi(x, y)$ corresponding to it? Indeed the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} =$ is necessary and sufficient for the existence of such $\psi(x, y)$.

Example

Consider $\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0 = M(x, y) + N(x, y)$. Since $\frac{\partial M}{\partial y} = \frac{1}{x}$ and $\frac{\partial N}{\partial x} = \frac{1}{x}$, the equation is exact. Notice that $\frac{\partial \psi}{\partial x} = M(x, y) = \frac{y}{x} + 6x$. Integrating with respect to *x*, we get $\psi(x, y) = y \ln x + 3x^2 + f(y)$.
Now, $\frac{\partial \psi}{\partial y} = N(x, y) = \ln x + f'(y) = \ln x - 2$, so $f'(y) = -2 \Rightarrow f(y) = -2y + c_0$. Hence $\psi(x, y) = y \ln x + 3x^2 - 2y + c_0$. The solution is $\psi(x, y) = C \Rightarrow y \ln x + 3x^2 - 2y = c$.

INTEGRATING FACTOR

- What if $M_y \neq N_x$? Does that mean $\psi(x, y)$ doesn't exists? Yes, but we may be able to change the differential equation $M(x, y)dx + N(x, y)dy = 0$ to a new "better" one.
- If we multiply $M(x, y)dx + N(x, y)dy = 0$ by $\mu(x, y)$, we may get $(\mu M)_y = (\mu N)_x$.
- How do we know if we should look for such μ? Finding μ is very difficult, unless μ is somehow a $\mu(x)$ or $\mu(y)$ only.

Example

- Suppose $(\mu M)dx + (\mu N)dy = 0$. We check if it is exact, and if it is, that fact should lead us to the answer.
- To be exact, we need $(\mu M)_y = (\mu N)_x \Rightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_x \Rightarrow \mu_y M - \mu_x N = \mu(N_x - M_y)$.
- Now if $\mu(x, y) = \mu(x)$, then $\mu_y = 0$. Then $-\mu_x N = \mu(N_x - M_y) \Rightarrow \frac{\mu_x}{\mu} = -\frac{(N_x - M_y)}{N}$.

Example

This tells us a criterion for finding integrating factors of different type: $\mu(xy)$. If $\frac{N_x - M_y}{xM - yN} = R(xy)$, then there exists a $\mu(xy)$.

Example

Consider $\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + 3\frac{y}{x}\right)\frac{dy}{dx} = 0$. It is obvious $M_y \neq N_x$, but

$$\frac{N_x - M_y}{xM - yN} = \frac{2\frac{x}{y} - 3\frac{y}{x^2} + \frac{6}{y^2}}{3x^2 + 6\frac{x}{y} - x^2 - 3\frac{y^2}{x}} = \frac{2\frac{x}{y} - 3\frac{y}{x^2} + \frac{6}{y^2}}{2x^2 + 6\frac{x}{y} - 3\frac{y^2}{x}} = \frac{1}{xy}\frac{2x^2 - 3\frac{y^2}{x} + 6\frac{x}{y}}{2x^2 + 6\frac{x}{y} - 3\frac{y^2}{x}} = \frac{1}{xy}$$

. So there is $\mu(xy)$ that makes this differential equation exact.

Note

We were looking for an integrating factor $\mu(xy)$ so that $\mu M + \mu N y' = 0$ is exact, i.e. $(\mu M)_y = (\mu N)_x$. This means $\mu_y M + \mu M_y = \mu_x N + \mu N_x$. But since we are requiring that $\mu(xy)$ is a function of *xy*, then by letting $w = xy$, $\mu(xy)$ becomes $\mu(w)$, and $\mu_x = \frac{d\mu}{dw}\frac{\partial w}{\partial x} = \mu' y$ and $\mu_y = \mu' x$. Therefore the condition of exactness becomes $\mu' x M + \mu M_y = \mu' y N + \mu N_x \Rightarrow \mu'(xM - yN) = \mu(N_x - M_y) \Rightarrow \frac{\mu'}{\mu} = \frac{N_x - M_y}{xM - yN} = R(w) = R(xy)$.

Now, we can easily solve for μ.

NUMERICAL APPROXIMATION SOLUTIONS TO DIFFERENTIAL EQUATIONS

If $g(t)$ is a solution to the differential equation $y' = f(t, y)$ and we need to know $y_1 = g(t_1)$ and an approximate value is good enough, then we can use the tangent line instead of $g(t)$. $y' = \frac{y_1 - y_0}{t_1 - t_0} \Rightarrow$ $y_1 - y_0 = y'(t_1 - t_0) \Rightarrow y_1 = y'(t_1 - t_0) + y_0 \Rightarrow y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$.

Euler suggested to evaluate $y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$, $y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$, $y_3 = y_2 + f(t_2, y_2)(t_3 - t_2)$.

So, to calculate $g(T)$, we can take $[t_0, T]$ instead of using $y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$ and subdivide it into $t_0, t_1, \ldots, t_{n-1}, T$. This way, the answer is a lot closer.

Convergence of Euler's Method

If we let $h \to 0$ (step size), then the approximate answer equals the actual answer. So for first order differential equations, it is a good idea to let $h \to 0$ to get a better solution.

Example

Prove that Euler's method converges for $y' = y, y(0) = 1$. We know the solution is $y(t) = e^t$.

- Let $h = \frac{t}{n}$.
- Now, $y_1 = y_0 + y'(0,1)h = 1 + h$, $y_2 = 1 + h + y'(t_2, 1 + h)h = 1 + h + (1 + h)h = (1 + h)^2$, so $y_n = (1 + h)^n$.
- If $n \to \infty$, then $(1 + h)^n = \left(1 + \frac{t}{n}\right)^n = e^t = y(t)$.

EXISTENCE AND UNIQUENESS OF A SOLUTION TO AN INITIAL VALUE PROBLEM

Technique

An initial value problem $y' = f(t, y), y(0) = y_0$ can be transformed into $w' = g(t, w), w(0) = 0$.

Example

Consider $y' = t^2 + y^2, y(1) = 2$. Let $t = s + 1$, then $y(t) = z(s)$, so the problem becomes $z' = (s - 1)^2 + z^2, z(0) = 2$. Now let $w = z - 2 \Leftrightarrow z = w + 2$, then $z' = w'$, so $w' = (s + 1)^2 + (w + 2)^2$. Now, $w(0) = z(0) - 2 = 2 - 2 = 0$.

Theorem

If $y' = f(y, t)$ and if *f* and $\frac{\partial f}{\partial y}$ are continuous on the rectangle $I \times J$, then for some $h > 0$ there is an unique solution of the form $y = \phi(t), \forall t_0 - h < t < t_0 + h$.

Note

To prove this theorem, we first showed it is alright to assume $y_0 = 0$ and $t_0 = 0$, so we assume that our IVP is of the form $y' = f(y,t), y(0) = 0$. The method of the proof is Picard's Method

Note

Observe that $y = \phi(t)$ is a solution to $y' = f(y,t), y(0) = 0$. So it also satisfies $y' = \phi'(t) = f(\phi(t),t)$ and $y = \phi(t) = \int_0^t \phi'(s)ds = \int_0^t f(\phi(s),s)ds$. So the IVP is equivalent to an integral equation.

Notice that $\phi(0) = \int_0^0 \phi'(s)ds = 0$, i.e. $y(0) = 0$.

Picard's Method

1) Find a function $\phi_0(t)$ that satisfies $\phi_0(0) = 0$ (ex: $\phi_0(t) = 0$).
2) Define $\phi_n(t)$ such that $\phi_{k+1} = \int_0^t f(\phi_k(s),t)ds, k = 0,1,\ldots$.

So we create a sequence of $\{\phi_n(t), n = 0,1,\ldots\}$. Now:

1) Is this an infinite sequence?
   - If at some point we stop producing new functions when $n = k$, then we already have the solution $\phi_k(t)$ of the integral equation.
   - If the sequence is infinite, then does it converge? To converge, we need $g_n(t) = |\phi_{n+1}(t) - \phi_n(t)|$ to converge, i.e. $\lim_{n\to\infty} g_n(t) = 0, \forall t \in [t_0 - h, t_0 + h]$, or $\lim_{n\to\infty} \phi_n(t) = \phi(t), \forall t \in [t_0 - h, t_0 + h]$.
2) Suppose $\{\phi_n(t), n = 0,1,\ldots\}$ is defined, and $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ exists, then what properties does $\phi(t)$ have?
   - $\phi(0) = \lim_{n\to\infty} \phi_n(0) = \lim_{n\to\infty} 0 = 0$.
   - $\int_0^t f(\phi(s),s)ds = \int_0^t f\left(\lim_{n\to\infty} \phi_n(s), s\right)ds = \int_0^t \lim_{n\to\infty} f(\phi_n(s),s)ds = \lim_{n\to\infty} \int_0^t f(\phi_n(s),s)ds = \lim_{n\to\infty} \phi_{n-1}(t) = \phi(t)$.
     So $\phi(t)$ is a solution.

Steps of the Proof of the Existence and Uniqueness Theorem

We constructed a sequence of functions $\{\phi_n(t), n = 0,1,\ldots\}$ with the hopes that the sequence converges to the solution $\phi(t)$. We saw that $\lim_{n\to\infty} \phi_n(t)$ satisfies the criteria for the solution.

1) When does this sequence converge? If $\frac{\partial f}{\partial y}$ exits and is continuous in *D*, then there is a number *K* such that $|f(t,y_1) - f(t,y_2)| \le K|y_1 - y_2| \Leftrightarrow \frac{|f(t,y_1) - f(t,y_2)|}{|y_1 - y_2|} \le K$.
2) $|f(t,\phi_n(t)) - f(t,\phi_{n-1}(t))| \le K|\phi_n(t) - \phi_{n-1}(t)|$.
3) Because $f(t,y)$ is continuous on *D*, then for some *h*, for all $t_0 - h < t < t_0 + h$, $f(t,y) \le M$. Then $|\phi_1(t)| = \left|\int_0^t f(s,\phi_0(s))ds\right| \le M|t|$, and because $\phi_0 = 0$, $|\phi_1(t) - \phi_0(t)| \le M|t|$. Also,

$|\phi_2(t)-\phi_1(t)| \le \int_0^t |f(s,\phi_1(s))-f(s,\phi_0(s))| ds \le \int_0^t K|\phi_1(s)-\phi_0(s)| ds \le \int_0^t (MKs) ds = MK\frac{t^2}{2}$. So, $|\phi_n(t)-\phi_{n-1}(t)| \le \frac{MK^2 t^{n+1}}{(n+1)!}$.

4) Now, notice that $\phi_n(t)-\phi_1(t) = (\phi_n(t)-\phi_{n-1}(t))+(\phi_{n-1}(t)-\phi_{n-2}(t))+\cdots+(\phi_2(t)-\phi_1(t))$, so by the triangle inequality, $|\phi_n(t)-\phi_1(t)| \le |\phi_n(t)-\phi_{n-1}(t)|+\cdots+|\phi_2(t)-\phi_1(t)|$. Finally, $|\phi_n(t)| \le |\phi_n(t)|+|\phi_n(t)-\phi_{n-1}(t)|+\cdots+|\phi_2(t)-\phi_1(t)| \le M|t|+MK\frac{t^2}{2}+\cdots = \frac{M}{K}\left(Kh+\frac{(Kh)^2}{2!}+\cdots\right)$. As $n\to\infty$, it converges to $e^{Kh}\frac{M}{K}$, so all the ϕ_n are bounded by $e^{Kh}\frac{M}{K}$.

5) We need to prove the uniqueness of this solution. Suppose there are two solutions, $\phi(t)$ and $\psi(t)$. Then $|\phi(t)-\psi(t)| = \left|\int_0^t f(s,\phi(s))-f(s,\psi(s))ds\right| \le \int_0^t |f(s,\phi(s))-f(s,\psi(s))| ds \le K\int_0^t |\phi(s)-\psi(s)| ds$. Now let $U(t)=\int_0^t |\phi(s)-\psi(s)| ds$, where $U(t)\ge 0$. Notice $U'(t)=|\phi(t)-\psi(t)| \le K\int_0^t |\phi(s)-\psi(s)| ds = KU(t)$. So $U'-KU\le 0 \Rightarrow e^{-kt}U'-e^{-kt}KU\le 0 \Rightarrow \left(e^{-kt}U\right)' \le 0$, therefore, $\int_0^t \left(e^{-ks}U(s)\right)' ds = e^{-kt}U(t)\le 0 \Rightarrow U(t)\le 0$.
But $\left.\begin{matrix} U(t)\ge 0 \\ U(t)\le 0 \end{matrix}\right\} \Rightarrow U(t)=0 \Rightarrow \phi(t)=\psi(t)$.

First Order Difference Equations

Examples of famous discrete process are stochastic processes $y_{n+1}=f(n,y_n,y_{n-1},\ldots,y_0,b)$.

However, we can only study simple ones $y_{n+1}=f(n,y_n)$. Compare it with $y'=f(t,y)$. Notice that $\frac{y_{n+1}-y_n}{(n+1)-n}=\frac{f(n,y_n)-y_n}{(n+1)-n}=g(n,y_n)\approx y_n' = g(n,y_n)$.

A solution to a difference equation is a sequence of numbers $y=y_1,\ldots,y_n$ that satisfies $y_{n+1}=f(n,y_n)$. Notice that the sequence defines a function with domain $\{0,1,2,\ldots\}$. So, $y_0=y(0)$, $y_1=y(1)$, etc.

Notice that if the increments become small, the difference equation becomes a differential equation.

Example

Solve $y_{n+1}=\sqrt{\frac{n+3}{n+1}}y_n$ with respect to y_0.

- $y_1=\sqrt{\frac{0+3}{0+1}}y_0=\sqrt{\frac{3}{1}}y_0$, $y_2=\sqrt{\frac{1+3}{1+1}}y_1=\sqrt{\frac{4\times 3}{2\times 1}}y_0$, $y_3=\sqrt{\frac{2+3}{2+1}}y_2=\sqrt{\frac{5\times 4\times 3}{3\times 2\times 1}}y_0$.
- So, $y_n=\sqrt{\frac{(n+2)(n+1)\cdots(4)(3)}{n(n-1)\cdots(2)(1)}}y_0=\sqrt{\frac{(n+2)(n+1)}{2}}y_0$.

Definitions

1) We say the annual rate is r compounding monthly to mean the period is one monthly and the rate on that period is $\frac{r}{12}$.
2) Effective annual rate equivalent to rate r compounding monthly is the rate r_e (compounding annually) such that the effect of r_e over a year is the same as the effect of $\frac{r}{12}$ for 12 periods. That is, if $Y_1 = (1+r_e)y_0$ and $y_{12} = \left(1+\frac{r}{12}\right)^{12} y_0$, then $Y_1 = y_{12}$.

Note

To calculate the effective annual rate, $1+r_e = \left(1+\frac{r}{12}\right)^{12}$.

AUTONOMOUS DIFFERENCE EQUATIONS

The simplest of difference equations is autonomous linear $y_{n+1} = f(y_n)$ where $f(y_n) = \rho y_n + b$.

Solution

To solve an autonomous linear difference equation, note that:

- $y_1 = f(y_0) = \rho y_0 + b$.
- $y_2 = f(y_1) = \rho y_1 + b = \rho(\rho y_0 + b) + b = \rho^2 y_0 + \rho b + b$.
- $y_3 = f(y_2) = \rho(\rho^2 y_0 + \rho b + b) + b = \rho^3 y_0 + \rho^2 b + \rho b + b$.

So $y_n = \rho^n y_0 + \rho^{n-1} b + \cdots + \rho b + b = \rho^n y_0 + b(\rho^{n-1} + \cdots + 1) = \rho^n y_0 + b\frac{\rho^n - 1}{\rho - 1}$.

Note

Notice that if $|\rho| < 1$, then as $n \to \infty$, $y_n = \rho^n y_0 + b\frac{\rho^n - 1}{\rho - 1} \to b\frac{1}{1-\rho}$. However, if $|\rho| > 1, \rho > 0$, as $n \to \infty$, $y_n = \rho^n y_0 + b\frac{\rho^n - 1}{\rho - 1} \to b\frac{1}{1-\rho}$; but if $|\rho| > 1, \rho < 0$, we are confused (the limit doesn't exist)!

Example

Consider a 20 year mortgage at rate 10% (compounded semi-annually) with monthly payment of $b = 1000$. What will the maximum loan be under the these restrictions?

- Model: $y_{n+1} = (1+i)y_n - 1000$. At $n = 12 \times 20 = 240$, $y_{240} = 0$.
- What is the periodic rate i? $\left(1+\frac{r}{2}\right)^2 = (1+i)^{12} \Rightarrow (1.05)^{\frac{1}{6}} = 1 + i = 1.00816$.

- So $y_n = (1-i)^n y_0 - b\dfrac{(1-i)^n - 1}{(1-i)-1}$. $y_{240} = 0 \Rightarrow (1-i)^n y_0 - b\dfrac{(1-i)^n - 1}{(1-i)-1} = 0 \Rightarrow$
  $y_0 = 1000\dfrac{(1.00816)^{240} - 1}{0.00816 \times (1.00816)^{240}} = 105121.37$.
- So the maximum loan is \$105121.37.

Logistic Growth Equation

Logistic growth equations have the form $u_{n+1} = \rho u_n (1 - u_n)$.

Equilibrium

Equilibrium solutions are obtained when $u_{n+1} = u_n$, i.e. $u_n = \rho u_n (1 - u_n) \Rightarrow u_n = \rho u_n - \rho {u_n}^2 \Rightarrow$
$\rho u_n - \rho {u_n}^2 - u_n = 0 \Rightarrow u_n(\rho - \rho u_n - 1) = 0 \Rightarrow \begin{cases} u_n = 0 \\ u_n = \dfrac{\rho - 1}{\rho} \end{cases}$. So different values of ρ present different equilibriums: if $\rho = 1$, then the two solutions are the same; if $\rho \approx 1$, then the two equilibriums are very close.

Stability of the Equilibriums

1) Suppose a system starts near the 0 equilibrium, and suppose for simplicity that it is linear, i.e. $u_{n+1} = \rho u_n$. If $u_n \approx 0$ (very near the equilibrium 0), then we may assume any logistic growth would be $u_{n+1} = \rho u_n - \rho {u_n}^2$. Then ${u_n}^2$ is closer to 0, so we can assume $u_{n+1} \approx \rho u_n$. We know as $n \to \infty$, $u_n \to \rho^n u_0$, so $u_n \to 0$ if $|\rho| < 1$. Therefore the logistic growth difference equation $u_{n+1} = \rho u_n (1 - u_n)$ will converge to 0 (the equilibrium) if we start near 0 and $|\rho| < 1$. More difficult analysis guarantees this situation is true for non-linear ones.
2) The behavior near the $\dfrac{\rho - 1}{\rho}$ equilibrium. Let v_n be an amount of perturbation that defines a new solution near $\dfrac{\rho - 1}{\rho}$, that is $u_n = \dfrac{\rho - 1}{\rho} + v_n, v_n \approx 0$. This is a solution and needs to be reformulated to a process. Notice that $u_{n+1} = \dfrac{\rho - 1}{\rho} + v_{n+1}$, $u_n = \dfrac{\rho - 1}{\rho} + v_n$, and $u_{n+1} = \rho u_n (1 - u_n)$, so $v_{n+1} = u_{n+1} - \dfrac{\rho - 1}{\rho} \Rightarrow$
   $v_{n+1} = \rho u_n (1 - u_n) - \dfrac{\rho - 1}{\rho} \Rightarrow v_{n+1} = (2 - \rho) v_n - \rho {v_n}^2$. Notice that since $v_n \approx 0$, $v_{n+1} \approx (2 - \rho) v_n$, so this new process converges if $|2 - \rho| < 1 \Leftrightarrow 1 < \rho < 3$.

Second Order Differential Equations

A second order differential equation is of the form $y'' = f(t, y, y')$.

Note

Because of y, y' in *f*, we have two degrees of freedom: at $t = t_0$, we have $\begin{cases} y_0 = y(t_0) \\ y_0' = y'(t_0) \end{cases}$.

Example

Consider $y'' + y = 0$. Solutions to this are $y = \sin x$, $y = \cos x$, $y = 0$,…there are infinitely many.

Specifying $y_0 = y(0) = 0$, we have $y = \sin x$, $y = 2\sin x$, $y = -\sin x$,…there are still infinite number of solutions.

To specify one solution, we need to fix y' also. If $\begin{cases} y_0 = y(0) = 0 \\ y_0' = y'(0) = -\frac{1}{3} \end{cases}$, then the solution is $y = -\frac{1}{3}\sin x$.

Note

Sometimes, we can point to a solution by specifying two points on the plane.

- If $y'' + y = 0, \begin{cases} y(0) = 0 \\ y\left(\frac{\pi}{2}\right) = 3 \end{cases}$, then $y(t) = 3\sin t$ is the solution. This is a boundary value problem.
- If $y'' + y = 0, \begin{cases} y(0) = y_0 \\ y'(0) = y_0' \end{cases}$, then this is a initial value problem.

Example

$y'' + y = 0, \begin{cases} y(0) = 0 \\ y(\pi) = 2 \end{cases}$ may have no solutions, but $y'' + y = 0, \begin{cases} y(0) = 0 \\ y(\pi) = 0 \end{cases}$ have infinitely many solutions. So BVP are not guaranteed to give a unique solution, but IVP will.

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

Definitions

1) A second order linear differential equation occurs when $y'' = f(t, y, y') = g(t) - p(t)y' - q(t)y$ or $y'' + p(t)y' + q(t)y = g(t)$.
2) If $p(t)$ and $q(t)$ are constants, when we have a constant coefficient second order linear differential equation.

Note

Notice that if $y(t) = e^{rt}$ is a solution to $ay'' + by' + cy = 0$, then $ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \Rightarrow ar^2 + br + c = 0$.

So if *r* is a solution to $ar^2 + br + c = 0$ (characteristic equation of the differential equation), then surely $y(t) = e^{rt}$ is a solution to $ay'' + by' + cy = 0$.

If the solution to $ar^2 + br + c = 0$ is imaginary, then $e^{it} = \cos t + i\sin t$.

Note

Notice that because the input is 0, any linear combination of y_1 and y_2 is also a solution, i.e. $y(t)=c_1y_1(t)+c_2y_2(t)$ is the general solution to the differential equation. Since $\begin{cases} y'(t)=c_1r_1e^{r_1t}+c_2r_2e^{r_2t} \\ y''(t)=c_1r_1^{\,2}e^{r_1t}+c_2r_2^{\,2}e^{r_2t} \end{cases}$, so $ay''+by'+cy\ =\ a\left(c_1r_1^{\,2}e^{r_1t}+c_2r_2^{\,2}e^{r_2t}\right)+b\left(c_1r_1e^{r_1t}+c_2r_2e^{r_2t}\right)+c\left(c_1e^{r_1t}+c_2e^{r_2t}\right)=$
$c_1\left(ar_1^{\,2}e^{r_1t}+br_1e^{r_1t}+cre^{r_1t}\right)+c_2\left(ar_2^{\,2}e^{r_2t}+br_2e^{r_2t}+ce^{r_2t}\right)=\ y_1(t)+y_2(t)\ =0.$

Example

The second order differential equation $y''-2y'-2y=0$ has the characteristic equation $r^2-2r-2=0$ with roots $\begin{cases} r_1=1+\sqrt{3} \\ r_2=1-\sqrt{3} \end{cases}$ and general solution $y(t)=c_1e^{r_1t}+c_2e^{r_2t}$ (the solution space, since all possible vectors in a vector space can be written as a linear combination of basis vectors).

To solve the IVP $y''-2y'-2y=0,\begin{cases} y(0)=1 \\ y'(0)=2 \end{cases}$, we solve $\begin{cases} y(t)=c_1e^{(1+\sqrt{3})t}+c_2e^{(1-\sqrt{3})t} \\ y'(t)=(1+\sqrt{3})c_1e^{(1+\sqrt{3})t}+(1+\sqrt{3})c_2e^{(1-\sqrt{3})t} \end{cases}\Rightarrow$
$\begin{cases} y(0)=c_1+c_2=1 \\ y'(t)=(1+\sqrt{3})c_1+(1+\sqrt{3})c_2=2 \end{cases}$. The augmented matrix is $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1+\sqrt{3} & 1-\sqrt{3} & 2 \end{array}\right]$, so by Cramer's Rule,

$$c_1=\frac{\begin{vmatrix} 1 & 1 \\ 2 & 1-\sqrt{3} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1+\sqrt{3} & 1-\sqrt{3} \end{vmatrix}}=\frac{1-\sqrt{3}-2}{1-\sqrt{3}-1-\sqrt{3}}=\frac{\sqrt{3}+1}{2\sqrt{3}}\ \text{and}\ c_1=\frac{\begin{vmatrix} 1 & 1 \\ 1+\sqrt{3} & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1+\sqrt{3} & 1-\sqrt{3} \end{vmatrix}}=\frac{2-1-\sqrt{3}}{1-\sqrt{3}-1-\sqrt{3}}=\frac{\sqrt{3}-1}{2\sqrt{3}}.$$

Example

The solution to $y''-2y'-2y=0$ is $y(t)=c_1e^{(1+\sqrt{3})t}+c_2e^{(1-\sqrt{3})t}$. We can analyze the behavior of the family, ex: where are the solutions positive, 0, negative; where are the max/min points; the behavior as $t\to\pm\infty$. To answer these questions, it is better to factor.

To find $y(t)=0$, let $y(t)=e^{(1+\sqrt{3})t}\left(c_1+c_2e^{-2\sqrt{3}t}\right)=0\Rightarrow t=-\frac{1}{2\sqrt{3}}\ln\left(\frac{c_2}{c_1}\right)$.

As $t\to\infty$, $y(t)=e^{(1+\sqrt{3})t}\left(c_1+c_2e^{-2\sqrt{3}t}\right)\to c_1e^{(1+\sqrt{3})t}$. As $t\to-\infty$, $y(t)=e^{(1-\sqrt{3})t}\left(c_1e^{2\sqrt{3}t}+c_2\right)\to c_2e^{(1-\sqrt{3})t}$.

EQUILIBRIUM SOLUTIONS

Equilibrium solutions are the constant solutions, i.e. $y=c$.

Example

The equilibrium solution to $y''-2y'-2y=3$ is $y_e=-\frac{3}{2}$.

We can transform this non-homogenous differential equation to a homogenous one. Let
$$\left.\begin{array}{l} Y = y - y_e \\ Y' = y' \\ Y'' = y'' \end{array}\right\} \Rightarrow Y'' - 2Y' - 2Y = 0$$. This transformation is shift.

Example

Consider $y'' + y' = e^{-t}$. To transform it, let $\begin{array}{l} v = y' \\ v' = y'' \end{array}$. Then $v' + v = e^{-t}$. Now solve
$v = f(t) + c_1 \Rightarrow \dfrac{dy}{dt} = f(t) + c_1$, then $y = \int f(t)dt + c_1 t + c_2$.

Example

Consider $2y^2 y'' + 2y(y')^2 = 1$ (t is not involved). Let $v = y' = \dfrac{dy}{dt}$, then $y'' = \dfrac{d}{dt}\left[\dfrac{dy}{dt}\right] = \dfrac{dv}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt} = v\dfrac{dv}{dy}$.

So we have $2y^2 v\dfrac{dv}{dy} + 2yv^2 = 1$. Now y is independent, i.e. $\dfrac{d}{dy}\left[y^2 v^2\right] = 1$.

FUNDAMENTAL SOLUTIONS OF LINEAR HOMOGENEOUS EQUATIONS

Solutions to a second order differential equation are real valued functions which are at least twice differentiable.
The collection of such functions is denoted by C^2.
We know from linear algebra that C^2 and addition of functions $(f+g)(x) = f(x) + g(x)$ and scalar multiplication $(\alpha f)(x) = \alpha f(x)$ form a vector space.

Linear Transformations

- The derivative $D(f) = f'$, $D^2(f) = D(D(f)) = f''$.
- $p(t)D(f) = p(t)f'$, $p(t)D(f+g) = p(t)(f+g)' = p(t)f' + p(t)g' = p(t)D(f) + p(t)D(g)$.
- $q(t): \mathbf{R}^2 \to \mathbf{R}^2, q(t) = q(t)1 \Leftrightarrow q(t)f = q(t)1f$.

Note

Any linear differential equation can be captured as a linear transformation. So $y'' + p(t)y' + q(t)y = 0$ can be written as $L[y] = \left(D^2 + p(t)D + q(t)\right)(y) = D^2(y) + p(t)D(y) + q(t)y = 0$.

Note

Notice that if y_1 and y_2 are solutions to $L[y]$, then so is $y = c_1 y_1 + c_2 y_2$.

Proof: $\left.\begin{array}{l} L[y_1] = 0 \\ L[y_2] = 0 \end{array}\right\} \Rightarrow L[c_1 y_1 + c_2 y_2] = c_1 L[c_1 y_1] + c_2 L[y_2] = 0 + 0 = 0$.

Theorem: Existence and Uniqueness Theorem

The initial value problem $L[y]=g(t), \begin{cases} y(t_0)=y_0 \\ y'(t_0)=y_0' \end{cases}$ has a unique solution on any interval *I* on which $t_0 \in I$ and $p(t)$, $q(t)$, $g(t)$ are all continuous.

Example

Determine the largest interval *I* on which the IVP $(x-2)y''+y'+((x-2)\tan x)y=0, \begin{cases} y(3)=1 \\ y'(3)=2 \end{cases}$ has a unique solution.

- At $x=2$, $y'=0$, so we have no solution at $x=2$.
- Since $x \neq 2$, the IVP is now $y''+\frac{1}{x-2}y'+(\tan x)y=0$. Note that $\tan x$ is discontinuous at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.
- So $\left(2, \frac{3\pi}{2}\right)$ is the largest interval on which we have a unique solution.

LINEAR INDEPENDENCE AND THE WRONSKIAN

Recall that in the case of a first order differential equation, whenever we found a solution, we added a constant to it in order to included all possible solutions (the general/family of solutions).
In the case of second order differential equations, we find two solutions y_1 and y_2 and claim that $c_1 y_1 + c_2 y_2$ is the general solution.
How do we decide if two functions are linearly independent on an interval *I*? Use Wronskian of the two functions.

Wronskian

If the Wronskian $W(y_1, y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is non zero for some $t \in I$, then y_1 and y_2 are linearly independent.

Theorem: Abel's Theorem

If y_1 and y_2 are solutions to $L[y]=y''+p(t)y'+q(t)y=0$ where $p(t)$ and $q(t)$ are continuous on an open interval *I*, then the Wronskian $(W(y_1, y_2))(t)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}(t)=ce^{-\int p(t)dt}$ where *c* is a constant that depends on y_1 and y_2 but not on *t*.

COMPLEX ROOTS OF THE CHARACTERISTIC EQUATION

Recall: To find solutions to $ay''+by'+cy=0$ we decided to find the roots r_1 and r_2 of $ar^2+br+c=0$, and $y_1(t)=e^{r_1 t}$, $y_2(t)=e^{r_2 t}$.

What if $r_1 = r_2$ (repeated root), or there are no real roots (complex roots)?

If $\Delta = b^2 - 4ac < 0$, then $r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-b \pm \sqrt{(-\Delta)(-1)}}{2a} = \frac{-b \pm \sqrt{(-1)}\sqrt{(-\Delta)}}{2a} = -\frac{b}{2a} \pm \frac{i\sqrt{(-\Delta)}}{2a} = \lambda \pm i\mu$.

As usual we say $\begin{cases} y_1(t) = e^{r_1 t} = e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}[\cos(\mu t) + i\sin(\mu t)] \\ y_2(t) = e^{r_2 t} = e^{(\lambda - i\mu)t} = e^{\lambda t} e^{-i\mu t} = e^{\lambda t}[\cos(\mu t) - i\sin(\mu t)] \end{cases}$ are the solutions.

Real-Valued Solutions

But we are looking for a real solution. Indeed, $\begin{cases} y_1(t) + y_2(t) = 2e^{\lambda t}\cos(\mu t) \\ -iy_1(t) + iy_2(t) = 2e^{\lambda t}\sin(\mu t) \end{cases}$ are solutions too. So the general solution is $y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t) = e^{\lambda t}[c_1\cos(\mu t) + c_2\sin(\mu t)]$.

If $\lambda = 0$ i.e. $-\frac{b}{2a} = 0 \Rightarrow b = 0$, then the solution becomes

$y(t) = c_1\cos(\mu t) + c_2\sin(\mu t) = c_2\left[\frac{c_1}{c_2}\cos(\mu t) + \sin(\mu t)\right]$. By letting $\frac{c_1}{c_2} = \tan\alpha = \frac{\sin\alpha}{\cos\alpha}$, we get

$y(t) = c_2\left[\frac{\sin\alpha}{\cos\alpha}\cos(\mu t) + \sin(\mu t)\right] = \frac{c_2}{\cos\alpha}[(\sin\alpha)(\cos(\mu t)) + (\cos\alpha)(\sin(\mu t))] = A\sin(\mu t + \alpha)$.

Note

Given $y'' + p(t)y' + q(t)y = 0$ we can change the variable t to x so $y(t) \mapsto y(x)$. Now, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$,

$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dx}{dt} + \frac{dy}{dx}\cdot\frac{d}{dt}\left(\frac{dx}{dt}\right) = \left(\frac{d}{dt}\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right) + \left(\frac{dy}{dx}\right)\left(\frac{d^2x}{dt^2}\right)$. It turns out that if we let $x(t) = \int |q(t)|^{\frac{1}{2}}\,dt$, then $y'' + p(t)y' + q(t)y = 0$ becomes $y'' + by' + c(t)y = 0$.

Example

Consider $ty'' + (t^2 - 1)y' + t^3 y = 0$, or $y'' + \frac{t^2 - 1}{t}y' + t^2 y = 0$. So $q(t) = t^2$. Now let

$x(t) = \int |q(t)|^{\frac{1}{2}}\,dt = \int t\,dt = \frac{t^2}{2}$, and so $\frac{dx}{dt} = t$, $\frac{d^2x}{dt^2} = 1$. Now, $y'' + \frac{t^2 - 1}{t}y' + t^2 y = 0$ becomes

$(y''t^2 + y') + \frac{t^2 - 1}{t}(y't) + t^2 y = 0 \Rightarrow y''t^2 + y' + t^2 y' - y' + t^2 y = 0 \Rightarrow y'' + y' + y = 0$.

REPEATED ROOTS

Recall that $r = \frac{-b \pm \sqrt{\Delta}}{2a}$. If $\Delta = b^2 - 4ac = 0$, then $r = -\frac{b}{2a}$. So only one solution exists: $y_1 = e^{-\frac{b}{2a}t}$. But if we let $y(t) = v(t)y_1$, then maybe we can come up with some other solutions as well.

If $y(t)$ is to satisfy $ay'' + by' + cy = 0$, then:

- $y(t) = v(t)y_1$
- $y'(t) = v'(t)y_1 + v(t)y_1'$
- $y''(t) = v''(t)y_1 + v'(t)y' + v'(t)y' + v(t)y_1'' = v''(t)y_1 + 2v'(t)y' + v(t)y_1''$

and $ay'' + by' + cy = 0$ becomes $a(v''(t)y_1 + 2v'(t)y' + v(t)y_1'') + b(v'(t)y_1 + v(t)y_1') + c(v(t)y_1) = 0 \Rightarrow$ $[av''(t)y_1 + 2av'(t)y' + bv'(t)y_1] + v(t)[ay_1'' + by_1' + cy_1] = 0 \Rightarrow av''(t)y_1 + 2av'(t)y' + bv'(t)y_1 = 0 \Rightarrow$ $av''(t)e^{-\frac{b}{2a}t} - bv'(t)e^{-\frac{b}{2a}t} + bv'(t)e^{-\frac{b}{2a}t} = 0 \Rightarrow v''(t) = 0 \Rightarrow v'(t) = c_1 \Rightarrow v(t) = c_1 t + c_2$. So another solution is $y_2(t) = (c_1 t + c_2)e^{-\frac{b}{2a}t}$.

So the general solution is $y(t) = C_1 e^{-\frac{b}{2a}t} + C_2(c_1 t + c_2)e^{-\frac{b}{2a}t} = (C_1 + C_2 c_1 t + c_2)e^{-\frac{b}{2a}t} = (at + b)e^{-\frac{b}{2a}t}$.

NON-HOMOGENOUS DIFFERENTIAL EQUATIONS

A non-homogenous differential equation has the form $L[y] = y'' + p(t)y' + q(t)y = g(t)$. $L[y] = 0$ decides the nature of the system.

General Solution

Suppose y_1 and y_2 are two solutions to $L[y] = g(t)$, i.e. $L[y_1] = g(t) = L[y_2]$. Then $L[y_1] - L[y_2] = g(t) = g(t) = 0$, so $y_1 - y_2$ is a solution to $L[y] = 0$. So if $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is the general solution to $L[y] = 0$, then $y_1 - y_2 = c_1 y_1 + c_2 y_2$ for some c_1 and c_2. Therefore, $y_1 = y_2 + c_1 y_1 + c_2 y_2$, which means if one solution to $L[y] = g(t)$ is known (y_p a particular solution) then the general solution to $L[y] = g(t)$ is $y(t) = y_p(t) + c_1 y_1(t) + c_2 y_2(t)$.

So to find the generals solution $y(t)$ to $L[y] = g(t)$, we need to first guess $y_p(t)$ then solve $L[y] = 0$.

How to Guess a Particular Solution?

Notice that a (linear) differential equation does not completely disfigure the input $g(t)$ (at least the "reasonable" inputs will be recognizable). "Reasonable" inputs: exponential e^{rt}, polynomial $p(t)$, trigonometric functions $\sin(\alpha t)$, $\cos(\alpha t)$, $\tan(\alpha t) = \frac{\sin(\alpha t)}{\cos(\alpha t)}$, roots $\sqrt[n]{x}$.

Example

Consider $L[y] = y'' + 2y' + 5y = 3\sin(2t)$. Guess a $y_p(t)$.

Let $\begin{cases} y_p(t) = A\sin(2t) + B\cos(2t) \\ y_p'(t) = 2A\cos(2t) - 2B\sin(2t) \\ y_p''(t) = -4A\sin(2t) - 4B\cos(2t) = -4y_p(t) \end{cases}$, then $L[y_p] = 3\sin(2t) \Rightarrow$

$-4y_p + 2[2A\cos(2t) - 2B\sin(2t)] + 5y_p = 3\sin(2t) \Rightarrow y_p + 4A\cos(2t) - 4B\sin(2t) = 3\sin(2t) \Rightarrow$

$y_p + 4A\cos(2t) - 4B\sin(2t) = 3\sin(2t) \Rightarrow A\sin(2t) + B\cos(2t) + 4A\cos(2t) - 4B\sin(2t) = 3\sin(2t) \Rightarrow$ $(A-4B)\sin(2t) + (4A+B)\cos(2t) = 3\sin(2t)$. Since $\sin(2t)$ and $\cos(2t)$ are linearly independent,

$$\begin{cases} A-4B=3 \\ 4A+B=0 \end{cases} \Rightarrow \begin{cases} A = \dfrac{3}{17} \\ B = -\dfrac{12}{17} \end{cases}.$$

So guess $y_p(t) = \frac{1}{17}[3\sin(2t) - 12\cos(2t)]$.

More Complicated Inputs

What if $L[y] = g(t)$, but $g(t) = g_1(t) + g_2(t) + g_3(t)$? It can be reduced to finding particular solutions to $\begin{cases} L[y] = g_1(t) \\ L[y] = g_2(t) \\ L[y] = g_3(t) \end{cases}$ because $L[y_1 + y_2 + y_3] = L[y_1] + L[y_2] + L[y_3] = g_1(t) + g_2(t) + g_3(t) = g(t) \Rightarrow$ $y_p(t) = y_1(t) + y_2(t) + y_3(t)$.

Example

Consider $y'' + y' = e^{-t}$. Let $\begin{cases} y_p(t) = Ae^{-t} \\ y_p'(t) = -Ae^{-t} \\ y_p''(t) = Ae^{-t} \end{cases} \Rightarrow Ae^{-t} - Ae^{-t} = e^{-t} \Rightarrow 0 = e^{-t}$! This problem happens because y_p is already a solution to $L[y_p] = 0$.

Try replacing y_p with ty_p.

Repeated Roots

If $L[y] = y'' + p(t)y' + q(t)y = g(t)$, then $L[ty_p] = g(t) \Rightarrow (tp_y)'' + p(t)(ty_p)' + q(t)(ty_p) = g(t) \Rightarrow$ $[y_p' + y_p' + ty_p''] + p(t)[y_p + ty_p'] + q(t)t[y_p] = g(t) \Rightarrow t[ty_p'' + p(t)y_p' + q(t)y_p] + 2y_p' + p(t)y_p = g(t) \Rightarrow$ $2y_p' + p(t)y_p = g(t)$. Now solve for $y_p(t)$.

METHOD OF UNDETERMINED COEFFICIENTS

Example

Solve the IVP $y'' + 4y = t^2 + 3e^t, \begin{cases} y(0) = 0 \\ y'(0) = 2 \end{cases}$.

1) Solve $y'' + 4y = 0$. $r^2 + 4 = 0 \Rightarrow r = 0 \pm 2i$, so the fundamental solution is $y_c(t) = c_1\cos(2t) + c_2\sin(2t)$.

2) Look for a particular solution $y_p(t)$. Since $g(t)=t^2+3e^t$, let $\begin{cases} y_p(t)=At^2+Bt+C+De^t \\ y_p'(t)=2At+B+De^t \\ y_p''(t)=2A+De^t \end{cases}$. So

$$y_p''+4y_p=t^2+3e^t \Rightarrow \left[2A+De^t\right]+4\left[At^2+Bt+C+De^t\right]=t^2+3e^t \Rightarrow$$

$$4At^2+4Bt+(2A+4C)+5De^t=t^2+3e^t \Rightarrow \begin{cases} 4A=1 \\ 4B=0 \\ 2A+4C=0 \\ 5D=3 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{4} \\ B=0 \\ C=-\frac{1}{8} \\ D=\frac{3}{5} \end{cases}.$$

3) The general solution is $y(t)=y_c(t)+y_p(t)=c_1\cos(2t)+c_2\sin(2t)+\frac{1}{4}t^2-\frac{1}{8}+\frac{3}{5}e^t$.

4) Now, $y\begin{cases}(t)=c_1\cos(2t)+c_2\sin(2t)+\frac{1}{4}t^2-\frac{1}{8}+\frac{3}{5}e^t \\ y'(t)\end{cases}=-2c_1\sin(2t)+2c_2\cos(2t)+\frac{1}{2}t+\frac{3}{5}e^t$, so

$\begin{cases} y(0)=0\Rightarrow c_1=\frac{-19}{40} \\ y'(0)=2\Rightarrow c_2=\frac{7}{10} \end{cases}$. So the solution to the IVP is $y(t)=-\frac{19}{40}\cos(2t)+\frac{7}{10}\sin(2t)+\frac{1}{4}t^2-\frac{1}{8}+\frac{3}{5}e^t$.

Example

Consider $y''+4y=3\sin 2t$. We already know the fundamental solutions $y_c(t)=c_1\cos(2t)+c_2\sin(2t)$. So when we "guess" the particular solution, use $y_p(t)=At\sin(2t)+Bt\cos(2t)$. Then the general solution is

$$y(t)=-\frac{3}{4}t\cos(2t)+c_1\cos(2t)+c_2\sin(2t).$$

VARIATION OF PARAMETERS

This technique works for general linear differential equations with arbitrary input $g(t)$.

Idea

If $y_1(t)$, $y_2(t)$ are fundamental solutions to $L[y]=y''+p(t)y'+q(t)y=0$, then the general solution to $L[y]=g(t)$ "can" be $y(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$.

We want more restrictions on u_1 and u_2. If $u_1'y_1+u_2'y_2=0$, then $(u_1'y_1+u_2'y_2)'=0 \Rightarrow$
$u_1''y_1+u_1'y_1'+u_2''y_2+u_2'y_2'=0$. So $\begin{cases} u_1'y_1+u_2'y_2=0 \\ u_1'y_1'+u_2'y_2'=g(t) \end{cases}$. Now, we can solve for $u_1(t)$ and $u_2(t)$.

Example

Consider $x^2 y'' - 3xy' + 4y = x^2 \ln x$, and $\begin{cases} y_1 = x^2 \\ y_2 = x^2 \ln x \end{cases}$ fundamental solutions. Rewrite as

$y'' - \frac{3}{x} y' + \frac{4}{x^2} y = \ln x$. Let $\begin{cases} u_1' x^2 + u_2' x^2 \ln x = 0 \\ u_1'(2x) + u_2'(2x \ln x + x) = \ln x \end{cases} \Rightarrow \begin{cases} u_1' x^2 + u_2' x^2 \ln x = 0 \\ u_1' + u_2'\left(\ln x + \frac{1}{2}\right) = \frac{\ln x}{2x} \end{cases}$.

Observation

To solve $\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(t) \end{cases}$, use Cramer's rule. So $u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 g(t)}{W(y_1, y_2)}$ and

$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 g(t)}{W(y_1, y_2)}$. Thus, the general solution is

$$y(t) = u_1(t) y_1(t) + u_2(t) y_2(t) = y_1(t)\left[\int_{t_0}^{t} \frac{-y_1(s) g(s)}{W(y_1, y_2)(s)}\, ds + c_1\right] + y_2(t)\left[\int_{t_0}^{t} \frac{y_2(s) g(s)}{W(y_1, y_2)(s)}\, ds + c_2\right].$$

Reduction of Order

If only $y_1(t)$ is given as a solution to $L[y] = y'' + p(t)y' + q(t)y = 0$, we claim that $y(t) = v(t) y_1(t)$ is a solution to $L[y] = g(t)$.

So $v'' y_1 + 2v' y_1' + p(t) v' y_1 = g(t)$ or $v'' y_1 + (2y_1' + p(t) y_1) v' = g(t)$, which we can solve!

Spring Vibrations

Hooke's Law

"The more the spring stretches, the more force." $F_s = -kL$.

Equilibrium

$mg = -kL \Leftrightarrow mg + kL = 0$.

Example

Given a spring, a 100g mass is hanged on it. The spring stretches by 2cm. What is *k*?

$mg = -kL \Rightarrow k = -\frac{mg}{L} = -\frac{(0.1)(-9.8)}{0.02} = 49$.

General Solution

On the way down, the mass passes through $u(t)$ at time *t*. So
$m(u''(t)+g)+k(L+u(t))=0 \Rightarrow mu''(t)+mg+kL+ku(t)=0 \Rightarrow mu''(t)+ku(t)=0$. Therefore, at any point *u* (with respect to any reference point), we have $mu''+ku=0$ as the differential equation form of Hooke's Law.

Now, $u''+\frac{k}{m}u=0$, so $\begin{cases}\lambda=0\\ \mu=\sqrt{\frac{k}{m}}=\omega_0\end{cases}$. The solution is $u(t)=c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)=\cdots=A\cos(\omega_0 t-\delta)$ (*A* is the amplitude, δ is the phase). The period is $T=\frac{2\pi}{\omega_0}=2\pi\sqrt{\frac{m}{k}}$.

Fluid Resistance

The fluid that the mass is moving in causes resistance. $F_r \sim v$, so $F_r=\lambda u'$. So now, the differential equation becomes $mu''+\lambda u'+ku=0$. So $r=-\frac{\lambda}{2m}\pm\frac{\sqrt{\lambda^2-4mk}}{2m}=-\frac{\lambda}{2m}\pm\sqrt{\frac{\lambda^2}{4m^2}-\frac{k}{m}}$.

If $\lambda=0$, we have undamped motion with frequency $\omega_0=\sqrt{\frac{k}{m}}$.

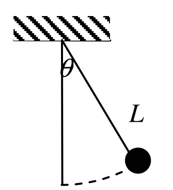
If $\lambda^2<4mk$, then $r=-\frac{\lambda}{2m}\pm i\sqrt{\frac{k}{m}-\frac{\lambda^2}{4m^2}}$. So $u(t)=Ae^{-\frac{\lambda}{2m}}\cos\left(\left(\sqrt{\frac{k}{m}-\frac{\lambda^2}{4m^2}}\right)t-\delta\right)$, where $\omega_q=\sqrt{\frac{k}{m}-\frac{\lambda^2}{4m^2}}=\sqrt{{\omega_0}^2-\frac{\lambda^2}{4m^2}}<\omega_0$ is the quasi-frequency.

Change in Mass

$\omega_q=\sqrt{\frac{k}{m}-\frac{\lambda^2}{4m^2}}, \frac{k}{m}>\frac{\lambda^2}{4m^2}$. As *m* increases, ω_q increases. As *m* decreases, $\frac{\lambda^2}{4m^2}$ increases faster than $\frac{k}{m}$, and there will eventually be no oscillation.

Pendulum

The motion of the pendulum is governed by $\theta''+\frac{g}{L}\theta=0$.



$\omega_0=\sqrt{\frac{g}{L}}$ is the number of oscillation per second.

$T=\frac{2\pi}{\omega_0}=2\pi\sqrt{\frac{L}{g}}$ is the period.

Electric Circuits

Components

1) Resistor: $I = \frac{1}{R}V = \frac{dQ}{dt} \Leftrightarrow V_R = R\frac{dQ}{dt}$, R is resistance in ohms.

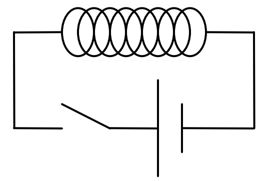
2) Capacitor: $V_L = \frac{1}{C}Q$, C is capacity in Faraday.

3) Coil: $V_L = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$, L is in Henry.

LC Circuit

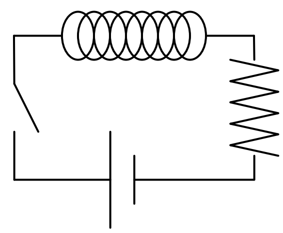
$V_C + V_L = 0$, $V_L = LQ''$, $V_C = \frac{1}{C}Q$.

So $LQ'' + \frac{1}{C}Q = 0$ with $\omega_0 = \sqrt{\frac{1}{LC}}$.



LCR Circuit

$V_C + V_R + V_L = 0$. So $LQ'' + RQ' + \frac{1}{C}Q = 0$.



FORCED VIBRATION

Vibration with input is $mu'' + \gamma u' + ku = F(t)$.

Cases of Interest

1) $F(t) = F_0 \cos(\omega t)$ where $\omega \neq \omega_0$. Then $u(t) = Ae^{-rt}\cos(\omega_0 t - \delta) + R\cos(\omega t - \delta)$, where $R = \frac{F_0}{\sqrt{m^2(\omega_0{}^2 - \omega^2)^2 + \gamma^2\omega^2}}$. Since $\omega_0{}^2 = \frac{k}{m}$, so if $\omega \approx 0$ then $R \approx \frac{F_0}{\sqrt{m^2\omega_0{}^4}} = \frac{F_0}{m\omega_0{}^2} = \frac{F_0}{k}$; if $\omega >> \omega_0$ then $R \approx \frac{F_0}{\sqrt{m^2\omega^4 + \gamma^2\omega^2}} = \frac{F_0}{\omega\sqrt{m^2\omega^2 + \gamma^2}}$ which approaches 0 as ω approaches infinity.

2) $F(t) = F_0 \cos(\omega t)$ where $\omega \approx \omega_0$. Then $u(t) = Ae^{-rt}\cos(\omega_0 t - \delta) + R\cos(\omega t - \delta)$, where $R = \frac{F_0}{\sqrt{m^2(\omega_0{}^2 - \omega^2)^2 + \gamma^2\omega^2}}$. Since $\omega \approx \omega_0$, so $R \approx \frac{F_0}{\sqrt{\gamma^2\omega^2}} = \frac{F_0}{\gamma\omega}$; if $\gamma \approx 0$ then R is large (momentary resonance).

3) $F(t) = F_0 \cos(\omega t)$ where $\gamma = 0$ and $\omega = \omega_0$. Then $u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0}t\sin(\omega_0 t)$. If $\begin{cases} u(0) = 0 \\ u'(0) = 0 \end{cases}$, then we have resonance.

4) $F(t)=F_0\cos(\omega t)$ where $\gamma=0$ and $\omega\neq\omega_0$. Then $u(t)=c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)+\dfrac{F_0}{m(\omega_0{}^2-\omega^2)}\cos(\omega t)$. If

$$\begin{cases} u(0)=0\Rightarrow c_1=-\dfrac{F_0}{m(\omega_0{}^2-\omega^2)} \\ u'(0)=0\Rightarrow c_2=0 \end{cases}, \text{ then}$$

$u(t)=\dfrac{F_0}{m(\omega_0{}^2-\omega)}[\cos(\omega t)-\cos(\omega_0 t)]=\dfrac{F_0}{m(\omega_0{}^2-\omega)}2\sin\left(\dfrac{\omega_0{}^2-\omega^2}{2}t\right)\sin\left(\dfrac{\omega_0{}^2+\omega^2}{2}t\right)$. We let the actual frequency to be $\omega_0{}^2+\omega^2$, and then the amplitude keeps changing with time (with frequency $\omega_0{}^2-\omega^2<\omega_0{}^2+\omega^2$.

Series Solution of Second Order Linear Equations

Definition: Regular Point

A point x_0 is said to be a regular point of the differential equation $P(x)y''+Q(x)y'+R(x)y=0$ if $P(x_0)\neq 0$. That is, if $\dfrac{Q(x)}{P(x)}$ and $\dfrac{R(x)}{P(x)}$ are continuous at x_0, then the IVP has a solution defined near $y(0)=x_0$.

Example

Consider the IVP $(1-x)^2y''-2xy'+12y=0,\begin{cases} x(0)=0 \\ x'(0)=1\end{cases}$. Assuming y exists near $x_0=0$, we have

$y=\sum_{n=0}^{\infty}a_n(x-0)^n=\sum_{n=0}^{\infty}a_nx^n$. Then $-2xy'=-2x\sum_{n=0}^{\infty}na_nx^{n-1}=\sum_{n=1}^{\infty}-2na_nx^n$ and

$(1-x^2)y''=(1-x^2)\sum_{n=0}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_nx^n$.

So $(1-x)^2y''-2xy'+12y=0$ becomes

$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=2}^{\infty}-n(n-1)a_nx^n+\sum_{n=1}^{\infty}-2na_nx^n+\sum_{n=0}^{\infty}12a_nx^n=0$. Making the powers look alike, we have $\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n+\sum_{n=2}^{\infty}-n(n-1)a_nx^n+\sum_{n=1}^{\infty}-2na_nx^n+\sum_{n=0}^{\infty}12a_nx^n=0$.

So the DE becomes

$$[2a_2+12a_0]+[6a_3x+10a_1x]+\sum_{n=2}^{\infty}[(n+1)(n+2)a_{n+2}-n(n-1)a_n-2na_n+12a_n]x^n=0\Rightarrow$$

$$\begin{cases} 6a_0 + a_2 = 0 \\ 5a_1 + 3a_3 = 0 \\ (n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + 12a_n = 0 \end{cases} \Rightarrow \begin{cases} a_2 = -6a_0 \\ a_3 = -\dfrac{5}{3}a_1 \\ a_{n+2} = \dfrac{n(n-1)+2n-12}{(n+1)(n+2)}a_n = \dfrac{(n+3)(n-3)}{(n+1)(n+2)}a_n \end{cases}$$

. Notice that $\begin{cases} y(0) = a_0 = 0 \\ y'(0) = a_1 = 1 \end{cases}$, so $\begin{cases} a_2 = 0 \\ a_{n+2} = \dfrac{(n+3)(n-3)}{(n+1)(n+2)}a_n \end{cases} \Rightarrow a_{2n} = 0, \forall n = 0,1,\ldots$ and

$$\begin{cases} a_3 = -\dfrac{5}{3} \\ a_5 = a_{3+2} = 0a_3 = 0 \\ a_{n+2} = \dfrac{(n+3)(n-3)}{(n+1)(n+2)}a_n \end{cases} \Rightarrow a_{2n+1} = 0, \forall n = 2,3,\ldots$$

. So the solution is $y_1(x) = x - \frac{5}{3}x^3$.

Now, if $\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$, then $\begin{cases} a_1 = 0 \\ a_3 = 0 \end{cases} \Rightarrow a_{2n+1} = 0, \forall n = 0,1,\ldots$ and $\begin{cases} a_0 = 1 \\ a_2 = -6 \\ a_4 = 3 \end{cases}$. So $y_2(x) = 1 - 6x^2 + 3x^4 + \cdots$.

Therefore, the general solution is $y(x) = c_1 y_1 + c_2 y_2$. Notice that if $\begin{cases} y(0) = b_1 \\ y'(0) = b_2 \end{cases}$, then $y(x) = b_2 y_1 + b_1 y_2$.

Definition: Regular Point

A regular point is a point where all the derivatives of the solution function are defined. In other words, the Taylor series exists.

RESTRICTIONS ON POWER SERIES SOLUTION

Example: Radius of Convergence

Consider $(x^2 - 2x - 3)y'' + xy' + 4y = 0$ at various points: $x_0 = 4$, $x_0 = -4$, and $x_0 = 0$. Find a lower bound for the radius of convergence of the given points.

- The issues arise when the coefficient of y'' becomes 0. $x^2 - 2x - 3 = (x-3)(x+1)$, so at $x = 3$ and $x = -1$, we have issues.
- The solution shall make sense on the intervals $(-\infty, -1)$, $(-1, 3)$, $(3, \infty)$.
- At $x_0 = 4$, the radius of convergence is at least 1 ($\rho \geq 1$).
  At $x_0 = -4$, the radius of convergence is at least 3 ($\rho \geq 3$).
  At $x_0 = 0$, the radius of convergence is at least 1 ($\rho \geq 1$).

Another technique for solving differential equations (by series) is by only calculating a few terms of the solution.

Example

Evaluate four terms of the solution to $(\cos x)y'' + xy' - 2y = 0$.

Rewriting, we get $\left(1 - \dfrac{x^2}{2} + \cdots\right)\left(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^5 + \cdots\right) + x\left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots\right)$
$- 2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) = 0$ $\Rightarrow$

$\left[\left(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^5 + \cdots\right) - \left(a_2 x^2 + 3a_3 x^3 + \cdots\right)\right] + \left[a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + \cdots\right]$
$- \left[2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + \cdots\right] = 0$ $\Rightarrow$

$(2a_2 - 2a_0) + (6a_3 - a_1)x + (12a_4 - a_2)x^2 + (20a_5 - 2a_3)x^3 + \cdots = 0 \Rightarrow \begin{cases} -a_0 + a_2 = 0 \\ -a_1 + 6a_3 = 0 \\ -a_2 + 12a_4 = 0 \\ -a_3 + 10a_5 = 0 \end{cases}$. Solve in terms of a_0 and a_1.

System of First Order Linear Equations

INTRODUCTION

A system of differential equation is like $\begin{cases} x_1' = F_1(t, x_1, x_2, x_3) \\ x_2' = F_2(t, x_1, x_2, x_3) \\ x_3' = F_3(t, x_1, x_2, x_3) \end{cases}$.

Example

Consider the system $\begin{cases} x_1' = 2x_1 - x_2 + \cos t \\ x_2' = 3x_1 \end{cases}$. Such a system is written as $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \cos t \\ 0 \end{bmatrix}$ or $X' = AX + g(t)$, which is a first order differential equation.

Example

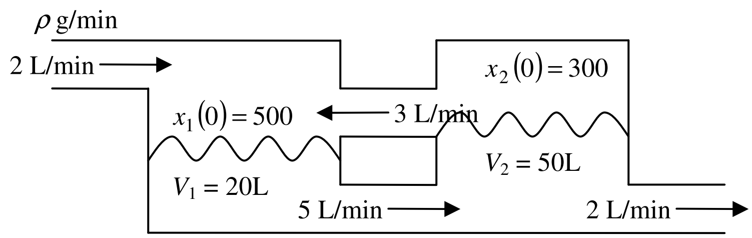
We can use a system approach to solve any differential equation of any order. Consider $u''' + 2u' - u = 1$. Let $\begin{cases} x_1 = u \\ x_2 = x_1' = u' \\ x_3 = x_2' = u'' \end{cases}$, then $\begin{cases} x_1' = x_2 \\ x_2' = x_1 \\ x_3' = u''' = u - 2u' + 1 = x_1 - x_2 + 1 \end{cases}$. So we have $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ or $X' = AX + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Example

Conversely, we one can translate a system to an ordinary differential equation. Consider $\begin{cases} x_1' = 2x_2 \\ x_2' = -2x_1 \end{cases}$ where $\begin{cases} x_1(0) = 3 \\ x_2(0) = 4 \end{cases}$. We have $\begin{cases} u = x_1 \\ u' = x_1' = 2x_2 \\ u'' = 2x_2' = -4x_1 = -4u \end{cases} \Rightarrow u'' + 4u = 0$ where $\begin{cases} u(0) = 3 \\ u'(0) = 2x_2(0) = 8 \end{cases}$.

Applications in Modeling

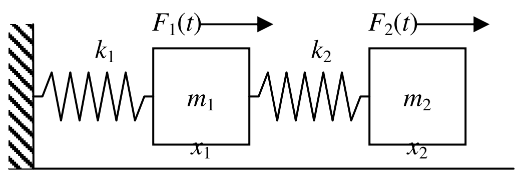
Example: A Double Tank Mixing Problem



$\begin{cases} \dfrac{dx_1}{dt} = 2\rho + 3\dfrac{x_2(t)}{V_2} - 5\dfrac{x_1(t)}{V_1} \\ \dfrac{dx_2}{dt} = 5\dfrac{x_1(t)}{V_1} - 5\dfrac{x_2(t)}{V_2} \end{cases}$, so we have $\begin{cases} x_1' = 2\rho + -\dfrac{5}{V_1}x_1 + \dfrac{3}{V_2}x_2 \\ x_2' = \dfrac{5}{V_1}x_1 - \dfrac{5}{V_2}x_2 \end{cases}$ where $\begin{cases} x_1(0) = 500 \\ x_2(0) = 300 \end{cases}$. So

$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -\frac{5}{20} & \frac{3}{50} \\ \frac{5}{20} & -\frac{5}{50} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2\rho \\ 0 \end{bmatrix} \Leftrightarrow X' = \begin{bmatrix} -\frac{1}{4} & \frac{3}{50} \\ \frac{1}{4} & -\frac{1}{10} \end{bmatrix} X + \begin{bmatrix} 2\rho \\ 0 \end{bmatrix}$ where $X(0) = \begin{bmatrix} 500 \\ 300 \end{bmatrix}$.

Example: Double Mass-Spring Problem



The variables are x_1 and x_2. So $\begin{cases} m_1 \dfrac{d^2x_1}{dt^2} = F_1(t) - k_1x_1 - k_2(x_2 - x_1) \\ m_2 \dfrac{d^2x_2}{dt^2} = F_2(t) - k_2(x_2 - x_1) \end{cases}$.

Eigenvalues and Eigenvectors

Recall

Two solutions y_1 and y_2 to a differential equation are independent on an interval I if $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$ on I (where I contains the initial values).

For systems $X' = AX$ we say solution $X^{(1)}(t) = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix}$, $X^{(2)}(t) = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix}$, $X^{(3)}(t) = \begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix}$ are independent on I if $W\left(X^{(1)}, X^{(2)}, X^{(3)}\right) = \begin{vmatrix} X^{(1)} & X^{(2)} & X^{(3)} \end{vmatrix} \neq 0$.

Recall

If r is an eigenvalue of A and ξ is the corresponding eigenvectors, i.e. $A\xi = r\xi$, then in particular

$$X' = AX = rX \Rightarrow \begin{cases} x_1' = rx_1 \\ x_2' = rx_2 \end{cases} \Rightarrow \begin{cases} x_1(t) = ae^{rt} \\ x_2(t) = be^{rt} \end{cases} \text{ or } X = e^{rt}\begin{bmatrix} a \\ b \end{bmatrix}.$$

Now, if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector then $x_2 = \alpha x_1$, then $be^{rt} = \alpha ae^{rt}$.

Notice that we may have two independent eigenvectors for A. In this case, two solutions in two independent directions can be found so that if $\xi^{(1)}$ and $\xi^{(2)}$ are independent eigenvectors of A corresponding to eigenvalues r_1 and r_2, then we have two independent solutions $\begin{cases} X^{(1)} = e^{r_1 t}\xi^{(1)} \\ X^{(2)} = e^{r_2 t}\xi^{(2)} \end{cases}$ and the general solution is $X(t) = c_1 X^{(1)}(t) + c_2 X^{(2)}(t)$.