Graphs and Subgraphs

Definition: Graph

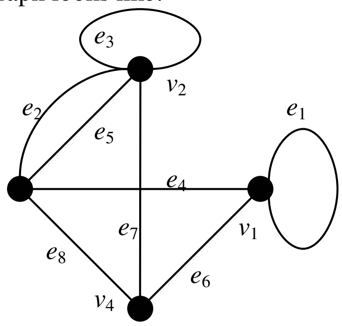
A graph G is a triple $\left(V(G), E(G), \psi_G\right)$ where

1) $V(G)$ is a finite set. The elements of $V(G)$ are called the vertices of G.
2) $E(G)$ is a finite set. The elements of $E(G)$ are called the edges.
3) ψ_G is a function which associates to every edge of an unordered pair of not necessarily distinct vertices. ψ_G is called the incidence function.

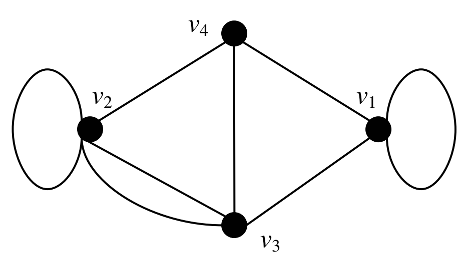
Example

Let $V(G) = \{v_1, v_2, v_3, v_4\}$.
Let $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$.
Let $\psi_G(e_1) = v_1 v_1$, $\psi_G(e_2) = v_3 v_2$,
$\psi_G(e_3) = v_2 v_2$, $\psi_G(e_4) = v_1 v_3$,
$\psi_G(e_5) = v_2 v_3$, $\psi_G(e_6) = v_4 v_1$,
$\psi_G(e_7) = v_2 v_4$, $\psi_G(e_8) = v_3 v_4$.

The graph looks like:



Could have drawn it like this:

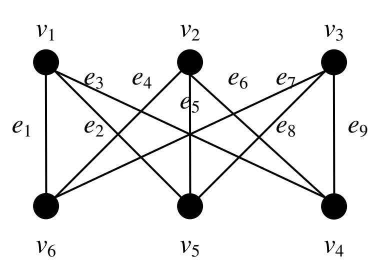


Definition: Planar

If a graph G can be drawn such that edges only intersect at their ends, then G is called planar.

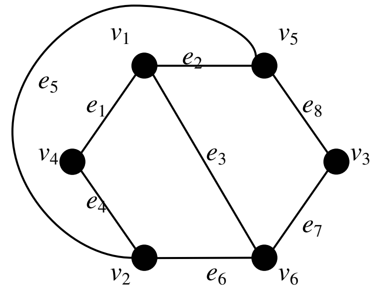
Example

Consider $k_{3,3}$. Prove that it's not planar.



$k_{3,3}$

Cycle-Chord technique. Observe that $k_{3,3}$ contains a cycle which covers every vertex: $v_1 e_1 v_4 e_4 v_2 e_6 v_6 e_9 v_3 e_8 v_5 e_2 v_1$.



There is nowhere to place e_9 !

Vocabulary

1) If $\psi(e) = uv$, then u and v are the ends of e.
2) e is said to join u and v.
3) u and v are said to be incident to the edge.
4) u and v are said to be adjacent.
5) If $u = v$, then e is said to be a loop.
6) If $u \neq v$, then e is said to be a link.

Definition: Simple

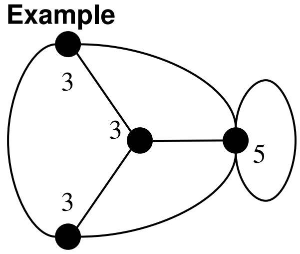
A graph is said to be simple if it has no loops and no two edges join the same pair of vertices.

VERTEX DEGREES

Definition: Vertex Degrees

The degree of a vertex v of a graph G, denoted $d_G(v)$, is defined to be the number of edges that end at the vertex (loops count twice).

Example



Fact

$$\sum_{v \in V(G)} d_G(v) = 2\varepsilon$$, where ε is the number of edges.

Proof:

- To begin, cut each edge into two pieces. Count the resulting half-edges in two different ways.

- Count 1: Label every half edge by the vertex it ends at. The number of edges labeled with some vertex v is clearly $d_G(v)$. So the number of half-edges is $\sum_{v \in V(G)} d_G(v)$.
- Count 2: There are clearly twice as many half-edges as edges.

Fact

In a graph, the number of vertices with odd degree is even.

Proof:

- Let $V(G)\text{even} \le V(G)$ be the set of vertices of G with even degrees. Similarly define $V(G)\text{odd} \le V(G)$.
- $2\varepsilon = \sum_{v \in V(G)} d_G(v) = \sum_{v \in V(G)\text{even}} d_G(v) + \sum_{v \in V(G)\text{odd}} d_G(v)$, so $\sum_{v \in V(G)\text{odd}} d_G(v)$ must be even, so the number of vertices with odd degree must be even.

Problem

1) In a group of 8 persons, is it possible that everyone knows exactly 3 others? Yes.
2) In a group of 7 persons, is it possible that everyone knows exactly 3 others? No.

APPLICATION: THE MOUNTAIN CLIMBERS PUZZLE

Consider 2 mountaineers approaching a mountain range from opposite sides. Is it possible for the climbers to always be at the same height and reach the summit?

Theorem

Two mountaineers can indeed clime in the required fashion to the summit of any mountain range.

Proof:

Construct a graph (the ascent graph) with vertices corresponding to configurations of the climbers where:

1) The two climbers are at the same height.
2) There is one climber on each side of the summit.
3) At least one of the climbers is at a local maxima or minima.

Join the vertices by an edge when the climbers can move between the corresponding configurations by strictly ascending or strictly descending together.

We must show that no matter for what mountain range, the corresponding ascent graph G has a path from the initial configuration (A, Z) to the summit (M, M).

We'll assume no such path exists and produce a contradiction. The proof follows from two observations on G.

Observation 1: G has exactly 2 vertices of degree 1 (namely, the initial configuration and the summit), and the rest are either degree 2 or 4 (2 when one is on a local max/min, 4 when both are on local max/min at the same height).

Observation 2: Let $V_{(A,Z)} \in V(G)$ denote the set of vertices that can be reached by a path from (A, Z). Note that the summit $(M, M) \notin V_{(A,Z)}$. It is easy to see that there are no edges in G with one end in $V_{(A,Z)}$ and one end in $V(G) \setminus V_{(A,Z)}$.

But Observation 1 now implies the graph with the vertices in $V_{(A,Z)}$ is a graph with exactly one odd vertex, which is absurd.

PATH ON GRAPHS

Definition: Walk
Let *v* and *w* be vertices of a graph $G = (V(G), E(G), \psi_G)$. A walk of length *k* from *v* to *w* on *G* is a sequence $v_0 e_1 v_1 \cdots v_{k-1} e_k v_k$ where $\psi_G(e_i) = v_{i-1}v_i$, $v_0 = v$, $v_k = w$, $v_i \in V(G)$, $e_i \in E(G)$.

Definition: Trail
A trail is a walk in which no edge appear twice.

Definition: Path
A path is a trail in which no vertex appear twice.

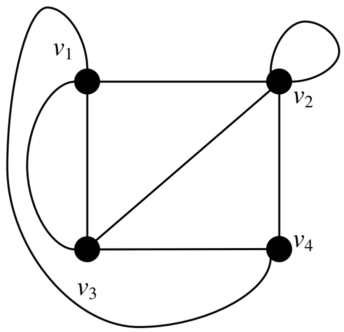
Fact
Let $G = (V(G), E(G), \psi_G)$ be a graph with ν vertices, where $V(G) = \{v_1, \ldots, v_\nu\}$. Let $W(k)_{ij}$ be the number of walks of length *k* from v_i to v_k. Then $W(k)_{ij} = \sum_{m=1}^{\nu} W(k-1)_{im} W(1)_{mj}$.

Fact
$W(k)_{ij} = \sum_{m=1}^{\nu} W(k-n)_{im} W(n)_{mj}$.

Matrix Interpretation
Let $W(k)$ be the matrix whose entry in row *i* and column *j* is $W(k)_{ij}$. Then $W(k) = W(k-n)W(n), 1 \le n \le k-1$.

Fact
$W(k) = W(1)^k$.
Notation: $W(1)$ is the adjacency matrix, denoted $A(G)$.

Example



The adjacency matrix is $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

DIJKGRA'S SHORTEST PATH ALGORITHM

Definition: Edge Weighted Graph

An edge-weighted graph is a graph $G = (V(G), E(G), \psi_G)$ with weight function $W : E(G) \to \{r \in \mathbf{R} \mid r > 0\}$.

Definition: Weight of Path

The weight of a path $v_0 e_1 v_1 \cdots v_{k-1} e_k v_k$ is $\sum_{j=1}^{k} W(e_j)$. If we interpret weight as a distance, this is just the length of a path.

Definition

Define $d(u, v)$ to be the minimum weight of the paths from u to v.

Basic Idea

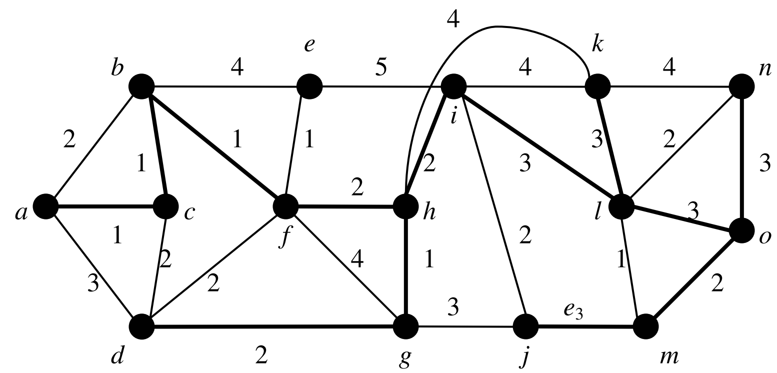
Dijkgra's Algorithm takes $r-1$ steps to determine a shortest path to a fixed point T for every vertex of the graph. Just say there is a subset $S \subset V(G)$ which are "solved" (for which we know the answer). Dijkgra observed a simple procedure for enlarging S.

The Algorithm

1) Let S denote the set of vertices for which we have already calculated the shortest path.
2) For $v \in S$, let P_v denote the shortest path to the origin o.
3) Let ∂S be the edges with one end in S (denoted $s(e)$) and one end out of S (denoted $u(e)$).
4) Select the edge $e \in \partial S$ which minimizes $W(e) + d(s(e), o)$. Then $u(e)P_{s(e)}$ is a shortest path from $u(e)$ to o.

Problem

What is the shortest path from o to a?



Proof of Dijkgra's Shortest Path Algorithm

- Just say e_b minimizes $W(e)+d(s(e),o)$.
- Why is $u(e_b)P_{s(e_b)}$ the shortest path from $u(e_b)$ to *o*? Well…just say there is a shorter path *P* from $u(e_b)$ to *o*.
- What does *P* have to look like? There are two possibilities:
  - Case 1: *P*'s first step is into *S* (ex: $P=u(e_c)P_{s(e_c)}$). This implies that $W(e_c)+d(s(e_c),o)<W(e_b)+d(s(e_b),o)$, but we chose e_b to minimize $W(e)+d(s(e),o)$. So this is impossible.
  - Case 2: *P*'s first step is into $V \setminus S$ (ex: $P=P'P_{s(e_a)}$ where P' is a path in $V \setminus S$ from $u(e_b)$ to $u(e_a)$). This implies that $W(e_a)+d(s(e_a),o)<$ length of $P<W(e_b)+d(s(e_b),o)$, but we chose e_b to minimize $W(e)+d(s(e),o)$. So this is impossible.

Subgraph Operations

Definition: Subgraph

Let $G=(V(G),E(G),\psi_G)$ and $H=(V(H),E(H),\psi_H)$ be graphs. *H* is said to be a subgraph of *G* if:

- $V(H)\subset V(G)$,
- $E(H)\subset E(G)$,
- ψ_H is just ψ_G restricted to $E(H)$.

Definition: Proper Subgraph

H is a proper subgraph if $H \neq G$.

Definition: Spanning Subgraph

H is a spanning subgraph if $V(H)=V(G)$.

Definition: Induced Subgraph

Let $G=(V(G),E(G),\psi_G)$ be a graph and let V' be a non-empty subset of $V(G)$. The subgraph of *G* induced by V' is the graph whose vertices are V' and whose edges are those with ends in V'.
Notation: $G[V']$.

Definition
Let $V' \subset V(G)$ where $V' \neq V(G)$. Define $G - V' := G[V(G) \setminus V']$.

ISOMORPHISM

Definition: Isomorphism
An isomorphism between graphs $G = (V(G), E(G), \psi_G)$ and $H = (V(H), E(H), \psi_H)$ is a pair of functions (θ, ϕ) where $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that:

- θ is a bijection (1-1 and onto),
- ϕ is a bijection (1-1 and onto),
- If $\psi_G(e) = uv$, then $\psi_H(\phi(e)) = \theta(u)\theta(v)$.

Definition: Simple Isomorphism
Two simple graphs $G = (V(G), E(G), \psi_G)$ and $H = (V(H), E(H), \psi_H)$ are simple isomorphisms if there is a bijection $\theta : V(G) \to V(H)$ such that *u* and *v* are joined by an edge in *G* iff $\theta(u)$ and $\theta(v)$ are joined by an edge in *H*.

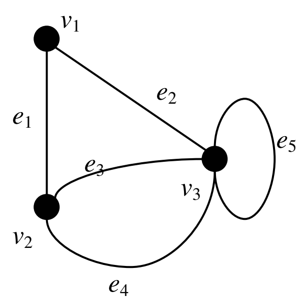
Notes
1) To prove $G \cong H$, we must exhibit an isomorphism.
2) To prove $G \not\cong H$, we either:
   - Test all possible isomorphisms, or
   - Find a property of *G* that is invariant under isomorphism not held by *H*.

Technique: Two Tools For Studying Isomorphisms
Let $G = (V(G), E(G), \psi_G)$ and $H = (V(H), E(H), \psi_H)$ be graphs, and (θ, ϕ) be an isomorphism from *G* to *H*. Then:
1) V and $\theta(V)$ have the same degrees.
2) Let $W \subset V(G)$ and $\theta(W) = \{v \in H \mid v \in \theta(W), w \in V(H)\}$. Then $G[W] \cong H[\theta(W)]$.

CYCLE AND TREES

Definition: Cycle
A cycle in a graph *G* is a walk $v_0 e_1 v_1 \cdots e_k v_k$ such that $e_i \neq e_j \forall i \neq j$ and $v_m = v_n \Leftrightarrow \{m, n\} = \{0, k\}$.

Example



- $v_3e_3v_2e_1v_1e_2v_3$ is a cycle.
- $v_3e_5v_3$ is a cycle.
- $v_3e_3v_2e_4v_3$ is a cycle.
- $v_3e_5v_3e_3v_2e_4v_3$ is a not cycle; it's a closed walk.

Definition: Connected

A graph *G* is connected iff for every pair $\{u, v\}$ there is a path in *G* from *u* to *v*.

Definition: Tree

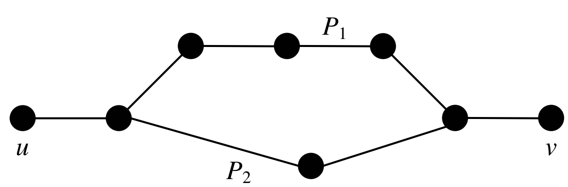
A graph is a tree if:

1) It is connected.
2) It contains no cycles ("acyclic").

Theorem

Let *u* and *v* be vertices of a tree *T*. There is one and only one path from *u* to *v*.

Proof:



If there were two different paths from *u* to *v* then *T* would have a cycle.

Theorem

Let *T* be a tree. Then $\varepsilon(T) = \nu(T) - 1$.

Proof:

It is obviously true for a tree with one vertex.

Assume this is true for all trees with less than ν vertices. Let *e* be an edge of *T*. Note that $T - e$ is disconnected. $T - e$ has two components, T_1 and T_2, both of which are trees. Then

$\varepsilon(T) = \varepsilon(T - e) + 1 = \varepsilon(T_1) + \varepsilon(T_2) + 1 = \nu(T_1) - 1 + \nu(T_2) - 1 + 1 = \nu(T) - 1$.

Definition: Spanning Tree

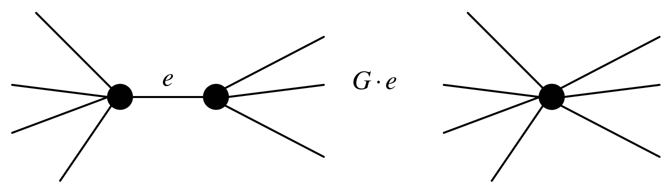
A spanning tree of a graph is a spanning subgraph which is a tree.

Theorem

Let *T* be a spanning tree of a graph *G* and let *e* be an edge of *G* not in *T*. Then $T + e$ has one and only one cycle.

Definition: Tree Number
The tree number of a graph is the number of spanning trees it has, denoted by $\tau(G)$.

Definition: Contraction
Let G be a graph and let e be an edge of G. The contraction of G along e, denoted $G \cdot e$ is defined by:



Cayley's Formula
Let G be a graph and let e be an edge of G. Then $\tau(G) = \tau(G-e) + \tau(G \cdot e)$.

SPANNING TREES OF K_N

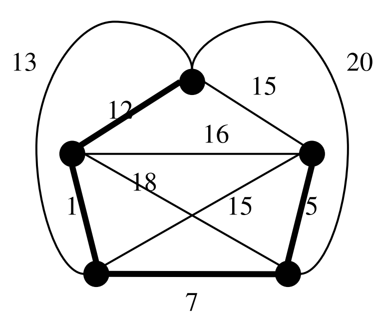
K_N is the simple graph with vertices $\{1, \ldots, N\}$ and an edge between ever pair of vertices.

Theorem
$\tau(K_N) = N^{N-2}$.

Kruskal's Algorithm: Minimal Weight Spanning Tree
Let G be a connected weighted graph with ν vertices.

1) Let e_1 be the edge with the smallest weight.
2) Assume we have already chosen $\{e_1, \ldots, e_{n-1}\}$. Let $E^{(n)} = \{e \in E(G) \setminus \{e_1, \ldots, e_{n-1}\} G[\{e_1, \ldots, e_{n-1}, e\}] \text{acyclic} \mid \}$.
   Let e_n be the edge in $E^{(n)}$ of smallest weight.
3) Continue for $\nu - 1$ steps.

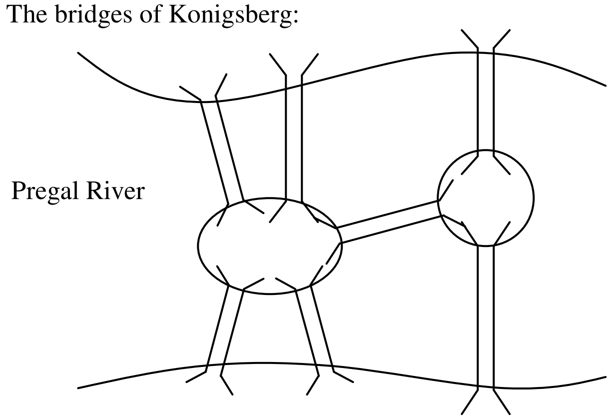
Example
A minimal weight spanning tree:



EULER TOURS

Problem: Start of Graph Theory

The bridges of Konigsberg:



Can you walk around town such that you cross every bridge only once?

Definition: Euler Tour

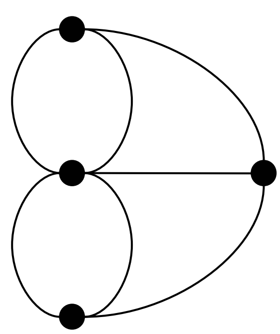
An Euler tour of G is a closed trail that visits every edge.

Theorem

A connected graph has an Euler tour iff every vertex has even degree.

Solution to Problem

A graph representation of the map:



Since not every edge has an even degree, there is no Euler tour.

Construction of Euler Tour: Smooth and Splice

We can decompose the set of edges into closed trails. Then we can splice them together.

Construction of Euler Tour: Fleury's Algorithm

Assume that $v_0 e_1 v_1, \ldots, e_j v_j$ chosen. For e_{j+1}, choose an edge only from $E(G) - \{e_1, \ldots, e_j\}$ with the proviso that you don't choose an edge which connects $G[E(G) - \{e_1, \ldots, e_j\}]$ unless you have to.

Fact

This algorithm solves a more general problem: Let *G* be a connected graph with one vertex labeled *o* (origin) and one labeled *d* (destination). If $o = d$, then *o* and *d* are of even degree; if $o \neq d$, then *o* and *d* are of odd degree. All other vertices are even.
If $o = d$, then there is an Euler Tour from *o* to *d*.
If $o \neq d$, then there is an Euler Trail from *o* to *d*.

Hamiltonian Cycles

Definition: Hamiltonian Cycle

A Hamiltonian cycle is a cycle which visits every vertex of a graph *G*.

Theorem

Let *G* be a connected graph. If $\deg(u) + \deg(v) \geq \nu$ for every pair of non-adjacent vertices *u* and *v*, then *G* has a Hamiltonian Cycle.

Techniques

Sometimes, the following three rules are enough:

1) If a vertex has two edges (but not a loop), then a Hamiltonian Cycle must use the edges connected to this vertex.
2) If a vertex is already used by two edges, then we can ignore the other edges at that vertex.
3) Edges which finish cycles but don't span can be ignored.

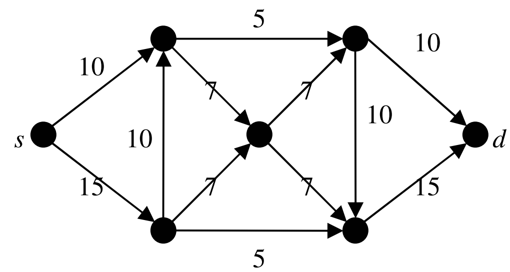
Networks and Flows

Definition: Network

A network $N = (V(N), A(N), \psi_N, c)$ is a diagraph (directed graph) where:

- $V(N)$ is the set of vertices of *N*.
- $A(N)$ is the set of arcs of *N*.
- $\psi_N : A(N) \to V(N) \times V(N)$ is an incident function which associates each arc with two vertices with a direction.
- $c : A(N) \to \mathbf{N}$ is a function which assigns each arc with a capacity.

Example



Definition: Flow

A flow on a network N is a function $f: A(N) \to \{0,1,2,\ldots\}$ such that:

1) For every arc a $f(a) < c(a)$.
2) For every vertex v $f^{+}(\{v\}) = f^{-}(\{v\})$ (conservation condition), except s and d.

Notation/Vocabulary

1) If $S_1, S_2 \subset V(N)$, then (S_1, S_2) denote the set of arcs originating in S_1 and ending in S_2.
2) If $S \subset V(N)$, then $\overline{S} = V(W) \setminus S_1$ (i.e. the set of vertices not in S).
3) If $S \subset V(N)$, then $f^{+}(S) - f^{-}(S)$ is the resultant flow out of S.

Fact

Let f be a flow on the network N. Take $S_1, S_2 \subset V(N)$ such that $s \in S_1, S_2$ and $d \notin S_1, S_2$. Then $f^{+}(S_1) - f^{-}(S_1) = f^{+}(S_2) - f^{-}(S_2)$.

Proof: Consider the case where S_1 and S_2 differ by a single vertex v.
$f^{+}(S_2) = f^{+}(S_1) + \sum_{a \in (v, \overline{S}_1)} f(a) - \sum_{a \in (S_1, v)} f(a)$ and $f^{-}(S_2) = f^{-}(S_1) + \sum_{a \in (\overline{S}_1, v)} f(a) - \sum_{a \in (v, S_1)} f(a)$. So
$f^{+}(S_2) - f^{-}(S_2) = \left(f^{+}(S_1) - f^{-}(S_1)\right) + \left(\sum_{a \in (v, \overline{S}_1)} f(a) + \sum_{a \in (v, S_1)} f(a)\right) - \left(\sum_{a \in (\overline{S}_1, v)} f(a) + \sum_{a \in (S_1, v)} f(a)\right)$, but
$\sum_{a \in (v, \overline{S}_1)} f(a) + \sum_{a \in (v, S_1)} f(a) = f^{+}(\{v\})$ and $\sum_{a \in (\overline{S}_1, v)} f(a) + \sum_{a \in (S_1, v)} f(a) = f^{-}(\{v\})$, so
$f^{+}(S_2) - f^{-}(S_2) = f^{+}(S_1) - f^{-}(S_1)$.

Definition: Value of a Flow

The value of a flow f is $f^{+}(S) - f^{-}(S), S \subset V(N), s \in S, d \notin S$.

The Bottleneck Principle

Take $S \subset V(N)$ with $s \in S$ and $d \notin S$; $(S, \overline{S})$ is called a cut of the network. A flow on a network N has value less than or equal to the capacity of any cut.

Reason: $\left.\begin{array}{l} f^+(S) \le \sum\limits_{a \in (S,\bar{S})} c(a) \\ f^-(S) \ge 0 \end{array}\right\} \Rightarrow \max\left(f^+(S) - f^-(S)\right) = \sum\limits_{a \in (S,\bar{S})} c(a)$.

Theorem
Let N be a network. Let m denote the minimum capacity of a cut. Then there exists a flow with value m.

INCREASING FLOWS

How to maximize a flow?

Principle
Increase the flow of an arc and restore the conservation at a vertex by adjusting exactly one other arc at that vertex.

Definition: *f*-Incremental Path
Take a flow on a network N and take a path P from s to d. Define the potential increment along a with respect to P to be $\iota_p(a) = \begin{cases} c(a) - f(a), \text{if } a \text{ has same direction as } P \\ f(a), \text{if } a \text{ has opposite direction as } P \end{cases}$. Define the potential increment along P to be $\iota(P) = \min(\{\iota_P(a) \mid a \text{ is an arc of } P\})$. If $\iota(P) > 0$, it is called an *f*-incrementing path.

Algorithm: Find Maximum Flow
We have in memory some flow *f*.
1) If there exists an *f*-incrementing path, use it to increase the flow. Then start again.
2) If no such path exists, stop.

Theorem
A network N has a flow whose value is the minimum capacity of a cut.

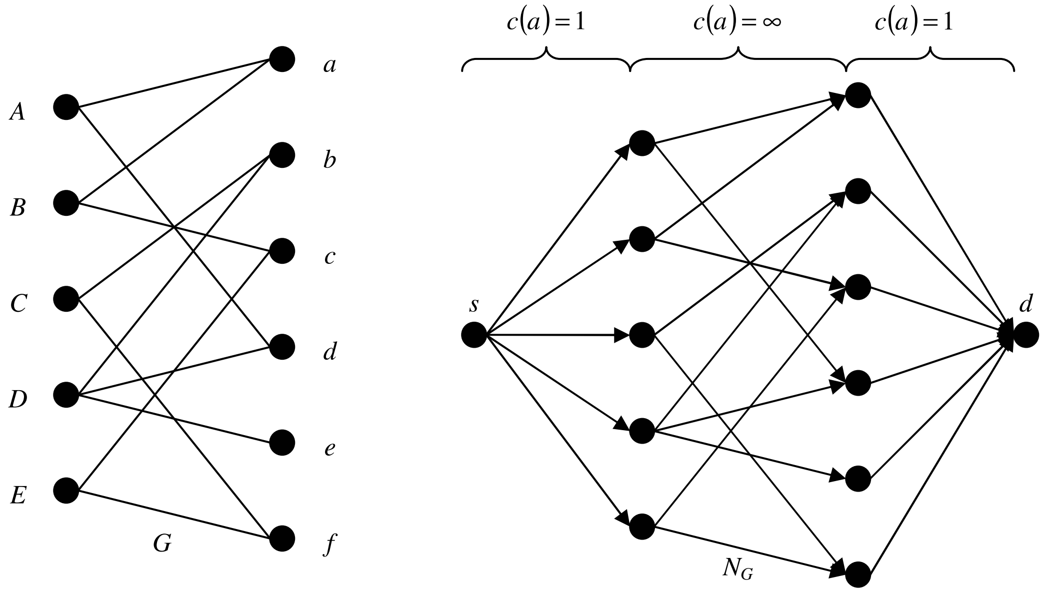
APPLICATION: MATCHING

Definition: Matching
Let G be a bipartite graph. A matching is a set of edges no two of which share an end.

Example: A Typical Problem
Just say there are five graduates $A, \ldots, E$ and six advertised jobs $a, \ldots, f$. Assume
- A is qualified for a and d.
- B is qualified for a and c.
- C is qualified for b and f.
- D is qualified for b, d, and e.
- E is qualified for c and f.

Can they all get a job?

To find a maximal matching, construct from a bipartite graph G a matching network N_G.



Fact

Let G be a bipartite graph and N_G be the associated matching network. There exists a bijection $\phi : \{\text{flows on } N_G\} \to \{\text{matchings on } G\}$; given f, define $\phi(f)$ to be the set of edges of G with $f > 0$. Furthermore, the number of edges in the matching $\phi(f)$ equals to the value of the flow.

Definition: Edge Cover

Let G be a bipartite graph. A subset S of vertices is said to be an edge cover of G if every edge has at least one end in S.

Fact

Let $G = (X, Y)$ be a bipartite graph, and let N_G denote the associated network. There exists a bijection $\psi : \{\text{edge covers of } G\} \to \{\text{sets of vertices } S \subset (N_G) \text{ such that } s \in S, d \notin S, \operatorname{cap}(S) < \infty\}$ given by $\psi(E) = S \cup (X \setminus E) \cup (Y \cap E)$ such that $\operatorname{cap}(\psi(E)) = |E|$ where E is an edge cover.

Theorem: König's Theorem

The size of a maximal matching equals the size of a minimal edge cover.

Definition: Range

Let $G = (X, Y)$ be a bipartite graph. Let $J \subset X$. The range of J, denoted $R(J)$, is $R(J) := \{v \in Y \mid uv \text{ is an edge for some } u \in J\}$.

Definition: *X*-complete Matching

An X-complete matching is a matching such that every vertex in X is an end of an edge in the matching.

Theorem: Hall's Marriage Theorem

The following is a necessary and sufficient condition for a bipartite graph $G = (X, Y)$ to have an X-complete matching: $|R(J)| \geq J, \forall J \subset X$.

Corollary: Hall's Marriage Theorem

Let $G = (X, Y)$ be a bipartite graph which is k-regular (every vertex has the same degree k). G has an X-complete matching.

LATIN SQUARES

Definition: Latin Square

An $n \times n$ matrix is a Latin square iff every element of $\{1, \ldots, n\}$ appears in every row and column.

Example

$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}$ is a Latin square. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ & & & & \\ & & & & \end{bmatrix}$ is a partial Latin square.

Basic Problem

Can every partial Latin square be completed to a full Latin square?
We'll try to complete it one row at a time. The problem of adding a single row is an example of a System of Distinct Representation (SDR) problem.

General SDR Problem

Take a finite set S, and subsets $S_i \subset S, 1 \leq i \leq n$. Choose elements $e_i \in S$ such that for all $e_i \in S_i, 1 \leq i \leq n$ $e_i = e_j \Leftrightarrow i = j$.

Fact

Every partial Latin square can be completed.
Just say we have k rows of an $n \times n$ Latin square. We have to solve the following SDR problem. Let $S = \{1, \ldots, n\}$, let S_i be the elements of $\{1, \ldots, n\}$ not appearing in column i.
Observation 1: Each S_i contains $n - k$ elements.
Observation 2: Each element appears in $n - k$ of the S_i's.
This fits into the Hall's Marriage Theorem Corollary.

Basic Enumeration

Fact

The number of ways of ordering n distinct elements is $n!$.

Definition

$\binom{n}{k}$ denotes the number of k element subsets of a set S of n distinct objects.

Theorem

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Pascal's Triangle

$$\begin{array}{ccccccc} & & & \binom{0}{0} & & & \\ & & \binom{1}{0} & & \binom{1}{1} & & \\ & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\ \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\ & & & \vdots & & & \end{array} \quad \text{gives} \quad \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & & & & \vdots & & & & & \end{array}.$$

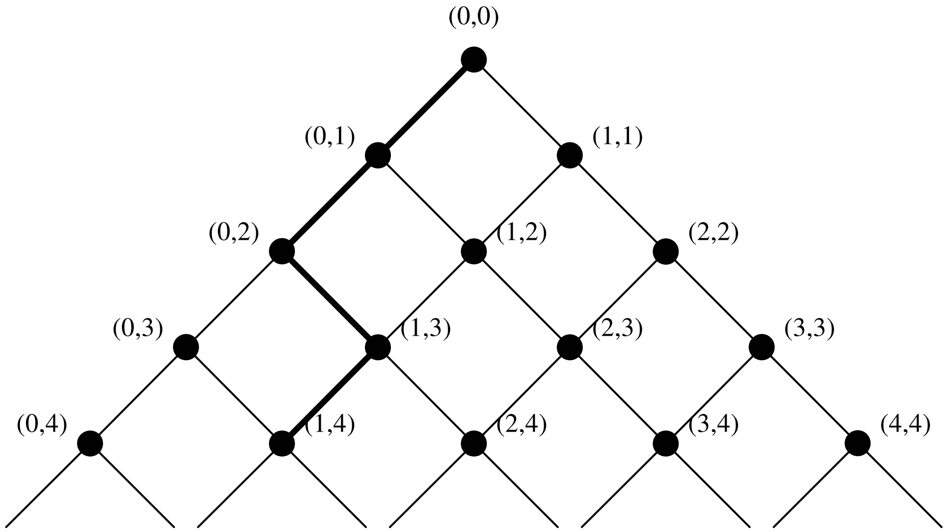
Note that:

1) $\binom{n}{0} = 1$.
2) $\binom{n}{n} = 1$.
3) $\binom{n}{k}, k \neq 0, n$ is the sum of the two numbers directly above it.

Descending Walk Model

Pascal Graph

The following is called the Pascal graph, with a descend walk from $(0,0)$ to $(1,4)$.



Fact

$\binom{n}{k}$ is the number of descending walks from $(0,0)$ to (k,n).

THE BINOMIAL THEOREM

Theorem: Binomial Theorem

$$(1+t)^n = \sum_{m=0}^{n} \binom{n}{m} t^m \text{ .}$$

Proof 1: $(1+t)^n = \sum_{i=0}^{n} \left| \left\{ \begin{matrix} \text{Words that can be formed using} \\ i \text{ copies of } t \text{ and } n\text{-}i \text{ copies of } 1 \end{matrix} \right\} \right| t^i = \sum_{i=0}^{n} \binom{n}{i} t^i$.

Proof 2:
$$\begin{aligned}(1+t)^n &= (1+t)(1+t)^{n-1} = (1+t)\sum_{i=0}^{n-1}\binom{n-1}{i}t^i = \sum_{i=0}^{n-1}\binom{n-1}{i}t^i + \sum_{i=0}^{n-1}\binom{n-1}{i}t^{i+1} \\ &= \sum_{i=0}^{n}\binom{n-1}{i}t^i + \sum_{i=1}^{n}\binom{n-1}{i-1}t^i = \sum_{i=0}^{n}\left(\binom{n-1}{i-1}+\binom{n-1}{i}\right)t^i = \sum_{i=0}^{n}\binom{n}{i}t^i\end{aligned}.$$

Fact

A set with n elements has 2^n subsets because $\left| \left\{ \begin{matrix} \text{Subsets of} \\ \{1,\ldots,n\} \end{matrix} \right\} \right| = \sum_{i=0}^{n} \left| \left\{ \begin{matrix} \text{Subsets} \\ \text{of size } i \end{matrix} \right\} \right| = \sum_{i=0}^{n} \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i} 1^i = 2^n$.

SELECTIONS

How many ways of selecting k elements from a set of n objects?

	Order Significant	Order Not Significant
Repetition Allowed	n^k	$\binom{n+k-1}{k-1}$
Repetition Not Allowed	$n(n-1)\cdots(n-k+1)$	$\binom{n}{k}$

GENERATING COMBINATORIAL OBJECTS

Problem

Subroutine to generate all simple graphs with n vertices. Let E_n denote the set of 2-element subsets of $\{1,\dots,n\}$. We need a bijection $\{\text{Simple graphs on } \{1,\dots,\text{n}\}\} \to \{\text{Subsets of } E_n\}$.

General Problem

A finite set $S=\{e_1,\dots,e_n\}$. Generate all subsets.

Method A: Recursive

Assume that $\{T_1,\dots,T_k\}$ are the subsets of $\{e_1,\dots,e_i\}$. Then $\{T_1,\dots,T_k,T_1\cup\{e_{i+1}\},\dots,T_k\cup\{e_{i+1}\}\}$ is the list of subsets of $\{e_1,\dots,e_i,e_{i+1}\}$.

Method B: Sequential

$T_1,\dots,T_{2^n}$ order list of the set we are generating. We need a procedure for constructing T_{i+1} from T_i.

1) Set $T_0=\varnothing$.
2) Now assume $T_i \in S=\{e_1,\dots,e_n\}$ has been defined. Find the largest m such that $e_m \notin T_i$. Define $T_{i+1}=(T_i\cup\{e_m\})\setminus\{e_{m+1},\dots,e_n\}$.

PERMUTATION

Definition: Permutation

A bijection $\phi:\{1,\dots,n\}\to\{1,\dots,n\}$ is called a permutation.

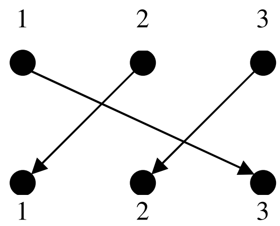
Problem

Is there a sequential construction of the permutations?

Example

$\phi(1)=3$, $\phi(2)=1$, $\phi(3)=2$.

$(3,1,2)$ is a passive representation.



is a graphical representation.

Note that there is a natural ordering. Take the passive representation and use the lexicographic ordering. $(1,2.3) < (1,3,2) < (2,1,3) < (2,3,1) < (3,1,2) < (3,2,1)$.

Claim

Let $(x_1, \ldots, x_n)$ be some permutation *p* of $\{1, \ldots, n\}$. The permutation p' which follows *p* with respect to lexicographic ordering is obtained as follows:

1) Let *j* be the largest number such that $x_j < x_{j+1}$.
2) Let $k > j$ such that x_k is the smallest possible satisfying $x_k > x_j$.
3) Swap x_k and x_j.
4) Now reverse the subsequence $x_{j+1}, \ldots, x_n$.

MULTINOMIAL COEFFICIENT

Definition: The Multinomial Coefficient

$\binom{n}{k_1, \ldots, k_m}$ denotes the number of ways of distributing *n* distinct objects into *m* labeled "boxes" such that box 1 gets k_1 objects,…, box *m* gets k_m objects.

Fact

$$\binom{n}{k_1, \ldots, k_m} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{k_m}{k_m}$$
$$= \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{k_2!(n-k_1-k_2)!}\frac{(n-k_1-k_2)!}{k_3!(n-k_1-k_2-k_3)!}\cdots\frac{k_m!}{k_m!(k_m-k_m)!} = \frac{n!}{k_1!\cdots k_m!}.$$

OR

Define $\psi : \left\{\begin{matrix}\text{orderings}\\ \text{of } \{1, \ldots, n\}\end{matrix}\right\} \to \left\{\begin{matrix}\text{distributions of } \{1, \ldots, \mathrm{n}\} \text{into } m \text{ labelled boxes}\\ \text{such that box } i \text{ gets } k_i \text{ elements, } i = 1, \ldots m\end{matrix}\right\}$. It is easy to see that ψ is $k_1!\cdots k_m!$-to-1.

Summary: $\binom{n}{k_1, \ldots, k_m} = \begin{cases}\dfrac{n!}{k_1!\cdots k_m!} \text{ if } k_1 + \cdots + k_m = n \\ 0 \text{ otherwise}\end{cases}$.

Theorem: Multinomial Theorem

$$(x_1+\cdots+x_m)^n = \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1,\ldots,k_m} x_1^{k_1}\cdots x_m^{k_m}\ .$$

Proof:

$$(x_1+\cdots+x_m)^n = (x_1+\cdots+x_m)\cdots(x_1+\cdots+x_m) = \sum_{\substack{\text{choices of}\\ \text{one element}\\ \text{from each set}}} \underbrace{\begin{Bmatrix} x_1\\ \vdots \\ x_m\end{Bmatrix}\cdots\begin{Bmatrix} x_1\\ \vdots \\ x_m\end{Bmatrix}}_{n\text{ copies}}$$

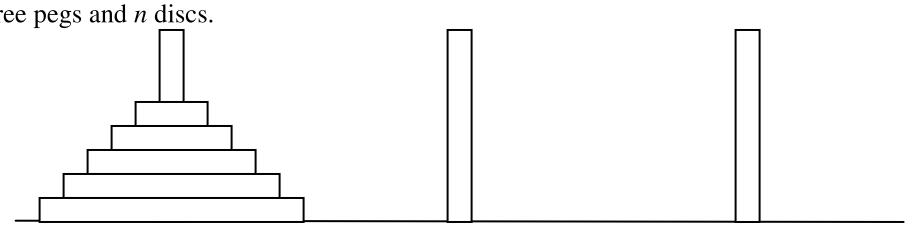
$$= \sum_{k_1+\cdots+k_m=n}\left(\sum_{\substack{\text{choices of one}\\ \text{element from each}\\ \text{set such that } x_1 \text{ is}\\ \text{chosen } k_i \text{ times}}} \underbrace{\begin{Bmatrix} x_1\\ \vdots \\ x_m\end{Bmatrix}\cdots\begin{Bmatrix} x_1\\ \vdots \\ x_m\end{Bmatrix}}_{n\text{ copies}}\right) = \sum_{k_1+\cdots+k_m=n}\left|\begin{Bmatrix}\text{words of length } n \text{ in}\\ \{x_1,\ldots,x_m\}\text{such that}\\ x_i \text{ is chosen } k_i \text{ times}\end{Bmatrix}\right|.$$

$$= \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1,\ldots,k_m} x_1^{k_1}\cdots x_m^{k_m}$$

Recursion Relations

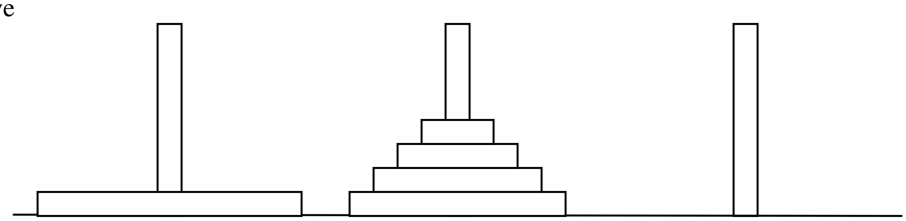
Example: Tower of Hanai

Consider three pegs and n discs.

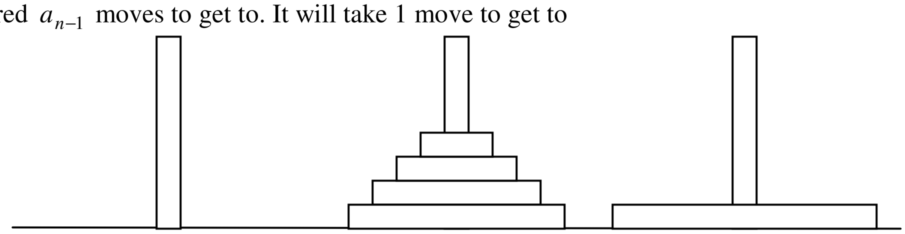


Must move all the disc to the last peg such that a bigger disc cannot be on top of a small disc on any peg at any time. Is there a way to solve the puzzle? If there is, what is the least number of moves?

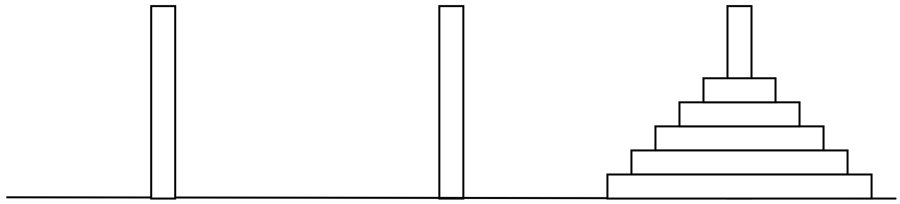
Assume there is a solution. Let a_n be the least number of moves required to solve the puzzle. At some point, we must have



which required a_{n-1} moves to get to. It will take 1 move to get to



and another a_{n-1} moves to get to



which is the solution. Thus, $a_n = 2a_{n-1} + 1$ is the recursion relation.

Now, $a_n = 2a_{n-1} + 1 = 2(2a_{n-2} + 1) + 1 = \cdots = 2^{n-1} a_1 + 2^{n-2} + 2^{n-3} + \cdots + 2^0$; since $a_1 = 1$, we have $a_n = 2^{n-1} + 2^{n-1} - 1 = 2^n - 1$.

Example: Fibonacci Sequence

Fix *n* a positive integer. How many ways are there to express *n* as an ordered sum of 1's and 2's? For example,

$$\begin{aligned} 4 &= 1+1+1+1 \\ &= 2+1+1 \\ &= 1+2+1 \quad \text{, so } F_4 = 5. \\ &= 1+1+2 \\ &= 2+2 \end{aligned}$$

$F_n = \begin{pmatrix} \text{\# of such} \\ \text{expressions} \end{pmatrix} = \begin{pmatrix} \text{\# of such expressions} \\ \text{ending in "1"} \end{pmatrix} + \begin{pmatrix} \text{\# of such expressions} \\ \text{ending in "2"} \end{pmatrix} = F_{n-1} + F_{n-2}$. This is the Fibonacci sequence.

What is F_n? We know $F_0 = 1$, $F_1 = 1$, $F_2 = 2$. Guess $F_n = \alpha^n$.

$F_n = F_{n-1} + F_{n-2} \Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2} \Rightarrow \alpha^{n-2}(\alpha^2 - \alpha - 1) = 0 \Rightarrow \alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$. Using the boundary conditions, we get $F_n = \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right)\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^n + \left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right)\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^n$.

CLASS

Linear Recursion

Take constants $c_1, \ldots, c_n$. An equation $a_n = c_1 a_{n-1} + \cdots + c_m a_{n-m} + f(n)$ is called a linear recursion with constant coefficients.

Homogeneous and Non-Homogeneous

Take a linear recursion $a_n = c_1 a_{n-1} + \cdots + c_m a_{n-m} + f(n)$. If $f(n) = 0$, the relation is homogenous. Otherwise, it is non-homogenous.

SOLVING RECURSION RELATIONS

Definition: Breadth

The homogeneous linear recursion relation with constant coefficients $a_n = c_1 a_{n-1} + \cdots + c_m a_{n-m}, c_m \neq 0$ has breadth *m*.

Note
The space of solutions of this recursion relation is a vector space of dimension equal to the breadth of the relation.

Definition: Characteristic Equation
The algebraic equation $\alpha^m - c_n \alpha^{m-1} - \cdots - c_m = 0$ is called the characteristic equation of the recursion relation.

Fact
If α is a solution to the characteristic equation, then α^n is a solution of the recursion relation.

Problem: Repeated Roots
When the characteristic equation has repeated roots the expression $\sum_{\text{roots } \alpha} k_\alpha \alpha^n$ does not give every solution.

The General Solution
Consider some homogeneous linear recursion relation with constant coefficients $a_n = c_1 a_{n-1} + \cdots + c_m a_{n-m}$. Just say the characteristic equation has roots:

- α_1 with multiplicity d_1,
- …
- α_p with multiplicity d_p.

In this situation, the general solution is $\sum_{i-1}^{p} \sum_{j=0}^{d_i-1} k_{ij} n^j (a_i)^n$.

Example
Solve $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ where $a_1 = 1$, $a_2 = 2$, $a_3 = 5$.

The characteristic equation is $\alpha^3 - 3\alpha^2 + 3\alpha - 1 = 0 \Leftrightarrow (\alpha - 1)^3 = 0$. So the general solution is $k_1 1^n + k_2 n 1^n + k_3 n^2 1^n$. Using the boundary conditions, $\begin{cases} a_1 = 1 = k_1 + k_2 + k_3 \\ a_2 = 2 = k_1 + 2k_2 + 4k_3 \\ a_3 = 5 = k_1 + 3k_2 + 9k_3 \end{cases}$, we get $a_n = 2 - 2n + n^2$.

Generating Functions

Definition: Generating Function
A sequence $\{c_i\}_{i=0}^{\infty}$ is often written as $c_0 + c_1 t + c_2 t^2 + \cdots$. This formal power series is called the generating function for the sequence.

The Ring of Formal Power Series

$\mathbf{C}[[t]]$ denotes the ring of formal power series with complex coefficients.

Addition $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is defined as $\left(\sum_{n=0}^{\infty} c_n t^n \right) + \left(\sum_{n=0}^{\infty} d_n t^n \right) = \sum_{n=0}^{\infty} (c_n + d_n) t^n$.

Multiplication $\cdot : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is defined as $\left(\sum_{n=0}^{\infty} c_n t^n \right) \cdot \left(\sum_{n=0}^{\infty} d_n t^n \right) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (c_i d_{n-i}) t^n$.

Example

The coefficient of t^5 in $\left(\sum_{n=0}^{\infty} t^n \right) \cdot \left(\sum_{n=0}^{\infty} n t^n \right) = \left(1 + t + t^2 + t^3 + t^4 + t^5 \cdots\right) \cdot \left(1 + t + 2t^2 + 3t^3 + 4t^4 + 5t^5 \cdots\right)$ is $(5 + 4 + 3 + 2 + 1) t^5 = 15 t^5$.

Example: Inverse

Consider $(1-t)^m$. Let $p(t)$ be such that $p(t) \cdot \left((1-t)^m\right) = 1$. Then $p(t) = (1-t)^{-m} = \left(1 + t + t^2 + \cdots\right)^m = \sum_{n=0}^{\infty} \binom{n+m-1}{n} t^n$ is the inverse of $(1-t)^m$.

General Binomial Coefficient

The General Binomial Coefficient

Let $r \in \mathbf{R}$. Then $\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}$.

Theorem

In $\mathbf{C}[[t]]$, $(1-t)^r = \sum_{n=0}^{\infty} \binom{r}{n} t^n$.

The Catalan Numbers

Exponential Generating Functions

Definition

Take a sequence $(s_0, s_1, s_2, \ldots)$. The formal power series $s_0 + s_1 t + \frac{s_2}{2!} t^2 + \cdots$ is called the exponential generating function associated to $(s_0, s_1, s_2, \ldots)$.

Note
The exponential generating function is usually useful for Combinatorics with labels.

Fact
Define e_r to be the number of arrangements of $\{1, \ldots, r\}$. The exponential generation function is $e_0 + e_1 t + \frac{e_2}{2!} t^2 + \cdots = 1 + t + \frac{2!}{2!} t^2 + \cdots = \frac{1}{1-t}$.

DIFFERENTIATION

Definition: Differentiation
If $p(t) = c_0 + c_1 t + c_2 t^2 + \cdots$, define $\frac{dp}{dt} = c_1 + 2c_2 t + 3c_3 t^2 + \cdots$.

Fact: Leibniz Rule
If $p(t)$ and $q(t)$ are two power series, then $\frac{d}{dt}(p \cdot q) = \frac{dp}{dt}(t) \cdot q(t) + p(t) \cdot \frac{dq}{dt}(t)$.

Example
Express $\sum_{r=1}^{\infty} rt^r$ as a rational function.

$\sum_{r=1}^{\infty} rt^r = t \cdot \sum_{r=1}^{\infty} rt^{r-1} = t \cdot \frac{d}{dt} \sum_{r=1}^{\infty} t^r = t \cdot \frac{d}{dt}\left(\frac{1}{1-t}\right) = \frac{t}{(1-t)^2}$.

More formally, we know $(1-t)\sum_{r=1}^{\infty} t^r = 1$. Differentiating both sides, we get

$-\sum_{r=1}^{\infty} t^r + (1-t)\sum_{r=1}^{\infty} rt^{r-1} = 0 \Rightarrow (1-t)^2 \sum_{r=1}^{\infty} rt^{r-1} = (1-t)\sum_{r=1}^{\infty} t^r = 1 \Rightarrow \sum_{r=1}^{\infty} rt^{r-1} = \frac{1}{(1-t)^2} \Rightarrow \sum_{r=1}^{\infty} rt^r = \frac{t}{(1-t)^2}$.

Partitions

Depends on whether the elements of the set are distinct or not.

PARTITIONS WITH DISTINCT OBJECTS

Definition: Bell Numbers
Define B_n, the n^{th} Bell number, to be the number of partitions of $\{1,\dots,n\}$. Declare $B_0 = 1$.

Example
$B_3 = 5$ because $\{1,2,3\}$ has 5 partitions: $\{\{1,2,3\}\}$, $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, $\{\{3\},\{1,2\}\}$, $\{\{1\},\{2\},\{3\}\}$.

Recursion Relation for The Bell Numbers
$$B_n = \left|\begin{Bmatrix}\text{partitions}\\ \text{of the set}\\ \{1,\dots,n\}\end{Bmatrix}\right| = \sum_{j=1}^{n}\left|\begin{Bmatrix}\text{partitions of the set}\\ \{1,\dots,n\}\text{such that 1}\\ \text{lies in a subset}\\ \text{with } j \text{ elements}\end{Bmatrix}\right| = \sum_{j=1}^{n}\left|\begin{matrix}\text{ways of choosing}\\ \text{the rest of the}\\ \text{subset}\end{matrix}\right|\left|\begin{matrix}\text{ways of}\\ \text{partitioning}\\ \text{the remainder}\end{matrix}\right| = \sum_{j=1}^{n}\binom{n-1}{j-1}B_{n-j}.$$

Differential Equation for The Bell Numbers
Let $\beta(t) = \sum_{n=1}^{\infty}\frac{B_n}{n!}t^n$, so $\frac{d\beta}{dt}(t) = \sum_{n=1}^{\infty}\frac{B_n}{(n-1)!}t^{n-1}$.

Now applying the recursion relation, $\frac{d\beta}{dt}(t) = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(\sum_{j=1}^{n}\binom{n-1}{j-1}B_{n-j}\right)t^{n-1} = \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\frac{1}{(j-1)!}\frac{B_{n-j}}{(n-j)!}\right)t^{n-1}$.

Setting $m = n-1$, $\frac{d\beta}{dt}(t) = \sum_{m=0}^{\infty}\left(\sum_{j=1}^{m+1}\frac{1}{(j-1)!}\frac{B_{m+1-j}}{(m+1-j)!}\right)t^m = \sum_{m=0}^{\infty}\left(\sum_{s=0}^{m}\frac{1}{s!}\frac{B_{m-s}}{(m-s)!}\right)t^m = e^t\beta(t)$.

Fact
If $f(t)$ satisfy $\frac{df}{dt} = e^t f(t)$ and $f(0) = 1$, then $f(t) = \beta(t)$.

The Formal Power Series of the Bell Numbers
Guess $\beta(t) = ke^{(e^t)}$. But since $\beta(0) = 1$, $k = \frac{1}{e} = e^{-1}$. So

$$\beta(t) = e^{-1}e^{(e^t)} = e^{e^t-1} = 1 + \sum_{n=1}^{\infty}\frac{t^n}{n!} + \cdots + \frac{1}{i!}\left(\sum_{n=1}^{\infty}\frac{t^n}{n!}\right)^i + \cdots.$$

PARTITIONS WITH IDENTICAL OBJECTS

Definition: Partition Numbers
Define $p(n)$, the n^{th} partition number, to be the number of partitions of *n*, where a partition of *n* into *k* parts is such that $x_1 + \cdots + x_k = n, x_1 \geq \cdots \geq x_k = 0$.

Definitions

1) $P_1 = 1 = (\bullet)$, $P_2 = 5 = \begin{pmatrix} & \bullet & \\ \bullet & & \bullet \\ \bullet & & \bullet \end{pmatrix}$, $P_3 = 12 = \begin{pmatrix} & & \bullet & & \\ & \bullet & & \bullet & \\ \bullet & & \bullet & & \bullet \\ \bullet & & \bullet & & \bullet \\ \bullet & & \bullet & & \bullet \end{pmatrix}$, etc.

2) $Q_1 = 1 = \begin{pmatrix} \bullet \\ \bullet \end{pmatrix}$, $Q_2 = 7 = \begin{pmatrix} & \bullet & \\ \bullet & & \bullet \\ \bullet & & \bullet \\ \bullet & & \bullet \end{pmatrix}$, $Q_3 = 15 = \begin{pmatrix} & & \bullet & & \\ & \bullet & & \bullet & \\ \bullet & & \bullet & & \bullet \\ \bullet & & \bullet & & \bullet \\ \bullet & & \bullet & & \bullet \\ \bullet & & \bullet & & \bullet \end{pmatrix}$, etc.

Recursion Relation of The Partition Numbers

$$p(n) = \sum_{i=1}^{\infty} (-1)^{k+1} \left(p(n - P_k) + p(n - Q_k) \right).$$

Fact: Euler's Formula

$\prod_{r=1}^{\infty} \frac{1}{1-t^r} = \pi(t)$, where $\pi(t) = p(0) + p(1)t + p(2)t^2 + \cdots$ is the generation function for the partition numbers.

<u>Proof</u>:

$$\prod_{r=1}^{\infty} \frac{1}{1-t^r} = \prod_{r=1}^{\infty} \left(1 + t^r + t^{2r} + t^{3r} + \cdots\right) = \sum_{\substack{\text{selections of one element} \\ \text{from each set such that} \\ \text{only finitely many of} \\ \text{the choices are not 1}}} \begin{Bmatrix} 1 \\ t^2 \\ t^3 \\ \vdots \end{Bmatrix} \begin{Bmatrix} 1 \\ t^{1\times 2} \\ t^{2\times 2} \\ \vdots \end{Bmatrix} \begin{Bmatrix} 1 \\ t^{1\times 3} \\ t^{2\times 3} \\ \vdots \end{Bmatrix} \cdots .$$

$$= \sum_{n=1}^{\infty} \left| \begin{Bmatrix} \text{infinite sequences } (s_1, s_2, \ldots) \\ \text{such that } 1\times s_1 + 2\times s_2 \times \cdots = n \end{Bmatrix} \right| = \sum_{n=1}^{\infty} p(n) t^n$$

The Generating Function of The Partition Numbers

We have $\pi(t) = \sum_{n=0}^{\infty} p(n) t^n$ and $\pi^{-1}(t) = 1 + \sum_{k \geq 1} (-1)^k \left(t^{\frac{k}{2}(3k-1)} + t^{\frac{k}{2}(3k+1)} \right)$. Thus, since $\pi \cdot \pi^{-1} = 1$, the coefficient of t^n is $p(n) + \sum_{k=1}^{\infty} (-1)^{k+1} \left(p\left(n - \frac{k}{2}(3k-1)\right) + p\left(n - \frac{k}{2}(3k+1)\right) \right) = 0$.

Principle of Inclusion-Exclusion

MOTIVATION

Example

Recall a function $f : X \to Y$ is onto if every $y \in Y$ is $f(x)$ for some $x \in X$. How many onto maps are there from $\{1,2,\ldots,7,8\}$ to $\{1,2,3,4\}$?

The total number of maps from $\{1,2,\ldots,7,8\}$ to $\{1,2,3,4\}$ is 4^8.

Let $S_n = \{1,\ldots,n\}$. We would like to say $\left|\begin{Bmatrix}\text{Onto maps}\\ \phi{:}S_8 \to S_4\end{Bmatrix}\right| = \underbrace{\left|\begin{Bmatrix}\text{All maps}\\ \phi{:}S_8 \to S_4\end{Bmatrix}\right|}_{A} - \underbrace{\left|\begin{Bmatrix}\text{Maps } \phi{:}S_8 \to S_4\\ \text{which are not onto}\end{Bmatrix}\right|}_{B}$.

First guess for *B*:

$$\left|\begin{Bmatrix}\text{All maps}\\ \phi{:}S_8 \to S_4 \setminus \{1\}\end{Bmatrix}\right| + \left|\begin{Bmatrix}\text{All maps}\\ \phi{:}S_8 \to S_4 \setminus \{2\}\end{Bmatrix}\right| + \left|\begin{Bmatrix}\text{All maps}\\ \phi{:}S_8 \to S_4 \setminus \{3\}\end{Bmatrix}\right| + \left|\begin{Bmatrix}\text{All maps}\\ \phi{:}S_8 \to S_4 \setminus \{4\}\end{Bmatrix}\right| = \sum_i \left|\begin{Bmatrix}\text{All maps}\\ \phi{:}S_8 \to S_4 \setminus \{i\}\end{Bmatrix}\right|.$$

Wrong because it over-counts ($\phi{:}S_8 \to S_4 \setminus \{1,2\}$ is counted twice).

Second guess for *B*: $\sum_i \left|\begin{Bmatrix}\text{All maps}\\ \phi{:}S_8 \to S_4 \setminus \{i\}\end{Bmatrix}\right| - \sum_{i<j} \left|\begin{Bmatrix}\text{All maps}\\ \phi{:}S_8 \to S_4 \setminus \{i,j\}\end{Bmatrix}\right|$. Wrong because it under-counts ($\phi{:}S_8 \to S_4 \setminus \{1,2,3\}$).

Third guess for *B*: $\sum_i \left|\begin{Bmatrix}\text{All maps}\\ \phi{:}S_8 \to S_4 \setminus \{i\}\end{Bmatrix}\right| - \sum_{i<j} \left|\begin{Bmatrix}\text{All maps}\\ \phi{:}S_8 \to S_4 \setminus \{i,j\}\end{Bmatrix}\right| + \sum_{i<j<k} \left|\begin{Bmatrix}\text{All maps}\\ \phi{:}S_8 \to S_4 \setminus \{i,j,k\}\end{Bmatrix}\right|$. This is correct.

So $\left|\begin{Bmatrix}\text{Onto maps}\\ \phi{:}S_8 \to S_4\end{Bmatrix}\right| = \left|\begin{Bmatrix}\text{All maps}\\ \phi{:}S_8 \to S_4\end{Bmatrix}\right| - \left|\begin{Bmatrix}\text{Maps } \phi{:}S_8 \to S_4\\ \text{which are not onto}\end{Bmatrix}\right| = 4^8 - \left(4 \cdot 3^8 - \binom{4}{2} \cdot 2^8 + 4 \cdot 1^8\right)$.

THE GENERAL INCLUSION-EXCLUSION PROBLEM

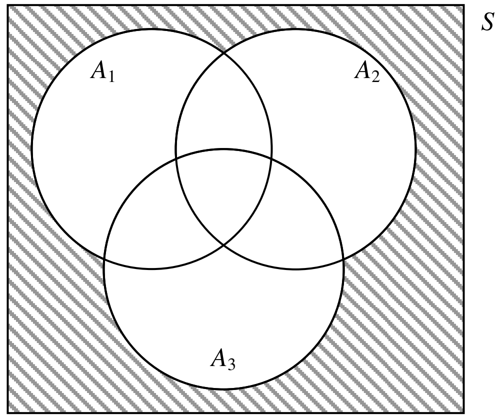
We are given a finite universe *S* and a collection of subsets $A_1,\ldots,A_n$.

Notation

If $I \subset \{1,\ldots,n\}$, then let $A_I = \bigcap_{i \in I} A_i$.

Problem

What we know is $|S|$ and $|A_I|$. Determine $|S \setminus (A_1 \cup \cdots \cup A_n)|$.



Theorem: Principle of Inclusion-Exclusion

$$\left|S \setminus (A_1 \cup \cdots \cup A_n)\right| = |S| + \sum_{\substack{\text{non empty} \\ \text{subsets} \\ I \subset \{1,\ldots,n\}}} (-1)^{|I|} |A_I|.$$

Application of Inclusion-Exclusion: Derangements

Definition: Derangements

A derangement on n elements is a permuatation $\phi : \{1,\ldots,n\} \to \{1,\ldots,n\}$ such that $\phi(i) \neq i, \forall i \in \{1,\ldots,n\}$.
Let $d(n)$ denote the number of derangements on n elements and define $d(0)=1$.

Example

If marked test papers are randomly returned to a class of 100 students, what is the probability no one gets the correct paper?

The answer is $\dfrac{d(n)}{n!}$. Apply inclusion-exclusion to calculate $d(n)$.
Let $S = \{\text{permuatations on } n \text{ elements}\}$. For every $i = 1,\ldots,n$ set $U_i = \{\text{permutations such that } \phi(i) = i\}$.
Clearly, $d(n) = |S \setminus U_1 \cup \cdots \cup U_n| = |S| - \sum_i |U_i| + \sum_{i<j} |U_{\{i,j\}}| - \sum_{i<j<k} |U_{\{i,j,k\}}| + \cdots + (-1)^n |U_1 \cap \cdots \cap U_n|$.
Now, $|U_1| = \cdots = |U_n| = (n-1)!$, $|U_{\{1,2\}}| = (n-2)!$, and in general $|U_I| = (n-|I|)!$. So
$d(n) = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \cdots = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$; as $n \to \infty$, this is approximately $n! e^{-1}$.

Proposition

There is an integer closest to $\dfrac{n!}{e}$ and it is $d(n)$.

Proof: The proposition is established once we show that $\left| d(n) - \frac{n!}{e} \right| < \frac{1}{2}$. Informally,

$$\left| \frac{n!}{e} - d(n) \right| = \left| n! \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} - n! \sum_{k=1}^{n} \frac{(-1)^k}{k!} \right| = n! \left| \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \right| < n! \frac{1}{(n+1)!} = \frac{1}{n+1} \le \frac{1}{2}.$$

APPLICATION OF INCLUSION-EXCLUSION: PERMUTATIONS WITH RESTRICTED POSITIONS

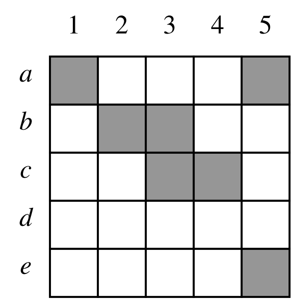
Example

How many rearrangements of the letters of the word "abcde" satisfies the following rules:

- "a" cannot end up in positions 1 or 5,
- "b" cannot end up in positions 2 or 3,
- "c" cannot end up in positions 3 or 4,
- "e" cannot end up in position 5?

It is useful to record these rules on a "chessboard".

Rearrangements of "abcde" corresponds to selections of 5 boxes such that in each row and column precisely one box is chosen. Admissible rearrangements correspond to choices squares not shaded.



Using inclusion-exclusion, let $S = \{\text{all selections}\}$ and $U_i = \left\{ \begin{array}{l} \text{selections which put an inadmissible} \\ \text{choice in the } i^{\text{th}} \text{ position} \end{array} \right\}$. Then

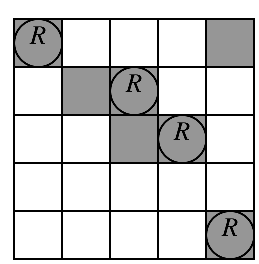
$$\left| \left\{ \begin{array}{l} \text{admissible} \\ \text{rearrangements} \end{array} \right\} \right| = \left| S \setminus (U_1 \cup \cdots \cup U_5) \right|.$$

What is $|U_1|$? This is the number of selections which put an inadmissible letter in the first position. So $|U_1| = 4!$. Also, $|U_2| = 4!$, $|U_3| = 2 \cdot 4!$, $|U_4| = 4!$, $|U_5| = 2 \cdot 4!$; $|U_{\{1,2\}}| = 3!$, $|U_{\{1,3\}}| = 2 \cdot 3!$, …, $|U_{\{3,5\}}| = 4 \cdot 3!$; etc.

Now, $\left| S \setminus (U_1 \cup \cdots \cup U_5) \right| = 1 \cdot 5! - 7 \cdot 4! + 16 \cdot 3! - 13 \cdot 2! + 3$. The coefficients are the rook numbers.

Definition: Mutually Non-Attacking Placement

A mutually non-attacking placement of k rooks on a chessboard is a placement of rooks on the shaded squares such that they don't attack each other.



Definition: Rook Number

Let $r_k(B)$ denote the k^{th} rook number of a chessboard *B*, which is the number of mutually non-attacking placements of *k* rooks on *B*. Declare $T_0(B)=1$.

Definition: Rook Polynomial
Define the rook polynomial of a chessboard to be the generating function of rook numbers.
$r(B,t)=r_0(B)+r_1(B)t+r_2(B)t^2+\cdots$.

Key Fact
$\left|\{\text{allowed permutations}\}\right|=\sum_{k=0}(-1)^k r_k(B)(n-k)!$.

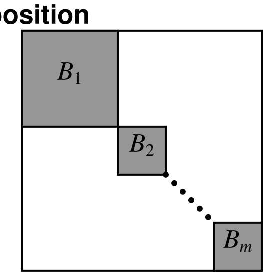
Lemma
The of choices of *k* non-consecutive elements from $\{1,\ldots,n\}$ is $\binom{n-(k-1)}{k}$.

Example: The Dinner Party Problem
Someone is holding a dinner party with *n* married couples. The seats are numbered $1,\ldots,2n$. How many ways are the there seat guests such that men and women alternate, and husbands and wives don't sit next to each other?
Let $s(n)$ denote the number of bijections $\phi:\{1,\ldots,n\}\to\{1,\ldots,n\}$ such that $\phi(i)\neq i,i+1 \bmod n$ (i.e. $\begin{cases}\phi(i)\neq i,i+1, 1\leq i\leq n+1\\ \phi(n)\neq n,1\end{cases}$). Then $\begin{pmatrix}\text{number of seating}\\ \text{arrangements}\end{pmatrix}=2(n!)s(n)$.
Let B_n denote the chessboard for $s(n)$. Then

$$r_k(B_n)=\begin{pmatrix}\#\text{ of selections of } k \text{ elements from } \{1,\ldots,2n\}\\ \text{such that no two in the selections are consecutive}\end{pmatrix}=\begin{pmatrix}\#\text{ of such selections}\\ \text{with } 2n \text{ chosen}\end{pmatrix}+\begin{pmatrix}\#\text{ of such selections}\\ \text{with } 2n \text{ not chosen}\end{pmatrix}$$
$$=\begin{pmatrix}\#\text{ of } k\text{-1 non-consecutive}\\ \text{selections from } \{2,\ldots,2n-2\}\end{pmatrix}+\begin{pmatrix}\#\text{ of } k \text{ non-consecutive}\\ \text{selections from } \{1,\ldots,2n-1\}\end{pmatrix}$$
$$=\binom{(2n-3)-(k-2)}{k-1}+\binom{(2n-1)-(k-1)}{k}=\binom{2n-1-k}{k-1}+\binom{2n-k}{k}=\frac{2n}{2n-k}\binom{2nk-k}{k}.$$

So $s(n)=\sum_{k=0}(-1)^k\frac{2n}{2n-k}\binom{2nk-k}{k}(n-k)!$.

Techniques For Calculating Rook Numbers: Decomposition
When a chessboard is in block form,
$r(B,t)=r(B_1,t)\cdots r(B_m,t)$



Techniques For Calculating Rook Numbers: Recursion Relation

Let B be a chessboard and (i, j) be a shaded square. $B-(i, j)$ means the chessboard you get from B by un-shading (i, j). $B*(i, j)$ means the chessboard you get from B by un-shading row i and column j.

Then for k an integer at least 1, $r_k(B) = \underbrace{r_k(B-(i, j))}_{\text{don't use }(i,j)} + \underbrace{r_k(B*(i, j))}_{\text{use }(i,j)}$. Also,

$r(B,t) = r(B-(i, j),t) + r(B*(i, j),t)t$.

Ramsey Theory

Philosophy

We look for some fixed pattern in data sets of some fixed type.

A typical theorem: "When the data set gets big enough, the pattern is guaranteed to appear".

Example

I am throwing a party. If I invite at least 6 guests, then one of the following two situations must occur:

1) There is a group of 3 guests, all of whom have previously met each other.
2) There is a group of 3 guests, all of whom have never met the other two.

Here, 6 is the threshold.

Pigeonhole Principle

The Pigeonhole Principle

If I distribute $n+1$ balls into n boxes, there is a box with at least 2 balls.

Example

Prove that if I select $n+1$ distinct integers from $\{1,\dots,2n\}$, then at least 2 must be consecutive.

Take boxes

$$\underset{1,\,2}{\llcorner\_\_\_\lrcorner} \quad \underset{3,\,4}{\llcorner\_\_\_\lrcorner} \quad \cdots\cdots \quad \underset{2n-1,\ 2n}{\llcorner\_\_\_\lrcorner} .$$

There must be a box with two elements.

Example

Prove that I select $n+1$ distinct integers from $\{1,\dots,2n\}$, then there is a pair one of which divides the other.

Note that every number is uniquely expressed by $n = 2^r s$. So distribute the $n+1$ selections into boxes according to s. By the pigeonhole principle, $2^{r_1} s$ and $2^{r_2} s$ are in the same box.

Generalization of the Pigeonhole Principle

If there are p boxes and $np+1$ balls, then there must be a box with $n+1$ balls.

Fact

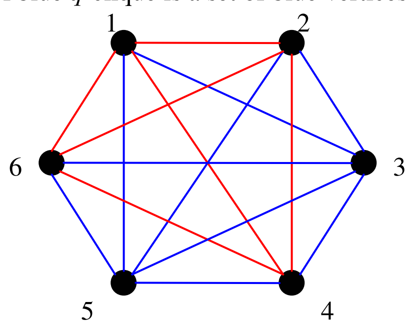
If there are p boxes and q balls, then there must be a box with $\left\lfloor \frac{q-1}{p} \right\rfloor + 1 = \left\lceil \frac{q}{p} \right\rceil + 1$ balls.

Ramsey Numbers

Recall that K_n denotes the complete simple graph on n vertices. Colour the edges red and blue.

Definition: Clique

A red p-clique is a set of red vertices such that every edge between them is red.
A blue q-clique is a set of blue vertices such that every edge between them is blue.



- $\{1,2,4,6\}$ is a red 4-clique.
- $\{2,3,5\}$ is a blue 3-clique.
- $\{1,3,5,6\}$ is not a blue 4-clique.

Definition: Ramsey Numbers

Let $p, q \geq 3$. Let $R(p,q)$ denote the least integer with the following property: Whenever $n \geq R(p,q)$, every colouring of K_n (with 2 colours, red and blue) contains one of:

- a red p-clique
- a blue q-clique.

Theorem

$R(3,3) = 6$.

Theorem

$R(3,4) = 9$.