Lecture #1 – Wednesday, September 10, 2003

SAMPLE SPACE

- Nothing is 100% sure – random phenomenon
- Definition: Consider a random action, phenomenon, or event whose outcome is unknown. Then the set of all possible outcomes is the sample space of the action, phenomenon, or event.

Examples

- Flip a coin with two sides: heads and tails
    - $S = \{H, T\}$
- Roll a die
    - $S = \{1,2,3,4,5,6\}$
- Horse race with 11 horses ($H_1 \ldots H_{11}$)
    - Who wins? – $S = \{H_1, H_2, ..., H_{11}\}$
    - Who are the first two? – S has 110 elements – $S = \{(H_1, H_2), (H_2, H_1), ..., (H_7, H_{11})\}$

Counting

- It is of interest the number of elements in S – therefore counting is important
- "Glancing a plate"
    - Assumptions: All cars have Ontario plates; 4 letters (26 in total); 3 numbers (10 in total)
    - "Watch a random plate"
        - S is not describable
        - We know $|S|$ ($|S|$ is the number of elements in S)

EVENTS

- subsets of the sample space

Examples

- Flip a coin
    - $E = \{\text{I flip a coin and get a head}\}$
      $E = \{H\} \subset S = \{H, T\}$
- Roll a die
    - $E = \{\text{I roll an even number}\}$
      $E = \{2,4,6\} \subset S = \{1,2,3,4,5,6\}$ – composite event
    - $F = \{2\}$ – single event
- Horse Race
    - First two – $S = \{(H_1, H_2), (H_2, H_1), ..., (H_7, H_{11})\}$ with 110 elements
    - $E = \{H_7 \text{ wins the race}\}$
      $E = \{(H_7, H_1), (H_7, H_2), ..., (H_7, H_{11})\}$ – 10 entries

OPERATIONS WITH EVENTS (SETS)

Reunion

- "Roll a 2 or 4"
  - $E_1 = \{2\}$
    $E_4 = \{4\}$
    $E = E_1 \cup E_2 = \{2,4\}$
- A, B are sets
- $A \cup B$ is the "reunion of A and B" – the set which contains the elements that are in either A or B

Intersection

- $E_1 = \{\text{roll an even number}\}$
  $E_2 = \{\text{roll a number small than } 5\}$
  $E = \{\text{roll an even number small than } 5\}$
- $E_1 = \{2,4,6\}$
  $E_2 = \{1,2,3,4\}$
  $E = E_1 \cap E_2 = \{2,4\}$
- A, B are sets
- $A \cup B$ is the "reunion of A and B" – the set which contains the elements that are in either A or B

Complement

- $E_1 = \{2,4,6\}$
  $E = \{\text{all the elements that are not in } E_1\} = \{1,3,5\}$
- Definition: If A is a set, then A^c is the complement of A – the set containing all the elements which are not in A, but in S

The Empty Set

- Denoted by Ø
- For any set A, $\emptyset \subset A$
- For any sample spaces S, $\emptyset \subset S$
- $\emptyset \cup A = A$
  $\emptyset \cap A = \emptyset$

Lecture #2 – Friday, September 12, 2003

OPERATION WITH SETS – CONTINUED

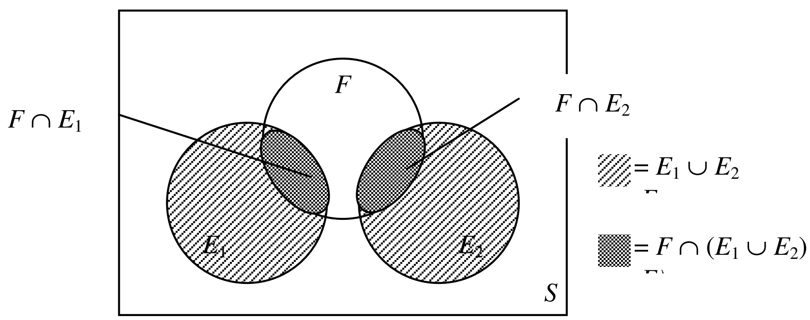
Infinite Sequences of Events (Notation)

- For $E_1, E_2, ..., E_n$,
  - $\bigcup_{i=1}^{n} E_i = E_1 \cup E_2 \cup ... \cup E_n$

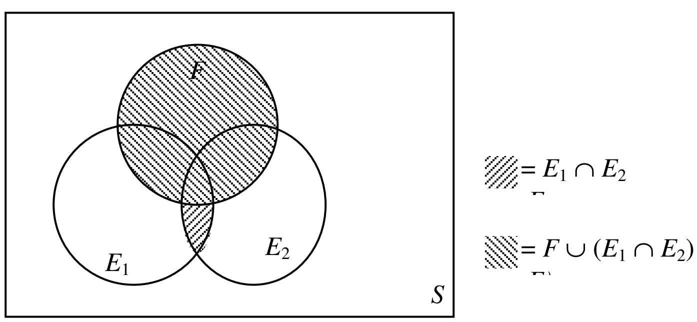
- $\bigcup_{i=1}^{\infty} E_i = E_1 \cup E_2 \cup \ldots \cup E_n \cup \ldots$
- $\bigcap_{i=1}^{n} E_i = E_1 \cap E_2 \cap \ldots \cap E_n$
- $\bigcap_{i=1}^{\infty} E_i = E_1 \cap E_2 \cap \ldots \cap E_n \cap \ldots$

Distributive Laws

- $x \cdot (a+b) = x \cdot a + x \cdot b$
- $F \cap (E_1 \cup E_2) = (F \cap E_1) \cup (F \cap E_2)$



- $F \cup (E_1 \cap E_2) = (F \cup E_1) \cap (F \cup E_2)$



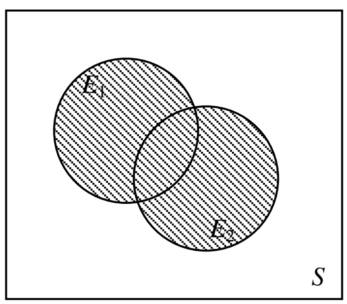
- For $F \cap (E_1 \cup E_2) = (F \cap E_1) \cup (F \cap E_2)$, two things are required to show the equality of two sets A and B
  - For each point $a \in A$, we need to show that $a \in B$
    $a \in F \cap (E_1 \cup E_2)$
    a is in F and a is in E_1 or E_2
    a is in F and is in E_1 or a is in F and is in E_2
    $\therefore a \in (F \cap E_1) \cup (F \cap E_2)$
  - For each point $b \in B$, we need to show that $b \in A$
    $b \in (F \cap E_1) \cup (F \cap E_2)$
    b is in F and is in E_1 or b is in F and is in E_2
    b is in F and b is in E_1 or E_2
    $\therefore b \in F \cap (E_1 \cup E_2)$

In General

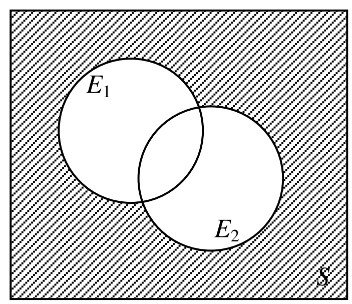
- $F \cap \left(\bigcup_{i=1}^{n} Ei \right) = \bigcup_{i=1}^{n} (E_i \cap F)$
- $F \cap (E_1 \cup E_2 \cup E_3) = (F \cap E_1) \cup (F \cap E_2) \cup (F \cap E_3)$
  $F \cap \left(\bigcup_{i=1}^{3} Ei \right) = \bigcup_{i=1}^{3} (F \cap E_i)$
- $F \cup \left(\bigcap_{i=1}^{n} E_i \right) = \bigcap_{i=1}^{n} (F \cup E_i)$

Complements

- For set A, A^c is the set of points which are not in set A
- $(A^c)^c = A$
- $(E_1 \cup E_2)^c = E_1{}^c \cap E_2{}^c$ – de Morgon's Law



= $E_1 \cup E_2$



= $E_1{}^c \cap E_2{}^c$

- $(E_1 \cap E_2)^c = E_1{}^c \cup E_2{}^c$
- $(E_1 \cap E_2)^c$
  $= \left((E_1{}^c)^c \cap (E_2{}^c)^c \right)^c$
  $= \left((E_1{}^c \cup E_2{}^c)^c \right)^c$
  $= E_1{}^c \cup E_2{}^c$

Probability

- For E_1, E_2, E_3,
  - E_1 is the set of possible outcomes
  - "E_1 is true" or "E_1 occurs"
  - The final outcome of the expression belongs to E_1

Examples

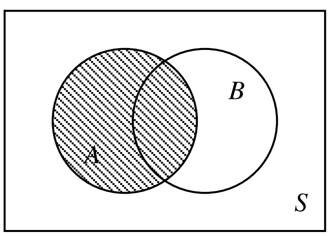
- "Roll a die"

- $E_1 = \{\text{roll an even number}\} = \{2,4,6\}$ – E_1 is true if the roll is either 2, 4, or 6
- $E_2 = \{\text{roll is less than 3}\}$
- If E_1 and E_2 are true, the outcome belongs to the intersection of $E_1 \cap E_2 = \{2\}$

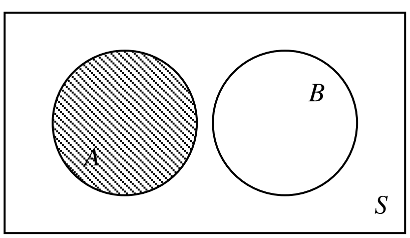
- If E_1, E_2, and E_3 are three events
  - Only E_1 is true
    E_1 and not in E_2 and not in E_3
    $E_1 \cap E_2{}^c \cap E_3{}^c$
  - Only E_2 or E_3 is true
    E_2 and not in E_1 and not in E_3 <u>or</u> E_3 and not in E_1 and not in E_2
    $\left(E_2 \cap E_1{}^c \cap E_3{}^c\right) \cup \left(E_3 \cap E_1{}^c \cap E_2{}^c\right)$
  - None of them is true
    $E_1{}^c \cap E_2{}^c \cap E_3{}^c$
  - Exactly two of them is true
    E_1, E_2 occur, E_3 doesn't <u>OR</u> E_1, E_3 occur, E_2 doesn't <u>OR</u> E_2, E_3 occur, E_1 doesn't
    $\left(E_1 \cap E_2 \cap E_3{}^c\right) \cup \left(E_1 \cap E_3 \cap E_2{}^c\right) \cup \left(E_2 \cap E_3 \cap E_1{}^c\right)$

Lecture #3 – Monday, September 15, 2003

- $A = (A \cap B) \cup \left(A \cap B^c\right)$



- If $A \cap B = \phi$, then $A \cap B^c = A$



Notation

- $E \cdot F = E \cap F$
- E happens but F doesn't: $E - F$
  - set containing those points which are in E but are not in F – $E \cap F^c$
- Symmetrical difference: $E \Delta F = (E - F) \cup (F - E)$
  - If $E \Delta F = \phi$, then $E = F$

DE MORGAN'S FIRST AND SECOND LAWS

1) $\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^{\,c}$

2) $\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^{\,c}$

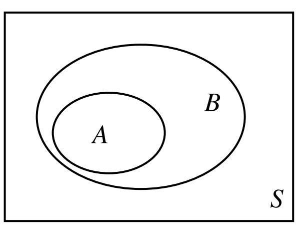
AXIOMS OF PROBABILITY

1) $\forall A \subset S, P(A) \geq 0$
2) $P(S) = 1$
3) For any infinite sequence of events E_i such that any two $E_i \cap E_j = \phi$ (mutually exclusive event),

$$P\left(\bigcup_{i=0}^{\infty} E_i\right) = \sum_{i=0}^{\infty} P(E_i)$$

- $[0,1]$
  $P(S) = 1$
  $P(A) \geq 0$, for any event $A \subset S$
  - For A and A^c,
    - $A \cup A^c = s$
    - $A \cap A^c = \phi$
    - $P(A \cup A^c) = P(S) = 1$
      $= P(A) + P(A^c)$
  - $P : \{\text{sebsets of } S\} = [0,1]$

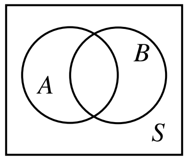
- If $A \subset B$, show $P(A) \leq P(B)$



  - $B = (B \cap A) \cup (B \cap A^c)$
    $P(B) = P((B \cap A) \cup (B \cap A^c))$
  - $= P(B \cap A) + P(B \cap A^c)$
    $= P(A) + P(B \cap A^c)$
  - $\therefore P(A) \leq P(B)$

Inclusion-Exclusion Formula

- What is $P(A \cap B)$ if $A \cap B \neq 0$?



- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
- In general, $P(A_1 \cup ... \cup A_n) = P(A_1) + ...P(A_n) - P(A_1 \cap A_2) - ...P(A_{n-1} \cap A_n) + (-1)^{n-1} P(A_1 \cap ... \cap A_n)$

COUNTING OUTCOMES

- S is a sample space with equal probability outcomes – $S = \{a_1, ..., a_n\}$
- $P(a_1) = P(a_n) = p = \frac{1}{n}$
- If $E_i = \{a_i\}$, for $i = 1,2,...,n$
  - $\bigcup_{i=1}^{n} E_i = S$
  - $E_i \cap E_j = \{a_i\} \cap \{a_j\} = \phi$
- $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) = n \cdot p = 1$
  $\therefore p = \frac{1}{n}$
- If all possible end outcomes are equally likely, the probability of each is $\frac{1}{n}$
  - $E \subset S$
    $P(E) = \frac{\text{\# of elements in set } E}{n}$

Lecture #4 – Wednesday, September 17, 2003

SAMPLE SURVEYS

Example

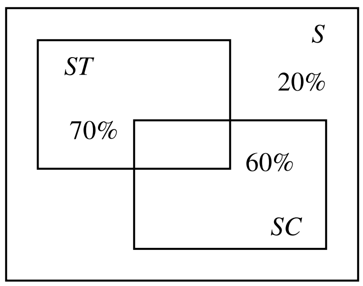
- Pick a person at random (from this university, city, country)
- Percentage of people in Toronto with "quality X" is 30%
- Sample space of this experiment – $S = \{(P_i, 1 \text{ or } 0)\}, i = 1,...,2000000$

- $E = \{\text{" A person has X"}\} = \{(P_i, 1), i = 1, \ldots, 2000000\}$ – size of E is the number of elements in E (smaller than 2000000)
- $P(E) = 30\%$

Example

- Second Cup (SC) = 60% – "person drinks from Second Cup"
  Starbucks (ST) = 70% – "person drinks from Starbucks"
  no coffee (NC) = 20% – "person doesn't drink coffee"
- Sample space: $S = \{(P_i, x, y), i = \text{all people in Toronto}\}$

- How many people drink from Second Cup and Starbucks?
  - $E = \{(P_i, 1, 1)\}$

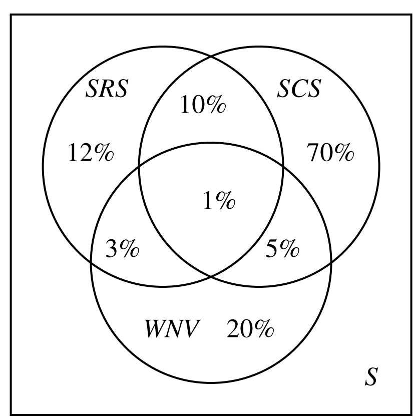


  - $$\begin{aligned} ST \cup SC &= \{\text{people who drink coffee}\} \\ P(ST \cup SC) &= 1 - 20\% = 80\% \\ P(ST \cup SC) &= P(ST) + P(SC) - P(ST \cap SC) \\ P(ST \cap SC) &= P(ST) + P(SC) - P(ST \cup SC) \\ &= 70\% + 60\% - 80\% \\ &= 50\% \end{aligned}$$
  - $\therefore$ 50% of the people drink from Second Cup and Starbucks.

- How many people drink from only Starbucks?
  - $70\% - 50\% = 20\%$

Example

- West Nile Virus (WNV) = 20%
  SARS (SRS) = 12%
  Without scars (SCS^c) = 30%
  SARS and West Nile Virus (SRS + WNV) = 3%
  Scars and SARS (SCS + SRS) = 20%
  Scars and West Nile Virus (SCS + WNV) = 20%
  All (WNV + SRS + SCS) = 1%



- $E = \{\text{He will leave the same way he came to Toronto}\}$

- $$\begin{aligned} P(E) &= 1 - P(E^c) \\ &= 1 - P(SRS \cup SCS \cup WNV) \\ &= 1 - \begin{pmatrix} P(SRS) + P(SCS) + P(WNV) \\ - P(SRS \cap SCS) - P(SRS \cap WNV) - P(SCS \cap WNV) \\ + P(SRS \cap SCS \cap WNV) \end{pmatrix} \\ &= 1 - (12\% + 70\% + 20\% - 10\% - 3\% - 5\% + 1\%) \\ &= 1 - 85\% \\ &= 15\% \end{aligned}$$

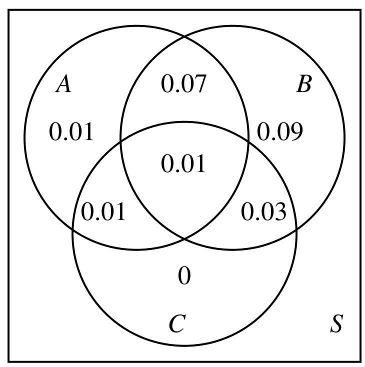
LECTURE #5 – FRIDAY, SEPTEMBER 19, 2003

SAMPLE SURVEYS (CONTINUED)

Example

Suppose that a town has three local newspaper (*A*, *B*, *C*). Suppose that

- 10% read newspaper *A*
- 20% read newspaper *B*
- 5% read newspaper *C*
- 8% read newspaper *A* and *B*
- 2% read newspaper *A* and *C*
- 4% read newspaper *B* and *C*
- 1% read newspaper *A*, *B*, and *C*



1) What is the percentage of people reading only one newspaper?
$$\begin{aligned} P(\text{only one newspaper}) &= 0.01 + 0.09 + 0 \\ &= 0.1 = 10\% \end{aligned}$$
2) What is the percentage of people reading exactly two newspapers?
$$\begin{aligned} P(\text{exactly two newspapers}) &= 0.07 + 0.01 + 0.03 \\ &= 0.11 = 11\% \end{aligned}$$
3) What is the percentage of people reading newspapers *A* or *B*?
$$\begin{aligned} P(\text{reading } A \text{ or } B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.1 + 0.2 - 0.08 \\ &= 0.22 = 22\% \end{aligned}$$

Example

We look at families with 4 children (it's equally probable to have a girl or a boy):

$$S = \{(b,b,g),(b,g,b),(g,b,b),(b,g,g),(g,b,g),(g,g,b),(g,g,g),(b,b,b)\}$$

- Each of these outcomes is equally likely

We are interested only in how many boys a family with kids has:

$$S = \{0b, 1b, 2b, 3b\}$$

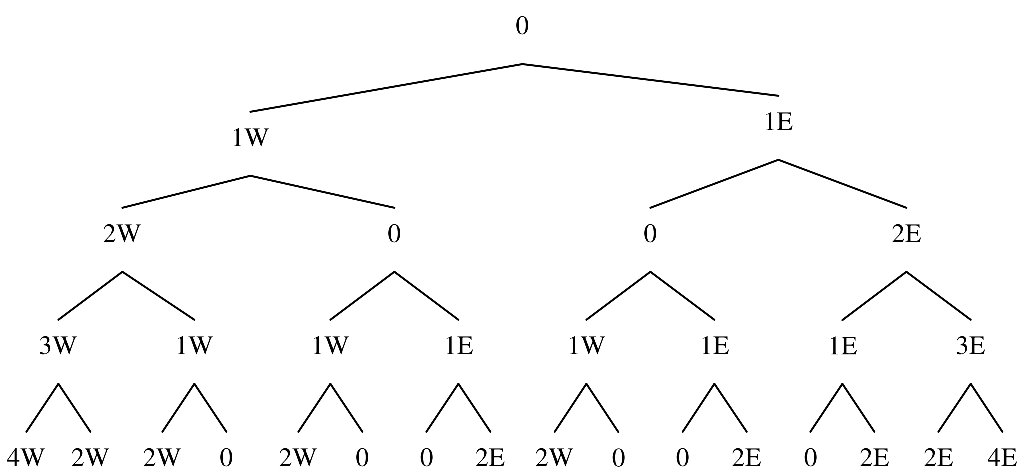
- $P\{0 \text{ boys}\} = \frac{1}{8}$
- $P\{1 \text{ boy}\} = \frac{3}{8}$
- $P\{2 \text{ boys}\} = \frac{3}{8}$
- $P\{3 \text{ boys}\} = \frac{1}{8}$

- Note: Look at the sample space with equally likely outcome – allows for probability formula

Example

A tourist wanders on Bloor St. and doesn't know where he is going. At each intersection he decides at random to go either east or west. Suppose that he chooses each time by flipping a coin.

- Two possible choices for sample space:
  - $S_1 = \{(1E, 2E, 3E, 4E), (1E, 2E, 3E, 2E), \ldots, (1W, 2W, 3W, 4W)\}$ – describes the whole path
  - $S_2 = \{0, 1E, 1W, 2E, 2W, \ldots, 4E, 4W\}$ – describes the final arrival points



1) After walking a length equal to four blocks, what is the probability that he is back at the starting point, say Yonge & Bloor?

   $P(\text{end where he stared}) = \frac{6}{16} = \frac{3}{8}$

2) What is the probability that he is one block away from the starting point?

   $P(\text{one block away}) = 0$

COUNTING TECHNIQUES

Fundamental Principle of Counting

- There are r experiments such that:
  - Experiment i has n_i possible outcomes no matter what happened in the previous $i-1$ experiment – number of outcomes in the i^{th} experiment does not depend on the particular outcome of the previous $i-1$, but can depend on i
  - Total number of possible outcome is then: $n_1 \times n_1 \times \ldots \times n_r$

Example

Six people, 3 girls (C, D, L) and 3 boys (J, T, X).

1) They are on a boat with six seats. How many ways can the sit on the boat?
   $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$
   - In general: $n! = 1 \times 2 \times \ldots (n-1) \times n$ and $0! = 1$ by definition

Lecture #6 – Monday, September 22, 2003

Counting Techniques (Continued…)

- In general, the total number of ordered arrangements of n items is $n! = 1 \times 2 \times \ldots \times n$

Example

In how many ways can 3 girls and 3 boys be seated?

$$\frac{G}{C} \quad \frac{B}{J} \quad \frac{B}{T} \quad \frac{G}{D} \quad \frac{B}{L} \quad \frac{B}{X}$$

- Switching C and D means:
  - different arrangement if names are considered
  - same arrangement if names are not considered

- $\dfrac{6!}{3! \times 3!} = 20$

Example

In a chess tournament, there are:
- 3 players from Canada (C_1, C_2, C_3)
- 2 players from Russia (R_1, R_2)
- 4 players from U.S.A. (U_1, U_2, U_3, U_4)
- 1 player from China (Ch)

- Individual final ranking: $10!$

- Final ranking by country: $\dfrac{10!}{3! \times 2! \times 4! \times 1!} = 12600$

- If n items are such that n_1 of them belong to Category 1, n_2 of them belong to Category 2,…, n_r of them belong to Category R, the number of possible arrangements is then $\dfrac{n!}{n_1! \times n_2! \times \ldots \times n_r!}$, $n = n_1 + n_2 + \ldots + n_r$

Combinations

I have n objects. I want to select r of them (in an unordered matter). In how many ways can I do this?

- Divide the n objects into 2 categories: the object is selected among the r, the object is not selected among the r

- $\frac{n!}{r!(n-r)!} = \binom{n}{r}$ – "*n* choose *r*"

Example

I am a contestant on Fear Factor. A jar in front of me contains 3 snakes (with venom), 5 spiders, and 4 bees. Magically, they do not attack each other but patiently wait for me to put my hand in the jar. I am crazy enough to do it. After 1 minute I take it out and see that it has 3 bites; what is chance that all the bites are from the bees and I'll get to see the episode aired?

- With labels:
  - Snakes: $S_{1,} S_{2,} S_3$
  - Spiders: $Sp_{1,} Sp_{2,...,} S_5$
  - Bees: $B_{1,} B_{2,} B_{3,} B_4$
  - $S = \{(S_1, S_2, S_3), (S_2, S_1, S_3), ..., (Sp_1, S_2, B_1),\}$
    - $|S| = 12 \times 11 \times 10$ – all outcomes equally likely
  - $E = \{3 \text{ bites are from } 3 \text{ bees}\}$
    - $$P(E) = \frac{\text{number of outcomes in } E}{12 \times 11 \times 10} = \frac{4 \times 3 \times 2}{12 \times 11 \times 10} = \frac{1}{55}$$

- Without labels, I lose the order to induce order:
  - $S = \{(S, Sp, S), ..., (B, B, B),\}$
    - $|S| = \binom{12}{3} = \frac{12!}{3!9!} = 2 \times 11 \times 10$
  - $|E| = \binom{4}{3} = \frac{4!}{3!1!} = 4$
    - $$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{4}{2 \times 11 \times 10} = \frac{1}{55}$$

Example

At various yard sales a woman has acquired 5 forks, 4 knives, and 7 spoons of which no two are alike. In how many different ways can she arrange a three place settings? (assume the place at the table is not important)

- In how many ways can she set P_1? $n_1 = 5 \times 4 \times 7$
- No matter the choice for P_1, the number of choices for P_2 is the same and equal to $n_2 = 4 \times 3 \times 6$
- Number of ways to set P_3: $n_3 = 3 \times 2 \times 5$
- In total, she has $n_1 \times n_2 \times n_3 = 302400$ ways to set the table.

Example

1) How many four digit numbers can be formed by using only digits 2, 4, 6, 8, and 9?
    - $5 \times 5 \times 5 \times 5 = 625$ – order is important
2) How many of the above have some digits repeated?
    - $5 \times 4 \times 3 \times 2 = 120$ have no digits repeated.
    - $625 - 120 = 505$ have some digits repeated.

Lecture #7 – Wednesday, September 24, 2003

Example

At various yard sales a woman has acquired 5 forks, 4 knives, and 7 spoons of which no two are alike. In how many different ways can she arrange a three place settings? (assume the place at the table is not important)

- On previous solution, we assumed the place at the table is important.

- P_1 can be set in $5 \times 4 \times 7$ ways
- P_2 can be set in $4 \times 3 \times 6$ ways
- P_3 can be set in $3 \times 2 \times 5$ ways
- If place is <u>not</u> important, the number of settings is $\frac{(5 \times 4 \times 7) \times (4 \times 3 \times 6) \times (3 \times 2 \times 5)}{3!} = 50400$.

Example

Professor X is somewhat familiar with six languages.

1) To translate text from one language to another directly, how many two-way dictionaries does he need?
    - If two languages, 1; if 3 languages, 3 – it is the number of possible pairs
    - Therefore, he needs $\binom{6}{2} = 15$ two-way dictionaries
2) To translate text from one language to another directly, how many one-way dictionaries does he need?
    - For every two-way dictionaries, he would need two one-way dictionaries
    - Therefore, he needs $2\binom{6}{2} = 30$ two-way dictionaries

Example

Mr. Jones has 10 books he wants to arrange on a shelf. Of these books, 3 are mathematics books, 4 are chemistry books, and 3 are chemistry books.

1) How many arrangements of the 10 books on the shelf are possible if Mr. Jones want to indulge in a small mania and wants the 3 mathematic books to be next to each other?
    - If 3 mathematic books are together, 10 books become 8 books
    - Therefore, $8! \times 3! = 241920$ arrangements
    - $P(3 \text{ math together}) = \frac{8! \times 3!}{10!} = \frac{1}{15}$
2) How many arrangements are possible if the mania is serious and she needs to have all the books dealing with the same subject together (next to each other) on the shelf?
    - 10 books become 3 bundles
    - Therefore, $3! \times 3! \times 4! \times 3! = 5184$ arrangements

Example

Eleven chairs are numbered 1 through 11. Seven boys and four girls sit on these at random. What is the probability that chair #5 is occupied by a boy?

$$\underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}\ \ \underset{5}{\underline{B}}\ \ \underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}$$

- $|S| = 11!$, $|E| = 10! \times 7$
- $P(E) = \dfrac{10! \times 7}{11!} = \dfrac{7}{11}$

What if there are two particular girls who don't want to separated?

$$\underbrace{\underline{\quad}\ \underline{\quad}}\underbrace{\underline{\quad}\ \underline{\quad}}\underbrace{\underline{\quad}\ \underline{\quad}}\underbrace{\underline{\quad}\ \underline{\quad}}\ \ \underset{5}{\underline{B}}\ \ \underbrace{\underline{\quad}\ \underline{\quad}}\underbrace{\underline{\quad}\ \underline{\quad}}\underbrace{\underline{\quad}\ \underline{\quad}}\underbrace{\underline{\quad}\ \underline{\quad}}\underbrace{\underline{\quad}\ \underline{\quad}}\underbrace{\underline{\quad}\ \underline{\quad}}$$

- $|S| = 11!$, $|E| = 7 \times 8 \times 8! \times 2!$
- $P(E) = \dfrac{7 \times 8 \times 8! \times 2}{11!} = \dfrac{56}{495}$

What if the girls don't want to sit next to each other?

- Event at #1: E
- Event at #2: F
- Event at #3: F^c
- $P(E) = P(E \cap F) + P(E \cap F^c)$
  $P(E \cap F^c) = P(E) - P(E \cap F)$

Lecture #8 – Saturday, September 27, 2003

Example

A box contains 5 blue and 8 red balls. Jim and jack start drawing balls from the box, one at a time, at random, and without replacement until a blue ball is drawn. Given that Jim starts first, what is the probability that Jack draws the blue ball?

- Assume that even after the game stops, they keep digging out balls
- $|S| = 13!$ (assume order is counted) – each is equally likely
- $E = \{\text{Jack draws the first blue}\}$
  $= \{\text{First blue is taken out in draw \#1}(A) \text{ or \#2}(B) \text{ or \#3}(C) \text{ or \#4}(D)\}$
- $P(E) = P(A \cup B \cup C \cup D) = P(A) \cup P(B) \cup P(C) \cup P(D)$
  - $P(A) = \dfrac{8 \times 5 \times 11!}{13!}$
  - $P(B) = \dfrac{8 \times 7 \times 6 \times 5 \times 9!}{13!}$
  - $P(C) = \dfrac{8 \times 7 \times 6 \times 5 \times 4 \times 5 \times 7!}{13!}$

- $P(D) = \dfrac{8\times7\times6\times5\times4\times3\times\times2\times5\times5!}{13!}$

Example

The elevator of a four-floor story building leaves the first floor with 6 passengers and stops at all of the remaining three floors. If it is equally likely that a passenger gets off at any of these floors, what is the probability that at each stop at least one passenger departs?

- $E = \{\text{At least one passenger leaves at each floor}\}$
- $P(E^c) = 1 - P(E)$
- $E^c = \{\text{Thereis at leastonefloor}(2,3,\text{or}\ 4)\ \text{on whichonegetsoff}\}$
  $= F_2 \cup F_3 \cup F_4$
- $P(F_2 \cup F_3 \cup F_4) = P(F_2) + P(F_3) + P(F_4) - P(F_2 \cap F_3) - P(F_2 \cap F_4) - P(F_3 \cap F_4) + P(F_2 \cap F_3 \cap F_4)$
  - $P(F_2) = \dfrac{2\times2\times2\times2\times2\times2}{3\times3\times3\times3\times3\times3} = \left(\dfrac{2}{3}\right)^6$, $P(F_3) = \left(\dfrac{2}{3}\right)^6$, $P(F_4) = \left(\dfrac{2}{3}\right)^6$
  - $P(F_2 \cap F_3) = P(F_2 \cap F_4) = P(F_3 \cap F_4) = \dfrac{1}{3^6}$
  - $P(F_2 \cap F_3 \cap F_4) = 0$
- $P(E) = 1 - 3\cdot\left(\dfrac{2}{3}\right)^6 + 3\cdot\dfrac{1}{3^6} - 0$

Example

Seven men throw their hats in the middle of the room. If each picks at random one hat from the stack, what is the probability that none of them gets his own hat?

- Counting principle cannot be applied directly here
- $E = \{\text{no one gets his own hat}\}$
- $E^c = \{\text{at least one gets his own hat}\}$
- $F_i = \{\text{the } i^{\text{th}} \text{ guy gets his hat}\}$
- $$\begin{aligned} P(E^c) &= P(F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5 \cup F_6 \cup F_7) \\ &= P(F_1) + P(F_2) + \ldots + P(F_7) \\ &\quad - P(F_1 \cap F_2) - \ldots - P(F_6 \cap F_7) \\ &\quad + P(F_1 \cap F_2 \cap F_3) + \ldots + P(F_5 \cap F_6 \cap F_7) \\ &\quad - P(F_1 \cap F_2 \cap F_3 \cap F_4) - \ldots - P(F_4 \cap F_5 \cap F_6 \cap F_7) \\ &\quad + P(F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5) + \ldots + P(F_3 \cap F_4 \cap F_5 \cap F_6 \cap F_7) \\ &\quad - P(F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5 \cap F_6) - \ldots - P(F_2 \cap F_3 \cap F_4 \cap F_5 \cap F_6 \cap F_7) \\ &\quad + P(F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5 \cap F_6 \cap F_7) \end{aligned}$$
  - $P(F_1) = \dfrac{1\times6!}{7!} = \dfrac{1}{7}$, $P(F_2) = P(F_3) = P(F_4) = P(F_5) = P(F_6) = P(F_7) = \dfrac{1}{7}$
  - two-way intersections: $\binom{7}{2}\cdot\left(\dfrac{1\times1\times5!}{7!}\right) = \binom{7}{2}\cdot\left(\dfrac{1}{42}\right)$

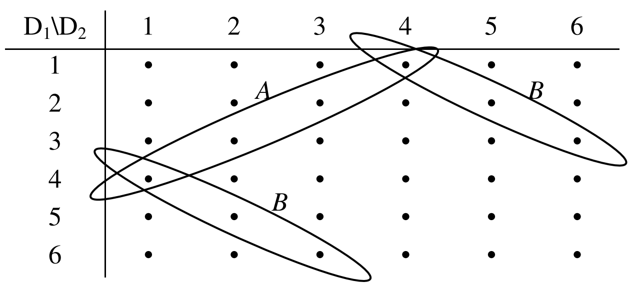
- four-way intersections: $\binom{7}{4} \cdot \left(\frac{3!}{7!}\right)$

Lecture #9 – Monday, September 29, 2003

Conditional Probability

Example

Suppose that two dice are rolled. Take x to be the number shown by the first die, and y to be the number shown by the second die.



1) What is the probability of $A = \{x + y = 5\}$?
   - $P(A) = \frac{4}{36} = \frac{1}{9}$
2) What is the probability of $B = \{|x - y| = 3\}$?
   - $P(A) = \frac{6}{36} = \frac{1}{6}$
3) Assume we know that $x + y = 5$; what is the probability of B?
   - If A is true, $P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{2}{4}$

In General

- Given an event A such that $P(A) > 0$ the conditional probability of another event B is defined

  $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$ ("probability of B given A")

Example

1) Roll two dice. What is the probability that the sum of the dice is 8?

   $A = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$
   - $P(A) = \frac{5}{36}$
2) Given that the first die is 3, what is now the probability to have sum equal to 8?
   - $B = \{\text{First die is 3}\}$
   - $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{6/36} = \frac{1}{6}$

Example

Suppose that the chances to have a daughter or to have a son are equal. The Jones' have two kids.

1) I ring their doorbell and the younger kid who happens to be a girl opens the door. What is the chance that the older kid is also a girl?
   - $S = \{(g,g),(g,b),(b,g),(b,b)\}$, where (g,b) means the youngest is a girl and eldest is a boy (no twins!)
   - $A = \{\text{older kid is a girl}\}$, $B = \{\text{younger kid is a girl}\}$
   - $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/4}{2/4} = \dfrac{1}{2}$
2) I ring their doorbell and their daughter opens the door. What is the probability that the other kid is a girl?
   - $A = \{\text{the other kid is a girl}\}$, $B = \{\text{kid is a girl}\}$
   - $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/4}{3/4} = \dfrac{1}{3}$

Example

Assume that in a family all sex distributions are equally likely.

1) From the set of all families with two children a family is selected at random and is found that one of the kids is a girl. What is the probability that the other child of the family is also a girl?
   - $S = \{(b,g),(g,b),(b,b),(g,g),\}$
   - $A = \{\text{two girls}\}$, $B = \{\text{one of them is a girl}\}$
   - $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/4}{3/4} = \dfrac{1}{3}$
2) From the set of all families with two children, a child is selected at random and is found to be a girl. What is the probability that the second child is also a girl?
   - It is like sampling from a set $S = \{4b, 4g\}$
   - $A = \{\text{her sibling is a girl}\}$, $B = \{\text{a girl is selected}\}$
   - $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/4}{1/2} = \dfrac{1}{2}$

A Little Theory

- Take an event A with $P(A) > 0$ and consider it fixed. For any other event $E \subset S$, I can define a function $Q(E) = P(E \mid A)$. It turns out Q is also a probability (it satisfy the three axioms):
  - $Q(E) > 0$ because $\dfrac{P(E \cap A)}{P(A)} \geq 0$
  - $Q(S) = 1$ – $P(S \mid A) = \dfrac{P(S \cap A)}{P(A)} = \dfrac{P(A)}{P(A)} = 1$

- If $E_1, E_2, ..., E_n$ are mutually exclusive, then $Q\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} Q(E_i)$.

  Take E_1, E_2, E_3 to be mutually exclusive ($E_1 \cap E_2 = \phi$, $E_2 \cap E_3 = \phi$, $E_2 \cap E_3 = \phi$), then

$$
\begin{aligned}
&Q(E_1 \cup E_2 \cup E_3) \\
&= P(E_1 \cup E_2 \cup E_3 \mid A) \\
&= \frac{P((E_1 \cup E_2 \cup E_3) \cap A)}{P(A)} \\
&= \frac{P((E_1 \cap A) \cup (E_2 \cap A) \cup (E_3 \cap A))}{P(A)} \\
&= \frac{P((E_1 \cap A))}{P(A)} + \frac{P(\cup (E_2 \cap A))}{P(A)} + \frac{P(\cup (E_3 \cap A))}{P(A)} \\
&= Q(E_1) + Q(E_2) + Q(E_3)
\end{aligned}
$$

Sample Space Reduction

Example

An urn contains 10 white, 5 yellow and 10 black marbles. A marble is chosen at random from the urn and it is noticed that it is not a black one. What is the probability that it is a yellow one?

- Rewrite Question: An urn contains 10 white and 5 yellow marbles. A marble taken out. What is the probability that it is a yellow one?
- $\frac{5}{15} = \frac{1}{3}$

Example

A chef mixes (by mistake!) 10 good and 5 bad eggs. A soccer team is being served the 15 eggs. If the first 4 players are still well, what is the probability that the fifth one is sick?

- I know that the first 4 eggs are ok.
- Rewrite Question: A chef mixes 6 good and 5 bad eggs. A client is served an egg. What is the probability the egg served is bad?
- $\frac{5}{11}$
- $A = \{5^{\text{th}} \text{ egg is bad}\}$, $B = \{\text{first four eggs are good}\}$
- $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\binom{10}{4}\binom{5}{1}}{\binom{10}{4}\binom{11}{1}} = \frac{5}{11}$

Lecture #10 – Wednesday, October 1, 2003

Multiplication Rule

- Conditional probability of an event *A* given an event *B* which satisfy $P(B) > 0$:

- $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{– multiplication rule for two event}$$
  $$\Rightarrow P(A \cap B) = P(A \mid B) \cdot P(B)$$

- Take events A, B, C such that $P(A) > 0$, $P(A \cap B) > 0$
  - Apply multiplication rule for two events using $A \cap B$ and C
    $$P(A \cap B \cap C) = P(C \mid A \cap B) \cdot P(A \cap B)$$
    $$= P(C \mid A \cap B) \cdot P(A \mid B) \cdot P(B) \text{ or } P(C \mid A \cap B) \cdot P(B \mid A) \cdot P(A)$$
  - I always have to make sure the events I condition on are with probability > 0

Example

- P = preview playing at Varsity
- C = choose a preview
- M = go to cinema
- $$\begin{aligned} P(M \cap C \cap P) &= P(M \mid C \cap P) \cdot P(C \cap P) \\ &= P(M \mid C \cap P) \cdot P(C \mid P) \cdot P(P) \\ &= \frac{10}{100} \cdot \frac{40}{100} \cdot \frac{10}{100} \\ &= \frac{1}{500} \end{aligned}$$

Example

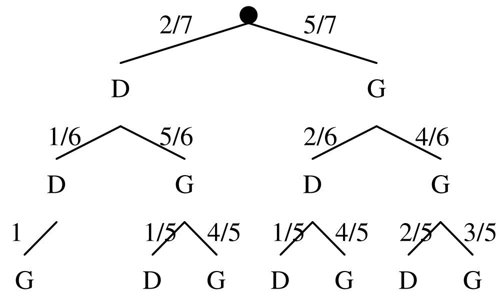
Suppose an urn contains 8 red marbles and 4 white marbles. Draw 2 marbles from the urn without replacement. What is the probability that both are red?

- $R_1 = \{\text{first marble is red}\}$, $R_2 = \{\text{second marble is red}\}$
- $P(R_1 \cap R_2) = P(R_1) \cdot P(R_2 \mid R_1) = \frac{8}{12} \cdot \frac{7}{11} = \frac{14}{33}$

- $$\begin{aligned} &P(\text{the third draw is white} \mid \text{the second draw is red}) \\ &= \frac{P(\text{third white, second red})}{P(\text{second red})} \\ &= \frac{8/12 \cdot 7/11 \cdot 4/10 + 4/12 \cdot 8/11 \cdot 3/10}{8/12 \cdot 7/11 + 4/12 \cdot 8/11} \end{aligned}$$

Example

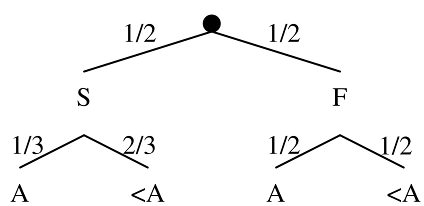
Suppose that 5 good and 2 defective fuses have been mixed up. To find the defective ones, we test them one-by-one, at random and without replacement. What is the probability that we find both of the defective fuses in exactly three tests?



- $P(\text{find fuses in exactly 3 tests}) = \frac{2}{7}\cdot\frac{5}{6}\cdot\frac{1}{5}+\frac{5}{7}\cdot\frac{2}{6}\cdot\frac{1}{5}$
- $P(\text{three is defective} \mid \text{second was good}) = \dfrac{P(\text{third is defective, second is good})}{P(\text{second is good})} = \dfrac{\frac{2}{7}\cdot\frac{5}{6}\cdot\frac{1}{5}+\frac{5}{7}\cdot\frac{4}{6}\cdot\frac{2}{5}}{\frac{2}{7}\cdot\frac{5}{6}+\frac{5}{7}\cdot\frac{4}{6}}$
- $$\begin{aligned} & P(\text{two defective in exactly 3 tests}) \\ &= P(G_1 \cap D_2 \cap D_3 \text{ or } \mathrm{D}_1 \cap G_2 \cap D_3) \\ &= P(G_1 \cap D_2 \cap D_3) + P(\mathrm{D}_1 \cap G_2 \cap D_3) \\ &= P(D_3 \mid G_1 \cap D_2)\cdot P(D_2 \mid G_1)\cdot P(G_1) + P(D_3 \mid D_1 \cap G_2)\cdot P(G_2 \mid D_1)\cdot P(D_1) \\ &= \frac{5}{7}\cdot\frac{2}{6}\cdot\frac{1}{5}+\frac{2}{7}\cdot\frac{5}{6}\cdot\frac{1}{5} \end{aligned}$$

Example

X is undecided as to whether take a French course or a Statistics course. Although she actually prefers Statistics, X estimates that her probability of getting an A in French would be 1/2 whereas it would be only 1/3 in Statistics. If X decides to make up her mind based on the flip of a coin, what is her chance to get an A in Statistics. What is her chance to get an A?



- $P(\text{she gets A in Statistics}) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$
- $P(\text{she gets an A}) = \frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{6}+\frac{1}{4} = \frac{5}{12}$

Lecture #11 – Friday, October 10, 2003

LAW OF TOTAL PROBABILITY

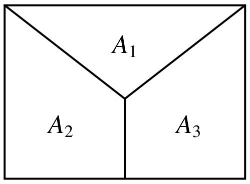
Partition

- A family of sets $\{A_1, A_2, ..., A_n\}$ is a partition of the sample space S if:
  - They are mutually exclusive – $A_i \cap A_j = \phi$ for $i, j \in 1,2,...,n, i \neq j$

- $\bigcup_{i=1}^{n} A_i = S$

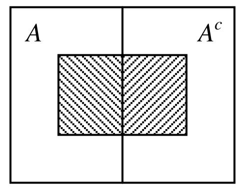
Examples

1) $\{S, \phi\}$
2) $\{A, A^c\}$ for any event A, $A \cap A^c = \phi$ and $A \cup A^c = S$



3) $\{A, A^c\}$, $P(B) = ?$

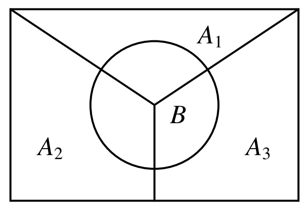
$$B = (A \cap B) \cup (A \cap A^c)$$
$$P(B) = P(B \cap A) + P(B \cap A^c)$$
$$= P(B \mid A)P(A) + P(B \mid A^c)P(A^c)$$



4) $\{A_1, A_2, A_3\}$ partition

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3)$$
$$= P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + P(B \mid A_3)P(A_3)$$

where $P(A_1), P(A_2), P(A_3) > 0$



Generalization

- If $\{B_1, \ldots, B_n\}$ is a partition of S and $P(B_i) > 0$ for $i = 1, \ldots, n$, then $P(A) = \sum_{i=1}^{n} P(A \mid B_i)P(B_i)$

Example

Suppose that 80% seniors, 70% juniors, 50% sophomores, and 30% freshmen of a college use the campus library frequently. If 30% of all students are freshmen, 25% sophomores, 25% juniors, and 20% seniors, what is the probability of all students use the library frequently?

- $A = \{\text{Students use library frequently}\}$
- $Sn = \{\text{Student is a senior}\}$, $J = \{\text{Student is a junior}\}$, $F = \{\text{Student is a freshman}\}$,
  $So = \{\text{Student is a sophomore}\}$

- $P(A) = ?$
- $P(A \mid Sn) = 80\%$, $P(A \mid J) = 70\%$, $P(A \mid So) = 50\%$, $P(A \mid F) = 30\%$

- Partition: $S = F \cup So \cup Jn \cup Sn$
  $P(A) = P(A \mid F)P(F) + P(A \mid So)P(So) + P(A \mid Jn)P(Jn) + P(A \mid Sn)P(Sn)$
- $= (30\%)(30\%) + (50\%)(25\%) + (70\%)(25\%) + (80\%)(20\%)$
  $= 55\%$

Example

An insurance company rents 35% of the cars for its customers from agency I and 65% from agency from agency II. If 8% of the cars of agency I and 5% of the cars of agency II break down during the rental periods, what is the probability that a car rented by this insurance company breaks down?

- $A = \{\text{Car breaks down}\}$, $B_1 = \{\text{Car rented from agency I}\}$, $B_2 = \{\text{Car rented from agency II}\}$
- B_1 and B_2 is a partition for all cars
- $P(B_1) = 35\%$, $P(B_2) = 65\%$; $P(A \mid B_1) = 8\%$, $P(A \mid B_2) = 5\%$
- $$\begin{aligned} P(A) &= P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) \\ &= (8\%)(35\%) + (5\%)(65\%) \\ &= 6.05\% \end{aligned}$$