# Lecture #12 – Wednesday, October 15, 2003

## PARTITIONS

1) The sets $\{A_1, A_2, ..., A_n\}$ are mutually exclusive
2) $\bigcup_{i=1}^{n} A_i = S$

- If $A_1, \ldots, A_n$ is a partition then for any event $B \subset S$, $B = \bigcup_{i=1}^{n} (B \cap A_i)$

### Example

- $B = (B \cap A) \cup (B \cap A^c)$

## LAW OF TOTAL PROBABILITY

- If $\{B, B^c\}$ is a partition, then $P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c)$
- If $\{B_1, B_2, B_3\}$ is a partition of $S$ and $P(B_i) > 0$ for $i = 1,2,3,...$, then $P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)$

### Generalization

- $P(A) = \sum_{i=1}^{n} P(A \mid B_i)P(B_i)$

### Example

A transport company needs to do safety checking for its cars. Suppose there are only two repair stores in town. One of them is owned by a rival firm and the other is owned by an allied firm so the diagnostics will not be objective. The friendly garage will lie with probability 20% if a car is unsafe and will tell the truth if the car is safe. The "rival" garage with probability 35% will declare a car unsafe even if the car is safe and will always declare the truth if the car is unsafe. Assume that 20% of the company's cars are actually unsafe and assume that the company will obtain a certificate from each garage for each of its cars. What is the percentage of cases in which the two conclusions are contradictory?

- $G = \{\text{Good cars}\}$, $B = \{\text{Bad cars}\}$
- $C = \{\text{Conclusions are contradictory}\}$

- $P(C) = P(C \cap G) + P(C \cap B)$ because $G$ and $B$ forms a partition of all cars
  $= P(C \mid G)P(G) + P(C \mid B)P(B)$
  - $P(C \mid G) = 35\%$ because the friendly garage tells the truth and the rival garage tells the lie
  - $P(C \mid B) = 20\%$ because the rival garage tells the truth and the friendly garage tells the lie
- $P(C) = \frac{35}{100} \times \frac{80}{100} + \frac{20}{100} \times \frac{20}{100} = \frac{32}{100}$

## BAYES RULE

- Start with a partition $\{B_1, B_2, B_3,\}$
- Take an event $A$

- Instead of being interested in $P(A \mid B_1)$, we are now interested in $P(B_1 \mid A)$

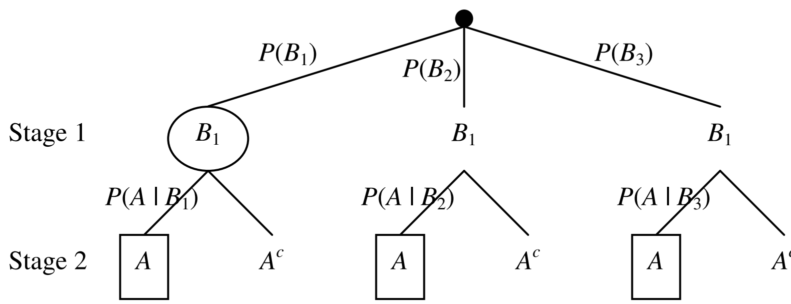
$$P(B_1 \mid A) = \frac{P(B_1 \cap A)}{P(A)}$$

$$= \frac{P(A \mid B_1)P(B_1)}{P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)}$$

$$= \frac{P(A \mid B_1)P(B_1)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)}$$

### Experiment In 2 Stages

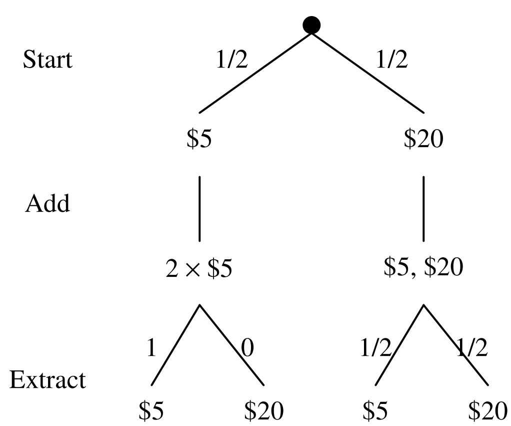
- First stage – one of the $B$'s happens
- Second stage – we check whether $A$ happens




- I know $A$ happened. What is the probability that $B_1$ happened?
  - $P(B_1 \mid A) = \dfrac{P(B_1 \cap A)}{P(A)} = \dfrac{P(A \mid B_1)P(B_1)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)}$
  - $P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)$
  - $P(B_1 \cap A) = P(A \cap B_1) = P(A \mid B_1)P(B_1)$

### Example

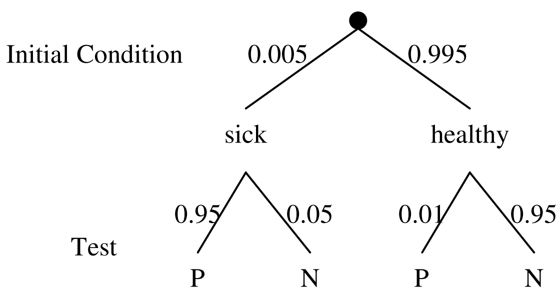
Say I have a wallet that contains either a \$5 bill or a \$20 bill (with equal probability), but I don't know which one. I add a \$5 bill. Later, I reach into my wallet (without looking) and remove a bill. It's a \$5 bill. What is the probability that the bill left in the wallet is a \$5 bill?



- $P(\text{start with \$5} \mid \text{extract a \$5}) = \dfrac{\frac{1}{2}\cdot 1}{\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{2}} = \dfrac{2}{3}$

#### Example

A lab blood test is 95% effective in detecting a certain disease when it is in fact, present. However the test also yields a “false positive” result for 1% of the healthy people tested.
If 0.5% of the population has the disease, what is the probability hat a person who was tested positive actually has the disease?




- $P(\text{sick} \mid \text{positive}) = \dfrac{P(\text{sick} \cap \text{positive})}{P(\text{positive})} = \dfrac{P(\text{positive} \mid \text{sick})P(\text{sick})}{P(\text{positive})}$
  - $P(\text{positive}) = 0.005 \times 0.95 + 0.995 \times 0.01$
  - $P(\text{sick} \cap \text{positive}) = 0.005 \times 0.95$
- $P(\text{sick} \mid \text{positive}) = \dfrac{0.005 \times 0.95}{0.005 \times 0.95 + 0.995 \times 0.01} = 32.31\%$

# Lecture #13 – Wednesday, October 22, 2003

## INDEPENDENCE

- Two events, $A$ and $B$ are independent if $P(A \cap B) = P(A) \cdot P(B)$.
- In general, $P(A \cap B) = P(A \mid B) \cdot P(B)$.
  - If $A$ and $B$ are independent, $P(A \mid B) = P(A)$ – The fact that $B$ happened doesn’t change the chance of the occurrence of $A$.
- In general, $\{A_1, A_2, \ldots, A_n\}$ are independent if any two $A_i$, $A_j$ are independent.

#### Example

An urn contains five red and seven blue balls. Suppose that two balls are selected at random and with replacement. Let $A$ and $B$ be the events that the first and the second balls are red, respectively. Check whether $A$ and $B$ are independent or not. Redo the calculation for the case of random selection without replacement.

- With replacement:
  - $P(A \cap B) = \dfrac{5 \times 5}{12 \times 12}$.

- $P(A)=\frac{5}{12}$, $P(B)=\frac{5}{12}$.
- So $A$, $B$ are independent ( $A \perp B$ ).

- Without replacement:
  - $P(A \cap B)=\frac{5\times 4}{12\times 11}=\frac{20}{12\times 11}$.
  - $P(A)=\frac{5}{12}$, $P(B)=P(B \mid A)P(A)+P(B \mid A^c)P(A^c)=\frac{4}{11}\cdot\frac{5}{12}+\frac{5}{11}\cdot\frac{7}{12}=\frac{55}{11\times 12}=\frac{5}{12}$.
  - So $A$ and $B$ are not independent.

### Exercises

1) Show that if $E$ and $F$ are independent then $E$ and $F^c$ are independent. What about then $E$ and $F^c$?
   - I want to show $P(E \cap F^c)=P(E)\cdot P(F^c)$.

     $$\begin{aligned} P(E \cap F^c) &= P(E \cap F^c)+P(E \cap F)-P(E \cap F) \\ &= P(E)-P(E \cap F) \\ &= P(E)-P(E)\cdot P(F) \\ &= P(E)\cdot(1-P(F)) \\ &= P(E)\cdot P(F^c) \end{aligned}$$
   - $E \perp F \Rightarrow E \perp F^c \Rightarrow E^c \perp F^c$

2) Show that if $E$ and $F$ are mutually exclusive with $P(E)>0$ and $P(F)>0$, then $E$, $F$ cannot be independent.
   - $P(E \cap F)=0$.
   - If $E$ and $F$ are independent, $P(E \cap F)=P(E)\cdot P(F)>0$.
   - So $A$ and $B$ are not independent if they are mutually exclusive.

3) True or False: If $E$ and $F$ are independent and $E$ and $G$ are independent, then $E$ and $F \cap G$ are independent.
   - False!

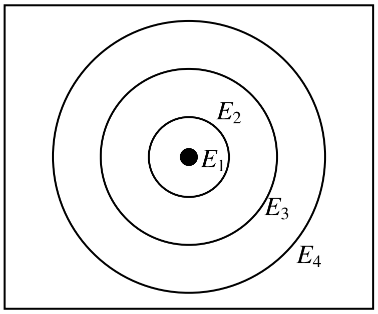
4) True or False: In the case of 3 events $E$, $F$, $G$, the equality $P(E \cap F \cap G)=P(E)\cdot P(F)\cdot P(G)$ doesn't imply that $E$, $F$, $G$ are independent.
   - True!

## PROBABILITY AS A CONTINUOUS FUNCTION

- $f:\mathbf{R}\to\mathbf{R}$.
- $x_n \xrightarrow[n\to\infty]{} x$.
- $f$ is continuous in $x$ iff $\lim\limits_{x\to\infty} f(x_n)=f(x)$
- $P$ : sets from $S \to [0,1]$

### Increasing Sequence of Events

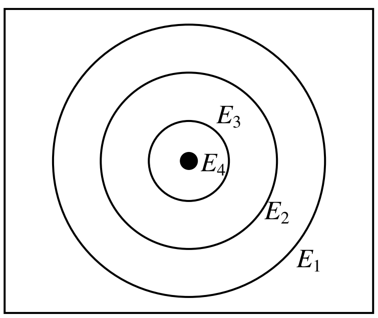
- $\{E_1, E_2, ..., E_n\}$ infinite sequence is increasing. $E_k \subset E_{k+1}, \forall k \geq 1$.




- The "limit" is the $\bigcup_{k=1}^{\infty} E_k$.

### Decreasing Sequence of Events

- $\{E_1, E_2, ..., E_n\}$ infinite sequence is decreasing. $E_k \supset E_{k+1}, \forall k \geq 1$.




- The "limit" is the $\bigcap_{k=1}^{\infty} E_k$.

- $\{E_n\}$ is decreasing or increasing. I can define $\lim E_n = \left(\bigcap E_n, \bigcup E_n\right)$.
- In both cases, $\lim_{n\to\infty} P(E_n) = P\left(\lim_{n\to\infty} E_n\right)$.

# Lecture #14 – Friday, October 24, 2003 10 26

- If sets are increasing, then $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$ – the smallest set containing all the sets.
- If sets are increasing, then $\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$ – the smallest set containing all the sets.

### Example

If the probability that the entire population will die before having offspring in the $n^{\text{th}}$ is $e^{-\frac{2n^2}{6n^2+3}}$, what is the probability that it will survive forever?

- Let $E_n = \{\text{population wiped out by the } n^{\text{th}} \text{ generation}\}$.
- So $E_1 \subset E_2 \subset \ldots \subset E_n \subset E_{n+1}$.
- $$\begin{aligned} P(\text{survives forever}) &= 1 - P(\text{dies out sometime in the future}) \\ &= 1 - P(E_1 \cup E_2 \cup \ldots \cup E_n) \\ &= 1 - P\left(\bigcup_{n=1}^{\infty} E_n\right) \\ &= 1 - \lim_{n\to\infty} P(E_n) \end{aligned}$$
- $$P(E_n) = \exp\left[-\frac{2n^2}{6n^2+3}\right]$$
- $$\lim_{n\to\infty} P(E_n) = \lim_{n\to\infty} \exp\left[-\frac{2n^2}{6n^2+3}\right] = \exp\left[\lim_{n\to\infty} \frac{-2n^2}{6n^2+3}\right] = \exp\left[-\frac{2}{6}\right] = e^{-\frac{1}{3}}$$

# Lecture #15 – Monday, October 27, 2003

## Random Variables

- In many situations when an experiment is performed the interest is in some numerical function of the outcome rather than the actual outcome itself.
- If $S$ is the sample space of an experiment then a map $X : S \to R$ is called a random variable.
- Additional Requirement: For any interval $I \subset R$, $X^{-1}(I)$ is an event in $S$.

### Example

- Tomorrow: rain, snow, shine.
- I bet \$5 that it's rain.
- $X = \text{my profit} = \{-\$5, \$5\}$
- $P(X = \$5)$
- $Y = \text{profit from betting } \$100 \text{ on snow}$

- If $S$ is a discrete sample space, then $X : S \to \mathbf{R}$ is a discrete random variable. The set of possible values of $X$ is also discrete – $X : S \to \{a_1, a_2, \ldots, a_n\}$
- Some $S$'s may not be discrete.

### Example

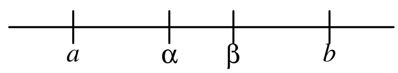
Suppose that three cards are drawn from an ordinary deck of 52 cards one-by-one at random and with replacement. Let $X$ be the number of spades drawn. Find $P(X = i), i = 0,1,2,3$.

- Possible values for random variable $X$ are $\{0,1,2,3\}$. $X : S \to \{0,1,2,3\}$.

- $P(X=0)=\dfrac{39\times39\times39}{52\times52\times52}$, $P(X=1)=\dfrac{13\times39\times39}{52\times52\times52}\times3$, $P(X=2)=\dfrac{13\times13\times39}{52\times52\times52}\times3$, $P(X=0)=\dfrac{13\times13\times13}{52\times52\times52}$

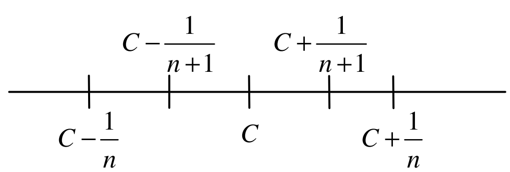
## Random Selection of Points In Intervals

- Fix $a<b$ and α, β such that $a\le\alpha<\beta\le b$. The probability that a point is randomly selected in the interval $(\alpha,\beta)$ is $\dfrac{\beta-\alpha}{b-a}$.




  - Sample space is not discrete but continuous (interval).

- Let $C$ be a fixed point in the interval $(a,b)$. If $X$ is a point randomly selected in the interval $(a,b)$ then the probability that $X$ is selected to be exactly $C$ is $P(X=C)=0$.
  - If $(a,b)$ was discrete $\{a_1,a_2,\dots,a_n\}$, then $P(X=a_i)=\dfrac{1}{n}$
  - Define $E_n=\left(C-\dfrac{1}{n},C+\dfrac{1}{n}\right)$.

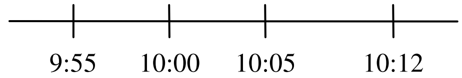



  - $E_n\supset E_{n+1}$, so $E_{n+1}=\left\{C-\dfrac{1}{n+1},C+\dfrac{1}{n+1}\right\}$. Sequence of events $E_n=\left\{C-\dfrac{1}{n},C+\dfrac{1}{n}\right\}$ which is decreasing.
  - $P(X=C)=P(X\in E_1 \text{ and...and } X\in E_n)=P\left(\bigcap_{n=1}^{\infty}X\in E_n\right)=\lim_{n\to\infty}P(X\in E_n)=\lim_{n\to\infty}\dfrac{\frac{2}{n}}{b-a}=0$

### Example

A train passes through a town at random time between 10:00 am and 10:12 pm. If I drive through town between 9:55 am and 10:05 am, what is the probability that I see the train?

- $(a,b)=(10:00\text{am},10:12\text{am})$




- $P(\text{I see the train})=P(\text{train goes through between }10:00\text{ and }10:05)=\dfrac{5}{12}$ – does not depend on the scale of measurement.

### Example

A patient with flu may have a fever between 39°C and 42°C. Let *X* be the temperature of a randomly selected flu patient. What is the probability that the temperature is less than 40°C?

- $S = \{\text{all possible temperatures}\} = \{39°\text{C}, 40°\text{C}\}$
- $(a,b) = (39,42)$, $(\alpha,\beta) = (39,40)$
- $P(X \in (39,40)) = \dfrac{40-39}{42-39} = \dfrac{1}{3}$
- $P(X = 39.5) = 0$
- $P(X \in (39.49, 39.51)) = \dfrac{0.02}{3}$

- For continuous random variable, we look at $P(\text{interval})$ instead of $P(\text{point})$.

### Definition

- For a discrete random variable $X : S \to \{a_1, a_2, ..., a_n\}$, we can define a function $P : \{a_1, a_2, ..., a_n\} \to [0,1]$ such that $P(a_i) = P(X = a_i)$.
- *P* is called probability function of the random variable *X*.

# Lecture #16 – Wednesday, October 29, 2003

## Distribution Functions

- The distribution function $F : \mathbf{R} \to [0,1]$ of a random variable *X* is defined as $F(t) = P(X \le t)$.
- For example, $F(1) = P(X \le 1)$.

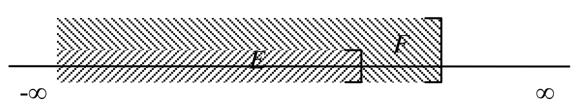
### Properties of *F*

1) *F* is non-decreasing – i.e. $t_1 \le t_2 \Rightarrow F(t_1) \le F(t_2)$. So F is increasing or constant.
2) $\lim\limits_{t \to \infty} F(t) = 1$
3) $\lim\limits_{t \to -\infty} F(t) = 0$
4) F is continuous to the right.

### Proof of (1)

$t_1 \le t_2$. I want to show $F(t_1) \le F(t_2)$

- $F(t_1) = P(X \le t_1)$, $F(t_2) = P(X \le t_2)$.
- $E = \{X \le t_1\}$, $F = \{X \le t_1\}$
- $E \subset F \Rightarrow P(E) \le P(F)$




- $\left.\begin{array}{l} P(E) = P(X \le t_1) = P(t_1) \\ P(F) = P(X \le t_2) = P(t_2) \end{array}\right\} \Rightarrow F(t_1) \le F(t_2)$
- Attention: $t_1 < t_2$ does not imply $F(t_1) < F(t_2)$; $t_1 < t_2$ implies $F(t_1) \le F(t_2)$

**Proof of (2)**

- $X : S \to \mathbf{R}$ – real random variable.
- Always, there will be $t$ large enough to have $P(X \le t)$ is almost 1.
- Let $E_n = \{X \le n\}, n \in \mathbf{Z}$.
- $\lim\limits_{t\to\infty} F(t) = \lim\limits_{n\to\infty} F(n) = \lim\limits_{n\to\infty} P(n)$.
- $E_n \subset E_{n+1}$ $\Rightarrow$ the sequence $E_n$ is increasing. I can use the continuity property of the probability function.
- $\lim\limits_{n\to\infty} P(E_n) = P\left(\lim\limits_{n\to\infty} E_n\right) = P\left(\bigcup\limits_{n=1}^{\infty} E_n\right) = P\left(\bigcup\limits_{n=1}^{\infty} X \le n\right) = 1$

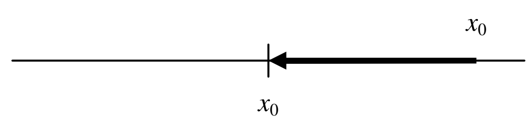
**Proof of (3)**

- The proof of $\lim\limits_{t\to-\infty} F(t) = 0$ is the mirror of the proof of $\lim\limits_{t\to\infty} F(t) = 1$.

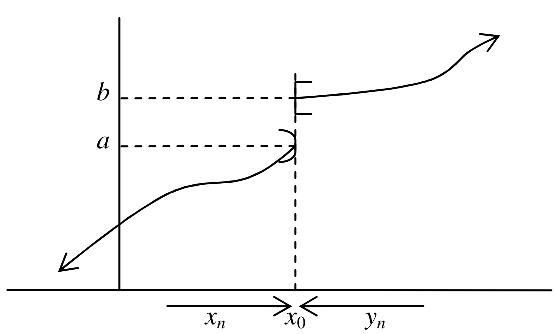
**Proof of (4)**

- Convergence from the left means for all $x_n \le x_0$ and $x_n \to x_0$. In fact, for all practical purposes, one can consider only increasing sequences. $x_n < x_{n+1} \forall n \in \mathbf{Z}$ and $x_n \le x_0$ and $\lim\limits_{n\to\infty} x_n = 0$.
- Notation: $x_n \uparrow x_0$ – converges to $x_0$ from the left. $\lim\limits_{x_n \uparrow x_0} f(x_n) = f(x_0 -)$.
- If $f(x_0 -) = f(x_0)$, then $f$ is continuous to the left.

- Continuity to the right.
- $\left.\begin{matrix} x_n \ge x_{n+1} \\ x_n \ge x_0 \end{matrix}\right\} \forall n \in \mathbf{Z}$



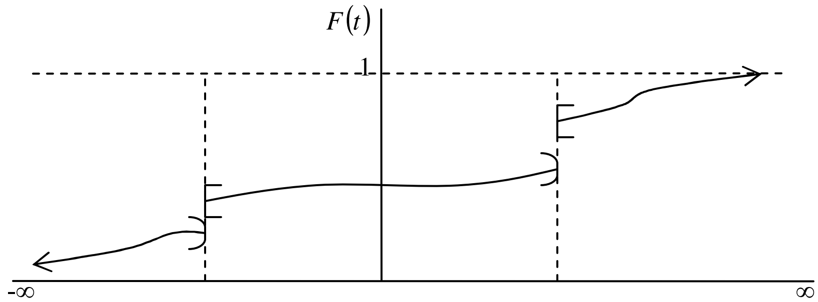

- The limit to the right is denoted $x_n \downarrow x_0$ – $\lim\limits_{x_n \downarrow x_0} f(x_n) = f(x_0 +)$.
- If $f(x_0 +) = f(x_0)$, then $f$ is called continuous to the right.




- $f(x_0) = b$
- $f(x_0 -) = \lim\limits_{x_n \uparrow x_0} f(x_n) = a$
- $f(x_0 +) = \lim\limits_{x_n \downarrow x_0} f(y_n) = b$

- $f(x_0) = f(x_0 +)$ $\Rightarrow f$ is continuous to the right, but $f$ is not continuous to the left.

**Generic *F*(*t*)**



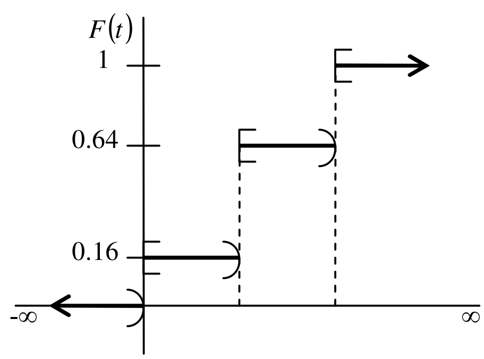

- Multiple jumps (or none).

**Example**

Suppose we flip twice a coin that has probability to land heads equal to 0.4. Let *X* be the number of tails. Calculate $F(t)$, the distribution function of *X*.

- $X : S \to \{0,1,2\}$
- $P(X=0)+P(X=1)+P(X=2)=1$
- $\left.\begin{array}{l} P(X=0)=0.4\cdot 0.4=0.16 \\ P(X=0)=0.4\cdot 0.6+0.6\cdot 0.4=0.48 \\ P(X=2)=0.36 \end{array}\right\}$ Probability function of $X$ .

- $F(t)=P(X\le t)$ for any $t\in \mathbf{R}$ :
    - If $t<0$, $P(X\le t)=F(0)=0$ .
    - If $t=0$, $P(X\le 0)=P(X=0)=0.16$ .
    - If $t\in(0,1)$, $P(X\le t)=0.16$ .
    - If $t=1$,
      $P(X\le 1)=P(X=0)+P(X=1)=0.64$ .
    - If $t\in(1,2)$, $P(X\le t)=0.64$ .
    - If $t\ge 2$, $P(X\le t)=1$ .




- Sometimes *F* called cumulative distributive function.
- Given a probability function for a discrete random variable *X*, one should be able to construct the distribution function for *X*.

# Lecture #17 – Friday, October 31, 2003

**Example**

From 18 potential women jurors and 28 potential men jurors, a jury of 12 is chose at random. Let *X* be the number of women selected. Find the probability function of *X*.

- $S=\{\text{set of all possible jurors}\}$
- $X : S \to \{0,1,\ldots,12\}$

- Probability function of $X$ is $P:\{0,1,2,...,12\}\rightarrow[0,1]$
  - $P(0)=P(X=0)=\dfrac{\binom{18}{0}\binom{28}{12}}{\binom{46}{12}}=0.000782$
  - $P(1)=P(X=1)=\dfrac{\binom{18}{1}\binom{28}{11}}{\binom{46}{12}}=0.001$
  - $P(4)=P(X=4)=\dfrac{\binom{18}{4}\binom{28}{8}}{\binom{46}{12}}=0.244$

### Connection Between Probability and Distribution

| Event concerning $X$ | Probability of the event in terms of $F$ |
|---|---|
| $X \le a$ | $F(a)$ |
| $X > a$ | $1-F(a)$ |
| $X \ge a$ | $1-F(a-)$ |
| $X = a$ | $F(a)-F(a-)$ |
| $a < X \le b$ | $F(b)-F(a)$ |
| $a < X < b$ | $F(b-)-F(a)$ |
| $a \le X \le b$ | $F(b)-F(a-)$ |
| $a \le X < b$ | $F(b-)-F(a-)$ |

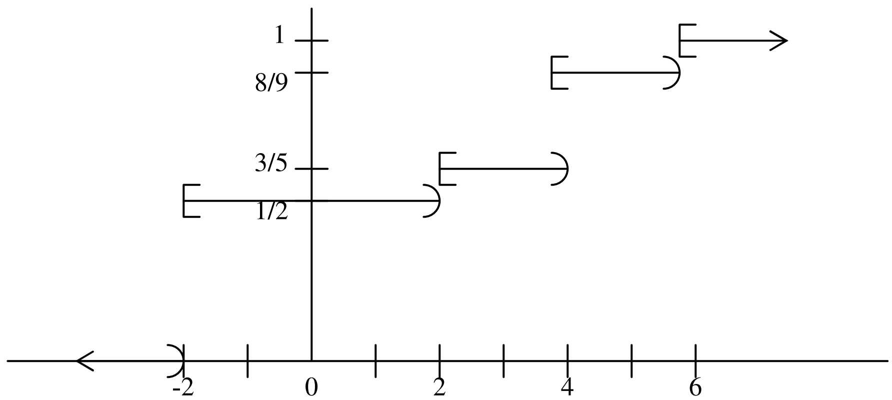
- $P(a<X\le b)=P(\{X\le b \text{ but not } \{X\le a\}\})=P(\{X\le b\}-\{X\le a\})$
  $P(X\in(a,b])=P(X\le b)-P(X\le a)=F(b)-F(a)$
- $P(a\le X<b)=P(X\in[a,b))=P(X\in\{(-\infty,b)-(-\infty,a)\})=P(X\in(-\infty,b))-P(X\in(-\infty,a))$
  $=P(X<b)-P(X\le a)=F(b-)-F(a-)$

### Example

The distribution function of a random variable $X$ is given by:

$$F(X)=\begin{cases}0 \text{ if } x<-2\\ \frac{1}{2} \text{ if } -2\le x<-2\\ \frac{3}{5} \text{ if } 2\le x<4\\ \frac{8}{9} \text{ if } 4\le x<6\\ 1 \text{ if } 6\le x\end{cases}$$

. Calculate the probability function of $X$.




- $P(X=-3)=0$. Why?
    - Take a sequence that converges to -3 from the left.
    - $P(X=-3)=P(X\le -3)-P(X<-3)=F(-3)-F((-3)-)=F(-3)-\lim_{X_n \uparrow -3} F(X_n)=0-0=0$.

- The set of all possible values that *X* can take is given, in general, by those points *t* such that $F(t-)\ne F(t+)=F(t)$.
- In this example the set is $\{-2,2,4,6\}$.
    - $P(X=-2)=F(-2)-F((-2)-)=\frac{1}{2}$.
    - $P(X=2)=F(2)-F(2-)=\frac{3}{5}-\frac{1}{2}=\frac{1}{10}$.
    - $P(X=4)=\frac{8}{9}-\frac{3}{5}=\frac{13}{45}$.
    - $P(X=6)=\frac{1}{9}$.

### Summary

- Both probability function and distribution function characterize completely the random variable *X*.
- Given the probability function *P*, we can construct the distribution function *F* and vice-versa.

# Lecture #18 – Monday, November 3, 2003

## Expectation of a Discrete Random Variable

### Example

If I flip a coin (fair: $P(\mathrm{T})=P(\mathrm{H})=\frac{1}{2}$ ). If T, I gain \$1. If H, I lose \$1.

- $X=\{\text{my gain}\}$
- $X:\{\mathrm{H},\mathrm{T}\}\to\{-1,1\}$

- If I repeat many times, the expected gain is 0 – $P(\mathrm{T})\cdot 1+P(\mathrm{H})\cdot(-1)=\frac{1}{2}-\frac{1}{2}=0$.

**Definition**

- $X : S \to \{a_1, a_2, ..., a_n\}$.
- The expected value of the discrete random variable $X$ is defined $E[X] = \sum_{i=1}^{\infty} P(X = a_i) \cdot a_i = \sum_{i=1}^{\infty} P(a_i) \cdot a_i$ .

**Properties of *E*[*X*]**

1) If $X$ is constant, $X = c$ – $E[X] = c \cdot P(X = c) = c \cdot 1 = c$ .

2) $E[X + c] = \sum_{n=1}^{\infty} P(a_i) \cdot (a_i + c)$ .
   - Define $Y = X + c$ .
   - $Y$ is also discrete and every time $X$ takes value $a_i$, $Y$ takes value $a_i + c$ .
   - $$E[X + c] = E[Y] = \sum_{i=1}^{\infty} (a_i + c) \cdot P(Y = a_i + c) = \sum_{i=1}^{\infty} (a_i + c) \cdot P(X = a_i) = \sum_{i=1}^{\infty} (a_i + c) \cdot P(a_i)$$
     $$= \sum_{i=1}^{\infty} (a_i) \cdot P(a_i) + c \sum_{i=1}^{\infty} P(a_i) = E[X] + c$$

3) $E[X \cdot C] = c \cdot E[X]$ .
   - Define $Y = X \cdot c$ .
   - If $X = a_i \Leftrightarrow Y = c \cdot a_i$ .
   - $E[X \cdot C] = E[Y] = \sum_{i=0}^{\infty} P(Y = c \cdot a_i) \cdot (c \cdot a_i) = c \sum_{i=0}^{\infty} P(X = a_i) \cdot a_i = c \cdot E[X]$.

**Example**

In a certain part of downtown Toronto parking lots charge \$10 per day. A car that is illegally parked on the street will be fined \$20 if caught and the chance of being caught is 70%. If money is the only concern, should one park illegally or not?

- $X = \text{What I pay per day if I park legally}$ , $Y = \text{What I pay per day if I park illegally}$ .
- $X = \$10$ is a constant variable.
- $Y = \{0, 20\}$ – $P(Y = 0) = 30\% = \frac{3}{10}$ , $P(Y = 20) = 70\% = \frac{7}{10}$ .
- $E[X] = 10$
- $E[Y] = P(Y = 0) \cdot 0 + P(Y = 20) \cdot 20 = \frac{3}{10} \cdot 0 + \frac{7}{10} \cdot 20 = 14$ .
- On average, we are better off if we park legally.

**Example**

In a lottery, a player pays \$1 and selects four distinct numbers from 0 to 9. Then from an urn containing 10 identical balls numbered 0 to 9, four balls are drawn at random and without replacement. If the numbers of three or all four of these balls matches the player's numbers, he wins \$5 and \$10, respectively. Otherwise he loses. On average, how much money does the player gain per game (gain = win – loss).

- $X = \text{gain after one round of the game}$ .
- $X : S \to \{-1, 4, 9\}$.

- $P(X=9)=\frac{\binom{4}{4}}{\binom{10}{4}}=0.005$, $P(X=4)=\frac{\binom{4}{3}\binom{6}{1}}{\binom{10}{4}}=0.114$, $P(X=-1)=1-P(X=9)-P(X=4)=0.881$.
- $E[X]=-1\cdot 0.881+4\cdot 0.114+9\cdot 0.005=-0.38$
- On average, he loses 38¢ per game.

### Example

In the US the number of twin births is approximately 1 in 90. Let *X* be the number of births until the first twins are born. Calculate distribution function of *X*.

- $X: S\to\{0,1,2,...,n\}$
- $P(X=0)=\frac{1}{90}$, $P(X=1)=\frac{89}{90}\cdot\frac{1}{90}$, $P(X=2)=\left(\frac{89}{90}\right)^2\cdot\frac{1}{90}$. So $P(X=n)=\left(\frac{89}{90}\right)^n\cdot\frac{1}{90}$.
- $\sum_{n=1}^{\infty}\left(\frac{89}{90}\right)^n=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{89}{90}\right)^i$

  $$\frac{89}{90}+\left(\frac{89}{90}\right)^2+...+\left(\frac{89}{90}\right)^n=R_n\Rightarrow\left(\frac{89}{90}\right)^2+\left(\frac{89}{90}\right)^3+...+\left(\frac{89}{90}\right)^{n+1}=\frac{89}{90}\cdot R_n$$

  - $$\Rightarrow R_n\left(1-\frac{89}{90}\right)=\frac{89}{90}-\left(\frac{89}{90}\right)^{n+1}\Rightarrow R_n=\frac{\left(\frac{89}{90}\right)-\left(\frac{89}{90}\right)^{n-1}}{1-\frac{89}{90}}$$
  - $\lim_{n\to\infty}R_n=\frac{\frac{89}{90}}{1-\frac{89}{90}}=89$
- $E[X]=0\cdot\frac{1}{90}+\left(\frac{89}{90}\right)\cdot\frac{1}{90}+...+n\left(\frac{89}{90}\right)^n\cdot\frac{1}{90}=\sum_{n=1}^{\infty}\left(\frac{89}{90}\right)^n\frac{n}{90}=\frac{1}{90}\cdot\frac{89}{90}\sum_{n=1}^{\infty}\left(\frac{89}{90}\right)^{n-1}\cdot n=\lim_{n\to\infty}\sum_{i=1}^{\infty}n\left(\frac{89}{90}\right)^{i-1}$.

# Lecture #19 – Wednesday, November 5 2003

- Sometimes we want $E[g(x)]$ where $g=\{a_1,a_2,...,a_n,...\}\to\mathbf{R}$ (a real function).
  - $g(x)=x^2$ – $E[X^2]$
  - $g(x)=\log x$ – $E[\log X]$

### Law of the Unconscious Probabilist

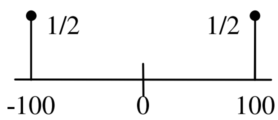
- $E[g(x)]=\sum_{i=1}^{\infty}g(a_i)\cdot p(a_i)$
- $E[X]=\sum_{i=1}^{\infty}a_i\cdot p(a_i)$

- If $X$ is a random variable $X : S \to \{a_1, a_2, ..., a_n, ...\}$ with probability mass function $p(\cdot)$ and $g$ is a map $g : \{a_1, a_2, ..., a_n, ...\} \to \mathbf{R}$ then $E[g(x)] = \sum_{i=1}^{\infty} g(a_i) \cdot p(a_i)$.

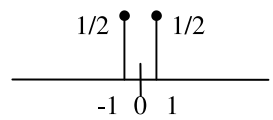
### Example: 3 Gamblers

1) Risk-lover: Flips a coin. If T, wins \$100; if H, loses \$100. ($X_1$).
2) Conservative-gambler: If T, wins \$1; if H, loses \$1. ($X_2$).
3) Boring: Does not gamble. ($X_3$).

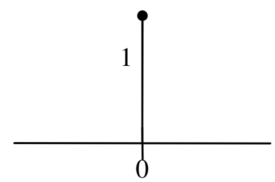
- $X_i$ = gain by player $i$.
- $E[X_1] = 100\frac{1}{2} - 100\frac{1}{2} = 0$




- $E[X_2] = 1\frac{1}{2} - 1\frac{1}{2} = 0$



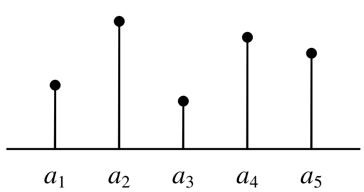

- $E[X_3] = 0\frac{1}{2} - 0\frac{1}{2} = 0$

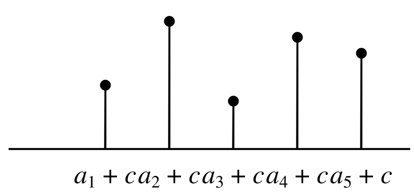



## VARIANCE

- Another interesting quantity we can look at is, beside the center, the spread of a distribution. How far apart the probability function "spikes" are.
- Formally, this is measured using the variance of a random variable. It is denoted $\mathrm{Var}(X)$.
  - $\mathrm{Var}(X) = E\left[(X - E(X))^2\right]$
- Let $E[X] = \mu$.
  - $$\mathrm{Var}(X) = E\left[(X - E(X))^2\right] = E\left[X^2 - 2\mu X + \mu^2\right] = E\left[X^2\right] - E[2\mu X] + E\left[\mu^2\right] = E\left[X^2\right] - 2\mu^2 + \mu^2$$ $$= E\left[X^2\right] - \mu^2 = E\left[X^2\right] - (E[X])^2$$

1) $X = c$ a constant:
   - $\mathrm{Var}(X) = E\left[(c - E(c))^2\right] = 0$.
   - The variance is never negative.
2) $\mathrm{Var}(X + c) = \mathrm{Var}(X)$
   - $\mathrm{Var}(X + c) = E\left[((X + c) - E[X + c])^2\right] = E\left[(X - E[X])^2\right] = \mathrm{Var}(X)$.
   - Spread remains the same.



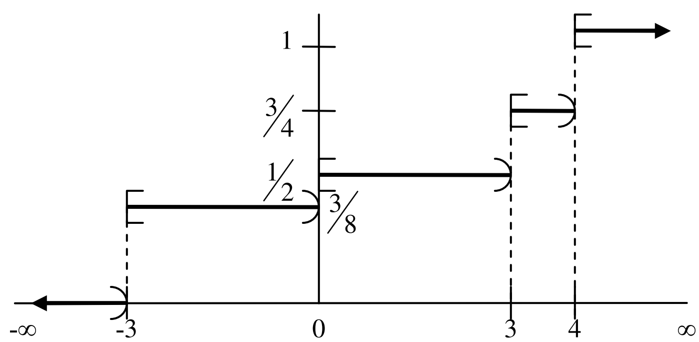




3) $\mathrm{Var}(c \cdot X) = c^2\ \mathrm{Var}(X)$

- $$\begin{aligned}\operatorname{Var}(c\cdot X) &= E\left[(c\cdot X)^2\right]-\left(E[c\cdot X]\right)^2 = E\left[c^2\cdot X^2\right]-\left(c\cdot E[X]\right)^2 = c^2\cdot E\left[X^2\right]-c^2\cdot\left(E[X]\right)^2\\ &= c^2\left(E\left[X^2\right]-\left(E[X]\right)^2\right)=c^2\operatorname{Var}(X)\end{aligned}$$

**Example**

The distribution function of a random variable $X$ is given by: $F(x)=\begin{cases}0 \text{ if } x<-3\\ \frac{3}{8}\text{ if } -3\le x<0\\ \frac{1}{2}\text{ if } 0\le x<3\\ \frac{3}{4}\text{ if } 3\le x<4\\ 1\text{ if } x\ge 4\end{cases}$. Calculate $E[X]$, $E\left[X^2-2|X|\right]$, and $\operatorname{Var}\left[X\cdot|X|\right]$.




- $\lim_{t\to-\infty} F(t)=0$
- $\lim_{t\to\infty} F(t)=1$
- $F$ is non-decreasing
- $F$ is continuous from the right

- The values $X$ can take are the values where $F$ has different limits at the left and the right. $X: S\to\{-3,0,3,4\}$.
  - $P(X=-3)=$ difference between limit to the right and limit to the left $=\frac{3}{8}$.
  - $P(X=0)=\frac{1}{2}-\frac{3}{8}=\frac{1}{8}$.
  - $P(X=3)=\frac{3}{4}-\frac{1}{2}=\frac{1}{4}$.
  - $P(X=4)=1-\frac{3}{4}=\frac{1}{4}$.

- $E[X]=-3\cdot\frac{3}{8}+0\cdot\frac{1}{8}+3\cdot\frac{1}{4}+4\cdot\frac{1}{4}=\frac{5}{8}$.

- $E\left[X^2-2|X|\right]=?$
  - $y(z)=z^2-2|z|$, $y:\mathbf{R}\to\mathbf{R}$
  - $E[y(X)]=\sum y(a_i)\cdot p(a_i)$
  - $y(-3)=3$, $y(0)=0$, $y(3)=3$, $y(4)=8$
  - $E\left[X^2-2|X|\right]=3\cdot\frac{3}{8}+0\cdot\frac{1}{8}+3\cdot\frac{1}{4}+4\cdot\frac{1}{4}=\frac{31}{8}$

- $\operatorname{Var}[X \cdot |X|] =$ always boils down to computing some expectations $= E\left[(X \cdot |X|)^2\right] - (E[X \cdot |X|])^2$
    - $h(t) = (t \cdot |t|)^2$, $h : \mathbf{R} \to \mathbf{R}$
        - $E[h(X)] = \sum h(a_i) \cdot p(a_i)$
        - $h(-3) = 81$, $h(0) = 0$, $h(3) = 81$, $h(4) = 256$
        - $E\left[(X \cdot |X|)^2\right] = 81 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 81 \cdot \frac{1}{4} + 256 \cdot \frac{1}{4} = \frac{755}{8}$
    - $c(v) = v \cdot |v|$
        - $v(-3) = -9$, $v(0) = 0$, $v(3) = 9$, $g(4) = 16$
        - $E[X \cdot |X|] = -9 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 9 \cdot \frac{1}{4} + 16 \cdot \frac{1}{4} = \frac{23}{8}$
    - $\operatorname{Var}[X \cdot |X|] = \frac{755}{8} - \left(\frac{23}{8}\right)^2 = \frac{5511}{64}$