Lecture #20 – November, 7, 2003

BERNOULLI DISTRIBUTION

- An experiment with only two possible outcomes. It is customary to call these "success" and "failure".
- $X : \{\text{success, failure}\} \to [0,1]$
  - $X(\text{failure}) = 0$, $X(\text{success}) = 1$.
- The probability function of X is perfectly $P(X = 1) = P(\text{success}) = p$.
- $P(X = 0) = 1 - p$
- $E[X] = p \cdot 1 + (1 - p) \cdot 0 = p$
- $\text{Var}[X] = E[X^2] - (E[X])^2 = p \cdot 1^2 + (1 - p) \cdot 0^2 - (p \cdot 1)^2 - ((1 - p) \cdot 0)^2 = p - p^2 = p(1 - p)$

BINOMIAL DISTRIBUTION

- Imagine you repeat independently n Bernoulli experiments. Each experiment has probability of success p.
- X = number of successes observed in the n experiments.
- For experiment i, I define $Y_i = \begin{cases} 1 \text{ if experiment is success} \\ 0 \text{ if experiment is failure} \end{cases}$.
  - Y_i has Bernoulli distribution – $P(Y_i = 1) = p$.
  - Then $X = \sum_{i=1}^{n} Y_i = \text{total number of successes in } n \text{ trials}$.
  - X has the Binomial distribution.
- X is determined by two numbers: n and p.
- Notation: $X \sim \text{Bin}(n, p)$ (n, p parameters).
- $X : S \to \{0, \ldots, n\}$.
- What is $P(n = k)$ for any $0 \le k \le n$?
  - $P(X = 0) = P(\text{no success}) = P(Y_1 = 0, Y_2 = 0, \ldots, Y_n = 0) = P(\{Y_1 = 0\} \cap \{Y_2 = 0\} \cap \ldots \cap \{Y_n = 0\})$ $= (1 - p)^n \cdot 1 = (1 - p)^n \binom{n}{0}$
  - $P(X = 1) = P(1 \text{ success}, n - 1 \text{ failures}) = p(1 - p)^{n-1} \binom{n}{1}$
  - $P(X = 2) = P(2 \text{ successes}, n - 2 \text{ failures}) = p^2 (1 - p)^{n-2} \binom{n}{2}$
  - $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$ – probability function of the Binomial Distribution.

Example

10 people with same skills shoot a basketball; independent shots; $P(\text{make a basket}) = 0.8$.

- $n = 10$, $p = 0.8$, shots independent $\Rightarrow$ X = number of successful shots has Binomial distribution $\Rightarrow$ $X \sim \text{Bin}(10, 0.8)$.

- $P(X=2)=\binom{10}{2}(0.8)^2(0.2)^8=0.000073728$.

Expected Value of *X* ~ Bin(n, p)

- $E[X]=\sum_{k=0}^{n}k\cdot P(X=k)=\sum_{k=0}^{n}k\cdot\frac{n!}{(n-k)!k!}\cdot p^k\cdot(1-p)^{n-k}=\sum_{k=1}^{n}\frac{n!}{(n-k)!(k-1)!}\cdot p^k\cdot(1-p)^{n-k}$
- Since $(n-k)!=((n-1)-(k-1))!$,
  $E[X]=\sum_{k=1}^{n}\frac{n\cdot(n-1)!}{((n-1)-(k-1))!k!}\cdot p\cdot p^{k-1}\cdot(1-p)^{(n-1)-(k-1)}=np\sum_{k=1}^{n}\binom{n-1}{k-1}\cdot p^{k-1}\cdot(1-p)^{(n-1)-(k-1)}$.
- By re-labeling indexes $k-1=l$, $E[X]=\sum_{l=0}^{n}\binom{n-1}{l}\cdot p^l\cdot(1-p)^{(n-1)-l}$.
- Since $\sum_{k=0}^{n}P(X=k)=\sum_{k=0}^{n}\binom{n}{k}\cdot p^k\cdot(1-p)^{n-k}=1=\sum P(X=k)$ when $X\sim\text{Bin}(n,p)$ $=\sum_{l=0}^{n-1}P(X=l)$ when $X\sim\text{Bin}(n-1,p)$, I express $E[X]$ when $X\sim\text{Bin}(n,p)$ as $n\cdot p\cdot\text{Sum}$, $\text{Sum}=\sum_{l=0}^{n-1}\binom{n-1}{l}\cdot p^l\cdot(1-p)^{n-1-l}=1$ because if I consider a random variable $Z\sim\text{Bin}(n-1,p)$, then $\sum_{l=0}^{n-1}P(Z=l)=1$.
- So $X\sim\text{Bin}(n,p)\Rightarrow E[X]=n\cdot p$.
- Using similar idea, $\text{Var}[X]=np(1-p)$

Example

Five fair coins are flipped. If the outcome are assumed independent, find the probability function of the number of heads.

- $n=5$; $\text{Success}=\text{"Head"}\Rightarrow p=\frac{1}{2}$; $X=\text{number of heads}\Rightarrow X\sim\text{Bin}(n,p)$.
- $E[X]=5\cdot 0.5=2.5$.
- $P(X=3)=\binom{5}{3}\cdot(0.5)^3\cdot(0.5)^2=\binom{5}{3}\cdot(0.5)^5=0.3125$.

Example

It is known that light bulbs produced by a certain company will be defected with probability 0.01 independent of each other. The company sells the bulbs in packages of 10 and offers money-back guarantee that at most 1 in 10 bulbs is defective. What percentage of the packages will the company have to refund?

- $\text{percentage}=P(\text{"I refund for a package selected at random"})$.
- Pick a package at random:
  - $X=\text{number of defectives in that package}=\{0,1,\ldots,10\}$.
  - $p=0.01$, $n=10$.
  - So $X\sim\text{Bin}(10,0.01)$

- $P(\text{" refund"}) = P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - \binom{10}{0}(0.01)^0(0.99)^{10} - \binom{10}{1}(0.01)^1(0.99)^9 = 0.00427$.

Lecture #21 – Friday, November 14, 2003

Example

Suppose a battleship has accuracy rate of 0.7. The ship fires 10 times at 10 different targets. Assuming the shots are independent from each other, what is the most likely number of targets hit?

- X = number of hits. $X \sim \text{Bin}(10, 0.7)$.
- The most likely number of hits is 7. $E[X] = n \cdot p = 10 \times 0.7 = 7$.

Theorem

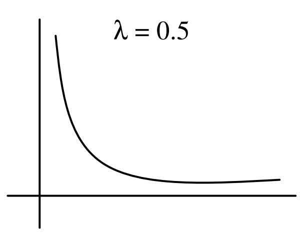
- For $X \sim \text{Bin}(n, p)$, the largest value (integer) is less than or equal to $(n+1)p$. This is not always equal to $E[X]$.

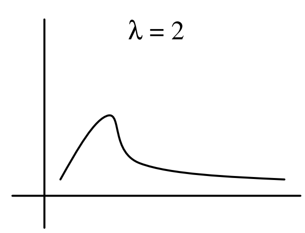
The Mode

- The most likely value of a random variable *X* is called the <u>mode</u> of the distribution of *X*.

Poisson Distribution

- The Poisson distribution is also called the “Law of Rare Events”.
- *X* with a Poisson distribution is still a discreet random variable, but with infinitely many possible values. $X : S \to \{0, 1, 2, \ldots, n, \ldots\}$.
- $P(X = k) = \exp(-\lambda) \cdot \frac{\lambda^k}{k!}$ – $\lambda > 0$ is the parameter of the distribution.
- $\sum_{k=0}^{\infty} P(X = k) = 1$
    - $e^x = \exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.
    - So $\sum_{k=0}^{\infty} P(X = k) = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$





- Centre of gravity = expectation – $E[X]$ is increasing in λ.
- Spread = variance – $\text{Var}[X]$ is increasing in λ..

Lecture #22 – Monday, November 17, 2003

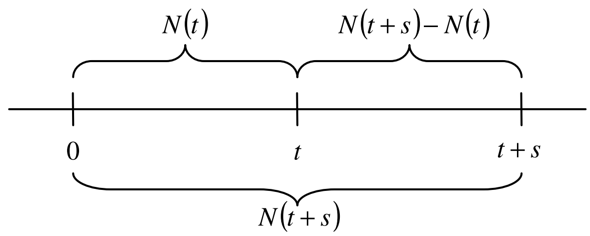
The Mean and Variance of a Poisson Distribution

- $E[X]=\sum_{k=0}^{\infty} P(X=k)\cdot k=\sum_{k=0}^{\infty} e^{-\lambda}\cdot\frac{\lambda^k}{k!}\cdot k=\sum_{k=1}^{\infty} e^{-\lambda}\cdot\frac{\lambda^k}{(k-1)!}=e^{-\lambda}\cdot\lambda\cdot\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}$. Define a new index $m=k-1$. $e^{-\lambda}\cdot\lambda\cdot\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}=e^{-\lambda}\cdot\lambda\cdot e^{\lambda}=\lambda$.

- $E[X^2]=\lambda+\lambda^2$. So $\text{Var}[X]=E[X^2]-(E[X])^2=\lambda$.

Law of Rare Events

- Let $N(t)$ denote the number of events occurring by time *t*. Assume that the following hold:
  - Stationarity – For two time intervals of equal length, the distribution of the number of events is the same within each interval.
  - Independent increment – The number of occurrences in an interval $(t, t+s)$ does not depend on the number of occurrences from previous times.



  - Orderness – No two (or more) events can occur simultaneously.
  - $N(0)=0$ – There is a time origin at which the counting of events starts.
- Then there exits a positive number $\lambda>0$ such that the distribution of $N(t)$ is $\text{Poisson}(\lambda t)$ for all $t>0$.

Example: Shark Attacks

In 2000, there six shark attacks. Data from 1990 to 1999 is given in the table.

Year	Number of shark attacks
1990	1
1991	0
1992	3
1993	2

1994	4
1995	2
1996	3
1997	3
1998	1
1999	1
Total	20

- The number of attacks in one year is $\text{Poisson}(\lambda)$.
- In 10 years, 20 attacks. So $\lambda = 2$.
- $X \sim \text{Poisson}(2)$.
- $P(X = 6) = e^{-2} \cdot \frac{2^6}{6!} = 0.012$.

Suppose you spend a 7-day vacation there. On average, how many years do you have to spend there in order to witness a shark attack?

- Instead of years, change to weeks: $\frac{2}{25}$/weeks = 0.038/week . So $\lambda_{\text{week}}\, 0.038$.
- $P(\text{I witness an attack during my week}) = 1 - P(\text{no attack}) = 1 - P(X = 0) = 1 - e^{-\lambda} \frac{\lambda^0}{0!} = 1 - e^{\lambda_{\text{week}}} = 0.0376$.
- Every week as a realization of a Bernoulli trial:
  - observe: $p = 0.0376$
  - not-observe: $1 - p$
- So $Y \sim (n, p)$.
- $E[X] = n \cdot p$, $n = \frac{1}{p} = 30$ years .

Example: The Great One

During Gretzky's season, he scored 1669 times in 696 games. What is the probability of scoreless Gretzky?

- Unit = 1 game
- Number of goals in a game: $X \sim \text{Poisson}(\lambda)$. In 696 games: $X_{696} \sim \text{Poisson}(696\lambda = 1669)$.

  $\lambda = \frac{1669}{696} = 2.398$.
- $P(X = 0) = e^{-\lambda} = 0.091$.
- So $0.091 \times 696 = 65.27$.

Lecture #23 – Wednesday, November 19, 2003

Example

Suppose that the number of finals with a maximum score in a statistics class has a Poisson distribution with $\lambda = \frac{1}{2}$. What is the probability of at least one such final in this class?

- X = number of finals with a maximum score.
- $X \sim \text{Poisson}\left(\frac{1}{2}\right)$.

- $P(X \ge 1) = \sum_{k=1}^{\infty} e^{-\frac{1}{2}} \frac{\left(\frac{1}{2}\right)^k}{k!} = 1 - P(X = 0) = 1 - e^{-\frac{1}{2}} = 0.394$.

Example

Accidents occur independently on University Ave. at a Poisson rate of 2/day. What is the probability that during March there are exactly 3 days without accidents?

- Combine Poisson and Binomial.
- X = number of accidents in a day in March. $X \sim \text{Poisson}(2)$.
- Y = number of accidents in March. $Y \sim \text{Poisson}(2 \times 31 = 62)$.
- Each day in March is a Bernoulli experiment: either we have an accident or not. So $Z_i = \begin{cases} 0 \text{ if accident occurs} \\ 1 \text{ if no accident} \end{cases}$.
- For the i^{th} day in March, $N = \text{number of days without accidents} = \sum_{i=1}^{31} Z_i$.
- $N \sim \text{Bin}(31, p)$, where $p = P(Z_i = 1) = P(\text{no accident in a day}) = P(X = 0) = e^{-2} = 0.135$.

Example: Poisson Process

Females customers arrive at a grocery store at a Poisson rate of 2/3 per minute. What is the probability that 15 females enter the store between 10:30 and 10:45?

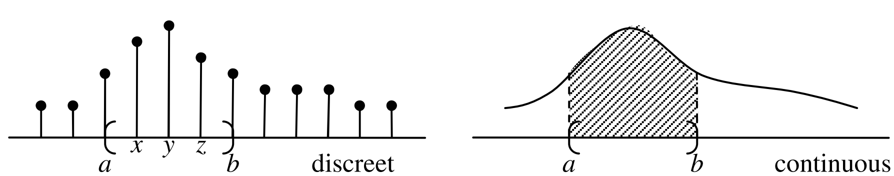
- $N_{15} \sim \text{Poisson}\left(\frac{2}{3} \times 15\right) = \text{Poisson}(10) = \text{number of customers entering between } 10:30 \text{ and } 10:45$.
- $P(N_{15} = 15) = e^{-10} \frac{10^{15}}{15!} 0.0347$.

Lecture #24 – Friday, November 21, 2003

CONTINUOUS RANDOM VARIABLE

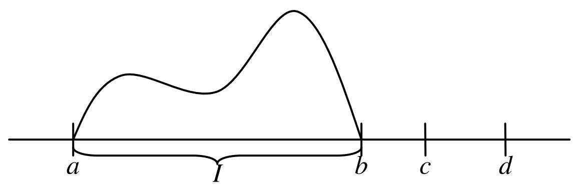
- $X : S \to I$
- I interval included in **R**.
- I bounded: (a,b), $(a,b]$, $(b,a]$
- I bounded: $(-\infty, a)$, $(-\infty, a]$, (a, ∞), $[a, \infty)$.
- The density function of a continuous random variable X is $f : \mathbf{R} \to \mathbf{R}_+$ such that

$$P(X \in (a,b)) = \int_a^b f(y)dy = \text{area under thecurve between } a \text{ and } b$$



- $P(X \in (a,b)) = P(X = x) + P(X = y) + P(X = z)$

- For a non-negative function *f*, certain conditions need to be satisfied to be a density:
  - *f* be integrable.
  - $\int_{\mathbf{R}} f(y)dy = \int_{-\infty}^{\infty} f(t)dt = P(X \in \mathbf{R}) = 1$
  - If $X : S \to I \subset \mathbf{R}$, then it's enough to have $\int_I f(x)dx = 1$.



  - No matter how I choose *c*, *d* outside interval *I*, *f* is always 0 outside *I*.

Connection Between Density Function and Distribution Function

- $F : \mathbf{R} \to [0,1]$, $F(t) = P(X \le t)$ – definition is same as the discreet case.
- $F(t) = P(X \in (-\infty, t]) = \int_{-\infty}^{t} f(y)dy$.
- In nice cases (say when *f* is continuous), $F'(t) = f(t)$.

Example: Tire Endurance

The lifetime of a tire selected at random from a used tire shop is $10000 \times X$ miles, where *X* is a random variable with density function $f(x) = \begin{cases} \frac{2}{x^2} \text{ if } 1 < x < 2 \\ 0 \text{ elsewhere} \end{cases}$.

1) Calculate the distribution function
- $X : S \to (1,2)$.
- Is *f* a density function?
  - $\int_1^2 \frac{1}{x^2} dx = 2 \cdot \left(\left(-\frac{1}{2} \right) - \left(-\frac{1}{1} \right) \right) = 2\left(\frac{1}{2} \right) = 1$.
- Distribution function:
  - $F(t) = P(X \le t) = \int_{-\infty}^{t} f(y)dy$.
  - If $t < 1$, $F(t) = 0$.
  - If $t = 1$, $F(t) = 0$.
  - If $t \in (1,2)$, then $F(t) = \int_{-\infty}^{t} f(y)dy = \int_1^t \frac{2}{y^2} dy = 2\int_1^t -\left(\frac{1}{y} \right)' dy = 2\left(\left(-\frac{1}{t} \right) - (-1) \right) = \frac{2(t+1)}{t}$.
  - If $t > 2$, $F(t) = 1$.

2) Calculate the mean and the variance.
- General formula for mean: $E[X] = \int_I x \cdot f(x)dx$.

- $E[X]=\int_1^2 y\cdot\frac{2}{y^2}dy=2\int_1^2\frac{1}{y}dy=2\int_1^2(\ln y)'dy=2(\ln 2-\ln 1)=2\ln 2$. So on average, the tire lasts $2\ln 2\times 10000=28000$ miles.

- $\text{Var}[X]=E[X^2]-(E[X])^2$
  - Law of Unconscious (continuous case): If $g:I\to\mathbf{R}$, then $E[g(x)]=\int_I g(x)\cdot f(x)dx$.
  - In this case $E[X^2]=\int_I y^2\cdot\frac{2}{y^2}=2(2-1)=2$.
  - So, $\text{Var}[X]=E[X^2]-(E[X])^2=2-(2\ln 2)^2=0.078$.

Example: Car Engines

Let *X* denote the time to failure of an engine in a certain type of car. The density function *X* is

$$f(x)=\begin{cases}\frac{\lambda e^{-\frac{x}{10}}}{10} \text{ if } 0\le x<\infty \\ 0 \text{ elsewhere}\end{cases}$$. Find λ.

- $\int_0^\infty f(x)dx=1\Leftrightarrow\int_0^\infty\frac{\lambda}{10}e^{-\frac{x}{10}}dx=1\Leftrightarrow\frac{\lambda}{10}\int_0^\infty-10\left(e^{-\frac{x}{10}}\right)'dx=1\Leftrightarrow-\lambda e^{-\frac{10}{x}}\Big|_0^\infty=1\Leftrightarrow-\lambda\left(0-e^{-\frac{0}{10}}\right)\Leftrightarrow\lambda=1$.

Lecture #25 – Monday, November 23, 2003

Connection Between Distribution Function and Density Function

- $F'(t)=f(t)$ for all *t* where $f(\cdot)$ is continuous.

Example: Tire Endurance

The lifetime of a tire selected at random from a used tire shop is $10000\times X$ miles, where *X* is a random variable with density function $f(x)=\begin{cases}\frac{2}{x^2} \text{ if } 1<x<2 \\ 0 \text{ elsewhere}\end{cases}$.

1) Calculate the distribution function

- $F(t)=\begin{cases}0, t\le 1 \\ \int_1^t\frac{2}{x^2}dx=2-\frac{2}{t}, t\in(1,2) \\ 1, t\ge 2\end{cases}$.

2) Calculate the mean and the variance.

- $E[X]=\int_1^2 y\cdot\frac{2}{y^2}dy=2\int_1^2\frac{1}{y}dy=2\int_1^2(\ln y)'dy=2(\ln 2-\ln 1)=2\ln 2$.
- $\text{Var}[X]=E[X^2]-(E[X])^2=2-(2\ln 2)^2=0.078$.

3) What percentage of tires in the shop have a lifetime shorter than 15000 miles?

- $P(X < 1.5) = P(X \le 1.5) = F(1.5) = 2 - \frac{2}{1.5} = \frac{2}{3}$.

4) What percentage of those having lifetime shorter than 15000 miles last between 10000 and 12000 miles?

- Reformulate question: Knowing that a tire lasts less than 15000 miles, what is the probability that it lasts between 10000 and 12000 miles?
- $P(X \in (1,1.2) \mid X \in (1,1.5)) = \frac{P(X \in (1,1.2) \cap X \in (1,1.5))}{P(X \in (1,1.5))} = \frac{P(X \in (1,1.2))}{P(X \in (1,1.5))} = \frac{F(1.2) - F(1)}{F(1.5) - F(1)} = \frac{F(1.2)}{F(1.5)} = \frac{1}{2}$.

Example: Car Engines

Let X denote the time to failure of an engine in a certain type of car. The density function X is

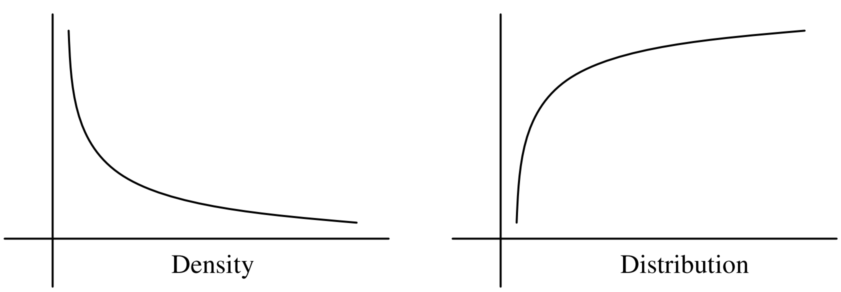
$$f(x) = \begin{cases} \frac{\lambda e^{-\frac{x}{10}}}{10} \text{ if } 0 \le x < \infty \\ 0 \text{ elsewhere} \end{cases}.$$

1) Find λ.

- $\int_0^\infty f(x)dx = 1 \Leftrightarrow \int_0^\infty \frac{\lambda}{10} e^{-\frac{x}{10}} dx = 1 \Leftrightarrow \lambda = 1$.

2) Find the distribution function.

- $F(t) = P(X \le t)$
- If $t < 0$, $F(t) = 0$.
- If, $t > 0$, $F(t) = P(X \le t) = \int_{-\infty}^{t} f(y)dy = \int_0^t \frac{e^{-\frac{y}{10}}}{10} dy = \frac{1}{10}\left(-10e^{-\frac{y}{10}} \Big|_0^t \right) = -e^{-\frac{y}{10}} + 1$.
- So $F(t) = \begin{cases} 0, t < 0 \\ 1 - e^{-\frac{y}{10}}, t \ge 0 \end{cases}$



3) What is the average time to failure for car engines of this type?

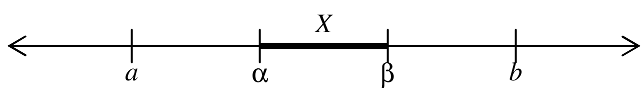
- $E[X] = \int_0^\infty x \frac{e^{-\frac{x}{10}}}{10} dx = \frac{1}{10} \int_0^\infty x e^{-\frac{x}{10}} dx$.

- Integrating by parts,
$$\int_0^\infty x\frac{e^{-\frac{x}{10}}}{10}dx=\frac{1}{10}\left[x\cdot\left(-10e^{-\frac{x}{10}}\right)\Big|_0^\infty-\int_0^\infty 1\cdot\left(-10e^{-\frac{x}{10}}\right)dx\right]=\frac{1}{10}\left(10\int_0^\infty e^{-\frac{x}{10}}dx\right)=-10e^{-\frac{x}{10}}\Big|_0^\infty=0-(-10)=10.$$
- So $E[X]=10$.

Lecture #26 – Wednesday, November 26, 2003

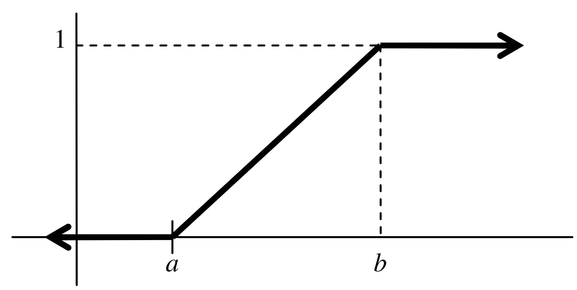
UNIFORM DISTRIBUTION

- Closely related to random section of point in intervals.
- Consider an interval (a,b), where *a*, *b* are fixed. Select a point at random in the interval $I=(a,b)$. If *X* is the coordinate of the chose point, then $P(X\in(\alpha,\beta))=\frac{\beta-\alpha}{b-a}$ for all $a<\alpha\le\beta<b$.



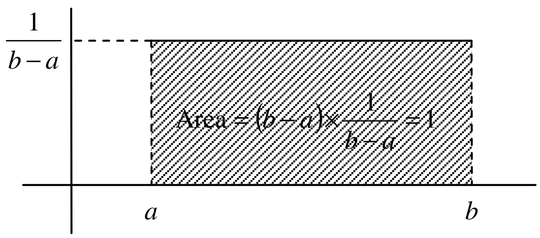
Distribution Function of *X*

- $F(t)=P(X\le t)$.
  - If $t\le a$: $P(X\le t)=0$.
  - If $t\ge a$: $P(X\le t)=1$.
  - If $t\in(a,b)$: $P(X\le t)=P(X\in(a,t])=\frac{t-a}{b-a}$.
  - So, $F(t)=\begin{cases}0\text{ if } t\le a\\ \frac{t-a}{b-a}\text{ if } t\in(a,b)\\ 1\text{ if } t\ge b\end{cases}$



Density Function of *X*

- *F* is continuous and F' is continuous, so $f=F'$ (density function for random variable *X*).
- If *F* is constant, then $f=0$.
- So, $F'(t)=f(t)=\begin{cases}0\text{ if } t\le a\\ \frac{1}{b-a}\text{ if } t\in(a,b)\\ 0\text{ if } t\ge b\end{cases}$.
- The density is 0 everywhere except between (a,b) where *f* is constant ($\frac{1}{b-a}$).

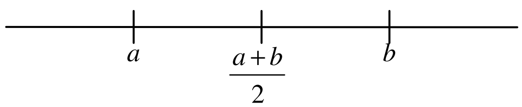


- $\text{Area} = \int_a^b f(y)dy = 1$

- A random variable with density *f* has uniform distribution on (a,b) is denoted $\mathrm{U}(a,b)$.

Expectation and Variance

- $\mathrm{E}[X] = \int_a^b y \cdot f(y)dy = \int_a^b \frac{y}{b-a}dy = \frac{1}{b-a}\int_a^b \left(\frac{y^2}{2}\right)' dy = \frac{1}{2(b-a)} y^2 \Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$.



- $\mathrm{Var}[X] = \mathrm{E}[X^2] + (\mathrm{E}[X])^2$
  - $\mathrm{E}[X^2] = \int_a^b y^2 f(y)dy = \frac{1}{b-a}\int_a^b \left(\frac{y^3}{3}\right)' dy = \frac{1}{3(b-a)} y^3 \Big|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{b^2+ab+a^2}{3}$
  - $\mathrm{Var}[X] = \frac{a^2+b^2+ab}{3} - \left(\frac{b+a}{2}\right)^2$
- $\sqrt{\mathrm{Var}[X]} = SD$ – standard deviation.

Example

Buses arrive at a specific stop at 15 minute intervals starting at 5 am. If a passenger arrives at a stop at a time that is uniformly distributed between 7:00 and 7:30 am, find the probability that he waits for the bus.

1) Less than 5 minutes.

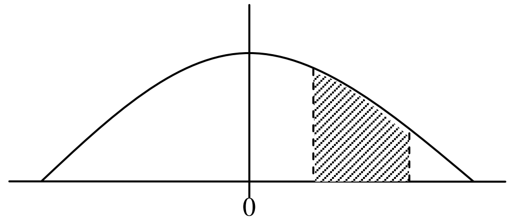
$$\begin{aligned} P(\text{waits less than 5 minutes}) &= P(X \in (7:10, 7:15) \cup X \in (7:25, 7:30)) \\ &= P(X \in (7:10, 7:15)) + P(X \in (7:25, 7:30)) \\ &= \int_I \frac{1}{30}dy + \int_I \frac{1}{30}dy = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \end{aligned}$$

NORMAL DISTRIBUTION (GAUSSIAN DISTRIBUTION)

- Most "natural" phenomenon – ex: distribution of grades in a class, characteristics of a group of organisms, errors of measurement.

Standard Normal Density

- $f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ for any $x \in \mathbf{R}$.
- f is symmetric about 0.
- $\mathrm{E}[X] = 0$.
- $\mathrm{Var}[X] = 1$.



Lecture #27 – Friday, November 28, 2003

Examples

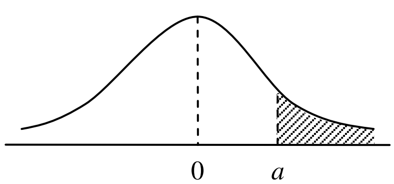
1) $P(X \le 1.34) = F(1.34) = \int_{-\infty}^{1.34} f(x)dx = 0.9099$
2) $P(X \le 0.58) = 0.7190$

Examples: Normal Table

1) $P(X \le 0.13) = 0.5517$
2) $P(X < -0.2) = 1 - P(X \le 0.2) = 1 - 0.5793 = 0.4207$
3) $P(-1 < X \le 0.4) = P(X \le 0.4) - P(X \le -1)$
$= F(0.4) - F(-1) = F(0.4) - (1 - F(1)) = F(0.4) + F(1) - 1 = 0.6556 + 0.8413 - 1 = 0.4969$
4) $P(-1.5 \le X \le -1) = F(1.5) - F(1) = 0.9332 - 0.8413 = 0.0919$

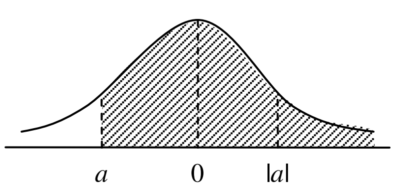
Example: Normal Table

Say $Z \sim \mathrm{N}(0,1)$. If $P(Z \ge a) = 0.2$, find a.



- $P(Z > a) = 0.2 \Rightarrow P(Z \le a) = 0.8$
- So $a = 0.84$.

What if $P(Z > a) = 0.6$?



- $P(Z \le |a|) = 0.6$
- So $|a| = 0.255 \Rightarrow a = -0.225$

GENERAL NORMAL DISTRIBUTION

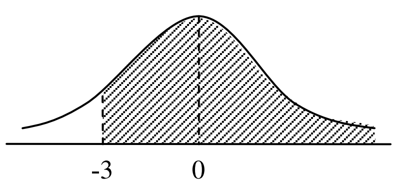
- The general normal distribution $\mathrm{N}(\mu, \sigma^2)$ has probability density function $f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ for all $x \in \mathbf{R}$.
- If $X \sim \mathrm{N}(\mu, \sigma^2)$, then $\mathrm{E}[X] = \mu$ and $\mathrm{Var}[X] = \sigma^2$.
- The square root of the variance is the standard deviation and is denoted *SD*.
- Probabilities like $P(X < a)$ or $P(a \le X < b)$ are calculated using the relationship between X and $Z = \frac{X-\mu}{\sigma}$ as well as the table for the standard normal.
  - Start with $X \sim \mathrm{N}(\mu, \sigma^2)$.
  - $\mathrm{E}[X-\mu] = \mathrm{E}[X] - \mu = 0$.
  - $\mathrm{Var}[X-\mu] = \mathrm{Var}[X] = \sigma^2$. So $\mathrm{Var}\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma^2}\mathrm{Var}[X] = 1$.
  - I can define $Z = \frac{X-\mu}{\sigma}$ such that $\mathrm{E}[Z] = 0$ and $\mathrm{Var}[Z] = 1$. Z is also normal.

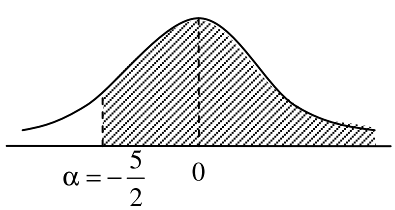
Example

1) $P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$, where $Z \sim (0,1)$.

Example

If $X \sim \mathrm{N}(2,4)$, calculate $P(X > -3)$.

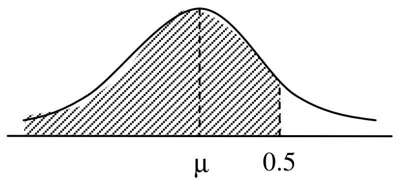




$$
\begin{aligned}
P(X > -3) &= P\left(\frac{X-2}{2} > \frac{-5}{2}\right) \\
&= P\left(Z > -\frac{5}{2}\right) \\
&= P\left(Z \le \frac{5}{2}\right) \\
&= F(2.5) \\
&= 0.9938
\end{aligned}
$$

Example: Standardized Normal

Say $X \sim \mathrm{N}(\mu, 2)$. If $P(X \le 0.5) = 0.8$, find μ.



- $P(X \le 0.5) = P\left(\frac{X-\mu}{\sqrt{2}} \le \frac{0.5-\mu}{\sqrt{2}}\right) = 0.8$
- $P\left(Z \le \frac{0.5-\mu}{\sqrt{2}}\right) = 0.8$
- $\frac{0.5-\mu}{\sqrt{2}} \approx 0.84 \Rightarrow \mu \approx 0.5 - \sqrt{2} \times 0.84 \approx \text{-}0.6879$

Lecture #28 – Monday, December 1, 2003

Example: Grading

Suppose that a professor finds a way to transform the grades in his class so that their distribution is $\mathrm{N}(0,1)$.
Suppose he then gives the final mark according to the following system:

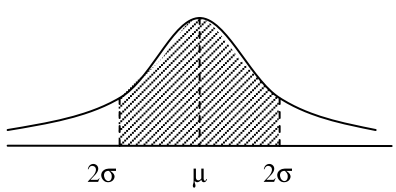
Range:	$X > 1.5$	$0.5 < X \le 1.5$	$-1 < X \le 0.5$	$-2 < X \le -1$	$X \le -2$
Grade:	A	B	C	D	F

1) What percentage of students will get A?
   - $X \sim \mathrm{N}(0,1)$
   - $P(X > 1.5) = 1 - P(X \le 1.5) = 1 - 0.9332 = 0.0668$

2) What percentage of students will get C?
   - $P(X \in (-1,0.5)) = F(0.5) - P(X \le -1) = F(0.5) - (1 - P(X \le 1)) = F(0.5) + F(1) - 1 = 0.5328$

Exercise

Say $X \sim \mathrm{N}(\mu, \sigma^2)$. Calculate $P(|X-\mu| < 2\sigma)$.

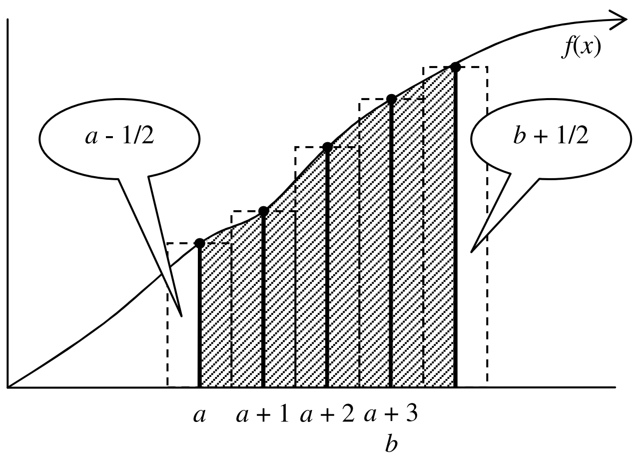


- $|X-\mu|$ is the distance between X and μ.
- $$\begin{aligned} P(|X-\mu| < 20) &= P(\mu - 2\sigma < X < \mu + 2\sigma) \\ &= P\left(-2 \le \frac{X-\mu}{\sigma} \le 2\right) \\ &= P(-2 \le Z \le 2), Z \sim \mathrm{N}(0,1) \\ &= 2(1-0.9772) = 0.0456 \end{aligned}$$

NORMAL APPROXIMATION OF A BINOMIAL

- Let $X \sim \mathrm{Bin}(n, p)$. <u>Rule of Thumb</u>: If $np \ge 5$ and $n(1-p) \ge 5$, approximate X with $Y \sim \mathrm{N}(np, np(1-p))$.

Illustration: Continuity Correction



- $X \in (a, a+1, a+2, a+3)$ is discreet. $P(a \le X \le b) = P(X = a) + \ldots P(X = b)$
- $= P\left(a - \frac{1}{2} \le Y \le b + \frac{1}{2}\right)$
- Y has density function f.

Continuity Correction

- A <u>continuity correction</u> is necessary. A continuity correction is an adjustment that we make by adding or subtracting $\frac{1}{2}$ to a discreet value when we use a continuous distribution to approximate the discreet one.
- To apply a continuity correction to the discrete values <u>included</u> in an interval, we subtract $\frac{1}{2}$ from the smallest value included in an interval, add $\frac{1}{2}$ tot eh largest value included in the interval, and then proceed.
- So if X is discreet, approximate $P(a \le X \le b)$ using the continuous Y with $P\left(a - \frac{1}{2} \le Y \le b + \frac{1}{2}\right)$.

Example: Normal Approximation of a Binomial

A factory which produces light bulbs estimates that the probability of a light bulb lighting continuously more than a week is 36%. What is the chance that out of 100 bulbs tested, the number of bulbs still working after a week is between 24 and 42 inclusive?

- Let X = Number of light bulbs out of 100 still working after one week.
- $X \sim \text{Bin}(100, 0.36)$
- Want: $P(24 \le X \le 42)$.
- I want to approximate X with a continuous distribution (normal). Check Rule of Thumb: $np = 100 \times 0.36 = 36 > 5$, $n(1-p) = 100 \times 0.64 = 64 > 5$. Ok!
- Take $Y \sim \text{N}(36, 4.8^2)$.
- $P(23.5 \le X \le 42.5) = P(-2.6 \le Z \le 1.35) = F(1.35) - F(-2.6) = F(1.35) - (1 - F(2.6)) = 0.9868$.

Lecture #29 – Wednesday, December 3, 2003

Example: Normal Approximation of a Binomial

A batch of $n = 80$ items is taken from a manufacturing process. The process creates a fraction of $p = 0.16$ defectives. What is the probability that a batch with 80 independent items will contain exactly 20 defectives?

- X = number of defectives out of 80. So $X \sim \text{Bin}(80, 0.16)$.
- $P(X = 20) = \binom{80}{20}(0.16)^{20}(0.84)^{60} = 0.0122$.
- Approximate X with $Y \sim \text{N}(\mu, \sigma^2)$.
  - $\mu = \text{E}[X] = 80 \cdot 0.16 = 12.8$.
  - $\sigma^2 = \text{Var}[X] = 80 \cdot 0.16 \cdot (1 - 0.16) = 10.752 \Rightarrow \sigma = 3.279$.
- $P(X = 20) = P(Y \in (19.5, 20.5)) = P(19.5 \le Y \le 20.5) = P(2.04 \le Z \le 2.35) = F(2.35) - F(2.04) = 0.0113$.