Introduction

Types of Partial Differential Equations

- Transport equation: $u_x(x,y)+u_y(x,y)=0$, where $u_x=\frac{\partial u}{\partial x}$, $u_y=\frac{\partial u}{\partial y}$, and $u(x,y)=?$.
- Shockwave equation: $u_x(x,y)+u(x,y)u_y(x,y)=0$.
- The vibrating string equation: $u_{tt}(x,t)=c^2 u_{xx}(x,t)$, where $u_t=\frac{\partial^2 u}{\partial t^2}$ and $u_{xx}=\frac{\partial^2 u}{\partial x^2}$.

  The wave equation: $u_{tt}(x,y,z,t)=c^2(u_{xx}(x,y,z,t)+u_{yy}(x,y,z,t)+u_{zz}(x,y,z,t))$.

  In general: $u_{tt}(x_1,\ldots,x_n,t)=c^2\Delta u(x_1,\ldots,x_n,t)$, where $\Delta=\text{the Laplacian}=\frac{\partial^2}{\partial x_1{}^2}+\cdots+\frac{\partial^2}{\partial x_n{}^2}$ and $\Delta u=\frac{\partial^2 u}{\partial x_1{}^2}+\cdots+\frac{\partial^2 u}{\partial x_n{}^2}$.
- Diffusion equation: $u_t(x,t)=c^2 u_{xx}(x,t)$.

  In general: $u_t(x_1,\ldots,x_n,t)=c^2\Delta u(x_1,\ldots,x_n,t)$.
- Steady state: $u_t=0$.
- Laplacian equation: $\Delta u=\frac{\partial^2 u}{\partial x_1{}^2}+\cdots+\frac{\partial^2 u}{\partial x_n{}^2}=0$.

Initial Conditions and Boundary Values for Ordinary Differential Equations

Consider $\frac{d^2y}{dt^2}=F(t,y,\frac{dy}{dt})$, and think of $y(t)$ as the position of the particle, $\frac{d^2y}{dt^2}$ as acceleration, and $F(t,y,\frac{dy}{dt})$ as force. The state/configuration space is $(x_1(t),x_2(t))$, where $x_1(t)=y(t)$, $x_2(t)=\frac{dy}{dt}$. Then the system of first order equations is

$$\begin{aligned}\frac{dx_1}{dt}&=\frac{dy}{dt}=x_2(t)\\ \frac{dx_2}{dt}&=\frac{d^2y}{dt^2}=F(t,y(t),\frac{dy}{dt})=F(t,x_1(t),x_2(t))\end{aligned}.$$

Theorem: Existence and Uniqueness of Solution

There exists one and only one solution $x(t)=(x_1(t),\ldots,x_n(t))$ that satisfies $x(t_0)=x_0(t_0)$ where $x_0(t_0)$ is the given intial condition.

Quasi-Linear Partial Differential Equations

Definition: Quasi-Linear Partial Differential Equation

$a(x,y,u)u_x(x,y)+b(x,y,u)u_y(x,y)=c(x,y,u)$ (*) where *a*, *b*, *c* are given functions.

Claim

Let *a* and *b* be constant functions, and $c=0$, so $au_x+bu_y=0$ (1) . Then every solution $u(x,y)$ of (1) is of the form $u(x,y)=f(bx-ay)$ for some function of one variable (ex: $f(\xi)=\xi^2\Rightarrow u(x,y)=(bx-ay)^2$).

Uniqueness and Initial Conditions

For initial condition, we prescribe u along a given curve $\varphi(x)$, so $u(x,\varphi(x))=u_0(x)$ is given. Note that when $u(x,y)=f(bx-ay)$, $u(x,y)$ is constant along the line $bx-ay=c$. So if $u_0(x)=f(bx-a\varphi(x))$, there is a unique f provided that $bx-a\varphi(x)=c$ is not constant.

Suppose that $\varphi(x)=Ax$. Then $u_0(x)=f(bx-aAx)\Rightarrow f(x)=u_0(\frac{x}{b-aA})$. In conclusion,

1) The solution $u(x,y)$ is unique for any $u_0(x)$ over the line $y=Ax$ provided that $A\neq\frac{b}{a}$.
2) When $A=\frac{b}{a}$ then there are infinitely many solutions provided that $u_0(x)$ is constant. If $u_0(x)$ is not constant, then there are no solutions.

Method of Characteristic

Define a vector field $V(x,y,z)=(a(x,y,z),b(x,y,z),c(x,y,z))$. Normal direction at $(x,y,z=u(x,y))$ is $\vec{n}=(u_x(x,y),u_y(x,y),-1)$, but $V\cdot\vec{n}=au_x+bu_y+c(-1)=0$ because $au_x+bu_y=c$. So V lies in the tangent plane.

If $(x(t),y(t),z(t))$ is a solution of (1) $\begin{matrix}\frac{dx}{dt}=a(x(t),y(t),z(t))\\ \frac{dy}{dt}=b(x(t),y(t),z(t))\\ \frac{dz}{dt}=c(x(t),y(t),z(t))\end{matrix}$ such that $(x(0),y(0),z(0))$ lies in $z=u(x,y)$, ie $u(x(0),y(0))=z(0)$, then $(x(t),y(t),z(t))$ lies in $z(t)=u(x(t),y(t))$.

Suppose now that $(x(t,x_0),y(t,y_0),z(t,z_0))$ is any solution of (1) such that $\begin{matrix}x(0,x_0)=x_0\\ y(0,y_0)=y_0\\ z(0,z_0)=z_0\end{matrix}$ where $\begin{matrix}x(t,s)=x(t,x_0(s))\\ y(t,s)=y(t,y_0(s))\\ z(t,s)=z(t,z_0(s))\end{matrix}$. In most situations, we can solve for t and s in terms of x and y. Then $u(x,y)=z(t(x,y),s(x,y))$.

Note: When the Jacobian $J=det\begin{bmatrix}\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s}\\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s}\end{bmatrix}\neq 0$, then we can solve for t and s in terms of x and y locally.

Note: If $J=0$, then if $u(x,y)=z$ that contains $u(x_0(s),y_0(s))=z_0(s)$ satisfies $\frac{dz_0(s)}{ds}=\lambda c(x_0(s),y_0(s),z_0(s))$, there are infinitely many solutions; if not, then there is no solution.

Second Order Equations

$a(x,y)\cdot u_{xx}+2b(x,y)\cdot u_{xy}+c(x,y)\cdot u_{yy}+d(x,y)\cdot u_x+e(x,y)\cdot u_y+f(x,y)\cdot u=0$ (1), where a, b, c, d, e, f are given functions.

Canonical Types

1. Hyperbolic type: $b^2-ac>0$.
2. Parabolic type: $b^2-ac=0$.
3. Elliptic type: $b^2-ac<0$.

Fact

If we make a (one-to-one) change in variables $\begin{matrix}\xi=\xi(x,y)\\ \eta=\eta(x,y)\end{matrix}$ and require that $\det\begin{bmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{bmatrix}\neq 0\Leftrightarrow \xi_x eta_y - xi_y \eta_x \neq 0$, then there is a transformation such that (1) is transformed into:

1. $u_{\xi\eta}+\text{lower order terms}=0$ in the hyperbolic type;

2. $u_{\xi\xi}+\text{lower order terms}=0$ in the parabolic type;
3. $u_{\xi\xi}+u_{\eta\eta}+\text{lower order terms}=0$ in the elliptic type;

Special Case: *a*, *b*, *c* constants

Linear change of coordinates $(x,y)\rightarrow(\xi,\eta)$ given by $\begin{matrix}\xi=\alpha x+\beta y\\ \eta=\gamma x+\delta y\end{matrix}$ such that

$$\det\begin{bmatrix}\xi_x & xi_y\\ \eta_x & \eta_y\end{bmatrix}=\det\begin{bmatrix}\xi_x & xi_y\\ \eta_x & \eta_y\end{bmatrix}=\alpha\delta-\beta\gamma\neq 0 .$$

Then (1) becomes $Au_{\xi\xi}+2Bu_{\xi\eta}+Cu_{\eta\eta}+\text{lower order terms}$, where $\begin{matrix}A=a\alpha^2+2b\alpha\beta+c\beta^2\\ B=a\alpha\gamma+b(\alpha\delta+\gamma\beta)+c\beta\delta\\ C=a\gamma^2+2b\gamma\delta+c\delta^2\end{matrix}$.

1. In the hyperbolic case, choose $\alpha=-b+\sqrt{(b^2-ac)},\alpha=-b-\sqrt{(b^2-ac)},\beta=\delta=a$, $\gamma=-b-\sqrt{(b^2-ac)}$, $\beta=\delta=a$. Then $A=C=0,B\neq 0$.
2. In the parabolic case, choose $\alpha=\gamma=-b,\beta=\delta=a$. Then $B=C=0,A\neq 0$ or $A=B=0,C\neq 0$.
3. In the elliptic case, choose $\alpha=\frac{c}{\sqrt{(ac-b^2)}},\alpha=\frac{-c}{\sqrt{(ac-b^2)}},\gamma=0,\delta=1$, then $A=C\neq 0,B=0$.

The Wave Equation

$u_{tt}(x,t)=c^2u_{xx}(x,t),-\infty<x<\infty$ with initial conditions $u(x,0)=\varphi(x),u_t(x,0)=\psi(x)$.

The solution is $u(x,t)=\frac{1}{2}(\varphi(x+ct)+\varphi(x-ct))+\frac{1}{2c}\left(\int_{x+ct}^{x-ct}\psi(z)\,dz\right)$.

Diffusion Equation

$u_t(x,y,z,t)=k\Delta u=k(u_{xx}+u_{yy}+u_{zz})$.

In one dimension, $u_t(x,t)=k\,u_{xx}(x,t)$ is a parabolic type.

In One Dimension

$u_t(x,t)=k\,u_{xx}(x,t),-\infty<x<\infty$ with given initial conditions $u(x,0)=\varphi(x)$ where $\varphi(x)$ is a given function.

The solution is $u(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}\varphi(y)e^{\frac{-(x-y)^2}{4kt}}dy$. If $S(x,t)=\frac{1}{\sqrt{4\pi kt}}e^{\frac{-x^2}{4kt}},t>0$, then $u(x,t)=\int_{-\infty}^{\infty}\varphi(y)S(x-y,t)\,dy$.

Properties of the Kernel

The heat kernel/Gaussian/diffusion kernel $S(x,t)$ has the following properties:

1. Symmetric: $S(x,t)=S(-x,t)$.
2. $\lim_{t\to 0}S(x,t)=\{\begin{matrix}\infty, x=0\\ 0, x\neq 0\end{matrix}$.
3. $\int_{-\infty}^{\infty}S(x,t)\,dx=1,\forall t>0$.
4. $\lim_{t\to 0}\int_{-\infty}^{\infty}\varphi(x)S(x,t)\,dx=\varphi(0),\forall\varphi$.

Evaluation Techniques

Useful formula: $\int_{-\infty}^{\infty} \varphi'(y) S(x-y,t)\,dy = \frac{1}{2kt}\left[\int_{-\infty}^{\infty} y\varphi(y)S(x-y,t)\,dy - x\int_{-\infty}^{\infty}\varphi(y)S(x-y,t)\,dy\right]$.

- If $\varphi=1, \varphi'=0$, then $\int_{-\infty}^{\infty} y S(x-y,t)\,dy = x$. So if $\varphi(x)=x$, then $u(x,t)=\int_{-\infty}^{\infty}\varphi(y)S(x-y,t)\,dy=\int_{-\infty}^{\infty} yS(x-y,t)\,dy=x$ and $u(x,0)=x=\varphi(x)$.
- If $\varphi=y$, then $\int_{-\infty}^{\infty} y^2 S(x-y,t)\,dy=x^2+2kt$. So if $\varphi(x)=x^2$, then $u(x,t)=x^2+2kt$ and $u(x,0)=x^2=\varphi(x)$.
- If $\varphi=y^2$, then $\int_{-\infty}^{\infty} y^3 S(x-y,t)\,dy=x^3+6ktx$. So if $\varphi(x)=x^3$, then $u(x,t)=x^3+6ktx$ and $u(x,0)=x^3=\varphi(x)$.

Theorem

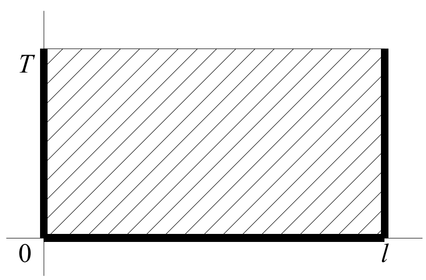
Suppose that $\varphi(x)$ is such that $\lim_{|x|\to\infty}\varphi(x)e^{-x^2}=\infty$, then $\lim_{t\downarrow 0}\int_{-\infty}^{\infty}\varphi(y)S(x-y,t)\,dy=\varphi(x), \forall x$. In that sense $\int_{-\infty}^{\infty}\varphi(y)S(x-y,t)\,dy$ is a solution with $u(x,0)=\varphi(x)$.

The Maximum Principle

Let $u(x,t)$ be a solution of $u_t=k\,u_{xx}$ on a rectangle $0\le x\le l, 0\le t\le T$. The maximum of $u(x,t)$ occurs only on the part of the boundary $\{(x,0): 0\le x\le l\}\cup\{(0,t): 0\le t\le T\}\cup\{(l,t): 0\le t\le T\}$.

Theorem: Uniqueness of Solution

Suppose that we seek a solution $u(x,t)$ that satisfies $u(x,0)=\varphi(x), 0\le x\le l$. Suppose further that $u(x,t)$ satisfies $u(0,t)=\alpha(t)$ and $u(l,t)=\beta(t)$, where $\alpha(t)$ and $\beta(t)$ are prescribed functions. Then the solution is unique, i.e. there is at most one solution.



Diffusion Equation On Half Line

Equation: $u_t(x,t)=k\,u_{xx}(x,t), 0<x<\infty$.
Initial data: $u(x,0)=\varphi(x), x>0$.
Boundary conditions:

- Dirichlet Condition: prescribe $u(0,t)=\alpha(t)$ (usually $\alpha(t)=0$).
- Neumann Condition: prescribe $u_x(0,t)=\alpha(t)$ (usually $\alpha(t)=0$).
- Robin Condition: prescribe $u(0,t)+a\,u_x(0,t)=0$.

Method of Solution: Dirichlet Boundary Condition

Take the case with $u_t(x,t)=k u_{xx}(x,t), x>0$, $u(x,0)=\varphi(x), x>0$, $u(0,t)=0, \forall t\geq 0$.

We want to extend φ to the entire line $-\infty<x<\infty$ such that the solution $u(x,t)$ induced by this extension satisfies $u(0,t)=0$.

Note that $\tilde{\varphi}(x)=\varphi(x), x>0$. Now, $u(0,t)=0$ for all $t>0$ iff $\tilde{\varphi}$ is an odd function ($\tilde{\varphi}(-x)=-\tilde{\varphi}(x)$).

Then $u(x,t)=\int_{-\infty}^{\infty}\tilde{\varphi}(y)S(x-y,t)dy=\int_{0}^{\infty}\varphi(y)(S(x-y,t)-S(x+y,t))dy$.

Method of Solution: Neumann Boundary Condition

Solve $u_t=k u_{xx}, x>0$, with initial data $u(x,0)=\varphi(x), x>0$ and Neumann condition $u_x(0,t)=0$.

If $u(x,t)$ is even (i.e. $u(-x,t)=-u(x,t)$), then $u_x(x,t)$ is odd (i.e. $u_x(-x,t)=-u_x(x,t)$).

The solution is $u(x,t)=\int_{-\infty}^{\infty}\tilde{\varphi}(y)S(x-y,t)dy=\int_{0}^{\infty}\varphi(y)(S(x-y,t)+S(x+y,t))dy$.

Wave Equation on Half Line

Solve $u_{tt}=c^2 u_{xx}, x>0$.

Initial data $u(x,0)=\varphi(x)$, and $u_t(x,0)=0$ for simplicity.

Dirichlet Boundary Condition

Dirichlet condition $u(0,t)=0$.

Extend φ to odd function $\tilde{\varphi}$. Then the solution is $u(x,t)=\frac{1}{2}(\tilde{\varphi}(x+ct)+\tilde{\varphi}(x-ct))$.

Note: $u(x,t)=-u(x,t)\Rightarrow u(0,t)=0$.

Wave Equation on Finite Interval

Solve: $u_{tt}=c^2 u_{xx}, 0<x<L$.

Initial data: $\varphi(x)=u(x,0), 0<x<L$ and $\psi(x)=u_t(x,0), 0<x<L$.

Dirichlet Boundary Condition

Dirichlet condition: $u(0,t)=u(L,t)=0$.

Extend φ to $\tilde{\varphi}$ and ψ to $\tilde{\psi}$ so that $u(x,t)$ is odd about $x=0$ (i.e. $u(-x,t)=-u(x,t)$) and odd about $x=L$ (i.e. $u(x+L,t)=-u(L-x,t)$). Then the solution is $u(x,t)=\frac{1}{2}(\tilde{\varphi}(x+ct)+\tilde{\varphi}(x-ct))+\frac{1}{2c}\left(\int_{x+ct}^{x-ct}\tilde{\psi}(z)dz\right)$.

Separation of Variables and Boundary Value Problems

Method of Separation of Variables

The method of separation of variables assumes that any solution $u(x,t)$ can be written as $u(x,t)=X(x)T(t)$.

Solutions

With the diffusion or wave equation, we need to solve $X''+\lambda X=0$, where λ is an unknown constant:

- For $\lambda>0$, $X(x)=A\cos(\sqrt{\lambda}\,x)+B\sin(\sqrt{\lambda}\,x)$.
- For $\lambda<0$, $X(x)=A\cosh(\sqrt{-\lambda}\,x)+B\sinh(\sqrt{-\lambda}\,x)$.
- For $\lambda=0$, $X(x)=Ax+B$.

In the Dirichlet case ($u(0,t)=u(L,t)=0$), $\lambda>0$ and $\lambda_n=\frac{n^2\pi^2}{L^2}$. So $X_n(x)=\sin\left(\frac{n\pi}{L}x\right)$.

In the Neumann case ($u_x(0,t)=u_x(L,t)=0$), $\lambda=0$ so $X(x)=\text{constant}$; or $\lambda<0$ and $\lambda_n=\frac{n^2\pi^2}{L^2}$, so $X_n(x)=\cos\left(\frac{n\pi}{L}x\right)$.

Dirichlet Boundary Condition

For the wave equation $u_{tt}(x,t)=c^2u_{xx}(x,t)$, we have $T_n(t)=a_n\cos\left(\frac{cn\pi}{L}t\right)+b_n\sin\left(\frac{cn\pi}{L}t\right)$. So $u_n(x,t)=X_n(x)T_n(t)=\left[a_n\cos\left(\frac{cn\pi}{L}t\right)+b_n\sin\left(\frac{cn\pi}{L}t\right)\right]\sin\left(\frac{n\pi}{L}x\right)$.

For the diffusion equation $u_t(x,t)=k\,u_{xx}(x,t)$, we have $T_n(t)=c_n\mathrm{e}^{-k\frac{n^2\pi^2}{L^2}t}$. So $u_n(x,t)=X_n(x)T_n(t)=c_n\mathrm{e}^{-k\frac{n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi}{L}x\right)$.

Neumann Boundary Condition

Wave equation: $u_n(x,t)=\left[a_n\cos\left(\frac{cn\pi}{L}t\right)+b_n\sin\left(\frac{cn\pi}{L}t\right)\right]\cos\left(\frac{n\pi}{L}x\right)$.

Diffusion equation: $u_n(x,t)=c_n\mathrm{e}^{-k\frac{n^2\pi^2}{L^2}t}\cos\left(\frac{n\pi}{L}x\right)$.

Mixed Boundary Condition

Mixed boundary condition $u(0,t)=u_x(L,t)=0$, then $X(0)=X'(L)=0$.

We have $\sqrt{\lambda_n}=\frac{\pi(2n+1)}{2L}$.

Robin Condition

Take $u_x(0,t)-h\,u(0,t)=0$ and $u_x(L,t)=0$. We have $X''+\lambda X=0$.

- Assume $\lambda>0$. Then $X(x)=A\cos(\sqrt{\lambda}\,x)+B\sin(\sqrt{\lambda}\,x)$, and we get $\tan(\sqrt{\lambda}\,L)=\frac{h}{\sqrt{\lambda}}$. Setting $y=\sqrt{\lambda}\,L>0$, we get $\tan y=\frac{Lh}{y}=\frac{c}{y}$ a transcendental equation. On $y>0$, we get infinitely many solutions $y_1<y_2<\cdots\Rightarrow\lambda_1<\lambda_2<\cdots$, with the difference approaching π .
- Assume $\lambda<0$. Then $X(x)=A\cosh(\sqrt{-\lambda}\,x)+B\sinh(\sqrt{-\lambda}\,x)$, and setting $y=\sqrt{-\lambda}\,L>0$ we get $\tanh y=-\frac{c}{y}$ a transcendental equation. We get no solution.

So there are infinitely many eigenvalues $\lambda_1<\lambda_2<\cdots$ with corresponding eigenfunctions $X_1(x), X_2(x), \ldots$.

Vector Spaces: Introduction to Fourier Series

Let V_n be the space of all linear combinations of $f=b_1\sin\left(\frac{\pi}{L}x\right)+b_2\sin\left(\frac{2\pi}{L}x\right)+\cdots+b_n\sin\left(\frac{n\pi}{L}x\right)$.

Define $L:V_n\to V_n$ $L(f)=-\frac{d^2f}{dx^2}=\sum_{k=1}^{n}b_k\frac{k^2\pi^2}{L^2}\sin\left(\frac{k\pi}{L}\right)$.

Choose basis: $v_1=\left\{\sin\left(\frac{\pi}{L}x\right),v_2=\sin\left(\frac{2\pi}{L}x\right),\ldots,v_n=\sin\left(\frac{n\pi}{L}x\right)\right\}$. Then the matrix of L relative to this basis

$$\begin{bmatrix} \left(\frac{\pi}{L}\right)^2 & 0 & \cdots & 0 \\ 0 & \left(\frac{2\pi}{L}\right)^2 & 0 & \vdots \\ \vdots & 2\pi^2/L^2 & \ddots & 0 \\ 0 & \cdots & 0 & \left(\frac{n\pi}{L}\right)^2 \end{bmatrix}$$

is a diagonal matrix since $L(v_k)=\left(\frac{k\pi}{L}\right)^2 v_k$.

Let $n\to\infty$ and consider the space of functions f on $0\le x<L$ which can be written as $f(x)=\sum_{k=1}^{\infty}b_k\sin\left(\frac{k\pi}{L}x\right)$ for Fourier coefficients $b_k=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)dx$ $k=1,2,3,\ldots$ of f relative to X_n .

Full Fourier Series

Definition

Let $-L<x<L$. The full Fourier series of $f(x)$ is $\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi}{L}\right)+b_n\sin\left(\frac{n\pi}{L}\right)$ ..

Coefficients

The coefficients are uniquely determined from orthogonality of functions $\cos\left(\frac{n\pi}{L}\right)$ and $\sin\left(\frac{n\pi}{L}\right)$:

- $\int_{-L}^{L}\sin\left(\frac{n\pi}{L}\right)\sin\left(\frac{m\pi}{L}\right)dx=\begin{cases}0 & n\neq m\\ L & n=m\end{cases}$.
- $\int_{-L}^{L}\sin\left(\frac{n\pi}{L}\right)\cos\left(\frac{m\pi}{L}\right)dx=0$.
- $\int_{-L}^{L}\cos\left(\frac{n\pi}{L}\right)\cos\left(\frac{m\pi}{L}\right)dx=\begin{cases}0 & n\neq m\\ L & n=m\end{cases}$.

These relations imply that:

- $a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi}{L}\right)dx$ $n=0,1,2,\ldots$.
- $b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi}{L}\right)dx$ $n=0,1,2,\ldots$.

Relation To Differential Equations

Take $0 \leq x \leq L$.

Dirichlet Condition: Take $f(x)$ an odd extension of $\varphi(x)$. Then $a_n = 0$ and $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)dx$ $n = 1, 2, \ldots$.

Neumann Condition: Take $f(x)$ an even extension of $\varphi(x)$. Then $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right)dx$ $n = 0, 1, 2, \ldots$ and $b_n = 0$.

General Eigenvalues and Eigenfunctions

$X'' + \lambda X = 0$ on $0 \leq x \leq L$.

1. If $X(0) = X(L) = 0$, then $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$.
2. If $X'(0) = X'(L) = 0$, then $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $X_n(x) = \cos\left(\frac{n\pi}{L}x\right)$.
3. If $X'(0) - hX(0) = 0$ and $X'(L) = 0$, then $\lambda_1 < \lambda_2 < \cdots$ (eigenvalues) and $X_1, X_2, \ldots$ (eigenfunctions).
4. If $X(0) = X(L) = 0$ and $X'(0) = X'(L) = 0$, then $\lambda = 0$ and $X(x) = \text{constant}$ or $\lambda_n = \left(\frac{2n\pi}{L}\right)^2$ and

   $X_n(x) = A_n \sin\left(\frac{2n\pi}{L}x\right) + B_n \cos\left(\frac{2n\pi}{L}x\right)$ where A_n and B_n are arbitrary constants.

General Boundary Conditions

Solve $X'' + \lambda X = 0$, $a \leq x \leq b$ subject to the boundary conditions $\begin{matrix} \alpha_1 X(a) + \alpha_2 X(b) + \alpha_3 X'(a) + \alpha_4 X'(b) = 0 \\ \beta_1 X(a) + \beta_2 X(b) + \beta_3 X'(a) + \beta_4 X'(b) = 0 \end{matrix}$ for some constants $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$.

Definition: Symmetric Boundary Conditions

Let f and g be any functions that satisfies the above boundary condition. Then conditions are called symmetric if

$f'(x)g(x) - f(x)g'(x)\Big|_{x=a}^{x=b} = 0 \Leftrightarrow f'(b)g(b) - f(b)g'(b) - (f'(a)g(a) - f(a)g'(a)) = 0$.

Fact

Conditions 1 to 4 are symmetric.

Theorem

Suppose that X_n and X_m are eigenfunctions on $[a, b]$ that corresponds to distinct eigenvalues λ_n and λ_m ($\lambda_n \neq \lambda_m$), and suppose that the boundary conditions are symmetric. Then X_n and X_m are orthogonal in the sense that

$\int_a^b X_n(x) X_m(x)\,dx = 0$.

Hilbert Space

Basic Space

$L^2[a, b] = \left\{ f : [a, b] \rightarrow \mathbb{R} \mid \int_a^b f^2(x)\,dx < \infty \right\}$.

Fact
$L^2[a,b]$ is a vector space.

Inner Product
Take f and g in L^2. Then define the inner product to be $(f,g)=\int_a^b f(x)g(x)dx$.

Norm
Define $\|f\|=\left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}}=(f,f)^{\frac{1}{2}}$ to be the norm of f.

Cauchy-Schwartz Inequality
$|(f,g)|\le\|f\|\|g\|$ or $\left|\int_a^b f(x)g(x)dx\right|\le\left|\int_a^b f^2(x)dx\right|^{\frac{1}{2}}\left|\int_a^b g^2(x)dx\right|^{\frac{1}{2}}$.

Note: This implies $\left|\left(\frac{f}{\|f\|},\frac{g}{\|g\|}\right)\right|\le 1$, so define $\cos\theta=\left|\left(\frac{f}{\|f\|},\frac{g}{\|g\|}\right)\right|$.

Definition: Convergence
$\{f_n\}\in L^2$ is said to converge to f if $\lim_{n\to\infty}\|f_n-f\|=0$.

Definition: Cauchy Sequence
$\{f_n\}\in L^2$ is called a Cauchy Sequence if $\|f_n-f_m\|\to 0$ as $n,m\to\infty$.

Basic Properties of Inner Product and Norm
1. Symmetric: $(f,g)=(g,f)$.
2. Bilinear: $(f,\alpha g+\beta h)=\alpha(f,g)+\beta(f,h)$ for $\alpha,\beta\in\mathbb{R}$ and $f,g,h\in L^2$.
3. $(f,f)\ge 0$ $\forall f\in L^2$; if $(f,f)=\int_a^b f^2(x)dx=0$ then $f=0$ "almost everywhere".
4. L^2 is complete in the sense that any Cauchy sequence in L^2 converges to an element in L^2 .

Definition: Hilbert Space
Any vector space H with an inner product $(\ ,\ )$ that satisfies properties 1 to 4 is called a Hilbert space.

Theorem
If $X_1, X_2, \ldots, X_n, \ldots$ are the eigenfunctions corresponding to symmetric boundary problem, then the Fourier series of any function f converges to f in L^2 norm.

Least Square Approximation
Let V_n denote the linear span of $X_1, X_2, \ldots, X_n$ (i.e. $f\in V_n \Leftrightarrow f=\alpha_1 X_1+\alpha_2 X_2+\cdots+\alpha_n X_n$.

Problem

Question: Let $f \in L^2$. For which values of $\alpha_1, \alpha_2, \dots, \alpha_n$ is the distance $\| f - \sum \alpha_i X_i \|$ minimum?

Answer: $\alpha_i = \frac{(f, X_i)}{\|X_i\|}$ $i = 1, \dots, n$.

Convergence of Fourier Series

Theorem

The Fourier series relative to $X_1, X_2, \dots$ of any element $f \in L^2$ converges to f. That is, if $S_N = \sum_{i=1}^{N} (f_i, X_i) X_i$, then $\lim_{N \to \infty} \| S_N - f \| = 0$.

Definition: Piecewise Continuous

A function is piecewise continuous if it is continuous at all but a finite number of points. At a point of discontinuity f has both a right and a left limit (ie f has a jump discontinuity).

So if c is a point of discontinuity of f, then both $f(c^+) = \lim_{\substack{x \to c \\ x > c}} f(x)$ and $f(c^-) = \lim_{\substack{x \to c \\ x < c}} f(x)$ exist.

Theorem: Point-wise Convergence of Fourier Series

Assume f is such that:

- f is periodic of period 2π .
- f and its derivative f' are "piecewise continuous".

Then $\lim_{n \to \infty} S_N(x) = \frac{1}{2}\left(f(x^+) + f(x^-) \right)$.

Note: If f is continuous at x , then $f(x^+) = f(x^-) = f(x)$, so $\lim_{n \to \infty} S_N(x) = f(x)$.

Auxiliary Results

1. Bessel's Inequality: $\|g\|^2 \geq \sum_{k=1}^{\infty} \frac{(g, X_k)^2}{\|X_k\|^2}$ where $X_1, X_2, \dots$ are eigenfunctions on $[a, b]$ with symmetric boundary values and $g \in L^2[a, b]$. This implies that $\lim_{k \to \infty} \frac{(g, X_k)^2}{\|X_k\|^2} = 0$.
2. Let $K_N(\theta) = 1 + 2 \sum_{k=1}^{N} \cos k\theta$. Then $\int_{-\pi}^{pi} K_N(\theta)\, d\theta = 2\pi \Leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{pi} K_N(\theta)\, d\theta = 1$.
3. $K_N(\theta) = \frac{\sin\left(\left(N + \frac{1}{2} \right) \theta \right)}{\sin\left(\frac{\theta}{2} \right)}$.

Definition: Uniform Convergence

f_n converges to f uniformly if $\lim_{n \to \infty} \max_{a < x < b} | f_n(x) - f(x) | = 0$.

Harmonic Functions and Laplace's Equation

Laplace's Equation

In n dimensions,

$$\Delta u = \operatorname{div}(\operatorname{grad} u) = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 \,.$$

Notes:

- Δ is the Laplacian, and it is an operator that acts on functions of n variables.
- The gradient of a scalar function $u(x_1, \ldots, x_n)$ is $\operatorname{grad} u = \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$.
- The divergence of a vector field $\vec{V}(x_1, \ldots, x_n) = (V_1(x_1, \ldots, x_n), \ldots, V_n(x_1, \ldots, x_n))$ is $\operatorname{div}(\vec{V}) = \frac{\partial V_1}{\partial x_1} + \cdots + \frac{\partial V_n}{\partial x_n}$.
- When $\vec{V} = \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$, then $\Delta u = \operatorname{div}(\operatorname{grad} u)$.

Definition: Harmonic

Any solution of Laplace's equation is called harmonic.

Properties

- $n=1$: $\frac{d^2 u}{d x^2} = 0$, so $u(x) = Ax + B$.
- $n=2$: Connection to complex functions $f(z) = u(x, y) + iv(x, y)$. Then f analytic (can be expressed as a Taylor series, i.e. differentiable) implies u and v are harmonic.

Maximum/Minimum Principle

Definitions

1. D is an open subset of $\mathbb{R}^n$ if for all $\vec{x} \in D$, there exists $r > 0$ such that for all $\vec{y} \in D$, $\|\vec{x} - \vec{y}\| < r$.
2. ∂D is the boundary of D . A point $\vec{b}$ is a boundary point if for all $\varepsilon > 0$, $B = \{\|\vec{x} - \vec{b}\| < \varepsilon\}$ has non empty intersection with both D and the complement of D in $\mathbb{R}^n$.
3. D is connected if there exists a polynomial curve joining any two points in D and is lying in D .
4. D is bounded if it is contained in some ball $B = \{\vec{x} \mid \|\vec{x}\| < R\}$ $0 < R < \infty$.

Maximum/Minimum Principle

Assume D to be an open, connected subset of $\mathbb{R}^n$ such that $D \cup \partial D$ is bounded. Let u be any solution of (the Laplace $\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$ equation) in D such that u is defined and continuous in $D \cup \partial D$. Then:

- Maximum Principle: $u(\vec{x}) \le \max_{\partial D} u(\vec{x}) \quad \forall \vec{x} \in D$.
- Minimum Principle: $u(\vec{x}) \ge \min_{\partial D} u(\vec{x}) \quad \forall \vec{x} \in D$.

That is, u attains its maximum/minimum on ∂D .

Boundary Value Problems

Dirichlet Problem

$\frac{\partial^2 u}{\partial x_1^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}=0$ subject to $u|_{\partial D}=\varphi(\vec{x})$ (given).

By the Maximum/Minimum Principle, the solution of the Dirichlet problem is unique.

Neumann Problem

$\frac{\partial^2 u}{\partial x_1^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}=0$ subject to $\frac{\partial u}{\partial \vec{n}}|_{\partial D}=\psi(\vec{x})$ (given). Here, $\vec{n}$ is the external normal.

Robin/Mixed Problem

$\frac{\partial^2 u}{\partial x_1^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}=0$ subject to $\frac{\partial u}{\partial \vec{n}}+a(x)u(\vec{x})=k(\vec{x})$ (given).

Basic Property

Solutions to the Laplace equation are invariant under rigid motions $\varphi(x)=T(\vec{x})+R(\vec{x})$, where

- $T(\vec{x})=\vec{a}+\vec{x}$ is a translation,
- $R(\vec{x})=A\vec{x}$ ($A^T=A^{-1}$ $det\, A=\pm 1$) is a rotation.

Rectangular Harmonics

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ on $D=\{(x,y): 0\le x\le a, 0\le y\le b\}$ with $\partial D=\begin{Bmatrix} u(x,0)=f(x) \\ u(a,y)=g(x) \\ u(x,b)=h(x) \\ u(0,y)=i(x)\end{Bmatrix}$ where f, g, h, i are given functions.

Separation of Variables

We assume $u(x,y)=X(x)Y(y)$.

When $\partial D=\begin{Bmatrix} u(x,0)=f(x) \\ u(x,b)=h(x) \\ u(0,y)=u(a,y)=0\end{Bmatrix}$, then $u_n(x,y)=\left(a_n\cosh\frac{n\pi}{a}y+b_n\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$.

When $\partial D=\begin{Bmatrix} u(0,y)=g(x) \\ u(a,y)=i(x) \\ u(x,0)=u(x,b)=0\end{Bmatrix}$, then $u_n(x,y)=\left(a_n\cosh\frac{n\pi}{b}x+b_n\sinh\frac{n\pi}{b}x\right)\sin\frac{n\pi}{b}y$.

Note: $\partial D=\begin{Bmatrix} u(x,0)=f(x) \\ u(a,y)=g(x) \\ u(x,b)=h(x) \\ u(0,y)=i(x)\end{Bmatrix}=\begin{Bmatrix} u(x,0)=f(x) \\ u(x,b)=h(x) \\ u(0,y)=u(a,y)=0\end{Bmatrix}+\begin{Bmatrix} u(0,y)=g(x) \\ u(a,y)=i(x) \\ u(x,0)=u(x,b)=0\end{Bmatrix}$.

Note: $u(x,y)=\sum_{n=1}^{\infty}u_n(x,y)$.

Case 1

Suppose that we want the solution with $\partial D=\begin{Bmatrix} u(x,0)=f(x) \\ u(x,b)=u(0,y)=u(a,y)=0\end{Bmatrix}$.

The sine Fourier coefficients of f are $a_n=\frac{2}{a}\int_0^a f(z)\sin\left(\frac{n\pi}{a}z\right)dz$.

Since $u(x,b)=0\Leftrightarrow\sum_{n=1}^{\infty}\left(a_n\cosh\frac{n\pi}{a}b+b_n\sinh\frac{n\pi}{a}b\right)\sin\frac{n\pi}{a}x=0\rightarrow a_n\cosh\frac{n\pi}{a}b+b_n\sinh\frac{n\pi}{a}b=0$, so

$$b_n=-\frac{a_n\cosh\frac{n\pi}{a}b}{\sinh\frac{n\pi}{a}b} .$$

Then

$$\begin{aligned}u(x,y)&=\sum_{n=1}^{\infty}\left(a_n\cosh\frac{n\pi}{a}y+b_n\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x\\&=\sum_{n=1}^{\infty}a_n\left(\cosh\frac{n\pi}{a}y-\frac{\cosh\frac{n\pi}{a}b}{\sinh\frac{n\pi}{a}b}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x .\\&=\sum_{n=1}^{\infty}\frac{a_n}{\sinh\frac{n\pi}{a}y}\left(\sinh\frac{n\pi}{a}(b-y)\right)\left(\sin\frac{n\pi}{a}x\right)\end{aligned}$$

Laplace's Equation on Circular Regions

1. Annulus: $D=\{(r,\theta),-\pi<\theta\leq\pi,a\leq r\leq b\}$, $\partial D=\{(a,\theta),-\pi<\theta\leq\pi\}\cup\{(b,\theta),-\pi<\theta\leq\pi\}$.
2. Disk: $D=\{(r,\theta),-\pi<\theta\leq\pi,0\leq r\leq b\}$, $\partial D=\{(b,\theta),-\pi<\theta\leq\pi\}$.
3. Wedge: $D=\{(r,\theta),0\leq\theta\leq\alpha,0\leq r\leq b\}$.

Polar Coordinates and Separation of Variables

Using polar coordinates $\begin{matrix}x=r\cos\theta\\y=r\sin\theta\end{matrix}$, $u_{xx}+u_{yy}=0$ becomes $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$.

Assuming $u(r,\theta)=R(r)\Theta(\theta)$, we get the two equations $r^2R''+rR'-\lambda R=0$ and $\Theta''+\lambda\Theta=0$.

Annulus

Eigenfunctions:

- When $n\neq0$, $\Theta_n(\theta)=A_n\cos n\theta+B_n\sin n\theta$ and $R_n(r)=C_nr^n+D_nr^{-n}$.
- When $n=0$, $\Theta_0(\theta)=A_n$ and $R_0(r)=c_0+c_1\ln r$.

So the solution is $u(r,\theta)=\sum_{n=0}^{\infty}R_n(r)\Theta_n(\theta)=c_0+c_1\ln r+\sum_{n=1}^{\infty}(C_nr^n+D_nr^{-n})(A_n\cos n\theta+B_n\sin n\theta)$. The coefficients are the Fourier coefficients of the boundary conditions $u(a,\theta)=f(\theta)$ and $u(b,\theta)=g(\theta)$.

Disk

The usual assumption is that $u(0,\theta)$ bounded. This forces $c_1=D_n=0$. So the solution is $u(r,\theta)=c_0+\sum_{n=1}^{\infty}C_nr^n(A_n\cos n\theta+B_n\sin n\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}r^n(a_n\cos n\theta+b_n\sin n\theta)$, where a_n and b_n are the Fourier coefficients determined by the boundary condition $u(b,\theta)=f(\theta)$.

Wedge

Consider the special case that $u(r,0)=u(r,\alpha)=0$ and $u(b,\theta)=f(\theta)$.

The solution is $u(r,\theta)=\sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\alpha}} \sin\frac{n\pi}{\alpha}\theta$, where $a_n=\frac{2}{b^{n\pi/\alpha}\alpha}\int_0^{\alpha} f(\varphi)\sin\frac{n\pi}{\alpha}\varphi\, d\varphi$ is the Fourier coefficient determined by $u(b,\theta)=f(\theta)$.

Poisson Formula and Poisson Kernel

Poisson Formula

On the disk $D=\{(r,\theta), -\pi<\theta\le\pi, 0\le r\le b\}$ with $u(b,\theta)=f(\theta)$,

$$\begin{aligned}u(r,\theta)&=\frac{1}{2\pi}\left[\int_{-\pi}^{\pi} f(\varphi)\left(1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\varphi-\theta)\right)d\varphi\right]\\&=\int_{-\pi}^{\pi} f(\varphi)P(\varphi-\theta)\,d\varphi\\&=\int_{-\pi}^{\pi}\frac{(b^2-r^2)f(\varphi)}{b^2-2br\cos(\varphi-\theta)+r^2}d\varphi\end{aligned}$$

where $P(\varphi-\theta)=\frac{1}{2\pi}\left(1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\varphi-\theta)\right)=\frac{1}{2\pi}\frac{b^2-r^2}{b^2-2br\cos(\varphi-\theta)+r^2}$.

Poisson Kernel

$P_a(r,\theta)=\frac{1}{2\pi}\frac{a^2-r^2}{a^2-2ar\cos\theta+r^2}$.

Basic Properties

1. $\int_0^{2\pi} P_a(r,\varphi-\theta)\,d\varphi=1\Leftrightarrow\frac{1}{2\pi}\int_0^{2\pi}\frac{a^2-r^2}{a^2-2ar\cos(\varphi-\theta)+r^2}d\varphi=1$. In this case $u(a,\theta)=f(\theta)=1$, but $u(r,\theta)=1\ \ \forall r,\theta$ also.
2. $\lim_{r\to a} P_a(r,\theta)=\{\begin{matrix}0 & \theta\neq 0\\ \infty & \theta=0\end{matrix}$.
3. $\lim_{r\to a}\int_0^{2\pi} f(\varphi)P_a(r,\varphi-\theta)\,d\varphi=f(\theta)$ whenever f is a continuous function of θ .
4. Averaging Property of Harmonic Functions: $x=y=0\Leftrightarrow r=0$, so $\underbrace{u(0,0)}_{\text{cartesian}}=\underbrace{u(0,\theta)}_{\text{polar}}=\frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2-0^2)f(\varphi)}{a^2-2a0\cos(\varphi-\theta)+0^2}d\varphi=\frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\,d\varphi$ is the average value of f.

Consequences of Poisson Representation

1. A harmonic function u defined on some domain D cannot attain a maximum (nor minimum) in the interior of D . Here the interior of D are the points p in D such that there exists a disk centered at p that is entirely contained in D .
2. $u(r,\theta)$ has partial derivatives of all orders, even when f is only continuous.

$$u(r,\theta)=\int_0^{2\pi} f(\varphi)P_a(r,\varphi-\theta)\,d\varphi \Rightarrow \frac{\partial u}{\partial r}=\int_0^{2\pi} f(\varphi)\frac{\partial}{\partial r}P_a(r,\varphi-\theta)\,d\varphi \ .$$