**Introduction**

Types of Partial Differential Equations

- Transport equation: \( u_x(x, y) + u_y(x, y) = 0 \), where \( u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y} \), and \( u(x, y) = \)
- Shockwave equation: \( u_x(x, y) + u(x, y)u_y(x, y) = 0 \).

The vibrating string equation: \( u_x(x, t) = c^2 u_{xx}(x, t) \), where \( u_x = \frac{\partial u}{\partial x} \) and \( u_{xx} = \frac{\partial^2 u}{\partial x^2} \).

The wave equation: \( u_{tt}(x, y, z, t) = c^2 (u_{xx}(x, y, z, t) + u_{yy}(x, y, z, t) + u_{zz}(x, y, z, t)) \).

In general: \( u_p(x_1, \ldots, x_n, t) = c^2 \Delta u(x_1, \ldots, x_n, t) \), where \( \Delta = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \).

Diffusion equation: \( u_y(x, t) = c^2 u_{xx}(x, t) \).

In general: \( u_i(x_1, \ldots, x_n, t) = c^2 \Delta u(x_1, \ldots, x_n, t) \).

Steady state: \( u = 0 \).

Laplacian equation: \( \Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 \).

**Initial Conditions and Boundary Values for Ordinary Differential Equations**

Consider \( \frac{d^2 y}{dt^2} = F(t, y, \frac{dy}{dt}) \), and think of \( y(t) \) as the position of the particle, \( \frac{d^2 y}{dt^2} \) as acceleration, and \( F(t, y, \frac{dy}{dt}) \) as force. The state/configuration space is \( (x_1(t), x_2(t)) \), where \( x_1(t) = y(t), \quad x_2(t) = \frac{dy}{dt} \). Then the system of first order

\[
\frac{dx_1}{dt} = \frac{dy}{dt} = x_2(t)
\]

equations is

\[
\frac{dx_2}{dt} - \frac{d^2 y}{dt^2} = F(t, y(t), \frac{dy}{dt}) = F(t, x_1(t), x_2(t))
\]

**Theorem: Existence and Uniqueness of Solution**

There exists one and only one solution \( x(t) = (x_1(t), \ldots, x_n(t)) \) that satisfies \( x(t_0) = x_0(t_0) \) where \( x_0(t_0) \) is the given initial condition.

**Quasi-Linear Partial Differential Equations**

**Definition:** Quasi-Linear Partial Differential Equation

\[ a(x, y, u)u_x(x, y) + b(x, y, u)u_y(x, y) = c(x, y, u) \quad (*) \]

where \( a, b, c \) are given functions.

**Claim**

Let \( a \) and \( b \) be constant functions, and \( c = 0 \), so \( au_x + bu_y = 0 \) \((1)\). Then every solution \( u(x, y) \) of \((1)\) is of the form \( u(x, y) = f(bx - ay) \) for some function of one variable (ex: \( f(\xi) = \xi^2 \Rightarrow u(x, y) = (bx - ay)^2 \)).
Uniqueness and Initial Conditions

For initial condition, we prescribe \( u(x) \), so \( u(x, \varphi(x)) = u_0(x) \) is given. Note that when \( u(x, y) = f(bx - ay) \), \( u(x, y) \) is constant along the line \( bx - ay = c \). So if \( u_0(x) = f(bx - a \varphi(x)) \), there is a unique \( f \) provided that \( bx - a \varphi(x) = c \) is not constant.

Suppose that \( \varphi(x) = Ax \). Then \( u_0(x) = f(bx - aAx) \Rightarrow f(x) = u_0\left(\frac{x}{b - aA}\right) \). In conclusion,

1) The solution \( u(x, y) \) is unique for any \( u_0(x) \) over the line \( y = Ax \) provided that \( A \neq \frac{b}{a} \).

2) When \( A = \frac{b}{a} \) then there are infinitely many solutions provided that \( u_0(x) \) is constant. If \( u_0(x) \) is not constant, then there are no solutions.

Method of Characteristic

Define a vector field \( V(x, y, z) = (a(x, y, z), b(x, y, z), c(x, y, z)) \). Normal direction at \((x, y, z = u(x, y))\) is \( \vec{n} = (u_x(x, y), u_y(x, y), -1) \), but \( V \cdot \vec{n} = au_x + bu_y + c(-1) = 0 \) because \( au_x + bu_y = c \). So \( V \) lies in the tangent plane.

If \( (x(t), y(t), z(t)) \) is a solution of (1) \( \frac{dx}{dt} = b(x(t), y(t), z(t)) \), then \( x(0), y(0), z(0) \) lies in \( z = u(x, y) \), ie \( \frac{dx}{dt} = a(x(t), y(t), z(t)) \).

Suppose now that \( (x(t, x_0), y(t, y_0), z(t, z_0)) \) is any solution of (1) such that \( y(0, y_0) = y_0 \) where \( y(t, s) = y(t, y_0(s)) \). In most situations, we can solve for \( t \) and \( s \) in terms of \( x \) and \( y \). Then \( u(x, y) = z(t(x, y), s(x, y)) \).

Note: When the Jacobian \( J = \det \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{vmatrix} \neq 0 \), then we can solve for \( t \) and \( s \) in terms of \( x \) and \( y \) locally.

Note: If \( J = 0 \), then if \( u(x, y) = z \) that contains \( u(x_0(s), y_0(s)) = z_0(s) \), satisfies \( \frac{dz_0(s)}{ds} = \lambda c(x_0(s), y_0(s), z_0(s)) \), there are infinitely many solutions; if not, then there is no solution.

Second Order Equations

\[ a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = 0 \]  

(1), where \( a, b, c, d, e, f \) are given functions.

Canonical Types

1. Hyperbolic type: \( b^2 - ac > 0 \).
2. Parabolic type: \( b^2 - ac = 0 \).
3. Elliptic type: \( b^2 - ac < 0 \).

Fact

If we make a (one-to-one) change in variables \( \xi = \xi(x, y) \) and \( \eta = \eta(x, y) \) and require that \( \det \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \Leftrightarrow \xi, \eta \text{ at } -x_i, \eta_i \neq 0 \), then there is a transformation such that (1) is transformed into:

1. \( u_{\xi \eta} + \text{lower order terms} = 0 \) in the hyperbolic type;
2. \( u_{tt} + \text{lower order terms}=0 \) in the parabolic type;
3. \( u_{tt} + u_{xx} + \text{lower order terms}=0 \) in the elliptic type;

Special Case: \( a, b, c \) constants

Linear change of coordinates \((x, y) \rightarrow (\xi, \eta)\) given by \(\xi = \alpha x + \beta y\) such that
\[
\det \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \det \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \alpha \delta - \beta \eta \neq 0
\]

Then (1) becomes \( Au_{tt} + 2 Bu_{t} + Cu_{y} + \text{lower order terms} \), where
\[
A = a \alpha^2 + 2 b \alpha \beta + c \beta^2
\]
\[
B = a \alpha y + b (\alpha \delta + \gamma \beta) + c \beta \delta
\]
\[
C = a y^2 + 2 b y \delta + c \delta^2
\]

1. In the hyperbolic case, choose \(\alpha = -b + \sqrt{b^2 - ac}, \beta = \delta = a\) \(A = C = 0, B \neq 0\).
2. In the parabolic case, choose \(\alpha = y = -b, \beta = \delta = a\) \(B = C = 0, A \neq 0\) or \(A = B = 0, C \neq 0\).
3. In the elliptic case, choose \(\alpha = \frac{c}{\sqrt{|ac-b^2|}}, \beta = \frac{-c}{\sqrt{|ac-b^2|}}, y = 0, \delta = 1\) \(A = C \neq 0, B = 0\).

The Wave Equation

\( u_t(x,t) = c^2 u_{xx}(x,t), -\infty < x < \infty \) with initial conditions \( u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \).

The solution is \( u(x,t) = \frac{1}{2} [\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(z) \, dz \).

Diffusion Equation

\( u_t(x,t) = k u_{xx}(x,t), -\infty < x < \infty \) with given initial conditions \( u(x,0) = \varphi(x) \) where \( \varphi(x) \) is a given function.

In One Dimension

\( u_t(x,t) = k u_{xx}(x,t), -\infty < x < \infty \) with given initial conditions \( u(x,0) = \varphi(x) \).

The solution is \( u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \varphi(y) e^{-\frac{(x-y)^2}{4kt}} \, dy \). If \( S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}, t > 0 \), then \( u(x,t) = \int_{-\infty}^{\infty} \varphi(y) S(x-y,t) \, dy \).

Properties of the Kernel

The heat kernel/Gaussian/diffusion kernel \( S(x,t) \) has the following properties:
1. Symmetric: \( S(x,t) = S(-x,t) \).
2. \( \lim_{t \to 0} S(x,t) = \begin{cases} \varphi(x) & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \).
3. \( \int_{-\infty}^{\infty} S(x,t) \, dx = 1, \forall t > 0 \).
4. \( \lim_{t \to 0} \int_{-\infty}^{\infty} \varphi(x) S(x,t) \, dx = \varphi(0), \forall \varphi \).
**Evaluation Techniques**

Useful formula: \[ \int_{-\infty}^{\infty} \varphi'(y) S(x-y, t) dy = \frac{1}{2kt} \left[ \int_{-\infty}^{\infty} y \varphi(y) S(x-y, t) dy - x \int \varphi(y) S(x-y, t) dy \right]. \]

- If \( \varphi=1, \varphi' = 0 \), then \( \int_{-\infty}^{\infty} y S(x-y, t) dy = x \). So if \( \varphi(x) = x \), then
  \[ u(x, t) = \int_{-\infty}^{\infty} \varphi(y) S(x-y, t) dy = \int y S(x-y, t) dy = x \quad \text{and} \quad u(x, 0) = x = \varphi(x). \]
- If \( \varphi = 0 \), then \( \int_{-\infty}^{\infty} y^2 S(x-y, t) dy = x^2 + 2kt \). So if \( \varphi(x) = x^2 \), then \( u(x, t) = x^2 + 2kt \) and \( u(x, 0) = x^2 = \varphi(x) \).
- If \( \varphi = y^2 \), then \( \int_{-\infty}^{\infty} y^3 S(x-y, t) dy = x^3 + 6ktx \). So if \( \varphi(x) = x^3 \), then \( u(x, t) = x^3 + 6ktx \) and \( u(x, 0) = x^3 = \varphi(x) \).

**Theorem**

Suppose that \( \varphi(x) \) is such that \( \lim_{|x| \to \infty} \varphi(x) e^{-x^2} = 0 \), then \( \lim_{t \to 0} \int_{-\infty}^{\infty} \varphi(y) S(x-y, t) dy = \varphi(x) \), \( \forall x \). In that sense \( \int_{-\infty}^{\infty} \varphi(y) S(x-y, t) dy \) is a solution with \( u(x, 0) = \varphi(x) \).

**The Maximum Principle**

Let \( u(x, t) \) be a solution of \( u_t = \Delta u \) on a rectangle \( 0 \leq x \leq L, 0 \leq t \leq T \). The maximum of \( u(x, t) \) occurs only on the part of the boundary \( \{(x, 0): 0 \leq x \leq L \} \cup \{(0, t): 0 \leq t \leq T \} \cup \{(L, t): 0 \leq t \leq T \} \).

**Theorem: Uniqueness of Solution**

Suppose that we seek a solution \( u(x, t) \) that satisfies \( u(0, t) = \varphi(t), 0 \leq x \leq l \). Suppose further that \( u(x, t) \) satisfies \( u(0, t) = \alpha(t) \) and \( u(L, t) = \beta(t) \), where \( \alpha(t) \) and \( \beta(t) \) are prescribed functions. Then the solution is unique, i.e. there is at most one solution.

![Diagram](image)

**Diffusion Equation on Half Line**

Equation: \( u_{x,x} + ku_{x} = 0 \).

Initial data: \( u(x, 0) = \varphi(x), x > 0 \).

Boundary conditions:
- Dirichlet Condition: prescribe \( u(0, t) = \alpha(t) \) (usually \( \alpha(t) = 0 \)).
- Neumann Condition: prescribe \( u_x(0, t) = \alpha(t) \) (usually \( \alpha(t) = 0 \)).
- Robin Condition: prescribe \( u(0, t) + a u_x(0, t) = 0 \).
Method of Solution: Dirichlet Boundary Condition
Take the case with \( u_x|\{x,t \}=k u_{xx}(x,t), x>0 \), \( u(x,0)=\varphi(x), x>0 \), \( u(0,t)=0, \forall t \geq 0 \).

We want to extend \( \Phi \) to the entire line \(-\infty < x < \infty \) such that the solution \( u(x,t) \) induced by this extension satisfies \( u(0,t)=0 \).

Note that \( \tilde{\Phi}(x)=\Phi(x), x>0 \). Now, \( u(0,t)=0 \) for all \( t>0 \) iff \( \tilde{\Phi} \) is an odd function (\( \Phi(-x)=-\Phi(x) \)).

Then \( u(x,t)=\int_{-\infty}^{\infty} \tilde{\Phi}(y) S(x-y,t)dy=\int_{0}^{\infty} \varphi(y) |S(x-y,t)-S(x+y,t)|dy \).

Method of Solution: Neumann Boundary Condition
Solve \( u_t=k u_{xx}, x>0 \), with initial data \( u(x,0)=\varphi(x), x>0 \) and Neumann condition \( u_x|\{0,t \}=0 \).

If \( u(x,t) \) is even (i.e. \( u(-x,t)=-u(x,t) \)), then \( u_x|\{x,t \} \) is odd (i.e. \( u_x|\{-x,t \}=-u_x|\{x,t \} \)).

The solution is \( u(x,t)=\int_{-\infty}^{\infty} \tilde{\Phi}(y) S(x-y,t)dy=\int_{0}^{\infty} \varphi(y) |S(x-y,t)+S(x+y,t)|dy \).

Wave Equation on Half Line
Solve \( u_t=c^2 u_{xx}, x>0 \).

Initial data \( u(x,0)=\varphi(x), x \geq 0 \) and \( u(x,0)=0 \) for simplicity.

Dirichlet Boundary Condition
Dirichlet condition \( u(0,t)=0 \).

Extend \( \Phi \) to odd function \( \tilde{\Phi} \). Then the solution is \( u(x,t)=\frac{1}{2}(|\tilde{\Phi}(x+ct)|+|\tilde{\Phi}(x-ct)|) \).

Note: \( u(x,t)=-u(x,t) \Rightarrow u(0,t)=0 \).

Wave Equation on Finite Interval
Solve: \( u_t=c^2 u_{xx}, 0<x<L \).

Initial data: \( \varphi(x)=u(x,0), 0<x<L \) and \( \psi(x)=u_x|\{x,0 \}, 0<x<L \).

Dirichlet Boundary Condition
Dirichlet condition: \( u(0,t)=u(L,t)=0 \).

Extend \( \varphi \) to \( \tilde{\varphi} \) and \( \psi \) to \( \tilde{\psi} \) so that \( u(x,t) \) is odd about \( x=0 \) (i.e. \( u(-x,t)=-u(x,t) \)) and odd about \( x=L \) (i.e. \( u(x+L,t)=-u(x+L,t) \)).

Then the solution is \( u(x,t)=\frac{1}{2}(|\tilde{\Phi}(x+ct)|+|\tilde{\Phi}(x-ct)|+\frac{1}{2}c \int_{x-ct}^{x+ct} \tilde{\psi}(z)dz) \).

Separation of Variables and Boundary Value Problems

Method of Separation of Variables
The method of separation of variables assumes that any solution \( u(x,t) \) can be written as \( u(x,t)=X(x)T(t) \).

Solutions
With the diffusion or wave equation, we need to solve \( X'''+\lambda X=0 \), where \( \lambda \) is an unknown constant:
So there are infinitely many eigenvalues.

- For $\lambda > 0$, $X(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x)$.
- For $\lambda < 0$, $X(x) = A \cosh(\sqrt{-\lambda}x) + B \sinh(\sqrt{-\lambda}x)$.
- For $\lambda = 0$, $X(x) = Ax + B$.

In the Dirichlet case ($u(0,t) = u(L,t) = 0$), $\lambda > 0$ and $\lambda_n = \frac{n^2 \pi^2}{L^2}$. So $X_n(x) = \sin \left( \frac{n \pi}{L} x \right)$.

In the Neumann case ($u_x(0,t) = u_x(L,t) = 0$), $\lambda = 0$ so $X(x) = \text{constant}$; or $\lambda < 0$ and $\lambda_n = \frac{n^2 \pi^2}{L^2}$, so $X_n(x) = \cos \left( \frac{n \pi}{L} x \right)$.

**Dirichlet Boundary Condition**

For the wave equation $u_{tt}(x,t) = c^2 u_{xx}(x,t)$, we have $T_n(t) = a_n \cos(\frac{c n \pi}{L} t) + b_n \sin(\frac{c n \pi}{L} t)$. So

$u_n(x,t) = X_n(x) T_n(t) = a_n \cos(\frac{c n \pi}{L} x) + b_n \sin(\frac{c n \pi}{L} x)$.

For the diffusion equation $u_t(x,t) = k u_{xx}(x,t)$, we have $T_n(t) = c_n e^{-\frac{L^2}{4k} t}$. So

$u_n(x,t) = X_n(x) T_n(t) = c_n e^{-\frac{L^2}{4k} t} \sin \left( \frac{n \pi}{L} x \right)$.

**Neumann Boundary Condition**

Wave equation: $u_n(x,t) = a_n \cos \left( \frac{c n \pi}{L} x \right) + b_n \sin \left( \frac{c n \pi}{L} x \right) \cos \left( \frac{n \pi}{L} x \right)$.

Diffusion equation: $u_n(x,t) = c_n e^{-\frac{L^2}{4k} t} \cos \left( \frac{n \pi}{L} x \right)$.

**Mixed Boundary Condition**

Mixed boundary condition $u(0,t) = u(L,t) = 0$, then $X(0) = X'(L) = 0$.

We have $\lambda_n = \frac{\pi (2n + 1)}{2L}$.

**Robin Condition**

Take $u(0,t) = hu(0,t) = 0$ and $u(L,t) = 0$. We have $X'' + \lambda X = 0$.

- Assume $\lambda > 0$. Then $X(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x)$, and we get $\tan(\sqrt{\lambda} L) = \frac{h}{\sqrt{\lambda}}$. Setting $y = \sqrt{\lambda} L > 0$, we get $\tan y = \frac{Lh}{y} = \frac{c}{\sqrt{\lambda}}$, a transcendental equation. On $y > 0$, we get infinitely many solutions $y_1 < y_2 < \cdots \Rightarrow \lambda_1 < \lambda_2 < \cdots$, with the difference approaching $\pi$.

- Assume $\lambda < 0$. Then $X(x) = A \cosh(\sqrt{-\lambda}x) + B \sinh(\sqrt{-\lambda}x)$, and setting $y = \sqrt{-\lambda} L > 0$ we get $\tanh y = \frac{c}{y}$, a transcendental equation. We get no solution.

So there are infinitely many eigenvalues $\lambda_1 < \lambda_2 < \cdots$ with corresponding eigenfunctions $X_1(x), X_2(x), \ldots$.

**Vector Spaces: Introduction to Fourier Series**
Let $V_n$ be the space of all linear combinations of $f=b_1 \sin \left( \frac{\pi}{L} x \right) + b_2 \sin \left( \frac{2\pi}{L} x \right) + \cdots + b_n \sin \left( \frac{n\pi}{L} x \right)$.

Define $L: V_n \rightarrow V_n$, $L(f) = \frac{d^2 f}{dx^2} = \sum_{k=1}^{n} b_k \frac{k^2 \pi^2}{L^2} \sin \left( \frac{k\pi}{L} x \right)$.

Choose basis: $v_1 = \sin \left( \frac{\pi}{L} x \right), v_2 = \sin \left( \frac{2\pi}{L} x \right), \ldots, v_n = \sin \left( \frac{n\pi}{L} x \right)$. Then the matrix of $L$ relative to this basis is a diagonal matrix since $L(v_k) = \left( \frac{k\pi}{L} \right)^2 v_k$.

Let $n \rightarrow \infty$ and consider the space of functions $f$ on $0 \leq x < L$ which can be written as $f(x) = \sum_{k=1}^{\infty} b_k \sin \left( \frac{k\pi}{L} x \right)$ for Fourier coefficients $b_k = \frac{2}{L} \int_{-L}^{L} f(x) \sin \left( \frac{k\pi}{L} x \right) dx$, $k = 1, 2, 3, \ldots$ of $f$ relative to $X_n$.

**FULL FOURIER SERIES**

**Definition**

Let $-L < x < L$. The full Fourier series of $f(x)$ is $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi}{L} x \right) + b_n \sin \left( \frac{n\pi}{L} x \right)$.

**Coefficients**

The coefficients are uniquely determined from orthogonality of functions $\cos \left( \frac{n\pi}{L} \right)$ and $\sin \left( \frac{n\pi}{L} \right)$:

- $\int_{-L}^{L} \sin \left( \frac{n\pi}{L} \right) \sin \left( \frac{m\pi}{L} \right) dx = \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}$
- $\int_{-L}^{L} \sin \left( \frac{n\pi}{L} \right) \cos \left( \frac{m\pi}{L} \right) dx = 0$.
- $\int_{-L}^{L} \cos \left( \frac{n\pi}{L} \right) \cos \left( \frac{m\pi}{L} \right) dx = \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}$

These relations imply that:
- $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi}{L} x \right) dx$, $n = 0, 1, 2, \ldots$.
- $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi}{L} x \right) dx$, $n = 0, 1, 2, \ldots$. 

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Relation To Differential Equations
Take $0 \leq x \leq L$.

Dirichlet Condition: Take $f(x)$ an odd extension of $\varphi(x)$. Then $a_n = 0$ and $b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n \pi x}{L} \right) dx \quad n = 1, 2, \ldots$.

Neumann Condition: Take $f(x)$ an even extension of $\varphi(x)$. Then $a_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{n \pi x}{L} \right) dx \quad n = 0, 1, 2, \ldots$ and $b_n = 0$.

**GENERAL EIGENVALUES AND EIGENFUNCTIONS**

$X'' + \lambda X = 0$ on $0 \leq x \leq L$.

1. If $X(0) = X(L) = 0$, then $\lambda_n = \left( \frac{n \pi}{L} \right)^2$ and $X_n(x) = \sin \left( \frac{n \pi x}{L} \right)$.

2. If $X'(0) = X'(L) = 0$, then $\lambda_n = \left( \frac{n \pi}{L} \right)^2$ and $X_n(x) = \cos \left( \frac{n \pi x}{L} \right)$.

3. If $X(0) = -hX(0) = 0$ and $X'(L) = 0$, then $\lambda_1 < \lambda_2 < \cdots$ (eigenvalues) and $X_1, X_2, \ldots$ (eigenfunctions).

4. If $X(0) = X(L) = 0$ and $X'(0) = X'(L) = 0$, then $\lambda = 0$ and $X(x)$ is constant or $\lambda_n = \left( \frac{2n \pi}{L} \right)^2$ and

$$X_n(x) = A_n \sin \left( \frac{2n \pi x}{L} \right) + B_n \cos \left( \frac{2n \pi x}{L} \right)$$

where $A_n$ and $B_n$ are arbitrary constants.

**General Boundary Conditions**

Solve $X'' + \lambda X = 0$, $a \leq x \leq b$ subject to the boundary conditions $\alpha_1 X(a) + \alpha_2 X(b) + \alpha_3 X'(a) + \alpha_4 X'(b) = 0$ and $\beta_1 X(a) + \beta_2 X(b) + \beta_3 X'(a) + \beta_4 X'(b) = 0$ for some constants $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$.

**Definition: Symmetric Boundary Conditions**

Let $f$ and $g$ be any functions that satisfies the above boundary condition. Then conditions are called symmetric if

$$f'(x)g(x) - f(x)g'(x)|_{a}^{b} = 0 \iff f'(b)g(b) - f(b)g'(b) - (f'(a)g(a) - f(a)g'(a)) = 0.$$

**Fact**

Conditions 1 to 4 are symmetric.

**Theorem**

Suppose that $X_n$ and $X_m$ are eigenfunctions on $[a, b]$ that corresponds to distinct eigenvalues $\lambda_n$ and $\lambda_m$ ($\lambda_n \neq \lambda_m$), and suppose that the boundary conditions are symmetric. Then $X_n$ and $X_m$ are orthogonal in the sense that

$$\int_a^b X_n(x)X_m(x)dx = 0.$$

**HILBERT SPACE**

**Basic Space**

$$L^2[a, b] = \left\{ f : [a, b] \to \mathbb{R} \mid \int_a^b |f|^2 dx < \infty \right\}.$$
Fact
$L^2[a, b]$ is a vector space.

Inner Product
Take $f$ and $g$ in $L^2$. Then define the inner product to be $\langle f, g \rangle = \int_a^b f(x)g(x)dx$.

Norm
Define $\|f\| = \left(\int_a^b f^2(x)dx\right)^{1/2} = \sqrt{\langle f, f \rangle}$ to be the norm of $f$.

Cauchy-Schwartz Inequality
$\|f \cdot g\| \leq \|f\| \|g\|$ or $\left| \int_a^b f(x)g(x)dx \right| \leq \left(\int_a^b f^2(x)dx\right)^{1/2} \left(\int_a^b g^2(x)dx\right)^{1/2}$.

Note: This implies $\left| \langle f, f \rangle \right| = \left| \langle f, f \rangle \right|^{1/2}$, so define $\cos \theta = \frac{\langle f, f \rangle}{\|f\| \|g\|}$.

Definition: Convergence
$f_n \in L^2$ is said to converge to $f$ if $\lim_{n \to \infty} \|f_n - f\| = 0$.

Definition: Cauchy Sequence
$f_n \in L^2$ is called a Cauchy Sequence if $\|f_n - f_m\| \to 0$ as $n, m \to \infty$.

Basic Properties of Inner Product and Norm
1. Symmetric: $\langle f, g \rangle = \langle g, f \rangle$.
2. Bilinear: $\langle f, \alpha g + \beta h \rangle = \alpha \langle f, g \rangle + \beta \langle f, h \rangle$ for $\alpha, \beta \in \mathbb{R}$ and $f, g, h \in L^2$.
3. $\langle f, f \rangle \geq 0$ for all $f \in L^2$; if $\langle f, f \rangle = \int_a^b f^2(x)dx = 0$ then $f = 0$ “almost everywhere”.
4. $L^2$ is complete in the sense that any Cauchy sequence in $L^2$ converges to an element in $L^2$.

Definition: Hilbert Space
Any vector space $H$ with an inner product $\langle \cdot, \cdot \rangle$ that satisfies properties 1 to 4 is called a Hilbert space.

Theorem
If $X_1, X_2, \ldots, X_n, \ldots$ are the eigenfunctions corresponding to symmetric boundary problem, then the Fourier series of any function $f$ converges to $f$ in $L^2$ norm.

Least Square Approximation
Let $V_n$ denote the linear span of $X_1, X_2, \ldots, X_n$ (i.e. $f \in V_n \iff f = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n$).

Problem
Question: Let \( f \in L^2 \). For which values of \( \alpha_1, \alpha_2, \ldots, \alpha_n \) is the distance \( \|f - \sum \alpha_i X_i\| \) minimum?

Answer: \( \alpha_i = \frac{(f, X_i)}{||X_i||} \quad i=1, \ldots, n \).

**Convergence of Fourier Series**

**Theorem**
The Fourier series relative to \( X_1, X_2, \ldots \) of any element \( f \in L^2 \) converges to \( f \). That is, if \( S_N = \sum_{i=1}^{N} (f, X_i) X_i \), then
\[
\lim_{N \to \infty} \|S_N - f\| = 0.
\]

**Definition: Piecewise Continuous**
A function is piecewise continuous if it is continuous at all but a finite number of points. At a point of discontinuity \( f \) has both a right and a left limit (ie \( f \) has a jump discontinuity).

So if \( c \) is a point of discontinuity of \( f \), then both \( f^+(x) = \lim_{x \to c^+} f(x) \) and \( f^-(x) = \lim_{x \to c^-} f(x) \) exist.

**Theorem: Point-wise Convergence of Fourier Series**
Assume \( f \) is such that:
- \( f \) is periodic of period \( 2\pi \).
- \( f \) and its derivative \( f' \) are “piecewise continuous”.

Then
\[
\lim_{n \to \infty} S_n(x) = \frac{1}{2} \left( f^+(x) + f^-(x) \right).
\]

Note: If \( f \) is continuous at \( x \), then \( f^+(x) = f^-(x) = f(x) \), so \( \lim_{n \to \infty} S_n(x) = f(x) \).

**Auxiliary Results**
1. Bessel's Inequality: \( \|g\|^2 \geq \sum_{k=1}^{N} \frac{(g, X_k)^2}{||X_k||^2} \) where \( X_1, X_2, \ldots \) are eigenfunctions on \([a, b]\) with symmetric boundary values and \( g \in L^2[\alpha, \beta] \). This implies that \( \lim_{k \to \infty} \frac{(g, X_k)^2}{||X_k||^2} = 0 \).

2. Let \( K_N(\theta) = 1 + 2 \sum_{k=1}^{N} \cos k\theta \) \( \quad \text{Then} \quad \int_{-\pi}^{\pi} K_N(\theta) d\theta = 2\pi \rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(\theta) d\theta = 1 \).

3. \( K_N(\theta) = \frac{\sin\left(\left(\frac{N+1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)} \).

**Definition: Uniform Convergence**
\( f_n \) converges to \( f \) uniformly if \( \lim_{n \to \infty} \max_{\alpha \in [a, b]} |f_n(x) - f(x)| = 0 \).
Harmonic Functions and Laplace's Equation

**LAPLACE'S EQUATION**
In $n$ dimensions,

$$\Delta u = \text{div} (\text{grad} u) = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$ 

Notes:
- $\Delta$ is the Laplacian, and it is an operator that acts on functions of $n$ variables.
- The gradient of a scalar function $u(x_1, \ldots, x_n)$ is $\text{grad} u = \nabla u = \left( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right)$.
- The divergence of a vector field $\mathbf{V}(x_1, \ldots, x_n) = (V_1(x_1, \ldots, x_n), \ldots, V_n(x_1, \ldots, x_n))$ is $\text{div} (\mathbf{V}) = \frac{\partial V_1}{\partial x_1} + \cdots + \frac{\partial V_n}{\partial x_n}$.
- When $\mathbf{V} = \nabla u = \left( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right)$, then $\Delta u = \text{div} (\text{grad} u)$.

**Definition: Harmonic**
Any solution of Laplace's equation is called harmonic.

**Properties**
- $n=1$: $\frac{d^2 u}{dx^2} = 0$, so $u(x) = Ax + B$.
- $n=2$: Connection to complex functions $f(z) = u(x, y) + iv(x, y)$. Then $f$ analytic (can be expressed as a Taylor series, i.e. differentiable) implies $u$ and $v$ are harmonic.

**MAXIMUM/MINIMUM PRINCIPLE**

**Definitions**
1. $D$ is an open subset of $\mathbb{R}^n$ if for all $\mathbf{x} \in D$, there exists $r > 0$ such that for all $\mathbf{y} \in D$, $\| \mathbf{x} - \mathbf{y} \| < r$.
2. $\partial D$ is the boundary of $D$. A point $\mathbf{b}$ is a boundary point if for all $\varepsilon > 0$, $B = \{ \| \mathbf{x} - \mathbf{b} \| < \varepsilon \}$ has non empty intersection with both $D$ and the complement of $D$ in $\mathbb{R}^n$.
3. $D$ is connected if there exists a polynomial curve joining any two points in $D$ and is lying in $D$.
4. $D$ is bounded if it is contained in some ball $B = \{ \| \mathbf{x} \| < R \}$, $0 < R < \infty$.

**Maximum/Minimum Principle**

Assume $D$ to be an open, connected subset of $\mathbb{R}^n$ such that $D \cup \partial D$ is bounded. Let $u$ be any solution of (the Laplace $\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$ equation) in $D$ such that $u$ is defined and continuous in $D \cup \partial D$. Then:

- **Maximum Principle:** $\max_{\partial D} u(\mathbf{x}) \leq u(\mathbf{x}) \quad \forall \mathbf{x} \in D$.
- **Minimum Principle:** $\min_{\partial D} u(\mathbf{x}) \geq u(\mathbf{x}) \quad \forall \mathbf{x} \in D$.

That is, $u$ attains its maximum/minimum on $\partial D$.

**BOUNDARY VALUE PROBLEMS**
Dirichlet Problem
\[ \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 \quad \text{subject to} \quad u|_{\partial \Omega} = \varphi(x) \quad \text{(given)}. \]
By the Maximum/Minimum Principle, the solution of the Dirichlet problem is unique.

Neumann Problem
\[ \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 \quad \text{subject to} \quad \frac{\partial u}{\partial n}|_{\partial \Omega} = \varphi(x) \quad \text{(given)}. \]
Here, \( n \) is the external normal.

Robin/Mixed Problem
\[ \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 \quad \text{subject to} \quad \frac{\partial u}{\partial n} + a(x)u(x) = k(x) \quad \text{(given)}. \]

**Basic Property**
Solutions to the Laplace equation are invariant under rigid motions \( \varphi(x) = T(x) + R(x) \), where
- \( T(x) = \bar{x} + \bar{x} \) is a translation,
- \( R(x) = A\bar{x} \) (\( A^T = A^{-1} \) \( \det A = \pm 1 \)) is a rotation.

**Rectangular Harmonics**
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on} \quad D = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq b\} \quad \text{with} \quad \partial D = \begin{cases} \quad u(x, 0) = f(x) \\ \quad u(x, b) = h(x) \\ \quad u(0, y) = u(a, y) = 0 \end{cases} \]
where \( f, g, h, i \) are given functions.

**Separation of Variables**
We assume \( u(x, y) = X(x)Y(y) \).

When \( \partial D = \begin{cases} \quad u(x, 0) = f(x) \\ \quad u(x, b) = h(x) \\ \quad u(0, y) = u(a, y) = 0 \end{cases} \), then
\[ u_n(x, y) = \left( a_n \cosh \frac{n\pi}{a} x + b_n \sinh \frac{n\pi}{a} x \right) \sin \frac{n\pi}{a} y. \]

When \( \partial D = \begin{cases} \quad u(x, 0) = f(x) \\ \quad u(x, y) = g(x) \\ \quad u(0, y) = u(a, y) = 0 \end{cases} \), then
\[ u_n(x, y) = \left( a_n \cosh \frac{n\pi}{b} x + b_n \sinh \frac{n\pi}{b} x \right) \sin \frac{n\pi}{b} y. \]

Note: \( \partial D = \begin{cases} \quad u(x, 0) = f(x) \\ \quad u(x, b) = h(x) \\ \quad u(0, y) = u(a, y) = 0 \end{cases} \) \( \begin{cases} \quad u(x, 0) = g(x) \\ \quad u(x, b) = h(x) \\ \quad u(0, y) = u(a, y) = 0 \end{cases} \). \( \begin{cases} \quad u(x, 0) = g(x) \\ \quad u(x, b) = h(x) \\ \quad u(0, y) = u(a, y) = 0 \end{cases} \)

Note: \( u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) \).

**Case 1**
Suppose that we want the solution with \( \partial D = \begin{cases} \quad u(x, 0) = f(x) \\ \quad u(x, b) = u(0, y) = u(a, y) = 0 \end{cases} \).
The sine Fourier coefficients of \( f \) are \( a_n = \frac{2}{a} \int_{0}^{a} f(z) \sin \left( \frac{n \pi z}{a} \right) dz \).

Since \( u(x, b) = 0 \) \( \Rightarrow \) \( \sum_{n=1}^{\infty} \left( a_n \cosh \frac{n \pi}{a} b + b_n \sinh \frac{n \pi}{a} b \right) \sin \frac{n \pi}{a} x = 0 \),

\[
b_n = -\frac{a_n \cosh \frac{n \pi}{a} b}{\sinh \frac{n \pi}{a} b}.
\]

Then

\[
u(x, y) = \sum_{n=1}^{\infty} \left( a_n \cosh \frac{n \pi}{a} y + b_n \sinh \frac{n \pi}{a} y \right) \sin \frac{n \pi}{a} x \]
\[
= \sum_{n=1}^{\infty} a_n \left( \cosh \frac{n \pi}{a} y - \frac{\cosh \frac{n \pi}{a} b}{\sinh \frac{n \pi}{a} b} \sin \frac{n \pi}{a} y \right) \sin \frac{n \pi}{a} x .
\]
\[
= \sum_{n=1}^{\infty} \frac{a_n}{\sinh \frac{n \pi}{a} y} \left( \sinh \frac{n \pi}{a} (b-y) \right) \left( \sin \frac{n \pi}{a} x \right).
\]

**Laplace’s Equation on Circular Regions**

1. Annulus: \( D = \{(r, \theta) : -\pi < \theta \leq \pi, a \leq r \leq b\} \), \( \partial D = \{(a, \theta), -\pi < \theta \leq \pi \} \cup \{(b, \theta), -\pi < \theta \leq \pi \} \).
2. Disk: \( D = \{(r, \theta) : -\pi < \theta \leq \pi, 0 \leq r \leq b\} \), \( \partial D = \{(b, \theta), -\pi < \theta \leq \pi \} \).
3. Wedge: \( D = |(r, \theta), 0 \leq \theta \leq \alpha, a \leq r \leq b\} \).

**Polar Coordinates and Separation of Variables**

Using polar coordinates \( x = r \cos \theta \), \( y = r \sin \theta \), \( u_x + u_y = 0 \) becomes \( u_r + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0 \).

Assuming \( u(r, \theta) = R(r) \Theta(\theta) \), we get the two equations \( r^2 R'' + r R' - \lambda R = 0 \) and \( \Theta'' + \lambda \Theta = 0 \).

**Annulus**

Eigenfunctions:

- When \( n \neq 0 \), \( \Theta_n(\theta) = A_n \cos n \theta + B_n \sin n \theta \) and \( R_n(r) = C_n r^n + D_n r^{-n} \).
- When \( n = 0 \), \( \Theta_0(\theta) = A_n \) and \( R_0(r) = c_n + c_1 \ln r \).

So the solution is \( u(r, \theta) = \sum_{n=0}^{\infty} R_n(r) \Theta_n(\theta) = c_0 + c_1 \ln r + \sum_{n=1}^{\infty} \left( C_n r^n + D_n r^{-n} \right) (A_n \cos n \theta + B_n \sin n \theta) \). The coefficients are the Fourier coefficients of the boundary conditions \( u(a, \theta) = f(\theta) \) and \( u(b, \theta) = g(\theta) \).

**Disk**

The usual assumption is that \( u(0, \theta) \) bounded. This forces \( c_1 = D_n = 0 \). So the solution is

\[
u(r, \theta) = c_0 + \sum_{n=1}^{\infty} C_n r^n (A_n \cos n \theta + B_n \sin n \theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n \theta + b_n \sin n \theta),
\]

where \( a_n \) and \( b_n \) are the Fourier coefficients determined by the boundary condition \( u(b, \theta) = f(\theta) \).
Wedge
Consider the special case that \( u(r,0) = u(r,\alpha) = 0 \) and \( u(b,\theta) = f(\theta) \).

The solution is \( u(r,\theta) = \sum_{n=1}^{\infty} a_n r^n \sin \frac{n\pi \theta}{\alpha} \), where \( a_n = \frac{2}{b^{n+1}/\alpha} \int_{0}^{\pi} f(\varphi) \sin \frac{n\pi \varphi}{\alpha} \varphi \, d\varphi \) is the Fourier coefficient determined by \( u(b,\theta) = f(\theta) \).

**Poisson Formula and Poisson Kernel**

**Poisson Formula**
On the disk \( D = \{(r,\theta), -\pi < \theta \leq \pi, 0 \leq r \leq b\} \) with \( u(b,\theta) = f(\theta) \),

\[
u(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ f(\varphi) \left( 1 + 2 \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n \cos n(\varphi - \theta) \right) \right] d\varphi,
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) P(\varphi - \theta) d\varphi
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(b^2 - r^2) f(\varphi)}{b^2 - 2br \cos(\varphi - \theta) + r^2} d\varphi
\]

where \( P(\varphi - \theta) = \frac{1}{2\pi} \left( 1 + 2 \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n \cos n(\varphi - \theta) \right) = \frac{1}{2\pi} \frac{b^2 - r^2}{b^2 - 2br \cos(\varphi - \theta) + r^2} \).

**Poisson Kernel**

\[
P_n(r,\theta) = \frac{a^2 - r^2}{2\pi a^2 - 2ar \cos \theta + r^2}.
\]

**Basic Properties**
1. \( \int_{0}^{2\pi} P_n(r,\varphi - \theta) d\varphi = 1 \) \( \forall r, \theta \). In this case \( u(a,\theta) = f(\theta) = 1 \), but \( u(r,\theta) = 1 \) \( \forall r, \theta \) also.
2. \( \lim_{r \to a} P_n(r,\theta) = \begin{cases} 0 & \theta \neq 0 \\ \infty & \theta = 0 \end{cases} \).
3. \( \lim_{r \to 0} \int_{0}^{2\pi} f(\varphi) P_n(r,\varphi - \theta) d\varphi = f(\theta) \) whenever \( f \) is a continuous function of \( \theta \).
4. Averaging Property of Harmonic Functions: \( x = y = 0 \Rightarrow r = 0 \), so
   \[
u(0,0) = u(0,\theta) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(a^2 - 0^2) f(\varphi)}{a^2 - 2a \cos(\varphi - \theta) + 0^2} d\varphi = \frac{1}{2\pi} \int_{0}^{2\pi} f(\varphi) d\varphi
   \]
   is the average value of \( f \).

**Consequences of Poisson Representation**
1. A harmonic function \( u \) defined on some domain \( D \) cannot attain a maximum (nor minimum) in the interior of \( D \).
2. \( u(r,\theta) \) has partial derivatives of all orders, even when \( f \) is only continuous.
\[ u(r, \theta) = \int_0^{2\pi} f(\varphi) P_\varphi(r, \varphi - \theta) \, d\varphi \Rightarrow \frac{\partial u}{\partial r} = \int_0^{2\pi} f(\varphi) \frac{\partial}{\partial r} P_\varphi(r, \varphi - \theta) \, d\varphi. \]