Asset Markets

What Are Assets

Something acquired today that will have purchasing power later.
1. Bonds, stocks, treasury-bills.
2. Physical assets – houses, machinery, etc.
3. Human capital.

Assumptions

1. No transaction costs.
2. Short sales of securities permitted.
3. Agents limited by wealth constraint – can't acquire liabilities today that they will be unable to honour in the future.
4. No arbitrage condition.

Rate of Return

Ex-Post Nominal Rate of Return

- Suppose t-bill price last year is \$955, and t-bill price this year (today) is \$1000. Let $r=\text{rate of return}$. Then $1+r=\frac{\text{payoff}}{\text{previous price}}=\frac{\$1000}{\$955}\Rightarrow r=0.04712=4.712\%$. Here, *r* is the interest rate a year ago.
- Suppose stock price last year is \$52, and stock price this year is \$50. Then $1+R=\frac{\text{current price}}{\text{previous price}}=\frac{\$50}{\$52}\Rightarrow R=-0.0385=-3.85\%$.
- *r* and *R* are the ex-post nominal rates of return.

Ex-Ante Nominal Rate of Return

Suppose current price $S_0=\$50$, and price one year from now $\tilde{S}_1=?$. Then $1+\tilde{R}=\frac{\tilde{S}_1}{\$50}$. Here $\tilde{R}=?$ is the ex-ante nominal rate of return (a random variable). It matters for decision making.

Ex-Ante Real Rate of Return

Let $P_0=\text{current consumption price}$ and $\tilde{P}_1=\text{future consumption price}$. Then $1+\tilde{\rho}=\frac{\tilde{S}_1/\tilde{P}_1}{S_0/P_0}=\frac{\tilde{S}_1}{S_0}\frac{P_0}{\tilde{P}_1}=(1+\tilde{R})\frac{P_0}{\tilde{P}_1}$.

Asset Pricing Under Perfect Certainty

Suppose we have:

- Two consumption periods, current period 0 and future period 1.
- Consumption price 0 = consumption price 1.
- $r=R=4\%$.

Suppose $S_0=\$90$, then $R=15.56\%$. This is impossible because one can borrow \$90 and buy stocks in period 0, and get \$104 and repay $\$90\times1.04=\$93.60$ in period 1. This is impossible because this is arbitrage.

Therefore, the equilibrium price for the stock $S_0=\$100=\frac{\$104}{1.04}=\text{present discounted value}$.

Discounting

Let $P_0^{(1)}$ denote the price today of a one-period discount bond that will pay \$1in period 1. Then

$P_0^{(1)}(1+r_0)=\$1 \Leftrightarrow P_0^{(1)}=\frac{\$1}{1+r_0}$.

Let $P_0^{(2)}$ denote the price today of a one-period discount bond that will pay \$1in period 2. Then

$P_1^{(2)}=\frac{\$1}{1+r_1} \Rightarrow P_0^{(2)}=\frac{P_0^{(2)}}{1+r_0}=\frac{\$1}{(1+r_0)(1+r_1)}$.

Let $r_0^{(2)}$ denote the 2-year rate of interest in period 0. $P_0^{(2)}=\frac{\$1}{\left(1+r_0^{(2)}\right)^2} \Rightarrow r_0^{(2)}=\sqrt{\frac{\$1}{P_0^{(2)}}}$.

Yield To Maturity (a.k.a. Internal Rate of Return)

The yield to maturity of asset y that pays $\$D_T$ in period T is $i_0^{(y)}$ such that $P_0^{(y)}=\frac{D_1}{\left(1+i_0^{(y)}\right)}+\frac{D_2}{\left(1+i_0^{(y)}\right)^2}+\cdots$.

Equity/Share Price

The ex-dividend price of a share in period 0 is $S_0^{\text{ex}}=\frac{D_1}{(1+r_0)}+\frac{D_2}{(1+r_0)(1+r_1)}+\cdots+\frac{D_T}{(1+r_0)\cdots(1+r_{T-1})}$.

The pre-dividend price of a share in period 0 is $S_0^{\text{pre}}=S_0^{\text{ex}}+D_0$.

If dividend D is constant and interest rate r is constant and the share pays forever, then $S_0^{\text{ex}}=\sum_{T=1}^{\infty}\frac{D}{(1+r)^T}=\frac{D}{r}$ and

$S_0^{\text{pre}}=\frac{D}{r}+D$. Note that the price-dividend ratio is $\frac{S_0^{\text{ex}}}{D}=\frac{1}{r}$.

If dividends grow at the constant rate g per time period and $g<r$, then $D_T=D_0(1+q)^T$ and

$S_0^{\text{ex}}=\sum_{T=1}^{\infty} D_0\left(\frac{1+g}{1+r}\right)^T=\frac{D_0(1+g)}{r-g}=\frac{D_1}{r-g}$.

Utility Maximization in a Two-Time Period

Lifetime utility: $U_0=U(c_{0,}c_1)$.

Endowment: $(y_{0,}y_1)$.

Initial wealth: $W_0=y_0+\frac{y_1}{1+r}$.

Budget constraint: $c_1=y_1+(1+r)(y_0-c_0)$ or $c_0+\frac{c_1}{1+r}=W_0$.

Maximum utility when $\underbrace{\text{MRS}}_{\text{slope of higest indifference curve}}=\underbrace{\text{MRT}}_{-(1+r)}$.

Fisher Separation Theorem

Corporate decisions independent of individual preference.

Asset Pricing Under Uncertainty

Expected Utility Theory

Utility is given by $U(C)$. An individual will choose between A and B according to highest utility, ie prefer A iff $E[U(A)]>E[U(B)]$.

Definitions

1. Certainty equivalent: The value CE that solves $U(CE)=E[U(W)]$.
2. Risk premimum: The maximum amount Π that an individual would pay to exchange the risky wealth W for $\bar{W}$; so $U(\bar{W}-\Pi)=E[U(W)]=U(CE)\Rightarrow\Pi=\bar{W}-CE$.

Types of Utility Functions

Let $\underbrace{W}_{\text{wealth}} = \underbrace{W_0}_{\text{certain wealth}} + \underbrace{X}_{\textit{random variable}}$. Then $E(W)=\bar{W}=W_0+\bar{X}$.

- Linear: $E[U(W)]=U(\bar{W})$, $CE=\bar{W}$, $\Pi=0$ risk neutral.
- Concave: $E[U(W)]<U(\bar{W})$, $CE<\bar{W}$, $\Pi>0$ risk averse.
- Convex: $E[U(W)]>U(\bar{W})$, $CE>\bar{W}$, $\Pi<0$ risk lover.

Markowitz Approximation

Let $W=W_0+X$, where X is a random variable with $E(X)=0$ and $\mathrm{var}(X)=\sigma_x^2>0$. Then the Markowitz approximation for π is $\pi\approx\frac{1}{2}\sigma_x^2\left[\frac{-U''(W)}{U'(W)}\right]$.

Measure of Risk Aversion

Pratt Measure of Absolute Risk Aversion (ARA)

Pratt measure of ARA: $\frac{-U''(W_0)}{U'(W_0)}$.

Note that this implies ARA decreases as W_0 increases.

Arrow Measure of Relative Risk Aversion (RRA)

Arrow measure of RRA: $-W_0\frac{U''(W_0)}{U'(W_0)}$.

This is much more reasonable.

Class of CRRA Utility Function

$U(W)=\frac{W^{1-\gamma}}{1-\gamma}$, for $\gamma\geq 0$ and $U(W)=\ln W$, for $\gamma=1$.

Then, $\mathrm{RRA}=-W\frac{U''}{U'}=\gamma$.

Portfolio Choice

- Assume two time periods: 0, 1. Household has initial wealth W at the start of period 0.
- Lifetime utility is $E[U(c_{0,}c_1)]=U(c_0)+\beta E[U(c_1)]$, where β is the time preference parameter ($\beta\approx 0.96$ empirically).
- Household chooses:
  - c_0, so $W_0=W-c_0$ is invested.
  - αW_0 invested in risky asset (r_1 with $E(r_1)=\bar{r}_1$ and $\mathrm{var}(r_1)=\sigma_1^2$), and $(1-\alpha)W_0$ invested in risk-free asset (r_f).

- Household has portfolio with return $r_p = \alpha r_1 + (1-\alpha) r_f$. So $c_1 = W_0 (1+r_p)$.
- Household will maximize $E[U(c_{0,} c_1)] = U(W - W_0) - \beta E[U(W_0(1+r_p))]$, or equivalently, $\max_\alpha E[U(W_0(1+\alpha r_1 + (1-\alpha) r_f))]$. The derivative is
  $E[U'(W_0(1+\alpha r_1+(1-\alpha)r_f))\cdot(r_1-r_f)] = 0 = E[U'(W_0(1+\alpha r_1+(1-\alpha)r_f))] E[(r_1-r_f)] + \text{cov}(\ ,\ )$
  - Case A: $\bar{r}_1 - r_f > 0 \Rightarrow \alpha > 0$.
  - Case B: $\bar{r}_1 - r_f < 0 \Rightarrow \alpha < 0$.
  - Case C: $\bar{r}_1 - r_f = 0 \Rightarrow \alpha = 0$.

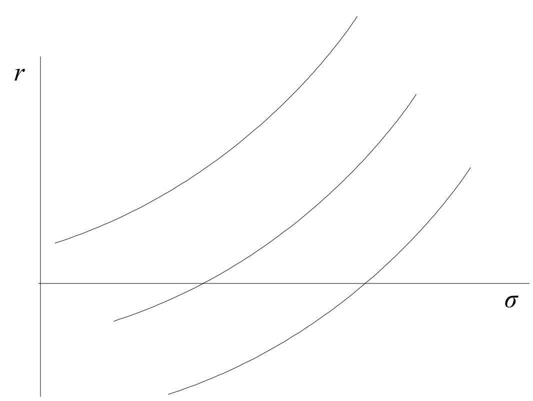
Result
If there are two households, the more risk averse one will have lower α , where $\bar{r}_1 > r_f$.

Approximation
Assume CRRA $U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \gamma \geq 0$, $U(c) = \ln c, \gamma = 1$. Then $\max_\alpha \beta E[U(c_1)] = \max_\alpha E[U(1+r_p)]$. Note that as γ increases, α decreases.

An approximate for $E[U(1+r_p)]$ is $E[U(1+r_p)] \approx U(1+r_p) + \frac{\sigma_p}{2}(1+\bar{r}_p) = V(\bar{r}_p, \sigma_p)$, where $\frac{\partial V}{\partial \bar{r}_p} > 0$ and $\frac{\partial V}{\partial \sigma_p} < 0$.

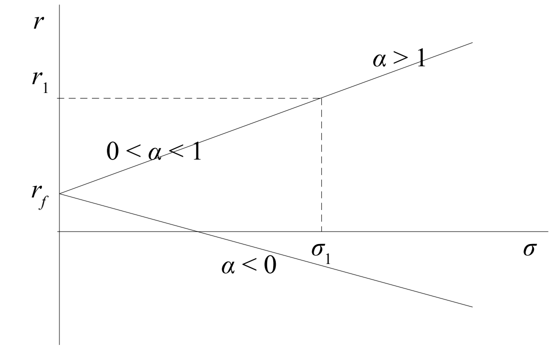
This is a good approximation because:
1. If r_1 is normal, then $E[U(1+r_p)] = V(\bar{r}_p, \sigma_p)$.
2. Most assets are very close to normal.

Preferences
Indifference curves of $V(\bar{r}_p, \sigma_p)$:



- The are upward sloping because σ is a “bad”.
- The slope is the MRS.
- The more risk averse an individual is, the steeper the curves.

Market Opportunities
If an individual invests α in risky asset, $1-\alpha$ in risk-free asset, then $\bar{r}_p = \alpha \bar{r}_1 + (1-\alpha) r_f = r_f + \alpha(\bar{r}_1 - r_f)$ and $\sigma_p^2 = \alpha^2 \sigma_1^2 \Leftrightarrow \sigma_p = \alpha \sigma_1$.



- An individual will only choose the effect set of portfolio (upper branch) – it has higher rate of return for each σ .
- The equation of the effective set is $\bar{r}_p = r_f + \left(\frac{\bar{r}_1 - r_f}{\sigma_1} \right) \sigma_p$.
- The slope is the MRT.

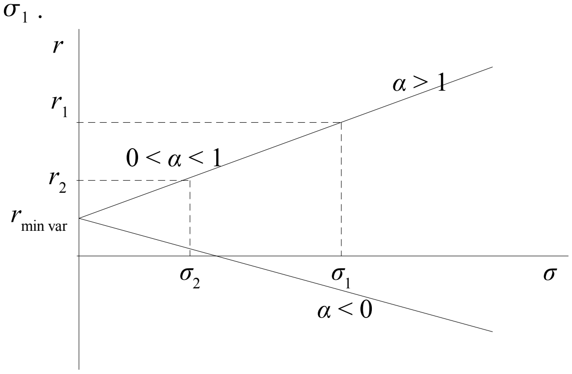
Equilibrium

The equilibrium is when $\mathrm{MRS} = \mathrm{MRT}$, i.e. when the indifference curve is tangent to the market opportunity curve.

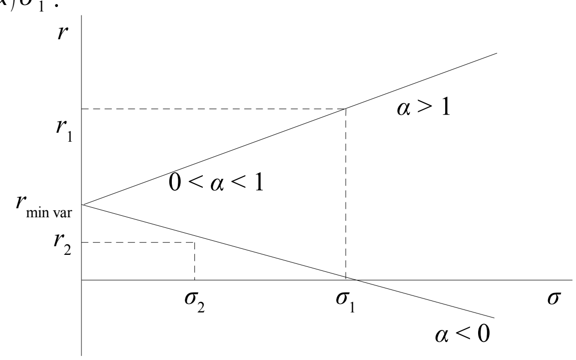
Market Opportunities: Two Risky Assets

Let α invested in riskier asset 1, and $1-\alpha$ invested in less risky asset 2. Then $r_p = \alpha r_1 + (1-\alpha) r_2$ with $\bar{r}_p = \alpha \bar{r}_1 + (1-\alpha) \bar{r}_2$ and $\sigma_p^2 = \alpha^2 \sigma_1^2 + (1-\alpha)^2 \sigma_2^2 + 2\alpha(1-\alpha)\rho_{1,2}\sigma_1\sigma_2$, where $\rho_{1,2} = \frac{\mathrm{cov}(r_1, r_2)}{\sigma_1 \sigma_2}$.

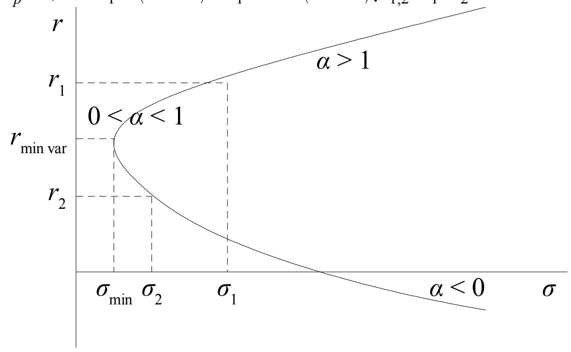
If $\rho_{1,2} = 1$, then $\sigma_p = \alpha\sigma_1 + (1-\alpha)\sigma_1$.



If $\rho_{1,2} = -1$, then $\sigma_p = \alpha\sigma_1 - (1-\alpha)\sigma_1$.



If $-1<\rho_{1,2}<1$ (usual case), then $\sigma_p=\sqrt{\alpha^2\sigma_1^2+(1-\alpha)^2\sigma_1^2+2\alpha(1-\alpha)\rho_{1,2}\sigma_1\sigma_2}$.

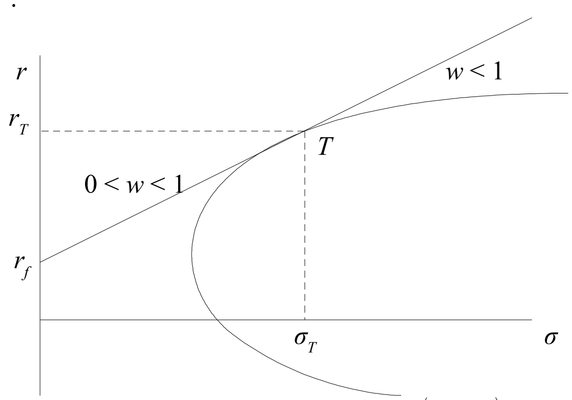


The efficient set is all portfolios with $r_p > \bar{r}_{\text{min var}}$.

Note: $\alpha_{\text{min var}}=\dfrac{\sigma_2^2-\rho_{1,2}\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2}$.

Market Opportunities: Two Risky Assets and a Risk-Free Asset

Suppose $\bar{r}_1>\bar{r}_2>r_f$ and $\bar{\sigma}_1>\bar{\sigma}_2$.



Tangency portfolio: α_T in asset 1, $1-\alpha_T$ in asset 2, and thus $\bar{r}_T=\alpha_T\bar{r}_1+(1-\alpha_T)\bar{r}_2$.

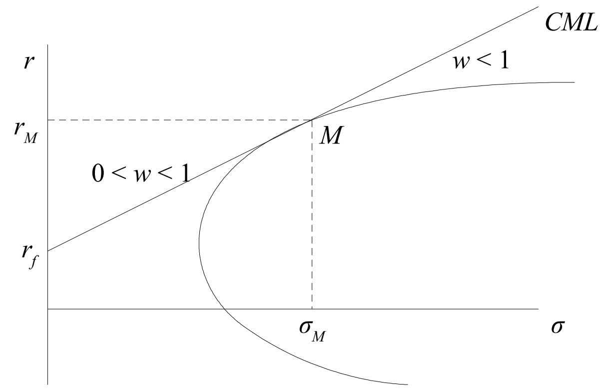
Agent's Portfolio: $r_p=wr_t+(1-w)r_f$, and $\sigma_p=w\sigma_T$.

Two-Fund Theorem

Any agent will have the same risky portfolio, the tangency portfolio.

Market Opportunities: All Risky Assets and a Risk-Free Asset

CML (capital market line) is the opportunity set available to every investor. This is the efficient set.



Note: The two-fund theorem still applies here.

Consider all risky assets $1,2,\ldots,N$. Let asset i have value V_i , and $V_M=\sum_{i=1}^{N} V_i$.

The market portfolio is $\frac{V_1}{V_M}$ in asset 1, $\frac{V_2}{V_M}$ in asset 2, etc. Then $\bar{r}_M=\sum_{i=1}^{N}\frac{V_i}{V_M}\bar{r}_i$.
Note: The tangency portfolio = market portfolio.
Note: Lending = borrowing, so average agent is at M.

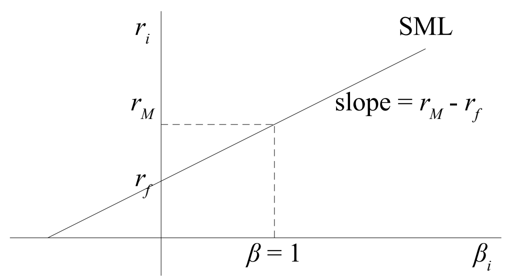
An agent's portfolio is $\bar{r}_p=w\bar{r}_M+(1-w)r_f$ with $\sigma_p=w\sigma_M$.

Equation of CML is $\bar{r}_p=r_f+\frac{\bar{r}_M-r_f}{\sigma_M}\sigma_p$.

Security Market Line

Now consider asset j . If an agent invests α in market portfolio and $1-\alpha$ in asset j , then $\bar{r}_p=\alpha\bar{r}_M+(1-\alpha)\bar{r}_j$ and $\sigma_p=\sqrt{\alpha^2\sigma_M^2+(1-\alpha)^2\sigma_j^2+2\alpha(1-\alpha)\sigma_{m,j}}$. When $\alpha=1$, the slope of CML is $\frac{\bar{r}_M-r_f}{\sigma_M}$. Thus

$\bar{r}_j=r_f+\frac{\operatorname{cov}(r_M,r_j)}{\sigma_M^2}(\bar{r}_M-r_f)=r_f+\beta_j(\bar{r}_M-r_f)$; the bigger β_j , the riskier it is!



Empirical Security Market Line: Since $E(r_j)=\bar{r}_j$ not observable, write $r_j=r_f+\beta_j(r_M-r_f)+\varepsilon_j$ with $E(\varepsilon_j)=0$ and $\operatorname{cov}(\varepsilon_j,r_M)=0$.

Diversifying Portfolio

Notice $\sigma_j^2=\beta_j^2\sigma_M^2+\sigma_{\varepsilon_j}^2$, i.e. total risk = systematic risk + unsystematic risk. An agent is only compensated for systematic risk.

If $r_p = w_1 r_1 + \cdots + w_k r_k, \sum w_i = 1$, then $r_p = r_f + \beta_p (r_M - r_f) + \sum w_i \varepsilon_i, \beta_p = \sum w_i \beta_i$ with $\bar{r}_p = r_f + \beta_p (\bar{r}_M - r_f)$ and $\sigma_p = \beta_p^2 \sigma_M^2 + \text{var}\left(\sum w_i \varepsilon_i\right)$. For large k, $\sigma_p \approx \beta_p^2 \sigma_M^2$; the unsystematic risk disappears!

Notice $\bar{r}_p = \alpha \bar{r}_m + (1-\alpha) r_f = r_f + \alpha (\bar{r}_m - r_f) = r_f + \beta_p (\bar{r}_m - r_f)$, so $\beta_p = \alpha$.

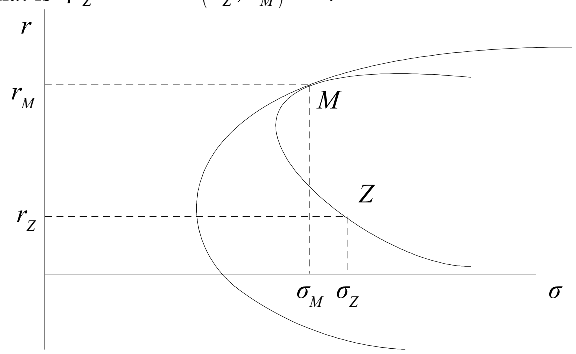
Application

Suppose the payoff is X in period 1, with $\bar{X}$ and σ_X^2. The price of X in period 0 is $P_X = \dfrac{\bar{X}}{1 + r_f + d}$.

What is d? Since $1 + r_f + d = \dfrac{\bar{X}}{P_X} = 1 + \bar{r}_x$ and $\bar{r}_x = r_f + \beta_j (\bar{r}_m - r_f)$, so $d = \beta_j (\bar{r}_m - r_f)$.

No Risk Free Asset

Let Z be the zero-beta portfolio, that is $\beta_Z = 0 \Leftrightarrow \text{cov}(r_Z, r_M) = 0$.



The slope at M is $\dfrac{\bar{r}_M - \bar{r}_z}{\sigma_m}$. So $\bar{r}_j = \bar{r}_Z + \beta_j (\bar{r}_M - \bar{r}_Z)$ (empirical: $r_j = r_Z + \beta_j (r_M - r_Z) + \varepsilon_j$).

Note that in this model, different agents will hold different risky assets, with the average agent holding M.

State Preference Theory

Suppose agents A and B has endowments:

Time 0	y_0^A	y_0^B
Time 1	y_1^A	y_1^B
	$U^A(c_0, c_1)$	$U^B(c_0, c_1)$

For each agent,

- Time 0: endowment y_0, utility $U(c_0) + E(U(c_1))$.
- Time 1: endowment y_1^1 with probability π_1 (state 1), endowment y_1^2 with probability π_2 (state 2), etc.

Suppose there are “pure securities”, with payoffs in time 1:

Security	Price at Time 0	Payoff in State 1	Payoff in State 2	Payoff in State 3
Pure 1	q_1	1	0	0
Pure 2	q_2	0	1	0
Pure 3	q_3	0	0	1
Pure A	$q_A = 3q_1 + 2q_2 + 4q_3$	3	2	4

- Since there is no arbitrage, the price any new addition of securities to a complete market can be determined.
- Since agents A and B buy and sell from each other, they share the risks.

Asset Pricing: Pure Securites

Pure Securities: Any security in the market is a linear combination of pure securities with price p_i and \$1 payout in the i-th state. So if S and any security and s_i is its payoff in the i-th state, then $P_S=p_1 s_1+\cdots+p_n s_n$.
Hedge Portfolio: A risky portfolio that replicated a risk-free portfolio. If it contains a_i amount of security i (priced P_i), then $a_1 P_1+a_2 P_2+\cdots+a_n P_n=\frac{1}{1+r_f}$.

Risk Neutral Probabilities: If an agent is risk neutral, then for any security S , $P_S=\frac{E(S)}{1+r_f}=\frac{\pi_1^{RN} s_1+\cdots+\pi_n^{RN} s_n}{1+r_f}$; π_i^{RN} is the risk neutral probability of state i . Notice $\pi_i^{RN}=(1+r_f)p_i$.

Continuous Time Process

Definition

The current period is date 0. \$1 in a risk-free debt at date 0 is worth \$ $e^{r_f T}$ at date T . Here r_f is the continuously compounded/instantaneous rate of interest.
Note: Time is in years, so $T=1$ is one year.

Value of Equity

The value of a share of equity at date T is $S_T=S_0 e^{\int_0^T r_s(t)dt}$.
Note: $\int_0^T r_s(t)dt\sim N\left(\left(\mu-\frac{1}{2}\sigma^2\right)T,\sigma^2 T\right)$, so S_T is log-normal (i.e $\ln\left(\frac{S_T}{S_0}\right)=\int_0^T r_s(t)dt\sim N\left(\left(\mu-\frac{1}{2}\sigma^2\right)T,\sigma^2 T\right)$).

Options

Call Option

Written on a stock with price S_0 , there is:

- exercise buying price X ,
- maturity date T .

European Call Options: At time T , one can “exercise” – sell at S_T and buy at X . So the payoff will be $\max(S_T-X,0)$.
American Call Options: One can exercise at any time up to time T .

Put Option

Written on a stock with price S_0 , there is:

- exercise selling price X ,
- maturity date T .

European Put Options: At time T , one can “exercise” – buy at X and sell at S_T . So the payoff will be $\max(X-S_T,0)$.
American Put Options: One can exercise at any time up to time T .

Put-Call Parity

If the price of a stock is S_0, c_0 is the price of a call option, p_0 is the price of a put option, and X is the payout of a risk free debt at time T, then

$$S_0+p_0=c_0+X\,\mathrm{e}^{-r_f T}\quad.$$

Note that for $0<t<T$,

- Any asset is a combination of calls, puts, and risk free debts.
- $c_t=p_t+S_t-X\,\mathrm{e}^{-r_f(T-t)}>p_t+S_t-X$ and $S_t-X\,\mathrm{e}^{-r_f(T-t)}>S_T-X$, so nobody will exercise a call early.
- $p_t=c_t+X\,\mathrm{e}^{-r_f(T-t)}-S_t$ and $X\,\mathrm{e}^{-r_f(T-t)}-S_t<X-S_T$, so for c_T small enough, one might exercise a put early.

Risk Neutral Probability Pricing of a Call Option

Consider a stock with price S_0 at date 0. Its payout is either $S_T^1=u\,S_0$ (state 1) or $S_T^1=d\,S_0$ (state 2) at date T. Then a call option's payout is $u\,S_0-X$ in state 1 and 0 in state 2. Let $1+r_f^{'}=\mathrm{e}^{r_f T}$.
If we assume $d<1+r_f^{'}<u$ (or else arbitrage) and $d\,S_0<X<u\,S_0$, then

$$c_0=\frac{\pi_1^{RN}\,u\,S_0-X}{1+r_f^{'}}$$

where $\pi_1^{RN}=\dfrac{1+r_f^{'}-d}{u-d}$.

Black-Scholes

The price of a call (European or American) is $c_0=S_0\,N(d_1)-\mathrm{e}^{-r_f T}X\,N(d_2)$, where

- $N(d_1)=P(z\le d_1)\quad z\sim N(0,1)$
- $N(d_2)=P(z\le d_2)\quad z\sim N(0,1)$
- $d_1=\dfrac{\ln\left(\dfrac{S_0}{X}\right)+r_f T}{\sigma\sqrt{T}}+\dfrac{1}{2}\sigma\sqrt{T}\quad(\sigma=\mathrm{SD}(\ln S_T))$
- $d_2=d_1-\sigma\sqrt{T}$.

Then by the put-call parity, the price of a put (European) is $p_0=c_0-S_0+X\,\mathrm{e}^{-r_f T}$.

Consumption Based Capital Asset Pricing Model

Every agent/household in the economy is identical. So only study a representative household.

Two Time Periods, Perfect Certainty

Suppose the representative household's lifetime utility is $U=U(C_0)+\beta\,U(C_1)$.
Suppose its income in period 0 is Y_0 and its income in period 1 is Y_1.
Consider a discount bond issued by the government that pays 1 unit of consumption in period 1 and sells for $p_0=\dfrac{1}{1+r_0}$.
The household will choose q_0 optimally to maximize its lifetime utility. Then we get

$$p_0=\beta\frac{U'(Y_1+q_0)}{U'(Y_0-p_0 q_0)}\quad.$$

p_0 is the equilibrium price if $q_0=q^s$ the quantity of bonds supplied per household.

Infinite Time Periods, Uncertainty

The representative household's lifetime utility (at period 0) is $U_0 = U(C_0) + E_0\left[\sum_{t=1}^{\infty} \beta^t U(C_t)\right]$.

One Period Security

Suppose security X has current price $P_{X,0}$. It can be redeemed in period 1 for x_1 . Then we get

$$P_{X,0} = \beta E_0\left[x_1 \frac{U'(C_1)}{U'(C_0)}\right] .$$

Since $\frac{x_1}{P_{X,0}} = 1 + r_{X,1}$, can rewrite $1 = \beta E_0\left[(1 + r_{X,1}) \frac{U'(C_1)}{U'(C_0)}\right]$.

Risk Free Asset

For risk-free asset, $1 = \beta(1 + r_{X,1}) E_0\left[\frac{U'(C_1)}{U'(C_0)}\right]$.

t Period Security

Suppose security Y has current price $P_{Y,0}$ and pays y_t in period *t*. (and nothing prior and after). Then

$$P_{Y,0} = \beta^t E_0\left[y_t \frac{U'(C_t)}{U'(C_0)}\right].$$

Price of a Stock

The price of a stock that pays dividend div_t in period t is

$$S_0 = E_0\left[\sum_{t=1}^{\infty} \beta^t \mathrm{div}_t \frac{U'(C_t)}{U'(C_0)}\right].$$