MAT1200: Modular Forms of Half Integral Weight

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Subgroups of $\mathrm{SL}_2(\mathbb{Z})$

We have $\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$. Define the following subgroups:

- $\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$
- $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$
- $\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}$.

The action of $\gamma \in \mathrm{SL}_2\mathbb{Z}$ on $z \in \mathbb{C}$ is given by

$$\gamma z := \frac{az + b}{cz + d}.$$

Proposition. $\Gamma_0(4)$ *is generated by*

$$-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad , \quad R = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}.$$

Also, if $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *, then* $ST^4S^{-1} = ST^4S = R$.

The map

$$\begin{aligned} \chi : \Gamma_0(4) rel\varepsilon\longrightarrow (\mathbb{Z}/4)^\times &\longrightarrow \{\pm 1\} \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d \pmod 4 &\longmapsto \chi(\gamma) \end{aligned}$$

defines the character of γ. Here, ε is a homomorphism.

Remark. $\chi(T) = 1$, $\chi(-I) = -1$, $\chi(R) = -1$.

Definition (Congruent Subgroup)**.** *A congruent subgroup is a subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *which contains* $\Gamma(N)$ *for some* $N \geq 1$.

Three Basic Functions

Theta Function

Let $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} \quad, 0 < t \in \mathbb{R}$.

Proposition. $\theta(t) = \frac{1}{\sqrt{t}}\theta(\frac{1}{t})$.

Proof. Apply Poisson Summation, i.e. $\sum f(n) = \sum \hat{f}(n)$, to $f_t(x) = e^{-\pi t x^2}$. Then $\hat{f}_t(y) = \frac{1}{\sqrt{t}} f_t(\frac{y}{\sqrt{t}})$ and

$$\sum e^{-\pi t n^2} = \sum f_t t(n) = \sum \hat{f}_t(n) = \frac{1}{\sqrt{t}} \sum f_t(\tfrac{n}{\sqrt{t}}) = \frac{1}{\sqrt{t}}\theta(\tfrac{1}{t}).$$

□

Proposition. *If* $\Phi(s) = \int_1^\infty t^{s/2} (\theta(t) - 1) \frac{dt}{t} + \int_0^1 t^{s/2} \left(\theta(t) - \frac{1}{\sqrt{t}}\right) \frac{dt}{t}$, *then* $\Phi(s) = \Phi(1-s)$ *and so*

$$\Lambda(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s) = \Lambda(1-s)$$

where $\Gamma(s) = \int_0^\infty e^{-u} u^{s-1} du$.

With the definition of θ, put $z = it \in \mathbb{H}$ and obtain the Theta function.

Definition (Theta Function)**.** *Define*

$$\Theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z} \quad , \Im(z) > 0.$$

Proposition.

1. $\Theta(z) = \frac{1}{\sqrt{-2iz}}\Theta(\frac{2i}{4z})$,
2. $\Theta(-\frac{1}{4z}) = \left(\frac{2z}{i}\right)^{1/2}\Theta(z)$, *and*
3. $\Theta^2(-\frac{1}{4z}) = \frac{2z}{i}\Theta^2(z)$.

Proposition. Θ^2 *satisfies*

$$\Theta^2(\gamma z) = \chi(\gamma) j(\gamma, z)\Theta^2(z)$$

where $j(\gamma, z) = (\det\gamma)^{-1/2}(cz + d)$.

Proposition. Θ *transforms as follows. For* $\gamma \in \Gamma_0(4)$,

$$\Theta(\gamma z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1}(cz+d)^{1/2}\Theta(z)$$

where z *has argument in* $(-\pi/2, \pi/2]$, $\left(\frac{c}{d}\right)$ *is the extended Jacobi symbol and* $\varepsilon_d = \begin{cases} 1 & d \equiv 1 \pmod 4 \\ i & d \equiv -1 \pmod 4 \end{cases}$.

Eisenstein Series

Let

$$G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}.$$

This converges absolutely for $k > 2$.

Proposition. *For all* $\gamma \in \mathrm{SL}_2\mathbb{Z}$,

$$G_k(\gamma z) = j(\gamma, z)^k G_k(z).$$

We normalize $G_k(z)$. Let

$$E_k(z) := \frac{1}{2\zeta(k)} G_k(z) \quad , k > 2, k\text{even}.$$

Setting $k = 2$, we get a conditionally convergent series $E_2(z)$. Note that

$$E_2(z) = \frac{1}{2\zeta(2)} \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^2} = 1 + \frac{1}{2\zeta(2)} \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2}.$$

Proposition. $E_2(z) = \frac{1}{z^2} E_2(-\frac{1}{z}) + \frac{6i}{\pi z}$.

To deal with convergence issues, set

$$\tilde{E}_2(z) := 1 - \frac{1}{2\zeta(2)} \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2(mz+n-1)}.$$

Proposition. $\tilde{E}_2(z) = E_2(z) + \frac{1}{2\zeta(2)} \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \left(\frac{1}{mz+n} - \frac{1}{mz+n-1} \right)$ *is absolutely convergent and converges to* $E_2(z)$.

Eta Function

Definition (Eta Function)**.** *For* $z \in \mathbb{H}$, *let*

$$\eta(z) := e^{2\pi i z/24} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right).$$

Proposition. $\eta(-\frac{1}{z}) = \left(\frac{z}{i}\right)^{1/2} \eta(z)$.

Relations of the Three Functions

Proposition. $\Theta(z) = \dfrac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}$.

Let $1 \leq N \in \mathbb{Z}$. Consider $(a_1, a_2) \in (\mathbb{Z}/N\mathbb{Z})^2$. Let $3 \leq k \in \mathbb{Z}$. Set

$$G_k^{(a_1,a_2)}(z) := \sum_{\substack{m_1 \equiv a_1 \pmod N \\ m_2 \equiv a_2 \pmod N}} \frac{1}{(m_1 z + m_2)^k}$$

and

$$h(z) := \prod_{a_2}^{N-1} G_3^{(0,a_2)}(z).$$

Proposition. *$h(z)$ is a constant multiple of* $\left(\dfrac{\eta^p(z)}{\eta(pz)}\right)^6$.

Proposition. *Let p be an odd prime. Set $\phi(z) = \Theta(\frac{z}{2})$ and $\psi(z) = \frac{\eta^p(z)}{\eta(pz)}$. Then*

$$\frac{\phi(pz)}{\phi^p(z)} = \frac{\psi(z)}{\psi^2(z+1/2)}.$$

Modular Forms of Integral and Half-Integral Weight

Cusps

Definition (Cusp)**.** *A cusp $z \in \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ is an element which is fixed by a parabolic element of Γ ($\gamma \in \Gamma$ is parabolic if $|\mathrm{tr}(\gamma)| = 2$).*

Example (Cusps of $\mathrm{SL}_2(\mathbb{Z})$)**.** *$i\infty$ is the only cust of $\mathrm{SL}_2(\mathbb{Z})$.*

This is because $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ fixes $i\infty$. If $\frac{a}{c} \in \mathbb{Q}$, $(a,b) = 1$, $c \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ fixes it.

In general, if $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is of finite index, then there are only a finite number of Γ-equivalent cusps.

Example (Cusps of $\Gamma_0(4)$)**.** *Note that* $\Gamma_0(4) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod 4\right\}$ $\cong \Gamma(2) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 4\right\}$

There are three $\Gamma_0(4)$-inequivalent cusps.

- *$i\infty$. If $4 \mid c$ and $(a,c) = 1$, then $\frac{a}{c}$ is $\Gamma_0(4)$-equivalent to $i\infty$.*
- *$\frac{1}{2}$. If $2 \parallel n$ and $(m,n) = 1$, then $\frac{m}{n}$ is $\Gamma_0(4)$-equivalent to $\frac{1}{2}$.*
- *0. If n is odd and $(m,n) = 1$, then $\frac{m}{n}$ is $\Gamma_0(4)$-equivalent to 0.*

Example (Cusps of $\Gamma_0(p)$)**.** *The cusps of $\Gamma_0(p)$, p prime, are $\{0, i\infty\}$.*

Modular Forms of Integral Weight

Definition. *Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. A modular form f of weight $k \in \mathbb{Z}$ is a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ such that:*

1. *$f(\gamma z) = (cz+d)^k f(z) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and*
2. *f is holomorphic at all cusps of Γ.*

Proposition. *$G_k^{(a_1,a_2)} \in M_k(\Gamma_1(N))$.*

Proposition. *$h(z) \in M_{3(p-1)}(\Gamma_1(p))$.*

Metapletic Group

Consider the extensions

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1.$$

If we can lift H to a subgroup of G, then G is the semi-direct product of H and K, and it is considered the trivial extension.

We want to consider the non-trivial element.

Consider a degree 4 extension of $\mathrm{GL}_2^+(\mathbb{Q})$

$$1 \longrightarrow \mu_4 \longrightarrow \tilde{G} \longrightarrow \mathrm{GL}_2^+(\mathbb{Q}) \longrightarrow 1$$

where $\mu_4 = \{\pm 1, \pm i\}$. Here,

$$\tilde{G} = \left\{ (\alpha, \phi) : \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}), \phi \text{ holomorphic function on } \mathbb{H} \text{ s.t. } \phi^2(z) = t\frac{cz+d}{\sqrt{\det \alpha}}, t = \pm 1 \right\}.$$

Proposition. *$\tilde{G}$ is a group under the operation $(\alpha, \phi)(\beta, \psi) = (\alpha\beta, z \mapsto \phi(\beta z)\psi(z))$.*

Remark. *$\tilde{G}$ is not an algebraic group.*

Example.

$$j(\gamma, z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} (cz+d)^{1/2} \quad , \gamma \in \Gamma_0(4)$$

is an example of ϕ since $j^2(\gamma, z) = \left(\frac{c}{d}\right)^2 \varepsilon_d^{-2}(cz+d) = 1(\pm 1)(cz+d)$.

Modular Forms of Half-Integral Weight

Let f be a function on $\mathbb{H}$. Let $\xi = (\alpha, \phi) \in \tilde{G}$. Set

$$(f|_k \xi)(z) = f(\alpha z)\phi^{-k}(z) \quad , k \in \frac{1}{2}\mathbb{Z}.$$

$\tilde{G}$ acts on the space of functions in this way.

Definition. *Let $k \in \frac{1}{2}\mathbb{Z}$. A modular form f of weight k for $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ such that:*

1. *$f|_k \gamma = f$ for all $\gamma \in \Gamma$, and*
2. *f is holomorphic at all cusps of Γ.*

Example. *η is a form of weight $\frac{1}{2}$ for $\mathrm{SL}_2(\mathbb{Z})$. $\Delta := \eta^{24}$ is a form of weight 12.*

Example. *Let $\Gamma \subseteq \Gamma_0(4)$. Let $\Gamma_\infty = \{\gamma \in \Gamma : \gamma(i\infty) = i\infty\}$. Then Γ_∞ is an infinite subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$.*

Consider

$$E_{k/2}(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} j(\gamma, z)^{-k} \qquad , j(\gamma, z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1}(cz+d)^{1/2}, 5 \leq k \text{ odd}$$

where $\Gamma_\infty \backslash \Gamma_0(4)$ have coset representatives $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) : 4|m, (m,n) = 1, n > 0 \right\}$. Then $E_{k/2}(z)$ is a modular form of weight $\frac{k}{2}$ for $\Gamma_0(4)$. Its Fourier coefficients are Dirichlet L-function values!

The Shimura Correspondence

To define the Shimura correspondence, we first need to discuss Hecke operators.

Hecke Operators

Let Γ be a congruent subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $f \in M_k(\Gamma), 0 \leq k \in \mathbb{Z}$.

Suppose $\alpha \in \mathrm{GL}_2^+(\mathbb{Z})$. We may consider

$$(f|_k\alpha)(z) = (cz+d)^{-k}(\det\alpha)^{k/2} f(\alpha z).$$

If $\alpha \in \Gamma$, then $f|_k\alpha = f$.

We have that $f|_k\alpha \in M_k(\Gamma')$ where $\Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$, since $(f|\alpha)|\alpha^{-1}\gamma\alpha = f|\gamma\alpha = f|\alpha$.

The groups Γ and $\alpha^{-1}\Gamma\alpha$ are commensurable, i.e $\Gamma \cap \alpha^{-1}\Gamma\alpha$ is of finite index in both Γ and $\alpha^{-1}\Gamma\alpha$. Suppose $\lambda := [\Gamma : \Gamma']$. We may decompose $\Gamma = \bigcup_{i=1}^{\lambda} \Gamma'\gamma_i'$. This gives a decomposition $\Gamma'\alpha\Gamma = \bigcup_{i=1}^{\lambda} \Gamma'\alpha\gamma_i'$.

We can define the action of the double cosets. Define

$$f|(\Gamma'\alpha\Gamma) := \sum_{i=1}^{\lambda} f|\alpha\gamma_i'.$$

This is well-defined. In fact, $f|(\Gamma'\alpha\Gamma) \in M_k(\Gamma)$.

Definition. *Suppose $(n, N) = 1$, $\Gamma = \Gamma_0(N)$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Define*

$$T_n(f) := n^{k/2-1} \sum_{\substack{\alpha \in M_2(\mathbb{Z}) \\ \det\alpha = n \\ N|c}} f|(\Gamma'\alpha\Gamma).$$

Lemma. *$T_n : M_k(\Gamma) \longrightarrow M_k(\Gamma)$ is a linear map.*

The Integral Weight Case

Consider the integral weight case.

Definition. *For $(n, N) = 1$, $f \in M_k(\Gamma_1(N))$, define*

$$T_n(f) := n^{k/2-1} \sum_{\alpha} f|(\Gamma_1(N)\alpha\Gamma_1(N))$$

where α ranges over the set of double coset representatives in

$$\Delta^n := \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : \det\alpha = n, \alpha \equiv \begin{pmatrix} 1 & * \\ 0 & n \end{pmatrix} \pmod{N} \right\}.$$

Note that $\Gamma_1(N)$ acts on Δ^n from the left and the right. Thus Δ^n can be expressed as the disjoint union of $\Gamma_1(N)$ double cosets.

What does T_p do to Fourier coefficients? Write $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$. Then $T_p f \in M_k(\Gamma_1(N))$ and $T_p f(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z}$ with

$$b_m = a_{pm} + p^{k-1} a_{\frac{m}{p}}$$

where $a_{\frac{m}{p}} = 0$ if $p \nmid m$.

Lemma. *If $f \in M_k(\Gamma_0(N), \chi)$ is an eigenform for all the T_m, say $T_m f = \lambda_m f$, then*

$$a_n = \lambda_n a_1$$

where $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$.

The Half-Integral Weight Case

For the half-integral weight case, we can define T_n by analogy. Let $\tilde{\Gamma} = \widetilde{\Gamma_1(N)}$; the double cosets are $\tilde{\Gamma}\xi\Gamma$. Set $\xi_n = \left(\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}, \cdot \right)$.

Proposition (Shimura)**.** *If $(n, N) = 1$, $4 \mid N$, k odd, $f \in M_{k/2}\left(\widetilde{\Gamma}\right)$, then*

$$f|(\tilde{\Gamma}\xi_n\tilde{\Gamma}) = 0$$

unless n is a square.

Corollary (Shimura)**.** *$T_n f = 0$ unless n is a square.*

Theorem. *Let $4 \mid N$, χ is a character modulo N. Suppose $f \in S_{k/2}(\widetilde{\Gamma_0(N)}, \chi)$ (i.e. a cusp form) with $0 < k \in \mathbb{Z}$ odd. Suppose f is an eigenform for $\{T_{p^2} : p \nmid N\}$, i.e. $T_{p^2} f = \lambda_p f$. Define*

$$g(z) := \sum_{n=1}^{\infty} c_n e^{2\pi i n z}$$

where $c_1 = 1$, $c_p = \lambda_p$, $c_{p^{i+1}} = c_p c_{p^i} - \chi(p)^2 p^{k-2} c_{p^{i-1}}$, $c_{mn} = c_m c_n$ if $(m, n) = 1$.

Then

$$g \in M_{k-1}\left(\Gamma_0\left(\frac{N}{2}\right), \chi^2\right).$$

If $k \geq 5$, then

$$g \in S_{k-1}\left(\Gamma_0\left(\frac{N}{2}\right), \chi^2\right).$$

Thus, if there is a basis for $S_{k/2}\left(\widetilde{\Gamma_0(N)}, \chi\right)$ consisting of Hecke eigenforms, this defines the Shimura map $S_{k/2}\left(\widetilde{\Gamma_0(N)}, \chi\right) \longrightarrow S_{k-1}\left(\Gamma_0\left(\frac{N}{2}\right), \chi^2\right)$.

Köhnen Subspace

We wish to define subspaces (Köhnen subspaces) $M^+_{k/2}\left(\widetilde{\Gamma_0(N)}, \chi\right)$ and $S^+_{k/2}\left(\widetilde{\Gamma_0(N)}, \chi\right)$ consisting of $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ with $a_n = 0$ unless $(-1)^{\frac{1-2}{2}} n \equiv 0, 1 \pmod 4$.

Petersson Inner Product

Definition. *Let $f, g \in M_k(\Gamma_0(N), \chi)$. Define*

$$\langle f, g \rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}$$

assuming at least one of f or g is a cusp form so that the integral converges.

This inner product is Hermitian with respect to the Hecke operators, i.e.

$$\langle f|T_n, g\rangle = \chi(n)\langle f, g|T_n\rangle$$

and

$$\langle g, f\rangle = \overline{\langle f, g\rangle}.$$

Proposition. *There is a basis of $S_k(\Gamma_0(N), \chi)$ consisting of eigenforms for all the T_n, $(n, N) = 1$.*

Lemma. *$S_k(\Gamma_0(N))$ has a $\mathbb{Q}$-structure. That is, there exists a basis of forms with rational Fourier coefficients, i.e. $S_k(\Gamma_0(N))_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = S_k(\Gamma_0(N))$. Thus, the Hecke operators are defined over $\mathbb{Q}$, i.e. they act on this space.*

Consider the characteristic polynomial which has coefficients in $\mathbb{Q}$.

Corollary. *The Fourier coefficients of normalized Hecke eigenforms are algebraic. In fact, they are algebraic integers. Moreover, they live in a number field.*

Remark. *This is not known in the half-integral weight case.*

L-Functions

We can define the L-function in two ways: Mellin transform and Hecke operators.

Mellin Transform

Let $f \in S_k(\Gamma_0(N), \chi)$. Consider

$$I := \int_0^\infty f(iy) y^s \frac{dy}{y}.$$

If $f(z) = \sum_{n=1}^\infty a_n e^{2\pi i n z}$, then

$$I = \sum_{n=1}^\infty a_n \int_0^\infty e^{-2\pi i n y} f(iy) y^s \frac{dy}{y}.$$

Letting $u = 2\pi n y$, we get

$$I = (2\pi)^{-s}\Gamma(s) \sum_{n=1}^\infty \frac{a_n}{n^s}.$$

Definition. *Define*

$$L(f, s) := \sum_{n=1}^\infty \frac{a_n}{n^s}.$$

Now suppose that $f|w_N = \varepsilon_N f$ where $w_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Then

$$\begin{aligned} \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(f, s) &= N^{s/2} \int_0^\infty f(iy) y^s \frac{dy}{y} \\ &= N^{s/2} \left(\int_0^{1/\sqrt{N}} \varepsilon_N^{-1} N^{-k/2} i^{-k} y^{-k} f\left(\frac{i}{Ny}\right) \frac{dy}{y} \right. \\ &\quad \left. + \int_{1/\sqrt{N}}^\infty \varepsilon_N^{-1} N^{-k/2} i^{-k} y^{-k} f\left(\frac{i}{Ny}\right) \frac{dy}{y} \right) \end{aligned}$$

Setting $u = \dfrac{1}{Ny}$ we get

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(f,s) = \int_{1/\sqrt{N}}^{\infty} f(iu)\left(N^{k/2}\varepsilon_N^{-1}i^{-k}N^{-s/2}u^{k-s} + N^{s/2}u^s\right)\frac{du}{u}$$

If $\varepsilon_N = i^{-k}$, then $N^{k/2}\varepsilon_N^{-1}i^{-k}N^{-s/2}u^{k-s}+N^{s/2}u^s = N^{\frac{k-s}{2}}u^{k-s}+N^{s/2}u^s$. This is invariant under $s \mapsto k-s$. This gives the functional equation

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(f,s) = \underbrace{\varepsilon_N i^k}_{=1}\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(k-s)L(f,k-s).$$

Hecke Operators

Definition. *Assume $T_pf = \lambda_p f$ if $p \nmid N$ and $U_p f = \mu_p f$ if $p \mid N$. Define*

$$L(f,s) := \prod_{p\mid N}\left(1-\frac{\mu}{p^s}\right)^{-1}\prod_{p\nmid N}\left(1-\lambda_p p^{-s}+\chi(p)p^{k-1-2s}\right).$$

If f is a normalized Hecke eigenform (including for w_N, then it is the same as the L-function defined by the Mellin transform. Moreover, there is a "multiplicity one" theorem.

Constant Term Function

Definition. *Suppose $F : \mathbb{H} \longrightarrow \mathbb{C}$ is $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ invariant. Define*

$$C(F,y) = \int_0^1 F(x+iy)dx \quad \text{for } y>0.$$

If we use the Mellin transform and set

$$\begin{aligned} L(F,s) &:= \int_0^{\infty} C(F,y)y^s\frac{dy}{y^2} \\ &= \int_0^{\infty}\int_0^{\infty} F(x+iy)y^s\frac{dxdy}{y^2} \end{aligned}$$

We have the action of $\Gamma_\infty = \left\{\pm\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}\right\}$ on $\mathbb{H}$ and we may consider the half strip of integration to be $\Gamma_\infty\backslash\mathbb{H}$. Also, $\Gamma_\infty \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. Using this, we have

$$\begin{aligned} L(F,s) &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma(\Gamma\backslash\mathbb{H})} F(z)y^s\frac{dxdy}{y^2} \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\Gamma\backslash\mathbb{H}} F(\gamma z)(\mathrm{Im}(\gamma z))^s\frac{dxdy}{y^2} \\ &= \int_{\Gamma\backslash\mathbb{H}} F(z)\left(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(\mathrm{Im}(\gamma z))^s\right)\frac{dxdy}{y^2} \end{aligned}$$

If we define $E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(\mathrm{Im}(\gamma z))^s = \sum_{(c,d)\neq(0,0)} \frac{y^s}{|cz+d|^{2s}}$, this is the Eisenstein series.

Consider the case $f \in S_k(\Gamma_0(N))$. Let $F(z) = (\mathrm{Im} z)^k |f(z)|^2$. If we write $F(z) = \sum_{n=1}^\infty a_n e^{2\pi i n z}$, then

$$C(F,y) = y^k \sum_n |a_n|^2 e^{-4\pi n y}$$

and

$$L(F,s) = (4\pi)^{-s-k+1}\Gamma(s+k-1)\sum_n \frac{|a_n|^2}{n^{s+k-1}}.$$

Notice that

$$L(F,s) = \int_{\Gamma\backslash\mathbb{H}} F(Z)E(z,s)\frac{dxdy}{y^2}.$$

Thus, if we have an analytic continuation of $E(z,s)$, then we deduce an analytic continuation for $L(F,s)$.

Remark. *Suppose we drop the Γ-invariance and consider instead an F satisfying $F(\gamma z) = j(\gamma z)^k F(z)$ with $\gamma \in \Gamma \in \Gamma_0(4)$. We can still define*

$$C(F,y) = \int_0^1 F(x+iy)dx \ \ \textit{for } y > 0.$$

The Mellin transform is

$$\begin{aligned} L(F,s) &= \int_0^\infty C(F,y)y^s\frac{dy}{y^2} \\ &\vdots \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\Gamma\backslash\mathbb{H}} F(\gamma z)(\mathrm{Im}(\gamma z))^s\frac{dxdy}{y^2} \\ &= \int_{\Gamma\backslash\mathbb{H}} F(z)\left(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^k\frac{(\mathrm{Im}(\gamma z))^s}{|cz+d|^{2s}}\right)\frac{dxdy}{y^2} \\ &= \int_{\Gamma\backslash\mathbb{H}} F(Z)E^*(z,s)y^s\frac{dxdy}{y^2}. \end{aligned}$$

In the previous case we had

$$L(F,s) = \int_{\Gamma\backslash\mathbb{H}} F(Z)E(z,s)\frac{dxdy}{y^2}$$

where $E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(\mathrm{Im}(\gamma z))^s = \sum_{(c,d)\neq(0,0)} \frac{y^s}{|cz+d|^{2s}}$.

To analytically continue all L, we only need to continue E. To continue E, continue its Fourier expansion.

Consider the Fourier expansion of $E(z,s)$:

$$E(z,s) = \sum_{r\in\mathbb{Z}} a_r(y,s)e^{-2\pi i r x}dx.$$

By definition,

$$a_r(y,s) = \int_0^1 E(x+iy,s)e^{-2\pi i r x}dx.$$

Normalize E and write $E(z,s)=\frac{1}{2}\pi^{-s}\Gamma(s)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(\mathrm{Im}(\gamma z))^s$. Then

$$\begin{aligned} a_r(y,s) &= \frac{1}{2}\pi^{-s}\Gamma(s)\int_0^1 \sum_{(c,d)\neq(0,0)} \frac{y^s}{|cz+d|^{2s}}e^{-2\pi i r x}dx \\ &= \pi^{-s}\Gamma(s)y^s\sum_{c=1}^{\infty}\sum_{d\in\mathbb{Z}}\int_0^1 \left((cx+d)^2+c^2y^2\right)^{-s}e^{-2\pi i r x}dx. \end{aligned}$$

Writing $d=cq+\rho$, we get $(cx+d)^2+c^2y^2=(c(x+q)+\rho)^2+c^2y^2$, and the double sum becomes

$$\sum_{c=1}^{\infty}\sum_{\rho=0}^{c-1}\int_{infty}^{\infty}\left((cx+\rho)^2+c^2y^2\right)^{-s}e^{-2\pi i r x}dx.$$

Now set $u=x+\frac{\rho}{c}$. Then $(cx+\rho)^2+c^2y^2=c^2(u^2+y^2)$. This makes the double sum become

$$\sum_{c=1}^{\infty}c^{-2s}\int_{infty}^{\infty}\left(u^2+q^2\right)^{-s}e^{-2\pi i r u}\left(\sum_{p \bmod c}e^{2\pi i r\rho/c}\right)du=\left(\sum_{c|r}c^{1-2s}\right)\int_{infty}^{\infty}(u^2+y^2)^{-s}e^{-2\pi i r u}du.$$

Set $\sigma_\alpha(r)=\sum_{\substack{c|r\\ c>0}}c^\alpha$.

The term with $r=0$ gives

$$\pi^{-s}\Gamma(s)y^s\left(\zeta(2s-1)\int_{\infty}^{\infty}(u^2+y^2)^{-s}du\right)=\cdots=\pi^{-s+1/2}\Gamma(s-1/2)y^{1-s}\zeta(2s-1).$$

Thus, the constant term is

$$\begin{aligned} a_0(y,s) &= \pi^{-s}\Gamma(s)\zeta(2s)y^s+\pi^{1/2-s}\Gamma(s-1/2)y^{1-s}\zeta(2s-1) \\ &= \pi^{-s}\Gamma(s)\zeta(2s)y^s+\pi^{-(1-s)}\Gamma(1-s)\zeta(2-2s)y^{1-s}. \end{aligned}$$

Therefore $a_0(y,s)=a_(0,1-s)$, and we have analytic continuation of the constant term.

In general,

$$a_r(y,s)=2\sigma_{1-2s}(|r|)|r|^{s-1/2}\sqrt{y}k_{s-1/2}(2\pi|r|y)$$

where $k_s(y):=\frac{1}{2}\int_0^\infty e^{-y(t+1/t)}t^s\frac{dt}{t}$ is the Bessel function. Note that $k_s(y)=k_{-s}(y)$ and has exponential decay in y. Thus $a_r(y,s)$ has exponential decay in $|r|$ and $a_r(y,1-s)=a_r(y,s)$.

This gives the analytic continuation and functional equation of $E(z,s)$. Note:

$$E(z,s)=\underbrace{a_0(y,s)}_{\text{pole at } s=0,1,\text{ residue}\neq 0}+\underbrace{\sum_{r\neq0}a_r(y,s)e^{2\pi i r x}}_{\text{entire}}.$$

Thus this gives the analytic continuation and functional equation of $L(F,s)=\int_{\Gamma\backslash\mathbb{H}}F(z)E(z,s)\frac{dx}{y^2}$. It has a pole at $s=0,1$ and its residue $\mathrm{Res}(L(F,s),s=1)=\mathrm{Res}(E(z,s),s=1)\int_{\Gamma\backslash\mathbb{H}}F(z)\frac{dxdy}{y^2}$.

Kohnen Subspace

We have
$$S_{k+1/2}(\Gamma_0(4)) \longrightarrow S_{2k}(\Gamma_0(2)) \quad \text{and} \quad M_{k+1/2}(\Gamma_0(4)) \longrightarrow M_{2k}(\Gamma_0(2)).$$

Niwa proved, under the Shimura Correspondence, forms of level $4N$ go to forms of level $2N$. The map is Hecke equivalent and an isomorphism:
$$S_{k+1/2}(\Gamma_0(4N)) \xrightarrow{\cong} S_{2k}(\Gamma_0(2N)).$$

Kohnen defines a subspace
$$S^+_{k+1/2}(\Gamma_0(4N)) \longrightarrow S_{2k}(\Gamma_0(N)).$$
For $N = 1$, this is an isomorphism.

Theorem. (Kohnen)

1. *$S^+_{k+1/2}(4)$ (and $M^+_{k+1/2}(4)$) is preserved by Hecke operators T_{p^2}.*
2. *There is a basis of Hecke eigenforms.*
3. *If $f \in S^+_{k+1/2}(4)$ is such an eigenform, say $T_{p^2}(f) = \lambda_p f$, then there is an $F \in S_{2k}(1)$ such that $T_p(f) = \lambda_p F$.*
4. *If $f(z) = \sum_{n=1}^\infty a_f(n)q^n$ and $F(z) = \sum_{n=1}^\infty a_F(n)q^n$, then $L(F,s) = \sum_{n=1}^\infty \frac{a_F(n)}{n^s}$ satisfies*
$$\begin{aligned} a_f(|D|)L(F,s) &= \left(\sum_{n=1}^\infty \frac{a_f(|D|n^2)}{n^s}\right) L\left(s-k+1, \left(\frac{D}{\cdot}\right)\right) \\ &= \sum_{n=1}^\infty \frac{1}{n^s}\left(\sum_{dm=n} a_f(|D|n^2/m^2)\left(\frac{D}{m}\right) m^{k-1}\right) \end{aligned}$$
5. *The map*
$$\begin{aligned} \mathcal{S}^+_D : S_{k+1/2}(4) &\longrightarrow S_{2k}(1) \\ \sum_{n=1}^\infty b(n)q^n &\longmapsto \sum_{n=1}^\infty \left(\sum_{d\mid n}\left(\frac{D}{d}\right) d^{2k-1} b(|D|n^2/d^2)\right) q^n \end{aligned}$$
is Hecke equivalent.
6. *There exists a linear combination of the $\mathcal{S}^+_D$ that defines an isomorphism.*

The proof relies heavily on explicit calculations involving U_4 and W_4:
$$T_p = \begin{cases} U_p + V_p & (V_pf)(z) = *f(pz) \text{ and } (U_pf)(z) = *\sum_{d=0}^{p-1} f\left(\frac{z+d}{p}\right) \\ W_d & \text{where } d \mid N \end{cases}.$$

Niwa showed U_4W_4 is a Hermitian operator on $S_{k+1/2}(4)$. It has eigenvalues
$$\begin{aligned} \alpha_1 &= 2^k\left(\frac{2}{2k+1}\right) = \pm 2^k \\ \alpha_2 &= -\frac{1}{2}\alpha_1 \end{aligned}$$

Kohnen essentially identifies

$$S_{k+1/2}(4)\big|_{U_4W_4=\alpha_1} = S_{k+1/2}(4).$$

Now, $L(F,s)$ satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)L(F,s) = (2\pi)^{-(k-s)}\Gamma(k-s)L(F,k-s).$$

The centre is at $s=k$. Also,

$$L(F,s) = \prod_p \left(1 - \frac{\lambda_p}{p^s} + \frac{p^{2k-1}}{p^2 s}\right)^{-1} = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1}\left(1 - \frac{\beta_p}{p^s}\right)^{-1}$$

where $\alpha_p + \beta_p = \lambda_p$, $\alpha_p\beta_p = p^{2k-1}$, and $|\alpha_p| = |\beta_p| = p^{k-1/2}$. Hence the Euler product converges absolutely for $\mathrm{Re}(s) > k - \frac{1}{2} + 1 = k + \frac{1}{2}$. Moreover, it is non-zero in the half plane $\mathrm{Re}(s) > k + \frac{1}{2}$.

Theorem. (Kohnen) *Consider*

$$\begin{array}{rcl} \mathcal{S}_1^+ : S_{k+1/2}(4) & \longrightarrow & S_{2k}(1) \\ f & \mapsto & F. \end{array}$$

The image of $\mathcal{S}_1^+$ is the space generated by $\{F \in S_{2k}(1) : L(F,k) \neq 0\}$.

Corollary. *For $k=6$, we have $S_{12}(1) \cong \mathbb{C}\Lambda$ and $L(\Lambda,6) \neq 0$. Also, $S^+_{13/2}(\Gamma_0(4)) \cong S_{12}(1)$.*

Waldspurger's Formula

The Waldspurger's formula relates Fourier coefficients of modular forms of weight $\frac{1}{2}$ to special values of L-function of (twisted) modular forms of integral weight. In the case when dealing with level 1 (i.e. forms for $\Gamma_0(4)$), the relation can be proven by explicit calculations (Kohen-Zagier).

Recall that we have a map

$$\begin{array}{rcl} \mathrm{Sh} : S^+_{k+1/2}(\Gamma_0(4)) & \xrightarrow{\cong} & S_{2k}(\mathrm{SL}_2(\mathbb{Z})) \\ g(z) = \displaystyle\sum_{n=1}^\infty a_g(q^n) & \longmapsto & f(z) = \displaystyle\sum_{n=1}^\infty a_f(q^n) \end{array}$$

We have $g \mid T^+_{k+1/2}(p^2) = f \mid T^+_{sk}(p)$, and for any fundamental discriminant D with $(-1)^k D > 0$ we have

$$a_g(n^2|D|) = a_g(|D|)\sum_{a|n}\mu(d)\left(\frac{D}{d}\right)d^{k-1}a_f\left(\frac{n}{d}\right).$$

Equivalently,

$$\sum_{n=1}^\infty \frac{a_g(n^2|D|)}{n^s} = a_g(|D|)\left(\sum_{n=1}^\infty \frac{a_f(n)}{n^s}\right)\frac{1}{L\left(s-k+1,\left(\frac{D}{\cdot}\right)\right)}.$$

Theorem. *Let $g \in S^+_{k+1/2}(\Gamma_0(4))$ and $f = \mathrm{Sh}(g)$. Let D be a fundamental discriminant with $(-1)^k D > 0$. then*

$$\frac{a_g(|D|)^2}{\langle g,g\rangle} = \frac{(k-1)!}{\pi^k}|D|^{k-1/2}\frac{L\left(f,\left(\frac{D}{\cdot}\right),k\right)}{\langle f,f\rangle}$$

Remark. *This amazing formula still leaves open the following two questions:*

1. *What is $a_g(|D|)$? Which square root? What does the sign mean?*
2. *Ramanujan Conjecture: For $f \in S_k(\Gamma_0(N))$, is $|a_f(n)| \leq d(n)n^{(k-1)/2}$ where $d(n) = \#$ of divisors of n? For example, a theorem of Delignc asserts that $|a_f(p)| \leq 2p^{(k-1)/2}$. What about $g \in S^+_{k+1/2}(\Gamma_0(N))$? We might expect the analogue of the Ramanujan Conjecture, that $|a_g(n)| \leq \alpha(n)n^{(k/2-1/4}$ where $\alpha(n)$ is an elementary function of n.*

Remark. *$a_g(n) \in \mathbb{R}$ since, in general, if $f \in S_k(\Gamma_0(N))$ (say, a normalized eigenform $f \mid T_n = a_f(n)f$, then*

$$\begin{aligned} a_f(n)\langle f, f\rangle &= \langle f \mid T_n, f\rangle \\ &= \langle f, f \mid T_n\rangle \\ &= \langle f, a_f(n)f\rangle \\ &= \overline{a_f(n)}\langle f, f\rangle. \end{aligned}$$

Therefore, $a_g(|D|)^2 \geq 0$. In particular, this implies that $L\left(f, \left(\frac{D}{\cdot}\right), k\right) \geq 0$; this is predicted by the Riemann Hypothesis for $L\left(f, \left(\frac{D}{\cdot}\right), s\right)$.

Suppose we assume the Riemann Hypothesis for $L\left(f, \left(\frac{D}{\cdot}\right), s\right)$. Then for any $\varepsilon > 0$,

$$\left|L\left(f, \left(\frac{D}{\cdot}\right), k + it\right)\right| \ll_\varepsilon \left(|D|^2 N(|t| + 2)\right)^\varepsilon.$$

In other words,

$$\left|L\left(f, \left(\frac{D}{\cdot}\right), s\right)\right| \ll_N |D|^\varepsilon.$$

From Waldspuger's formula $\frac{a_g(|D|)^2}{\langle g,g\rangle} = \frac{(k-1)!}{\pi^k}\frac{L\left(f,\left(\frac{D}{\cdot}\right),k\right)}{\langle f,f\rangle}|D|^{k-1/2}$,

$$\begin{aligned} |a_g(|D|)|^2 &\ll_{f,g,\varepsilon} |D|^{k-1/2+\varepsilon} \quad \text{(Lindelöf)} \\ \Rightarrow |a_g(|D|)| &\ll |D|^{k/2-1/4+\varepsilon}. \end{aligned}$$

Idea of Proof of Waldpurger's Formula

Consider the Eisenstein series

$$G_{k,D}(z) = \frac{1}{2}L(1-k, \chi_D) + \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{D}{d}\right)d^{k-1}\right)q^{n-1} \in M_k(\Gamma_0(D), \chi_D)$$

and

$$G_{k,4D}(z) = G_{k,D}(4z) - 2^{-k}\left(\frac{D}{2}\right)G_{k,D}(2z) \in M_k(\Gamma_0(4D), \chi_D)$$

where $\chi_D = \left(\frac{D}{\cdot}\right)$.

Define the trace operator: Let $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z}) = \bigcup \Gamma_0(D)\alpha_i$. Let

$$\begin{aligned} \mathrm{Tr} : M_k(\Gamma_0(D)) &\longrightarrow M_k(\Gamma_0(1)) \\ f &\longmapsto \mathrm{Tr} := \sum f \mid \alpha_i. \end{aligned}$$

Note that $\mathrm{Tr}(f) \mid \alpha = \mathrm{Tr}(f)\ \forall \alpha \in \mathrm{SL}_2(\mathbb{Z})$. Also, we have

$$\langle f, g \mid \alpha \rangle = \int_{\Gamma_0(1)\backslash\mathbb{H}} f(z)\overline{(g \mid \alpha)(z)} y^k \frac{dxdy}{y^2} = \cdots = \langle f \mid \alpha^{-1}, g \rangle,$$

and so

$$\langle f, g \mid \mathrm{Tr} \rangle = \sum \langle f, g \mid \alpha_i \rangle = \sum * \langle f \mid \alpha^{-1}, g \rangle = \sum * \langle f, g \rangle = \langle f, g \rangle.$$

Define the projection operator: Let $M_k^+(\Gamma_0(4)) \hookrightarrow M_k(\Gamma_0(4))$. Let

$$\begin{array}{rcl} \mathrm{pr}^+ : M_k(\Gamma_0(4)) & \longrightarrow & M_k^+(\Gamma_0(4)) \\ f & \longmapsto & \mathrm{pr}^+(f) = \left((-1)^{(k-1)/2} 2^{-k} W - 4U_4 + \dfrac{1}{3} \right) f. \end{array}$$

Then $\langle f, \mathrm{pr}^+ g \rangle = \langle \mathrm{pr}^+ f, g \rangle$. Also, if $f \in M^+$ and $g \in M$, then $\langle f, \mathrm{pr}^+ g \rangle = \langle f, g \rangle$.

Define

$$\mathcal{F}_D(z) := \mathrm{Tr}_{\Gamma_0(D) \to \mathrm{SL}_2(\mathbb{Z})}(G_{k,D}(z)^2).$$

Note that $\mathcal{F}_D(z) \in M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$.

Proposition. *Suppose $f \in S_{2k}(\Gamma_0(1))$ is a normalized eigenform. Then*

$$\langle f, \mathcal{F}_D \rangle = \frac{1}{2} \frac{(2k-2)!}{(4\pi)^{2k-1}} \frac{L(1-k, \chi_D)}{L(k, \chi_D)} L(f, 2k-1) L(f, \chi_D, k).$$

Here, $L(f, \chi_D, k)$ is $L(f, \chi_D, \cdot)$ evaluated at the critical line.

Define

$$\mathcal{G}_D(z) := \left(\frac{3}{2} \left(1 - \left(\frac{D}{2} \right) 2^{-k} \right) \right)^{-1} \mathrm{Tr}_{\Gamma_0(4D) \to \Gamma_0(4)}(G_{k,4D}(z)) \Theta(|D|z).$$

Note that $G_{k,4D}(z)\Theta(|D|z) \in M_{k+1/2}(\Gamma_0(4D))$ and $\mathcal{G}_D(z) \in M^+_{k+1/2}(\Gamma_0(4))$.

Proposition. *Suppose $g \in S^+_{k+1/2}(\Gamma_0(4))$, $g = \sum c(n) q^n$. Then*

$$\langle f, \mathcal{G}_D \rangle = \frac{1}{4} \frac{\Gamma(k-1/2)}{(4\pi)^{k-1/2}} \frac{L(1-k, \chi_D)}{L(k, \chi_D)} |D|^{k-1/2} L(f, 2k-1) c(|D|).$$

Proposition. *For $\mathcal{S}_D^+ : M^+_{k+1/2}(\Gamma_0(4)) \longrightarrow M_{2k}(\Gamma_0(1))$, we have*

$$\mathcal{S}_D^+(\mathcal{G}_D) = \mathcal{F}_D.$$

Using these propositions, the theorem is proved as follows.

Proof. $S^+_{k+1/2}(\Gamma_0(4))$ has an orthogonal basis $\{g_\nu\}$ consisting of Hecke eigenforms so that

$$\{f_\nu : f_\nu = \mathrm{Sh}(g_\nu)\}$$

is a basis of normalized orthogonal Hecke eigenforms of $S_{2k}(\Gamma_0(1))$.

Now, $M^+_{k+1/2}(\Gamma_0(4)) = S^+_{k+1/2} \oplus \mathbb{C} H^+_{k+1/2}$ where $H^+_{k+1/2}$ is an Eisenstein series. Also,

$$\mathcal{G}_D = \lambda H^+_{k+1/2} + \sum_\nu \lambda_\nu g_\nu$$

since $\langle \mathcal{G}_D, g_\nu \rangle = \lambda_\nu \langle g_\nu, g_\nu \rangle$. Moreover,

$$\mathcal{S}_D^+(g_\nu) = c_\nu(|D|) f_\nu$$

where $c_\nu(|D|)$ is the $|D|$-th Fourier coefficient of g_ν. Thus, applying $\mathcal{S}_D^+$,

$$\mathcal{F}_D = \lambda L(1-k, \chi_D) G_{2k}(z) + \sum_\nu \lambda_\nu c_\nu(|D|) f_\nu$$

and so

$$\begin{aligned} \langle \mathcal{F}_\nu, \mathcal{F}_D \rangle &= c_\nu(|D|) \langle f_\nu, f_\nu \rangle \lambda_\nu \\ &= c_\nu(|D|) \langle f_\nu, f_\nu \rangle \frac{\langle \mathcal{G}_D, g_\nu \rangle}{\langle g_\nu, g_\nu \rangle}. \end{aligned}$$

Now apply the first two propositions:

$$L(f_\nu, \chi_D, k) \cdots = c_\nu(|D|)^2 |D|^{k-1/2} \cdots .$$

□