Subgroups of $\text{SL}_2(\mathbb{Z})$

We have $\text{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$. Define the following subgroups:

- $\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$
- $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$
- $\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}$.

The action of $\gamma \in \text{SL}_2(\mathbb{Z})$ on $z \in \mathbb{C}$ is given by

$$\gamma z := \frac{az + b}{cz + d}.$$

**Proposition.** $\Gamma_0(4)$ is generated by

$$-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}.$$

Also, if $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $ST^4S^{-1} = ST^4S = R$.

The map

$$\chi : \Gamma_0(4) \rightarrow (\mathbb{Z}/4)^\times \rightarrow \{\pm 1\}$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \mod 4 \mapsto \chi(\gamma)$$

defines the character of $\gamma$. Here, $\varepsilon$ is a homomorphism.

**Remark.** $\chi(T) = 1$, $\chi(-I) = -1$, $\chi(R) = -1$.

**Definition (Congruent Subgroup).** A congruent subgroup is a subgroup of $\text{SL}_2(\mathbb{Z})$ which contains $\Gamma(N)$ for some $N \geq 1$. 

Three Basic Functions

Theta Function

Let \( \theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} \), \( 0 < t \in \mathbb{R} \).

Proposition. \( \theta(t) = \frac{1}{\sqrt{t}} \theta(\frac{1}{t}) \).

Proof. Apply Poisson Summation, i.e. \( \sum f(n) = \sum \hat{f}(n) \), to \( f(t) = e^{-\pi tx^2} \). Then \( \hat{f}(y) = \frac{1}{\sqrt{t}} f(t)\left(\frac{y}{\sqrt{t}}\right) \) and

\[
\sum e^{-\pi n^2} = \sum f(t(n)) = \frac{1}{\sqrt{t}} \sum \hat{f}(\frac{n}{\sqrt{t}}) = \frac{1}{\sqrt{t}} \theta(\frac{1}{t}).
\]

Proposition. \( \Phi(s) = \int_{1}^{\infty} t^{s/2} (\theta(t) - 1) \frac{dt}{t} + \int_{0}^{1} t^{s/2} (\theta(t) - \frac{1}{\sqrt{t}}) \frac{dt}{t} \), then \( \Phi(s) = \Phi(1 - s) \) and so

\[
\Lambda(s) = \pi^{-s/2} \Gamma(\frac{s}{2})\zeta(s) = \Lambda(1 - s)
\]

where \( \Gamma(s) = \int_{0}^{\infty} e^{-u} u^{s-1} du \).

With the definition of \( \theta \), put \( z = it \in \mathbb{H} \) and obtain the Theta function.

Definition (Theta Function). Define

\[
\Theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi in^2z}, \Im(z) > 0.
\]

Proposition.

1. \( \Theta(z) = \frac{1}{\sqrt{-2iz}} \Theta\left(\frac{\zeta}{z}\right) \),

2. \( \Theta(-\frac{1}{4z}) = (\frac{e}{\pi}) \frac{1}{\sqrt{z}} \Theta(z) \), and

3. \( \Theta^2(-\frac{1}{4z}) = \frac{2e}{\pi} \Theta^2(z) \).

Proposition. \( \Theta^2 \) satisfies

\[
\Theta^2(\gamma z) = \chi(\gamma) j(\gamma, z) \Theta^2(z)
\]

where \( j(\gamma, z) = (\det \gamma)^{-1/2} (cz + d) \).

Proposition. \( \Theta \) transforms as follows. For \( \gamma \in \Gamma_0(4) \),

\[
\Theta(\gamma z) = \left(\frac{\zeta}{\delta}\right) \Theta^{-1}(cz + d)^{1/2} \Theta(z)
\]

where \( z \) has argument in \((-\pi/2, \pi/2] \), \( \left(\frac{\zeta}{\delta}\right) \) is the extended Jacobi symbol and \( \varepsilon_d = \begin{cases} 1 & d \equiv 1 \pmod{4} \\ i & d \equiv -1 \pmod{4} \end{cases} \).
Eisenstein Series

Let

$$G_k(z) = \sum_{(m,n) \neq (0,0)} \frac{1}{(mz+n)^k}.$$ 

This converges absolutely for $k > 2$.

**Proposition.** For all $\gamma \in \text{SL}_2 \mathbb{Z}$,

$$G_k(\gamma z) = j(\gamma, z)^k G_k(z).$$

We normalize $G_k(z)$. Let

$$E_k(z) := \frac{1}{2\zeta(k)} G_k(z), \quad k > 2, \text{ even}.$$ 

Setting $k = 2$, we get a conditionally convergent series $E_2(z)$. Note that

$$E_2(z) = \frac{1}{2\zeta(2)} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^2}. $$

**Proposition.**

$$E_2(z) = \frac{1}{\pi^2} E_2\left(-\frac{1}{z}\right) + \frac{6i\pi}{\pi^2}.$$ 

To deal with convergence issues, set

$$\tilde{E}_2(z) := \frac{1}{\pi^2} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^2(mz+n-1)}.$$ 

**Proposition.** $\tilde{E}_2(z) = E_2(z) + \frac{1}{\pi^2} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \left(\frac{1}{mz+n} - \frac{1}{mz+n-1}\right)$ is absolutely convergent and converges to $E_2(z)$.

Eta Function

**Definition (Eta Function).** For $z \in \mathbb{H}$, let

$$\eta(z) := e^{2\pi i z/24} \prod_{n=1}^{\infty} \left(1 - e^{2\pi inz}\right).$$

**Proposition.** $\eta\left(-\frac{1}{z}\right) = \left(\frac{z}{\pi}\right)^{1/2} \eta(z)$.

Relations of the Three Functions

**Proposition.** $\Theta(z) = \frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}$.

Let $1 \leq N \in \mathbb{Z}$. Consider $(a_1, a_2) \in (\mathbb{Z}/N\mathbb{Z})^2$. Let $3 \leq k \in \mathbb{Z}$. Set

$$G_k^{(a_1, a_2)}(z) := \sum_{\substack{m_1 \equiv a_1 \pmod{N} \\ m_2 \equiv a_2 \pmod{N}}} \frac{1}{(m_1 z + m_2)^k}$$

and

$$h(z) := \prod_{a_2} G_3^{(0, a_2)}(z).$$
Proposition. $h(z)$ is a constant multiple of \( \left( \frac{\eta^p(z)}{\eta(pz)} \right)^6 \).

Proposition. Let $p$ be an odd prime. Set $\phi(z) = \Theta(\frac{z}{2})$ and $\psi(z) = \frac{\eta^p(z)}{\eta(pz)}$. Then
\[
\frac{\phi(pz)}{\phi(z)} = \frac{\psi(z)}{\psi^2(z + 1/2)}.
\]

Modular Forms of Integral and Half-Integral Weight

Cusps

Definition (Cusp). A cusp $z \in \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ is an element which is fixed by a parabolic element of $\Gamma$ ($\gamma \in \Gamma$ is parabolic if $|\text{tr}(\gamma)| = 2$).

Example (Cusps of $\text{SL}_2(\mathbb{Z})$). $i\infty$ is the only cusp of $\text{SL}_2(\mathbb{Z})$.

This is because $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ fixes $i\infty$. If $\frac{a}{b} \in \mathbb{Q}$, $(a,b) = 1$, $c \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ fixes it.

In general, if $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ is of finite index, then there are only a finite number of $\Gamma$-equivalent cusps.

Example (Cusps of $\Gamma_0(4)$). Note that $\Gamma_0(4) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{4} \right\}$

$\cong \Gamma(2) := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) : \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \pmod{4} \right\}$

There are three $\Gamma_0(4)$-inequivalent cusps.

- $i\infty$. If $4 \mid c$ and $(a,c) = 1$, then $\frac{a}{c}$ is $\Gamma_0(4)$-equivalent to $i\infty$.
- $\frac{1}{2}$. If $2 \parallel n$ and $(m,n) = 1$, then $\frac{m}{n}$ is $\Gamma_0(4)$-equivalent to $\frac{1}{2}$.
- $0$. If $n$ is odd and $(m,n) = 1$, then $\frac{m}{n}$ is $\Gamma_0(4)$-equivalent to $0$.

Example (Cusps of $\Gamma_0(p)$). The cusps of $\Gamma_0(p)$, $p$ prime, are $\{0, i\infty\}$.

Modular Forms of Integral Weight

Definition. Let $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$. A modular form $f$ of weight $k \in \mathbb{Z}$ is a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ such that:

1. $f(\gamma z) = (cz+d)^k f(z)$ \quad $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and

2. $f$ is holomorphic at all cusps of $\Gamma$.

Proposition. $G_k^{(a_1,a_2)} \in M_k(\Gamma_1(N))$.

Proposition. $h(z) \in M_{3(p-1)}(\Gamma_1(p))$. 
Metaplectic Group

Consider the extensions

\[ 1 \to K \to G \to H \to 1. \]

If we can lift \( H \) to a subgroup of \( G \), then \( G \) is the semi-direct product of \( H \) and \( K \), and it is considered the trivial extension.

We want to consider the non-trivial element.

Consider a degree 4 extension of \( \text{GL}_2^+ (\mathbb{Q}) \)

\[ 1 \to \mu_4 \to \tilde{G} \to \text{GL}_2^+ (\mathbb{Q}) \to 1 \]

where \( \mu_4 = \{ \pm 1, \pm i \} \). Here,

\[ \tilde{G} = \left\{ (\alpha, \phi) : \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+ (\mathbb{Q}), \phi \text{ holomorphic function on } \mathbb{H} \text{ s.t. } \phi^2(z) = t \frac{cz + d}{\sqrt{\det \alpha}}, t = \pm 1 \right\} \]

**Proposition.** \( \tilde{G} \) is a group under the operation \((\alpha, \phi)(\beta, \psi) = (\alpha \beta, z \mapsto \phi(\beta z) \psi(z))\).

**Remark.** \( \tilde{G} \) is not an algebraic group.

**Example.** \( j(\gamma, z) = \left( \frac{c}{d} \right) \varepsilon_d^{-1} (cz + d)^{1/2}, \gamma \in \Gamma_0(4) \)

is an example of \( \phi \) since \( j^2(\gamma, z) = (\frac{c}{d})^2 \varepsilon_d^{-2} (cz + d) = 1(\pm 1)(cz + d) \).

**Modular Forms of Half-Integral Weight**

Let \( f \) be a function on \( \mathbb{H} \). Let \( \xi = (\alpha, \phi) \in \tilde{G} \). Set

\[ (f|_k \xi)(z) = f(\alpha z) \phi^{-k}(z), k \in \frac{1}{2} \mathbb{Z}. \]

\( \tilde{G} \) acts on the space of functions in this way.

**Definition.** Let \( k \in \frac{1}{2} \mathbb{Z} \). A modular form \( f \) of weight \( k \) for \( \Gamma \subseteq \text{SL}_2(\mathbb{Z}) \) is a holomorphic function \( f : \mathbb{H} \to \mathbb{C} \) such that:

1. \( f|_k \gamma = f \) for all \( \gamma \in \Gamma \), and
2. \( f \) is holomorphic at all cusps of \( \Gamma \).

**Example.** \( \eta \) is a form of weight \( \frac{1}{2} \) for \( \text{SL}_2(\mathbb{Z}) \). \( \Delta := \eta^{24} \) is a form of weight 12.

**Example.** Let \( \Gamma \subseteq \Gamma_0(4) \). Let \( \Gamma_\infty = \{ \gamma \in \Gamma : \gamma(i\infty) = i\infty \} \). Then \( \Gamma_\infty \) is an infinite subgroup of \( \text{SL}_2(\mathbb{Z}) \) generated by \( \left( \begin{array}{cc} 1 & h \\ 0 & 1 \end{array} \right) \).

Consider

\[ E_{k/2}(z) := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(4)} j(\gamma, z)^{-k}, \quad j(\gamma, z) = \left( \frac{c}{d} \right) \varepsilon_d^{-1} (cz + d)^{1/2}, 5 \leq k \text{ odd} \]

where \( \Gamma_\infty \setminus \Gamma_0(4) \) have coset representatives \( \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) : 4|m, (m, n) = 1, n > 0 \right\} \). Then \( E_{k/2}(z) \) is a modular form of weight \( \frac{k}{2} \) for \( \Gamma_0(4) \). Its Fourier coefficients are Dirichlet \( L \)-function values!
The Shimura Correspondence

To define the Shimura correspondence, we first need to discuss Hecke operators.

Hecke Operators

Let $\Gamma$ be a congruent subgroup of $\text{SL}_2(\mathbb{Z})$. Let $f \in M_k(\Gamma), 0 \leq k \in \mathbb{Z}$.

Suppose $\alpha \in \text{GL}_2^+(\mathbb{Z})$. We may consider

$$(f|_k\alpha)(z) = (cz + d)^{-k}(\det \alpha)^{k/2} f(\alpha z).$$

If $\alpha \in \Gamma$, then $f|_k\alpha = f$.

We have that $f|_k\alpha \in M_k(\Gamma')$ where $\Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$, since $(f|\alpha)|\alpha^{-1}\gamma\alpha = f|\gamma\alpha = f|\alpha$.

The groups $\Gamma$ and $\alpha^{-1}\Gamma\alpha$ are commensurable, i.e $\Gamma \cap \alpha^{-1}\Gamma\alpha$ is of finite index in both $\Gamma$ and $\alpha^{-1}\Gamma\alpha$.

Suppose $\lambda := [\Gamma : \Gamma']$. We may decompose $\Gamma = \bigcup_{i=1}^{\lambda} \Gamma\gamma_i$. This gives a decomposition $\Gamma'\alpha\Gamma = \bigcup_{i=1}^{\lambda} \Gamma'\alpha_i\Gamma_i'$.

We can define the action of the double cosets. Define

$$f|(\Gamma'\alpha\Gamma) := \sum_{i=1}^{\lambda} f|\alpha_i\gamma_i'.$$

This is well-defined. In fact, $f|(\Gamma'\alpha\Gamma) \in M_k(\Gamma)$.

**Definition.** Suppose $(n, N) = 1$, $\Gamma = \Gamma_0(N)$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Define

$$T_n(f) := n^{k/2-1} \sum_{\alpha \in \text{M}_2(\mathbb{Z})} f|(\Gamma'\alpha\Gamma).$$

**Lemma.** $T_n : M_k(\Gamma) \rightarrow M_k(\Gamma)$ is a linear map.

The Integral Weight Case

Consider the integral weight case.

**Definition.** For $(n, N) = 1$, $f \in M_k(\Gamma_1(N))$, define

$$T_n(f) := n^{k/2-1} \sum_{\alpha \in \Delta^n} f|(\Gamma_1(N)\alpha\Gamma_1(N))$$

where $\alpha$ ranges over the set of double coset representatives in

$$\Delta^n := \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{M}_2(\mathbb{Z}) : \det \alpha = n, \alpha \equiv \begin{pmatrix} 1 & * \\ 0 & n \end{pmatrix} \pmod{N} \right\}.$$  

Note that $\Gamma_1(N)$ acts on $\Delta^n$ from the left and the right. Thus $\Delta^n$ can be expressed as the disjoint union of $\Gamma_1(N)$ double cosets.

What does $T_p$ do to Fourier coefficients? Write $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i nz}$. Then $T_p f \in M_k(\Gamma_1(N))$ and $T_p f(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i nz}$ with

$$b_m = a_{pm} + p^{k-1} a_{m/p}$$

where $a_{m/p} = 0$ if $p \nmid m$.  

Lemma. If \( f \in M_k(\Gamma_0(N), \chi) \) is an eigenform for all the \( T_m \), say \( T_m f = \lambda_m f \), then
\[
a_n = \lambda_n a_1
\]
where \( f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \).

The Half-Integral Weight Case

For the half-integral weight case, we can define \( T_n \) by analogy. Let \( \tilde{\Gamma} = \Gamma_1(N) \); the double cosets are \( \tilde{\Gamma}_n \Gamma \).

Set \( \xi_n = \left( \begin{array}{cc} 1 & 0 \\ 0 & n \end{array} \right) \).

Proposition (Shimura). If \( (n, N) = 1 \), \( 4 \mid N \), \( k \) odd, \( f \in M_{k/2}(\tilde{\Gamma}) \), then
\[
f|_{\tilde{\Gamma}_n \tilde{\Gamma}} = 0
\]
unless \( n \) is a square.

Corollary (Shimura). \( T_n f = 0 \) unless \( n \) is a square.

Theorem. Let \( 4 \mid N \), \( \chi \) is a character modulo \( N \). Suppose \( f \in S_{k/2}(\tilde{\Gamma}_0(N), \chi) \) (i.e. a cusp form) with \( 0 < k \in \mathbb{Z} \) odd. Suppose \( f \) is an eigenform for \( \{ T_p : p \nmid N \} \), i.e. \( T_p^2 f = \lambda_p f \). Define
\[
g(z) := \sum_{n=1}^{\infty} c_n e^{2\pi i n z}
\]
where \( c_1 = 1 \), \( c_p = \lambda_p \), \( c_{p+1} = c_p c_{p^i} - \chi(p) p^{k-2} c_{p^i-1} \), \( c_{mn} = c_m c_n \) if \( (m, n) = 1 \).

Then
\[
g \in M_{k-1} \left( \Gamma_0 \left( \frac{N}{2} \right), \chi^2 \right).
\]
If \( k \geq 5 \), then
\[
g \in S_{k-1} \left( \Gamma_0 \left( \frac{N}{2} \right), \chi^2 \right).
\]

Thus, if there is a basis for \( S_{k/2}(\tilde{\Gamma}_0(N), \chi) \) consisting of Hecke eigenforms, this defines the Shimura map
\[
S_{k/2} \left( \Gamma_0(N), \chi \right) \longrightarrow S_{k-1} \left( \Gamma_0 \left( \frac{N}{2} \right), \chi^2 \right).
\]

Köhnen Subspace

We wish to define subspaces (Köhnen subspaces) \( M_{k/2}^+ \left( \Gamma_0(N), \chi \right) \) and \( S_{k/2}^+ \left( \Gamma_0(N), \chi \right) \) consisting of \( f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} \) with \( a_n = 0 \) unless \( (-1)^{\frac{k-1}{2}} n \equiv 0, 1 \pmod{4} \).

Petersson Inner Product

Definition. Let \( f, g \in M_k(\Gamma_0(N), \chi) \). Define
\[
\langle f, g \rangle := \frac{1}{[SL_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N) \backslash \mathbb{H}} f(z) \overline{g(z)} y^k \frac{dx dy}{y^2}
\]
assuming at least one of \( f \) or \( g \) is a cusp form so that the integral converges.
This inner product is Hermitian with respect to the Hecke operators, i.e.
\[ \langle f|T_n,g \rangle = \chi(n)\langle f,g|T_n \rangle \]
and
\[ \langle g,f \rangle = \langle f,g \rangle. \]

**Proposition.** There is a basis of \( S_k(\Gamma_0(N),\chi) \) consisting of eigenforms for all the \( T_n, (n,N) = 1 \).

**Lemma.** \( S_k(\Gamma_0(N)) \) has a \( \mathbb{Q} \)-structure. That is, there exists a basis of forms with rational Fourier coefficients, i.e. \( S_k(\Gamma_0(N))_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = S_k(\Gamma_0(N)) \). Thus, the Hecke operators are defined over \( \mathbb{Q} \), i.e. they act on this space.

Consider the characteristic polynomial which has coefficients in \( \mathbb{Q} \).

**Corollary.** The Fourier coefficients of normalized Hecke eigenforms are algebraic. In fact, they are algebraic integers. Moreover, they live in a number field.

**Remark.** This is not known in the half-integral weight case.

**L-Functions**

We can define the L-function in two ways: Mellin transform and Hecke operators.

**Mellin Transform**

Let \( f \in S_k(\Gamma_0(N),\chi) \). Consider
\[ I := \int_0^\infty f(iy)y^s dy. \]
If \( f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i nz} \), then
\[ I = \sum_{n=1}^{\infty} a_n \int_0^\infty e^{-2\pi i ny} f(iy)y^s dy. \]
Letting \( u = 2\pi ny \), we get
\[ I = (2\pi)^{-s} \Gamma(s) \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \]

**Definition.** Define
\[ L(f,s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \]

Now suppose that \( f|w_N = \varepsilon_N f \) where \( w_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \). Then
\[ \left( \frac{\sqrt{N}}{2\pi} \right)^s \Gamma(s)L(f,s) = N^{s/2} \int_0^{1/\sqrt{N}} f(iy)y^s dy = N^{s/2} \left( \int_0^{1/\sqrt{N}} \varepsilon_N^{-1} N^{-k/2} i^{-k} y^{-k} f \left( \frac{i}{N} \right) y \frac{dy}{y} \right) \]
\[ + \int_{1/\sqrt{N}}^{\infty} \varepsilon_N^{-1} N^{-k/2} i^{-k} y^{-k} f \left( \frac{i}{N} \right) y \frac{dy}{y} \]
Setting \( u = \frac{1}{N} \) we get

\[
\left( \frac{\sqrt{N}}{2\pi} \right)^s \Gamma(s)L(f, s) = \int_{1/\sqrt{N}}^\infty f(iu) \left( N^{k/2} e^{-1} i^{-k} N^{-s/2} u^{k-s} + N^{s/2} u^s \right) \frac{du}{u}
\]

If \( \varepsilon_N = i^{-k} \), then \( N^{k/2} e^{-1} i^{-k} N^{-s/2} u^{k-s} + N^{s/2} u^s = N^{k/2} u^{k-s} + N^{s/2} u^s \). This is invariant under \( s \mapsto k - s \).

This gives the functional equation

\[
\left( \frac{\sqrt{N}}{2\pi} \right)^s \Gamma(s)L(f, s) = \varepsilon_N^k = \left( \frac{\sqrt{N}}{2\pi} \right)^s \Gamma(k-s)L(f, k-s).
\]

**Hecke Operators**

**Definition.** Assume \( T_p f = \lambda_p f \) if \( p \nmid N \) and \( U_p f = \mu_p f \) if \( p \mid N \). Define

\[
L(f, s) := \prod_{p \mid N} \left( 1 - \frac{\mu_p}{p^s} \right)^{-1} \prod_{p \nmid N} \left( 1 - \lambda_p p^{-s} + \chi(p) p^{k-1-2s} \right).
\]

If \( f \) is a normalized Hecke eigenform (including for \( w_N \), then it is the same as the L-function defined by the Mellin transform. Moreover, there is a "multiplicity one" theorem.

**Constant Term Function**

**Definition.** Suppose \( F : \mathbb{H} \rightarrow \mathbb{C} \) is \( \Gamma \subseteq \text{SL}_2(\mathbb{Z}) \) invariant. Define

\[
C(F, y) = \int_0^1 F(x + iy) dx \quad \text{for} \quad y > 0.
\]

If we use the Mellin transform and set

\[
L(F, s) := \int_0^\infty C(F, y) y^s \frac{dy}{y^2} = \int_0^\infty \int_0^\infty F(x + iy) y^s \frac{dxdy}{y^2}
\]

We have the action of \( \Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix} \right\} \) on \( \mathbb{H} \) and we may consider the half strip of integration to be \( \Gamma_\infty \backslash \mathbb{H} \). Also, \( \Gamma_\infty \subset \Gamma \subset \text{SL}_2(\mathbb{Z}) \). Using this, we have

\[
L(F, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma, \text{tr} \gamma = 0} \int_{\gamma(\mathbb{H})} F(z) y^s \frac{dxdy}{y^2}
\]

\[
= \int_{\Gamma \backslash \mathbb{H}} F(z) \left( \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left( \text{Im}(\gamma z) \right)^s \right) \frac{dxdy}{y^2}
\]
If we define $E(z, s) = \sum_{\gamma \in \Gamma} \left( \frac{y}{|cz + d|^2} \right)^s$, this is the Eisenstein series.

Consider the case $f \in S_k(\Gamma_0(N))$. Let $F(z) = (\text{Im} z)^k |f(z)|^2$. If we write $F(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$, then

$$C(F, y) = y^k \sum_n |a_n|^2 e^{-4\pi n y}$$

and

$$L(F, s) = (4\pi)^{-s-k+1} \Gamma(s+k-1) \sum_n \frac{|a_n|^2}{n^{s+k-1}}.$$ 

Notice that

$$L(F, s) = \int_{\Gamma \backslash \mathbb{H}} F(z) E(z, s) \frac{dx dy}{y^2}.$$ 

Thus, if we have an analytic continuation of $E(z, s)$, then we deduce an analytic continuation for $L(F, s)$.

**Remark.** Suppose we drop the $\Gamma$-invariance and consider instead an $F$ satisfying $F(\gamma z) = j(\gamma z)^k F(z)$ with $\gamma \in \Gamma \in \Gamma_0(4)$. We can still define

$$C(F, y) = \int_0^1 F(x + iy) dx \text{ for } y > 0.$$ 

The Mellin transform is

$$L(F, s) = \int_0^\infty C(F, y) y^{s} \frac{dy}{y^2}$$

$$= \sum_{\gamma \in \Gamma} \int_{\Gamma \backslash \mathbb{H}} F(\gamma z) (\text{Im}(\gamma z))^s \frac{dx dy}{y^2}$$

$$= \int_{\Gamma \backslash \mathbb{H}} F(z) \left( \sum_{\gamma \in \Gamma} j(\gamma z)^k \frac{(\text{Im}(\gamma z))^s}{|cz + d|^2} \right) \frac{dx dy}{y^2}$$

$$= \int_{\Gamma \backslash \mathbb{H}} F(Z) E^*(z, s) y^s \frac{dx dy}{y^2}.$$ 

In the previous case we had

$$L(F, s) = \int_{\Gamma \backslash \mathbb{H}} F(Z) E(z, s) \frac{dx dy}{y^2}$$

where $E(z, s) = \sum_{\gamma \in \Gamma} (\text{Im}(\gamma z))^s = \sum_{(c,d) \neq (0,0)} \frac{y^s}{|cz + d|^2}$.

To analytically continue all $L$, we only need to continue $E$. To continue $E$, continue its Fourier expansion.

Consider the Fourier expansion of $E(z, s)$:

$$E(z, s) = \sum_{r \in \mathbb{Z}} a_r(y, s) e^{-2\pi i r x} dx.$$ 

By definition,

$$a_r(y, s) = \int_0^1 E(x + iy) e^{-2\pi i r x} dx.$$
Normalize $E$ and write $E(z, s) = \frac{1}{2} \pi^{-s} \Gamma(s) \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} (\text{Im}(\gamma z))^s$. Then

$$a_r(y, s) = \frac{1}{2} \pi^{-s} \Gamma(s) \int_0^1 \sum_{(c, d) \neq (0, 0)} \frac{y^s}{|cz + d|^{2s}} e^{-2\pi i r y} \, dx$$

$$= \pi^{-s} \Gamma(s) y^s \sum_{c=1}^{\infty} \sum_{d \in \mathbb{Z}} \int_0^1 (cx + d)^2 + c^2 y^2 \, e^{-2\pi i r y} \, dx.$$ 

Writing $d = cq + \rho$, we get $(cx + d)^2 + c^2 y^2 = (c(x + q) + \rho)^2 + c^2 y^2$, and the double sum becomes

$$\sum_{c=1}^{\infty} \sum_{d=0}^{\infty} \int_{\text{infty}} (c(x + q) + \rho)^2 + c^2 y^2 \, e^{-2\pi i r y} \, dx.$$ 

Now set $u = x + \frac{\rho}{c}$. Then $(cx + \rho)^2 + c^2 y^2 = c^2(u^2 + y^2)$. This makes the double sum become

$$\sum_{c=1}^{\infty} c^{-2s} \int_{\text{infty}} (u^2 + q^2)^{-s} e^{-2\pi i ru} \left( \sum_{p \mod c} e^{2\pi i ru/p} \right) du = \left( \sum_{c|\rho} c^{1-2s} \right) \int_{\text{infty}} (u^2 + y^2)^{-s} e^{-2\pi i ru} du.$$

Set $\sigma_\alpha(r) = \sum_{c|\rho} c^\alpha$.

The term with $r = 0$ gives

$$\pi^{-s} \Gamma(s) y^s \left( \zeta(2s - 1) \int_{\text{infty}} (u^2 + y^2)^{-s} du \right) = \cdots = \pi^{-s + 1/2} \Gamma(s - 1/2) y^{1-s} \zeta(2s - 1).$$

Thus, the constant term is

$$a_0(y, s) = \pi^{-s} \Gamma(s) \zeta(2s) y^s + \pi^{1/2-s} \Gamma(s - 1/2) y^{1-s} \zeta(2s - 1)$$

$$= \pi^{-s} \Gamma(s) \zeta(2s) y^s + \pi^{-(1-s)} \Gamma(1-s) \zeta(2-2s) y^{1-s}.$$ 

Therefore $a_0(y, s) = a_0(0, 1 - s)$, and we have analytic continuation of the constant term.

In general,

$$a_r(y, s) = 2\sigma_{1-2s}(|r|) \Gamma(|r|)^{s-1/2} \sqrt{\pi} k_{s-1/2}(2\pi|r| y)$$

where $k_s(y) := \frac{1}{2} \int_0^{\infty} e^{-yt^{1/2}} t^{s} \, dt$ is the Bessel function. Note that $k_s(y) = k_{-s}(y)$ and has exponential decay in $y$. Thus $a_r(y, s)$ has exponential decay in $|r|$ and $a_r(y, 1 - s) = a_r(y, s)$.

This gives the analytic continuation and functional equation of $E(z, s)$. Note:

$$E(z, s) = \underbrace{a_0(y, s)}_{\text{pole at } s=0, 1, \text{ residue } 0} + \sum_{r \neq 0} a_r(y, s) e^{2\pi i r y}.$$ 

Thus this gives the analytic continuation and functional equation of $L(F, s) = \int_{\Gamma \setminus \mathbb{H}} F(z) E(z, s) \frac{dz}{y^s}$. It has a pole at $s = 0, 1$ and its residue $\text{Res}(L(F, s), s = 1) = \text{Res}(E(z, s), s = 1) \int_{\Gamma \setminus \mathbb{H}} F(z) \frac{dz}{y^s}$. 

11
Kohnen Subspace

We have
\[ S_{k+1/2}(\Gamma_0(4)) \rightarrow S_{2k}(\Gamma_0(2)) \quad \text{and} \quad M_{k+1/2}(\Gamma_0(4)) \rightarrow M_{2k}(\Gamma_0(2)). \]

Niwa proved, under the Shimura Correspondence, forms of level \(4N\) go to forms of level \(2N\). The map is Hecke equivalent and an isomorphism:
\[ S_{k+1/2}(\Gamma_0(4N)) \xrightarrow{\cong} S_{2k}(\Gamma_0(2N)). \]

Kohnen defines a subspace
\[ S_{k+1/2}^+(\Gamma_0(4N)) \rightarrow S_{2k}(\Gamma_0(N)). \]

For \(N = 1\), this is an isomorphism.

**Theorem.** (Kohnen)

1. \(S_{k+1/2}^+(4)\) (and \(M_{k+1/2}^+(4)\)) is preserved by Hecke operators \(T_{p^2}\).

2. There is a basis of Hecke eigenforms.

3. If \(f \in S_{k+1/2}^+(4)\) is such an eigenform, say \(T_{p^2}(f) = \lambda_p f\), then there is an \(F \in S_{2k}(1)\) such that \(T_p(f) = \lambda_p F\).

4. If \(f(z) = \sum_{n=1}^{\infty} a_f(n)q^n\) and \(F(z) = \sum_{n=1}^{\infty} a_F(n)q^n\), then \(L(F,s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}\) satisfies
   \[ a_f(|D|)L(F,s) = \left( \sum_{n=1}^{\infty} \frac{a_f(|D|n^2)}{n^s} \right) L\left( s - k + 1, \frac{|D|}{s} \right) \]
   \[ = \sum_{n=1}^{\infty} \frac{1}{n^s} \left( \sum_{d|m=n} a_f(|D|m^2/m) \left( \frac{|D|m}{m} \right) m^{k-1} \right) \]

5. The map
   \[ S_{k+1/2}^+: S_{k+1/2}(4) \rightarrow S_{2k}(1) \]
   \[ \sum_{n=1}^{\infty} b(n)q^n \rightarrow \sum_{n=1}^{\infty} \left( \sum_{d|m=n} \left( \frac{|D|}{d} \right) d^{2k-1}b(|D|n^2/d^2) \right) q^n \]

is Hecke equivalent.

6. There exists a linear combination of the \(S_{k+1/2}^+\) that defines an isomorphism.

The proof relies heavily on explicit calculations involving \(U_4\) and \(W_4\):
\[ T_p = \begin{cases} U_p + V_p \quad (V_p f)(z) = *f(pz) \text{ and } (U_p f)(z) = * \sum_{d=0}^{p-1} f \left( \frac{z+d}{p} \right) \\ W_d \quad \text{where } d \mid N \end{cases} \]

Niwa showed \(U_4 W_4\) is a Hermitian operator on \(S_{k+1/2}(4)\). It has eigenvalues
\[ \alpha_1 = 2^k \left( \frac{2}{2k+1} \right) = \pm 2^k \]
\[ \alpha_2 = -\frac{1}{2} \alpha_1 \]
Kohnen essentially identifies
\[ S_{k+1/2}(4) |_{U_4 W_4 = \alpha_1} = S_{k+1/2}(4). \]

Now, \( L(F, s) \) satisfies the functional equation
\[ (2\pi)^{-s} \Gamma(s) L(F, s) = (2\pi)^{-(k-s)} \Gamma(k-s) L(F, k-s). \]
The centre is at \( s = k \). Also,
\[
L(F, s) = \prod_p \left(1 - \frac{\lambda_p}{p^s} + \frac{p^{2k-1}}{p^{s+k}}\right)^{-1} = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1}
\]
where \( \alpha_p + \beta_p = \lambda_p \), \( \alpha_p \beta_p = p^{2k-1} \), and \( |\alpha_p| = |\beta_p| = p^{k-1/2} \). Hence the Euler product converges absolutely for \( \text{Re}(s) > k - \frac{1}{2} + 1 = k + \frac{1}{2} \). Moreover, it is non-zero in the half plane \( \text{Re}(s) > k + \frac{1}{2} \).

**Theorem.** (Kohnen) Consider
\[ S_1^+ : S_{k+1/2}(4) \rightarrow S_{2k}(1) \quad f \mapsto F. \]
The image of \( S_1^+ \) is the space generated by \( \{ F \in S_{2k}(1) : L(F, k) \neq 0 \} \).

**Corollary.** For \( k = 6 \), we have \( S_{12}(1) \cong \mathbb{C} \Lambda \) and \( L(\Lambda, 6) \neq 0 \). Also, \( S_{13/2}(\Gamma_0(4)) \cong S_{12}(1) \).

**Waldspurger’s Formula**

The Waldspurger’s formula relates Fourier coefficients of modular forms of weight \( \frac{1}{2} \) to special values of L-function of (twisted) modular forms of integral weight. In the case when dealing with level 1 (i.e. forms for \( \Gamma_0(4) \)), the relation can be proven by explicit calculations (Kohen-Zagier).

Recall that we have a map
\[
\text{Sh} : S_{k+1/2}^+(\Gamma_0(4)) \rightarrow S_{2k}(\text{SL}_2(\mathbb{Z}))
\]
\[ g(z) = \sum_{n=1}^{\infty} a_g(q^n) \mapsto f(z) = \sum_{n=1}^{\infty} a_f(q^n) \]
We have \( g | T_{k+1/2}(p^2) = f | T_{2k}(p) \), and for any fundamental discriminant \( D \) with \((-1)^k D > 0\) we have
\[ a_g(n^2 |D|) = a_g(|D|) \sum_{d | n} \mu(d) \left(\frac{D}{d}\right) d^{k-1} a_f \left(\frac{n}{d}\right). \]
Equivalently,
\[
\sum_{n=1}^{\infty} \frac{a_g(n^2 |D|)}{n^s} = a_g(|D|) \left(\sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}\right) \frac{1}{L(s-k+1, (\frac{D}{|D|})}.
\]

**Theorem.** Let \( g \in S_{k+1/2}^+(\Gamma_0(4)) \) and \( f = \text{Sh}(g) \). Let \( D \) be a fundamental discriminant with \((-1)^k D > 0\), then
\[
\frac{a_g(|D|)^2}{\langle g, g \rangle} = \frac{(k-1)!}{\pi^k} |D|^{k-1/2} L(f, (\frac{D}{|D|}), k) \frac{\langle f, f \rangle}{\langle g, f \rangle}.
\]
Remark. This amazing formula still leaves open the following two questions:

1. What is \(a_g(|D|)\)? Which square root? What does the sign mean?

2. Ramanujan Conjecture: For \(f \in S_k(\Gamma_0(N))\), is \(|a_f(n)| \leq d(n)n^{(k-1)/2}\) where \(d(n) = \# \text{ of divisors of } n\)?

   For example, a theorem of Delignc asserts that \(|a_f(p)| \leq 2p^{(k-1)/2}\). What about \(g \in S^*_k+1/2(\Gamma_0(N))\)?

   We might expect the analogue of the Ramanujan Conjecture, that \(|a_g(n)| \leq \alpha(n)n^{(k/2-1/4)}\) where \(\alpha(n)\) is an elementary function of \(n\).

Remark. \(a_g(n) \in \mathbb{R}\) since, in general, if \(f \in S_k(\Gamma_0(N))\) (say, a normalized eigenform \(f \mid T_n = a_f(n)f\), then

\[
 a_f(n)\langle f, f \rangle = \langle f | T_n, f \rangle = \langle f, f | T_n \rangle = \langle f, a_f(n)f \rangle = \frac{1}{a_f(n)}\langle f, f \rangle.
\]

Therefore, \(a_g(|D|)^2 \geq 0\). In particular, this implies that \(L(f, (\frac{D}{\cdot}), k) \geq 0\); this is predicted by the Riemann Hypothesis for \(L(f, (\frac{D}{\cdot}), s)\).

Suppose we assume the Riemann Hypothesis for \(L(f, (\frac{D}{\cdot}), s)\). Then for any \(\varepsilon > 0\),

\[
 |L(f, (\frac{D}{\cdot}), k + it)| \ll_\varepsilon (|D|^2N(|t| + 2)^{\varepsilon}).
\]

In other words,

\[
 |L(f, (\frac{D}{\cdot}), s)| \ll_N |D|^{\varepsilon}.
\]

From Waldspugger’s formula

\[
 \frac{a_g(|D|)^2}{(g,g)} = \frac{(k-1)!}{(2\pi)^{k+1}} \left| L(f, (\frac{D}{\cdot}), k) \right| |D|^{k-1/2},
\]

\[
 |a_g(|D|)|^2 \ll_{f,g,\varepsilon} |D|^{k-1/2+\varepsilon} \quad \text{Lindelöf}
\]

\[
 \Rightarrow |a_g(|D|)| \ll |D|^{k/2-1/4+\varepsilon}.
\]

**Idea of Proof of Waldpurger’s Formula**

Consider the Eisenstein series

\[
 G_{k,D}(z) = \frac{1}{2}L(1-k, \chi_D) + \sum_{n=1}^{\infty} \left( \sum_{d|n} \frac{D}{d} d^{k-1} \right) q^{n-1} \in M_k(\Gamma_0(D), \chi_D)
\]

and

\[
 G_{k,AD}(z) = G_{k,D}(4z) - 2^{-k} \left( \frac{D}{2} \right) G_{k,D}(2z) \in M_k(\Gamma_0(4D), \chi_D)
\]

where \(\chi_D = (\frac{D}{\cdot})\).

Define the trace operator: Let \(\Gamma_0(1) = SL_2(\mathbb{Z}) = \bigcup \Gamma_0(D)\alpha_i\). Let

\[
 \text{Tr} : M_k(\Gamma_0(D)) \rightarrow M_k(\Gamma_0(1))
\]

\[
 f \mapsto \text{Tr} := \sum f \mid \alpha_i.
\]
Note that \( \text{Tr}(f) \mid \alpha = \text{Tr}(f) \forall \alpha \in \text{SL}_2(\mathbb{Z}) \). Also, we have

\[
\langle f, g \mid \alpha \rangle = \int_{\Gamma_0(1) \backslash \mathbb{H}} f(z) g(z) \alpha(z) y^k dx \, dy \frac{y}{2} = \cdots = \langle f \mid \alpha^{-1}, g \rangle,
\]

and so

\[
\langle f, g \mid \text{Tr} \rangle = \sum \langle f, g \mid \alpha_i \rangle = \sum \ast \langle f \mid \alpha^{-1}, g \rangle = \sum \ast \langle f, g \rangle = \langle f, g \rangle.
\]

Define the projection operator: Let \( M_k^+(\Gamma_0(4)) \hookrightarrow M_k(\Gamma_0(4)) \). Let

\[
\text{pr}^+: M_k(\Gamma_0(4)) \rightarrow M_k^+(\Gamma_0(4))
\]

\[
f \mapsto \text{pr}^+(f) = \left((-1)^{(k-1)/2} 2^{-k} W - 4U_4 + \frac{1}{2}\right) f.
\]

Then \( \langle f, \text{pr}^+ g \rangle = \langle \text{pr}^+ f, g \rangle \). Also, if \( f \in M^+ \) and \( g \in M \), then \( \langle f, \text{pr}^+ g \rangle = \langle f, g \rangle \).

Define

\[
\mathcal{F}_D(z) := \text{Tr}_{\Gamma_0(4D) \rightarrow \text{SL}_2(\mathbb{Z})}(G_{k,D}(z))^2.
\]

Note that \( \mathcal{F}_D(z) \in M_{2k}(\text{SL}_2(\mathbb{Z})) \).

**Proposition.** Suppose \( f \in S_{2k}(\Gamma_0(1)) \) is a normalized eigenform. Then

\[
\langle f, \mathcal{F}_D \rangle = \frac{1}{2} \frac{(2k-2)!}{(4\pi)^{k-1}} L(1-k, \chi_D) L(f, 2k-1) L(f, \chi_D, k).
\]

Here, \( L(f, \chi_D, \cdot) \) is \( L(f, \chi_D, \cdot) \) evaluated at the critical line.

Define

\[
\mathcal{G}_D(z) := \left(\frac{3}{2} \left(1 - \frac{D}{2} \right) 2^{-k}\right)^{-1} \text{Tr}_{\Gamma_0(4D) \rightarrow \Gamma_0(4)}(G_{k,AD}(z)) \Theta(|D|z).
\]

Note that \( G_{k,AD}(z) \Theta(|D|z) \in M_{k+1/2}(\Gamma_0(4D)) \) and \( \mathcal{G}_D(z) \in M_{k+1/2}^+(\Gamma_0(4)) \).

**Proposition.** Suppose \( g \in S_{k+1/2}^+(\Gamma_0(4)) \), \( g = \sum c(n) q^n \). Then

\[
\langle f, \mathcal{G}_D \rangle = \frac{1}{4} \frac{\Gamma(k-1/2)}{(4\pi)^{k-1/2}} L(1-k, \chi_D) |D|^{k-1/2} L(f, 2k-1) c(|D|).
\]

**Proposition.** For \( S_D^+: M_{k+1/2}^+(\Gamma_0(4)) \rightarrow M_{2k}(\Gamma_0(1)) \), we have

\[
S_D^+(\mathcal{G}_D) = \mathcal{F}_D.
\]

Using these propositions, the theorem is proved as follows.

**Proof.** \( S_{k+1/2}^+(\Gamma_0(4)) \) has an orthogonal basis \( \{ g_\nu \} \) consisting of Hecke eigenforms so that

\[
\{ f_\nu : f_\nu = \text{Sh}(g_\nu) \}
\]

is a basis of normalized orthogonal Hecke eigenforms of \( S_{2k}(\Gamma_0(1)) \).

Now, \( M_{k+1/2}^+(\Gamma_0(4)) = S_{k+1/2}^+ \oplus \mathbb{C} H_{k+1/2}^+ \) where \( H_{k+1/2}^+ \) is an Eisenstein series. Also, \( \mathcal{G}_D = \lambda H_{k+1/2}^+ + \sum \lambda_\nu g_\nu \).
since ⟨G_D, g_ν⟩ = λ_ν ⟨g_ν, g_ν⟩. Moreover,

\[ S^+_D (g_ν) = c_ν(|D|) f_ν \]

where \(c_ν(|D|)\) is the \(|D|\)-th Fourier coefficient of \(g_ν\). Thus, applying \(S^+_D\),

\[ F_D = \lambda L(1 - k, \chi_D) G_{2k}(z) + \sum_ν \lambda_ν c_ν(|D|) f_ν \]

and so

\[ \langle F_ν, F_D \rangle = c_ν(|D|) \langle f_ν, f_ν \rangle \lambda_ν \]

\[ = c_ν(|D|) \langle f_ν, f_ν \rangle \frac{⟨G_D, g_ν⟩}{⟨g_ν, g_ν⟩} \]

Now apply the first two propositions:

\[ L(f_ν, \chi_D, k) \cdots = c_ν(|D|)^2 |D|^{k-1/2} \cdots . \]