Topological Spaces and Continuous Functions

**TOPOLOGICAL SPACES**

**Definition: Topology**

A topology on a set $X$ is a collection $T$ of subsets of $X$, with the following properties:

1. $\emptyset, X \in T$.
2. If $u_\alpha \in T, \alpha \in A$, then $\bigcup_{\alpha \in A} u_\alpha \in T$.
3. If $u_i \in T, i=1,\ldots,n$, then $\bigcap_{i=1}^n u_i \in T$.

The elements of $T$ are called open sets.

**Examples**

1. $T=\{\emptyset, X\}$ is an indiscrete topology.
2. $T=2^X$ = set of all subsets of $X$ is a discrete topology.

**Definition: Finer, Coarser**

$T_1$ is finer than $T_2$ if $T_2 \subset T_1$. $T_2$ is coarser than $T_1$.

**BASIS FOR A TOPOLOGY**

**Definition: Basis**

A collection of subsets $B$ of $X$ is called a basis for a topology if:

1. The union of the elements of $B$ is $X$.
2. If $x \in B_1 \cap B_2$, $B_1, B_2 \in B$, then there exists a $B_3$ of $B$ such that $x \in B_3 \subset B_1 \cap B_2$.

**Examples**

1. $B$ is the set of open intervals $(a, b)$ in $\mathbb{R}$ with $a < b$. For each $x \in \mathbb{R}$, $x \in (x - \frac{1}{2}, x + \frac{1}{2})$.
2. $B$ is the set of all open intervals $(a, b)$ in $\mathbb{R}$ where $a < b$ and $a$ and $b$ are rational numbers.
3. Let $T$ be the collection of subsets of $\mathbb{R}$ which are either empty or are the complements of finite sets. Note that $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. This topology does not have a countable basis.

**Claim**

A basis $B$ generates a topology $T$ whose elements are all possible unions of elements of $B$. That is, the topology generated by $B$ is the collection of arbitrary unions of the subsets of $B$.

**Proof:** Have to prove that if $u_1 \cup \cdots \cup u_n \in T$ then $u_1 \cap \cdots \cap u_n \in T$. By induction, it suffices to prove that if $u_1 \cup u_2 \in T$ then $u_1 \cap u_2 \in T$. If $x \in u_1 \cap u_2$, then there exists $u_1 = \bigcup \alpha \in A \cup B_1$ and $u_2 = \bigcup \beta \in B_2 \cap B_3$. Also there exists $B_3$ such that $B_3 = \bigcup \gamma \in B_3 \cap B_4$. For $y \in B_3 \cap B_4$ such that $x \in B_3 \cap B_4$, so $x \in B_{a,y}$ such that $B_{a,y} \cup B_{a,y}$. So $u_1 \cap u_2 = \bigcup \beta \in B_{a,y}$.

**Examples**

1. $S$, the standard topology on $\mathbb{R}$, is generated by the basis of open intervals $(a, b)$ where $a < b$.
2. The open sets of $F$ are complements of finite sets and $\emptyset$. 

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3. A basis for another topology on \( \mathbb{R} \) is given by half open intervals \((a,b)\) with \( a < b \). It generated the lower limit topology \( L \).
4. The open intervals \((a,b)\) with \( a \) and \( b \) rational is a countable basis. It generateds the same topology as \( S \).

Claim: \( S \) is finer than \( F \), and \( L \) is finer than \( S \).

**Proposition**
Suppose \( \beta \) and \( \beta' \) are bases for topologies \( T \) and \( T' \) on the same space \( X \). If they have the property that for every \( B \in \beta \) and \( x \in \beta \) there exists \( B' \in \beta' \) such that \( x \in B' \subseteq B \), then \( T' \) is finer than \( T \).

Proof: If \( B \in \beta \) and \( x \in \beta \), there exists \( B_x' \) such that \( x \in B_x' \subseteq B \) and \( B_x' \in \beta' \). \( B = \bigcup_{x \in B} B_x' \in T' \). So every \( B \in \beta \) is in \( T' \).

**Definition: Sub-Base**
A sub-base for a topology on \( X \) is a collection \( \beta \) of subsets on \( X \) satisfying \( \bigcup_{B \in \beta} B = X \).

We build a basis by taking all finite intersections of the elements of \( \beta \).

**The Subspace Topology**

**Relative Topology**
Given a topology \( T \) on \( X \) and a subset \( Y \) of \( X \), \( T \) induces a topology \( T_Y \) on \( Y \) called the relative topology.

\[ T_Y = \{ t \cap Y \mid t \in T \} \]

Check that \( T_Y \) is a topology:
1. \( \emptyset = \emptyset \cap Y \), \( X = X \cap Y \).
2. Let \( S_a \in T_Y \), \( S_a \cap T_a \cap Y \). So \( \bigcup S_a = (\bigcup T_a) \cap Y \).
3. \( S_1 \cap \cdots \cap S_n = (T_1 \cap Y) \cap \cdots \cap (T_n \cap Y) = (T_1 \cap \cdots \cap T_n) \cap Y \).

**Closed Sets and Limit Points**

**Definition: Closed**
A subset \( A \subset X \) a topological space is closed if \( A^c \) is open.

**Properties of Closed Sets**
1. \( \emptyset \), \( X \) are closed.
2. If \( A_\beta \in \beta \in B \) is closed, then \( \bigcap_{B \in \beta} A_\beta \) is closed.
   Proof: \( \left( \bigcap_{B \in \beta} A_\beta \right)^c = \bigcup_{B \in \beta} A_\beta^c \) is open.
3. If \( A_1, \ldots, A_n \) are closed, then \( \bigcup_{i=1}^n A_i \) is closed.

**Examples**
Consider the standard topology on \( \mathbb{R} \).
1. Let \( x \in \mathbb{R} \). \( \{ x \} \) is closed.
2. \( I = [a, b] \) is closed.

**Definition: Interior, Closure**

Let \( X \) be a topological space. Let \( A \subset X \). The interior of \( A \), denoted \( \overset{\circ}{A} \), is the largest open set in \( A \). The closure \( \overline{A} \) is the smallest closed set containing \( A \).

**Proposition**

\( x \in \overset{\circ}{A} \) if and only if there exists an open \( U \) such that \( x \in U \subset A \).

Proof:

(\( \Rightarrow \)) \( x \in \overset{\circ}{A} \), take \( U = A \).

(\( \Leftarrow \)) If \( x \in U \subset A \), \( U \) open, then \( \overset{\circ}{A} \cup U = A \) is open and contained in \( A \). So \( U \subset \overset{\circ}{A} \) and \( x \in \overset{\circ}{A} \).

**Proposition**

\( x \in \overline{A} \) if and only if for all open \( U \), \( x \in U \), \( U \cap A \neq \emptyset \).

Proof:

(\( \Rightarrow \)) If \( x \in \overline{A} \), \( x \in U \) and \( U \cap A = \emptyset \), then \( U \) is closed and contains \( A \). So \( A \cap U \) is closed and contains \( A \), but \( x \notin A \cap U \) which is smaller than \( A \).

(\( \Leftarrow \)) Now suppose \( x \in U \), \( U \cap A \neq \emptyset \). Consider \( A^c \), which is open. If \( x \in A^c \), then \( A \cap A^c \neq \emptyset \). Contradiction. So \( x \in \overline{A} \).

**Definition: Limit Point**

Let \( A \subset X \). \( x \) is a limit point of \( A \) iff every open set \( U \), \( x \in U \), intersects \( A \) in a point different from \( x \).

**Proposition**

Let \( A' \) be the set of limit points of \( A \). Then \( \overline{A} = A \cup A' \).

**Proposition**

\( f : X \to Y \) is continuous if and only if \( f(\overline{A}) \subset \overline{f(A)} \) for all \( A \subset X \).

Proof:

(\( \Rightarrow \)) Suppose \( f \) is continuous and \( x \in \overline{A} \). To prove \( f(x) \in \overline{f(A)} \), it suffices to prove that if \( V \) is open, \( f(x) \in V \), then \( V \cap f(A) \neq \emptyset \). Now \( f^{-1}(V) \) is open and \( x \in f^{-1}(V) \) so \( f^{-1}(V) \cap A \neq \emptyset \). So \( V \cap f(A) \neq \emptyset \).

(\( \Leftarrow \)) Suppose \( C \in Y \) is closed. Then \( f(f^{-1}(C)) \subset C \), and \( f(f^{-1}(C)) \subset \overline{f(f^{-1}(C))} \subset C \). So \( f^{-1}(C) = \overline{f^{-1}(C)} \). Hence \( f^{-1}(C) \) is closed.

**Lemma**

\( A \) is closed if and only if \( A = \overline{A} \).

**Definition: Hausdorff**

A topological space \( X \) is a Hausdorff space if given any two points \( x, y \in X \), \( x \neq y \), there exists neighbourhoods \( U_x \) of \( x \), \( U_y \) of \( y \) such that \( U_x \cap U_y \neq \emptyset \).

**Definition: T_1**
A topological space \( X \) is a T₁ if given any two points \( x, y \in X, x \neq y \), there exists neighbourhoods \( U_x \) of \( x \) such that \( y \notin U_x \).

**Proposition**

If the topological space \( X \) is T₁ or Hausdorff, points are closed sets.

### Continuous Functions

**Definition: Continuity**

Let \( X \) and \( Y \) be topological spaces. A function \( f: X \to Y \) is continuous if \( f^{-1}(U) \) is open in \( X \) for every open set \( U \) in \( Y \).

**Definition: Neighbourhood**

An open set containing \( x \) is called a neighbourhood of \( x \).

**Definition: Continuity Pointwise**

Let \( X \) and \( Y \) be topological spaces. A function \( f: X \to Y \) is continuous at \( x \in X \) iff \( f^{-1}(U(f(x))) \) contains a neighbourhood of \( x \) for all neighbourhoods \( U(f(x)) \) of \( f(x) \).

**Theorem**

Let \( X \) and \( Y \) be topological spaces. \( f: X \to Y \) is continuous if and only if it is continuous at every \( x \in X \).

**Proof:**

- \((\Rightarrow)\) Let \( x \in X \) and \( U(f(x)) \) be a neighbourhood of \( f(x) \). Then \( f^{-1}(U(f(x))) \) is a neighbourhood of \( x \).
- \((\Leftarrow)\) Let \( U \subset Y \) be open. Let \( x \in f^{-1}(U) \). Then \( f(x) \in U \), so \( f^{-1}(U) \) contains a neighbourhood \( V_x \) of \( x \).
  \[
  \bigcup_{x \in f^{-1}(U)} V_x = f^{-1}(U)
  \]
  is open.

**Example**

\( f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \geq 0 \end{cases} \) is continuous in the lower limit topology of \( \mathbb{R} \), but not in the standard topology.

### Identifying Some Continuous Function

Let \( X \) and \( Y \) be topological spaces.

1. \( \text{Id}: X \to X \) is continuous.
   **Proof:** \( \text{Id}^{-1}(U) = U \).
2. If \( f: X \to Y \) is continuous and \( f(x) \) is a constant function, then \( f: X \to Y \) is continuous.
3. If \( f: X \to Y \) is continuous and \( A \subset X \) has the relative topology, then \( (f|A): A \to Y \) is continuous.
   **Proof:** \( (f|A)^{-1}(U) = f^{-1}(U) \cap A \).
4. If \( f: A \to Y \) is continuous, then \( f: A \to Y \) is continuous.
5. If \( f(X) \) is given the relative topology in \( Y \) and \( f: X \to Y \) is continuous, then \( f: X \to f(X) \) is continuous.
   **Proof:** If \( U \) is open in \( f(X) \), there exists \( V \) open in \( Y \) such that \( U = V \cap Y \) so \( f^{-1}(U) = f^{-1}(V) \) is open.

**Proposition**

If \( f: X \to Y \) is continuous and \( g: Y \to Z \) is continuous, then \( g \circ f: X \to Z \) is continuous.
Proof: If $U$ is open in $Z$, then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open.

**Proposition**

Let $\pi_i, i=1,2$ be the projection on the $i$-th factor (so $\pi_1((y_1,y_2)=y_1$ and $\pi_2(y_1,y_2)=y_2$). The $\pi_i$'s are continuous.

Proof: If $U$ is open in $Y_1$, $\pi_i^{-1}(U)=U \times Y_2$ which is open.

**Proposition**

$f: X \to Y_1 \times Y_2$ is continuous if and only if $\pi_i \circ f = f_i$ is continuous for $i=1,2$.

Proof:

($\Rightarrow$) $f$ and $\pi_i$ are continuous, so $\pi_i \circ f$ is continuous.

($\Leftarrow$) $\pi_i \circ f$ is continuous means given $U_i$, $(\pi_i \circ f)^{-1}(U_i)$ is open, but $f^{-1}(U_1 \times U_2)=(\pi_1 \circ f)^{-1}(U_1) \cap (\pi_2 \circ f)^{-1}(U_2)$ is open (using basis, etc.).

**Proposition**

The product topology is the coarsest topology with the property that $f: X \to Y_1 \times Y_2$ is continuous if and only if $f_i: X \to Y_i$ is continuous.

Proof: $\text{Id}: Y_1 \times Y_2 \to Y_1 \times Y_2$ is continuous. So $\pi_i \circ \text{Id}$ is continuous if $U_1$ is open in $Y_1$ and $U_2$ is open in $Y_2$. Now $U_1 \times Y_2$ is open in $Y_1 \times Y_2$ and $Y_1 \times U_2$ is open in $Y_1 \times Y_2$.

**Proposition**

$f: X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed for all closed $C$ in $Y$.

Proof: $f^{-1}(C) = [f^{-1}(C)^c]^c$.

**Definition: Homeomorphism**

A 1-1 onto map $f: X \to Y$ whose inverse $f^{-1}: Y \to X$ is also continuous is called a homeomorphism. $X$ and $Y$ are said to be homeomorphic.

**Proposition**

Suppose $X$ is Hausdorff or $T_1$, and $x$ is a limit point of $X$. Then any neighbourhood $U$ of $x$ contains infinitely many distinct points of $X$.

Proof: Suppose $U$ is a neighbourhood of $x$, and $U$ has only $n$ distinct points $x_1,\ldots,x_n$ where $x_i \neq x$. Then there exists $U_i$, a neighbourhood of $x$ such that $x_i \notin U_i$. Then $U \cap U_1 \cap \cdots \cap U_n$ is a neighbourhood of $x$ and $U \cap U_1 \cap \cdots \cap U_n$ has only $x$ in it. Contradiction.

**Definition: Convergence of Sequences**

The sequence $x_i, i \in \mathbb{N}$ converges to $x$ in $X$ if for any neighbourhood $U$ of $x$, there exists $N$ such that $x_i \in U$ for all $i > N$. $x$ is called a limit point of $x_i, i \in \mathbb{N}$, written as $\lim x_i = x$ or $x_i \to x$.

**Proposition**

If $X$ is Hausdorff, then the limit points are unique; that is, if $x_i \to x$ and $x_i \to y$, then $x = y$. 

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Proof: Assume otherwise. Then there exists neighbourhoods \( U_x \) of \( x \) and \( U_y \) of \( y \) such that \( U_x \cap U_y = \emptyset \). So \( x_i, i \in \mathbb{N} \) can't converge to both \( x \) and \( y \).

**Proposition**

Suppose \( X \) is Hausdorff and \( x \to x \). Then \( \{ x \} \) is closed.

**Proposition**

Suppose \( x \in X \) and \( x \) is not a limit point of \( X \), then \( \{ x \} \) is open.

**Definition: Open Cover**

Let \( U_\alpha, \alpha \in A \) be open sets in \( X \). Then \( \bigcup \alpha \in A U_\alpha \) is called an open cover if \( X = \bigcup \alpha \in A U_\alpha \).

**Proposition**

Let \( f: X \to Y \) be a function. Let \( U_\alpha, \alpha \in A \) be an open cover. Then \( f \) is continuous if and only if \( f|U_\alpha \) is continuous for all \( \alpha \in A \).

Proof: Suppose \( f|U_\alpha \) is continuous for all \( \alpha \in A \) and \( V \subset Y \) is open. Then \( f^{-1}(V) = \bigcup \alpha \in A (f|U_\alpha)^{-1}(V) \).

\( (f|U_\alpha)^{-1}(V) \) is open in \( U_\alpha \), so it is open in \( X \). Hence \( f^{-1}(V) \) is open in \( X \).

**Pasting Lemma**

Suppose \( A, B \subset X \) are closed, \( f: A \to Y \) and \( g: B \to Y \), and \( f=g \) on \( A \cap B \). Let \( h \) be defined on \( A \cup B \), \( h=f \) on \( A \) and \( h=g \) on \( B \). Then \( h:A \cup B \to Y \) is continuous.

Proof: Let \( C \subset Y \) be closed. \( g^{-1}(C) \) is closed in \( B \) and \( f^{-1}(C) \) is closed in \( A \), so \( h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C) \) is closed since \( A \cup B \) is closed.

**Proposition**

\( f: X \to Y \) is continuous if and only if \( f^{-1}(U) \) is open for every element \( U \) of a basis \( B \) of the topology on \( Y \).

Proof: If \( U \) is open in \( Y \), then \( U = \bigcup \alpha \in A V_\alpha \) for some collection \( V_\alpha \) of basis elements in \( B \). Then \( f^{-1}(U) = f^{-1}(\bigcup \alpha \in A V_\alpha) = \bigcup \alpha \in A f^{-1}(V_\alpha) \), which is an union of open sets, so open.

### The Product Topology

#### Examples: Product Spaces

1. \( X, Y \) sets. \( X \times Y = \{(x, y) | x \in X, y \in Y \} \).
2. \( X_\alpha \) sets indexed by \( \alpha \in A \). The product \( \prod \alpha \in A X_\alpha \) is functions from \( A \) to \( X_\alpha \) such that \( \alpha \) goes into \( X_\alpha \).

**Definition: Product Topology**

Let \( X, Y \) be sets with topologies \( T_X \) and \( T_Y \). We define a topology \( T_{X \times Y} \) on \( X \times Y \) called the product topology by taking as basis all sets of the form \( U \times W \) where \( U \in T_X \) and \( W \in T_Y \).
Note: $\bigcup_{\alpha \in A} X_\alpha \times \bigcup_{\beta \in B} Y_\beta = \bigcup_{(\alpha, \beta) \in A \times B} X_\alpha \times Y_\beta$. So a basis for $T_X$ and a basis for $T_Y$ generate a basis for $T_{X \times Y}$.

**Examples**

1. Standard $\times$ standard on $\mathbb{R}^2$; basis is open rectangles.
2. Standard $\times$ standard $\times$ standard on $\mathbb{R}^3$; basis is open cubes.

**Definition: Box Topology**

Let $X_\alpha, \alpha \in A$ be topological spaces. A basis of open sets of a topology on $\prod_{\alpha \in A} X_\alpha$ is $\prod_{\alpha \in A} U_\alpha$ where $U_\alpha$ is open in $X_\alpha$. The topology it generates is called the box topology.

Note: If we allow $U_\alpha \neq X_\alpha$ for finitely many $\alpha \in A$, we get the product topology.

**Remarks**

- Box topology is finer than the product topology.
- For fixed $a \in A$, $\pi_a: \prod X_\alpha \rightarrow X_a$ is continuous.

**Proposition**

The product topology is the unique topology with the property that $f: Y \rightarrow \prod_{\alpha \in A} X_\alpha$ is continuous if and only if $\pi_a \circ f: Y \rightarrow X_a$ is continuous for each $a \in A$.

**Proposition**

If $X_\alpha, \alpha \in A$ are Hausdorff, so is $\prod_{\alpha \in A} X_\alpha$ with the product or box topology.

Proof: If $x, y \in \prod_{\alpha \in A} X_\alpha$ and $x \neq y$, then $x_\alpha \neq y_\alpha$ for some $\alpha \in A$. Let $U_x$ and $U_y$ be open sets in $X_\alpha$ such that $x_\alpha \in U_x$ and $y_\alpha \in U_y$ and $U_x \cap U_y = \emptyset$. Then $U_x \times \prod_{\alpha \neq \alpha_0} X_\alpha$ and $U_y \times \prod_{\alpha \neq \alpha_0} X_\alpha$ separate $x$ and $y$.

**Proposition**

If $A_\alpha \subseteq X_\alpha$ is closed, so is $\prod_{\alpha \in A} A_\alpha \subseteq \prod_{\alpha \in A} X_\alpha$.

Proof: Let $x \in (\prod_{\alpha \in A} A_\alpha)^c$. Then $x_\alpha \in (A_\alpha)^c$ for some $\alpha_0$. So $x \in (A_{\alpha_0})^c \times \prod_{\alpha \neq \alpha_0} X_\alpha = (\prod_{\alpha \in A} A_\alpha)^c$ is an open set.

**The Metric Topology**

**Definition: Metric**

A metric on a set $X$ is a function $d: X \times X \rightarrow \mathbb{R}_+$ such that for all $x, y, z \in X$:

1. $d(x, y) = d(y, x)$.
2. $d(x, y) = 0$ iff $x = y$.
3. $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality).

**Definition: Metric Space**

A metric space is a set $X$ with a distance function $d$. 

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Definition: Open Ball
The open ball of radius $r > 0$ around $x \in X$ is $B_r(x) = \{ y \in X | d(x, y) < r \}$.

Lemma
If $y \in B_s(x)$, then $B_y(s) \subset B_s(x)$ for $s < r - d(x, y)$.
Proof: If $z \in B_y(s)$, then $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + s < d(x, y) + r - d(x, y) = r$.

Proposition
The open balls are a basis of a topology on $X$.
Proof: Let $z \in B_s(x) \cap B_t(y)$. Let $\epsilon = \min(r - d(x, z), s - d(y, z))$. Then $B_\epsilon(z) \subset B_s(x) \cap B_t(y)$.

Example
Let $X$ be any non-empty set. Set $d(x, y) = 1$ if $x \neq y$ for all $x, y \in X$, and $d(x, x) = 0$. Then $B_1(x) = \{ x \}$. This metric generates the discrete topology.

Definition: Metrizable
A topological $X$ is metrizable if there exists a metric $d$ on set $X$ that induces the topology of $X$.

Proposition
A metric space is Hausdorff.
Proof: If $x \neq y$, then $d(x, y) > 0$. So $B_{d(x, y)}/3(x) \cap B_{d(x, y)}/3(y) = \emptyset$, since if $z \in B_{d(x, y)}/3(x) \cap B_{d(x, y)}/3(y)$ then $d(x, z) < 1/3 d(x, y)$ and $d(y, z) < 1/3 d(x, y)$, but $d(x, y) \leq d(x, z) + d(z, y) < 2/3 d(x, y)$, so contradiction.

Examples
• If $X$ has at least 2 points, then the indiscreet topology is not metrizable.
• The topology $F$ (complements of finite sets on $\mathbb{R}$) is not Hausdorff and not metrizable.

Example
On $\mathbb{R}$, $d(x, y) = |x - y|$ is a metric on $\mathbb{R}$ which gives the standard topology.

Proposition
If $d_1$ and $d_2$ are metrics on $X$, and for each $x \in X$, $r_1, r_2 \in \mathbb{R}$, there exists $s_1, s_2 \in \mathbb{R}$ such that $B_{r_1}(x) \subset B_{s_1}(x)$ and $B_{r_2}(x) \subset B_{s_2}(x)$, then these two metrics generate the same topologies.

Proposition
If $d_1$ and $d_2$ are metrics on $X$ and there exists $c_1, c_2 > 0$ such that $c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y)$ for all $x, y \in X$, then these two metrics generate the same topologies.
**Definition: Bounded**
A metric space $X$ is bounded (or has finite diameter) if there exists $k > 0$ such that $d(x, y) \leq k$ for all $x, y \in X$.

**Constructing a Bounded Metric**
Start with a metric $d$ on $X$. We can produce a new metric with diameter 1 which gives it the same topology. Let $\tilde{d}(x, y) = \min\{d(x, y), 1\}$. Then $B_d^\delta(y) \subseteq B_1^\delta(x)$ and $B_1^\delta(y) \subseteq B_d^\delta(x)$, so the topologies are the same. $\tilde{d}$ gives the standard bounded metric.

**Example: Uniform Topology**
If $X_\alpha, \alpha \in A$ are bounded metric spaces with bounded diameters (say $\leq 1$), then $\prod X_\alpha$ has $d(x, y) = \sup_\alpha d(x_\alpha, y_\alpha)$.

This gives the uniform topology.

**Remark**
The Sequence Lemma is true if $X$ satisfies the First Countability Axiom: For any $x \in X$, there exists $U_x, n \in \mathbb{N}$ open neighborhoods of $x$ such that if $V$ is a neighborhood of $x$, $V \supseteq U_x$ for some $n \in \mathbb{N}$.

**Theorem**
Let $X$ be a metric space. Then $f : X \to Y$ is continuous if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.

**Definition: Uniform Convergence**
Let $X$ and $Y$ be metric spaces. $f_n: X \to Y$ converges uniformly to $f: X \to Y$ if given $\varepsilon > 0$ there exists $N > 0$ such that for all $n > N$, $d (f_n(x), f(x)) < \varepsilon$ for all $x \in X$.

**Proposition**

Let $X$ and $Y$ be metric spaces. If $f_n: X \to Y$ converges uniformly to $f: X \to Y$ and $f_n$ continuous, then $f$ is continuous.

Proof: Given $x_0 \in X$ and $\varepsilon > 0$, we have to find $\delta > 0$ such that $d (f(x_0), f(y)) < \varepsilon$ whenever $d(x_0, y) < \delta$. Find $N > 0$ such that $d (f(x_i), f_n(x_i)) < \frac{\varepsilon}{3}$ for all $n > N$ and all $x_i \in X$. Now fix $n > N$. $f_n$ is continuous, so there exists $\delta > 0$ such that $d(x_0, y) < \delta \Rightarrow d(f_n(x_0), f_n(y)) < \frac{\varepsilon}{3}$. So

$$d(f(x_0), f(y)) \leq d(f(x_0), f_n(x_0)) + d(f_n(x_0), f_n(y)) + d(f_n(y), f(y)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$ 

**The Quotient Topology**

**Definition: Quotient Map**

Let $X$ and $Y$ be topological spaces and $f: X \to Y$ a continuous surjective (onto) map. Then $f$ is a quotient map if moreover $f^{-1}(U) \subseteq X$ is open if and only if $U \subseteq Y$ is open.

**Definition: Open Map**

$f: X \to Y$ is open iff $f(U) \subseteq Y$ is open for all open $U \subseteq X$.

**Proposition**

$\pi: X \times Y \to X$ is an open map.

Proof: Let $U \subseteq X \times Y$ be open. Then $U = \bigcup_{i \in I} (U_i \times V_i)$, so $\pi(U) = \bigcup_{i \in I} U_i$ which is open in $X$.

**Proposition**

If $f: X \to Y$ is continuous, surjective, and open, then $f$ is a quotient.

Proof: If $f^{-1}(U) \subseteq X$ is open, then $U = f(f^{-1}(U))$ is open in $Y$.

**Definition: Equivalence Relation**

A equivalence relation on a set $X$, $(X, \sim)$, satisfy:

1. $x \sim y \Leftrightarrow y \sim x$,
2. $x \sim x$,
3. $x \sim y, y \sim z \Rightarrow x \sim z$,

for all $x, y, z \in X$.

Note: $X$ is partitioned into equivalent classes. $X/\sim$ is the set of equivalent classes.

**Remark**

Let $f: X \to Y$ be surjective. Let $x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2)$. Then $X/\sim$ is in 1-1 correspondence with $Y$. 

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Proposition
Given \( (X, \sim) \), there is a unique topology on \( X/\sim \) which makes the natural map \( f : X \to X/\sim \) a quotient map and \( X/\sim \) a quotient space.

Proof: \( U \subset X/\sim \) is open if and only if \( f^{-1}(U) \) is open in \( X \).

Lemma
Let \( \{0,1\} \) have the topology \( \{\emptyset, \{0\}, \{0,1\}\} \). Let \( U \subset X \) be open. Then \( \Phi : X \to \{0,1\}, \Phi(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in U^c \end{cases} \) is continuous.

Proposition
Let \( f : X \to Y \) be surjective and continuous. Then \( f \) is a quotient map if and only if given \( g : Y \to Z \), \( g \) is continuous if and only if \( g \circ f \) is continuous for all \( g \) and \( Z \).

Proof:
\((\Rightarrow)\) Suppose \( f \) is a quotient map. Assume \( g : Y \to Z \) is continuous, so \( g^{-1}(V) \subset Y \) open for all \( V \subset Z \), and so \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \subset X \) is open since \( f \) is continuous. Assume \( g \circ f \) is continuous, so \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \subset X \) is open, and so \( g^{-1}(V) \subset Y \) is open since \( f \) is open.

\((\Leftarrow)\) It remains to show that if \( f^{-1}(V) \) is open in \( X \), then \( V \) is open in \( Y \). Let \( \Psi : Y \to \{0,1\}, \Psi(y) = \begin{cases} 0 & \text{if } y \in V \\ 1 & \text{if } y \in V^c \end{cases} \). \( \Psi \) is continuous if and only if \( \Psi \circ f \) is continuous, which it is since \( f^{-1}(V) \) is open.

Proposition
Let \( f : X \to Y \). Let \( \sim \) be the equivalence relation of \( X \) where \( x_1 \sim x_2 \iff f(x_1) = f(x_2) \). Then \( f \) is continuous if and only if the natural map \( \hat{f} : X/\sim \to Y \) is continuous.

Proposition
If \( f : X \to Y \) is a quotient map, then \( \hat{f} : X/\sim \to Y \) is a homeomorphism.

Proof: \( \hat{f} \) is 1-1, onto, and continuous (since \( f \) is continuous). Let \( \rho : X \to X/\sim \), which is continuous. Then \( \hat{f}^{-1} \circ \rho = \rho \) is continuous, and \( \hat{f}^{-1} \) is continuous since \( f \) is a quotient map.

Definition: Retraction
Let \( A \subset X \) and \( f : X \to A \). If \( f|A = Id_A \) and \( f \) is continuous, then \( f \) is called a retraction.

Proposition
A retraction is a quotient map.

Proof: If \( U \) is open in \( A \), \( f^{-1}(U) \) is open in \( X \) since \( f \) is continuous. It remains to show that if \( f^{-1}(U) \) is open in \( X \), then \( U \) is open in \( A \). Now \( U = f^{-1}(U) \cap A \) since \( f|A = Id_A \), so \( U \) is open in \( A \) since it is the intersection of an open set in \( X \) with \( A \).

Connectedness and Compactness
**Definition: Separation**

Let \( X \) be a topological space. A separation of \( X \) is a pair of nonempty open sets, \( U_1 \) and \( U_2 \), such that \( U_1 \cup U_2 = X \) and \( U_1 \cap U_2 = \emptyset \).

**Definition: Connected**

\( X \) is connected if there is no separation of \( X \).

**Theorem**

If \( f : X \to Y \) is continuous and \( X \) is connected, then so is \( f(X) \).

*Proof: Suppose \( f(X) \) is not connected. Let \( V_1 \) and \( V_2 \) open in \( f(X) \) be a separation. Then \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) is a separation of \( X \).*

**Proposition**

Suppose \( Y \subset \mathbb{R} \). If there exists \( a, b \in Y \) and \( c \notin Y \) such that \( a < c < b \), then \( Y \) is not connected.

*Proof: \((-\infty, c) \cap Y \) and \((c, \infty) \cap Y \) is a separation if \( Y \).*

**Corollary**

If \( f : X \to \mathbb{R} \) is continuous and \( X \) is connected, and \( f(x_1) = a \), \( f(x_2) = b \), \( a < c < b \), then there exists \( x \in X \) such that \( f(x) = c \).

*Proof: \( f(X) \) is connected, so it contains \( c \).*

**Theorem**

The non-empty connected sets in \( \mathbb{R} \) are precisely the intervals.

*Proof: Suppose \( Y \subset \mathbb{R} \) is a connected set. Let \( a = \text{glb} \ Y \) and \( b = \text{lub} \ Y \). \( Y \) is the interval between \( a \) and \( b \), perhaps within the endpoints. Suppose \( U_1 \) and \( U_2 \) is a separation of \( Y \). Let \( a_1, b_1 \in U_1 \), \( a_2, b_2 \in U_2 \), \( a < a_1 < b_2 < b \). Then we can find \( a_2 \) and \( b_2 \) such that \( [a_2, b_2] \) is separated by \( [a_2, b_2] \cap U_1 \) and \( [a_2, b_2] \cap U_2 \). Let \( c = \text{lub} \ [a_2, b_2] \cap U_1 \). Then \( c \notin U_2 \) since \( U_2 \) is open neighborhood \( [a_2, b_2] \cap U_2 \). Also \( c \notin U_1 \) since there exists \( c + \varepsilon \in U_1 \), so \( c \notin \text{lub} \ [a_2, b_2] \cap U_1 \). Similarly \( c \notin U_2 \) since there exists \( c - \varepsilon \in U_2 \), so \( c \notin \text{lub} \ [a_2, b_2] \cap U_1 \). Hence by the openness of \( U_1 \) and \( U_2 \), \( c \) can't be anywhere. Contradiction. So the intervals are connected.*

**Corollary**

Let \( I \) be an interval in \( \mathbb{R} \). If \( f : I \to X \) is continuous, then \( f(I) \) is connected.

**Theorem**

Suppose \( X_\alpha \subset X \), \( \alpha \in \mathcal{A} \) are connected, and there exists \( p \in X_\alpha \), \( \forall \alpha \in \mathcal{A} \). Then \( \bigcup_{\alpha \in \mathcal{A}} X_\alpha \) is connected.

*Proof: Suppose \( U_1 \) and \( U_2 \) is a separation. Then \( p \) is in \( U_1 \) or \( U_2 \); say \( U_1 \). Then \( X_\alpha \subset U_1 \), \( \forall \alpha \in \mathcal{A} \), for if \( U_2 \cap X_\alpha \neq \emptyset \) then \( U_1 \cap X_\alpha \) and \( U_2 \cap X_\alpha \) separate \( X_\alpha \).*

**Definition: Path Connected**
$X$ is path connected if given $x, y \in X$, there exists a continuous map $f : I \to X$ such that $f(0) = x$ and $f(1) = y$.

**Theorem**
If $X$ is path connected, then $X$ connected.

**Theorem**
If $A \subset X$ is connected and $A \subseteq B \subseteq \overline{A}$, then $B$ is connected.

Proof: Suppose $U_1$ and $U_2$ separate $B$. Then $U_1 \cap A \neq \emptyset$ and $U_2 \cap A \neq \emptyset$. So $U_1 \cap A$ and $U_2 \cap A$ separate $A$.

**Theorem**
The product of a finite number of connected spaces is connected.

Proof: It is sufficient to prove for two. Fix a point $(a, b) \in X \times Y$. Then $\{(x, b) \mid x \in X\}$ is connected. So for an arbitrary point $(x_0, y_0)$, $\{(x, b) \mid x \in X\} \cup \{(x_0, y) \mid y \in Y\}$ is connected. Hence their union $X \times Y$ is connected.

**Components and Local Connectedness**

**Definition: Component**
Given $X$ and $x \in X$, let the component of $x$, $C_x$, be the largest connected set containing $x$, i.e. the union of all connected sets containing $x$.

**Proposition**
For $x, y \in X$, either $C_x = C_y$ or $C_x \cap C_y = \emptyset$.

Proof: If $C_x \cap C_y \neq \emptyset$, then let $z \in C_x \cap C_y$. Now $z \in C_x \implies C_z = C_x$ and $z \in C_y \implies C_z = C_y$. So $C_z = C_x$ and $C_z = C_y$, so $C_x = C_y$.

**Theorem**
The connected components of $X$ partition $X$.

**Definition: Path Component**
Given $X$ and $x \in X$, let the path component of $x$ $(PC)_x = \{y \mid y$ can be joined to $x$ by a path $\}$. 

**Theorem**
The path components of $X$ partition $X$. In fact, they partition the connected components of $X$, i.e. $(PC)_x \subseteq C_x$.

**Definition: Local Connectivity**
$X$ is locally connected at $x$ iff for each neighborhood $U_x$ of $x$ there is a connected neighborhood $V_x$ of $x$ such that $V_x \subseteq U_x$.

$X$ is locally connected iff $X$ is locally connected at each point $x \in X$.

**Theorem**
If a space $X$ is locally connected, then the connected components of $X$ are open.
Proof: Let \( x \in C_x \). Then any \( U_x \) contains \( V_x \). Now each \( V_x \subseteq C_x \), so \( C_x \) is open.

**Lemma**
If a space \( X \) is locally connected, then for each open \( U \subseteq X \), \( U \) is connected.

**Theorem**
A space \( X \) is locally connected if and only if for every open set \( U \subseteq X \), each component of \( U \) is open in \( X \).

**Definition: Local Path Connectivity**
\( X \) is locally path connected at \( x \) iff for each neighborhood \( U_x \) of \( x \) there is a path connected neighborhood \( V_x \) of \( x \) such that \( V_x \subseteq U_x \).

\( X \) is locally path connected iff \( X \) is locally connected at each point \( x \in X \).

**Theorem**
A space \( X \) is locally path connected if and only if for every open set \( U \subseteq X \), each component of \( U \) is open in \( X \).

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**COMPACT SPACES**

**Definition: Open Cover**
Let \( U_\alpha \subseteq X \), \( \alpha \in A \) be open sets such that \( \bigcup_{\alpha \in A} U_\alpha = X \). \( \{ U_\alpha \}_{\alpha \in A} \) is an open cover of \( X \).

**Definition: Subcover**
If \( B \subseteq A \) and \( \{ U_\alpha \}_{\alpha \in B} \) is still a cover, then \( \{ U_\alpha \}_{\alpha \in B} \) is called a subcover of \( \{ U_\alpha \}_{\alpha \in A} \).

**Definition: Compact**
\( X \) is compact iff every open cover of \( X \) has a finite subcover.

**Theorem**
If \( X \) is compact and \( f : X \to Y \) is continuous, then \( f(X) \) is compact.

Proof: Let \( \{ U_\alpha \}_{\alpha \in A} \) be an open cover of \( f(X) \) . Let \( V_\alpha \) be open in \( Y \) and \( V_\alpha \cap f(X) = U_\alpha \). Then \( f^{-1}(V_\alpha) = f^{-1}(U_\alpha) \) is open in \( X \). \( \{ f^{-1}(U_\alpha) \}_{\alpha \in A} \) is an open cover of \( X \), so there exists \( \alpha_1, \ldots, \alpha_n \) such that \( f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_n}) = X \), but then \( U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} = f(X) \).

**Proposition**
A compact set in \( \mathbb{R} \) is bounded.

Proof: Cover \( \mathbb{R} \) by \( \{ (z-1, z+2), z \in \mathbb{Z} \} \). A finite collection of these is bounded.

**Proposition**
A compact subset \( A \) of a Hausdorff space \( X \) is closed.

Proof: Suppose there exists \( p \in \overline{A} - A \). Let \( x \in A \). Let \( U_x \) and \( V_x \) be neighbourhoods of \( x \) and \( p \) respectively such
Suppose \( U_i \cap V_i = \emptyset \) \( \{ U_i \}_{i \in A} \) form an open cover of \( A \). Now let \( U_{x_1}, \ldots, U_{x_n} \) be a finite subcover and \( U = U_{x_1} \cup \cdots \cup U_{x_n} \) and \( V = V_{x_1} \cap \cdots \cap V_{x_n} \). Then \( U \cap V = \emptyset \). But \( p \in V \) and \( V \cap A = \emptyset \). Contradiction. Hence \( p \notin A - A \) and \( A = \bar{A} \) so \( A \) is closed.

**Theorem**
Compact sets are closed and bounded in \( \mathbb{R} \), \( \mathbb{R}^n \), or any metric space.

**Theorem**
If \( f : X \to \mathbb{R} \) is continuous and \( X \) is compact, then \( f \) achieves its maximum and minimum, i.e. its lub and glb.

Proof: lub and glb are in \( \text{sup}(f(X)) \).

**Definition: Finite Intersection Property**
A collection of (nonempty) sets \( \{ B_i \}_{i \in A} \) has the finite intersection property if for every finite sub-collection \( \{ B_{i_1}, \ldots, B_{i_n} \} \) \( B_{i_1} \cap \cdots \cap B_{i_n} \neq \emptyset \).

**Theorem**
\( X \) is compact if and only if every collection of closed subsets of \( X \) with the finite intersection property has nonempty intersection.

Proof:
( \( \Rightarrow \) ) Assume \( X \) is compact. Let \( \{ C_i \}_{i \in A} \) be a collection of closed sets, so \( \{ C^c_i \}_{i \in A} \) are open. \( \{ C^c_i \}_{i \in A} \) does not cover \( X \) if there is no finite subcover, that is \( \bigcup_{i \in A} C^c_i \neq X \iff \bigcap_{i \in A} C_i \neq \emptyset \).

( \( \Leftarrow \) ) Let \( \{ U_i \}_{i \in A} \) be an open cover. Suppose \( \{ U_i \}_{i \in A} \) has no finite subcover. Then \( U_{i_1} \cap \cdots \cap U_{i_n} \neq \emptyset \) for all finite sub-collection, so \( \bigcap_{i \in A} U_i \neq \emptyset \) and hence \( \bigcup_{i \in A} U_i \neq X \). Contradiction since \( \{ U_i \}_{i \in A} \) is an open cover.

**Theorem**
If \( X \) is compact and \( A \subseteq X \) is closed, then \( A \) is also compact.

Proof: Let \( \{ U_\beta \}_{\beta \in B} \) be an open cover of \( A \), \( U_\beta = V_\beta \cap A \), \( V_\beta \) open in \( X \). Then \( \{ V_\beta \}_{\beta \in B} \cup A^c \) is an open cover of \( X \), so there exists a finite collection \( \beta_1, \ldots, \beta_n \) such that \( V_{\beta_1}, \ldots, V_{\beta_n}, A^c \) cover \( X \). Hence \( U_{\beta_1}, \ldots, U_{\beta_n} \) cover \( A \).

**Theorem**
If \( X \) is compact and Hausdorff, and \( A, B \) are closed sets in \( X \) such that \( A \cap B = \emptyset \), then there exists open sets \( U \) and \( V \), \( A \subseteq U \), \( B \subseteq V \), such that \( U \cap V = \emptyset \).

Proof: Let \( x \in A \). There exists neighbourhoods \( U_x \) of \( x \) and \( V_x \) of \( B \) such that \( U_x \cap V_x = \emptyset \). Let \( U_{x_1}, \ldots, U_{x_n} \) be a finite subcover of \( A \). Let \( U = \bigcup_{i=1}^n U_{x_i} \) and \( V = \bigcap_{i=1}^n V_{x_i} \). Then \( U \) and \( V \) are open and \( U \cap V = \emptyset \).

**Theorem**
Suppose \( X \) is compact. Let \( \{ C_i \}_{i \in N} \) be a collection of non-empty, closed, and nested sets (i.e. \( C_1 \supseteq C_2 \supseteq \cdots \)). Then \( \bigcap_{n \in N} C_n \neq \emptyset \).
Proof: $X$ is compact, so $\left\{ C_n \right\}_{n \in \mathbb{N}}$ has the finite intersection property. Suppose $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$. Then there exists a finite subsequence $n_i, \ldots, n_k$ such that $\bigcap_{i=n_i}^{n_k} C_{n_i} = \emptyset$, but $\bigcap_{i=n_i}^{n_k} C_{n_i} = C_{n_i}$. Contradiction.

**Theorem**

Suppose $X$ is compact and $Y$ is Hausdorff. If $f : X \to Y$ is bijective and continuous, then $f$ is a homeomorphism.

Proof: We need $f^{-1} : Y \to X$ to be continuous. Let $C \subset X$ be closed. Then $C$ is compact. So $f(C) \subset Y$ is compact, and so closed. Now $(f^{-1})^{-1}(C) = f(C)$. Hence if $C \subset X$ be closed, then $(f^{-1})^{-1}(C)$ is closed. Therefore $f^{-1}$ is continuous.

**Lemma: Tube Lemma**

Suppose $Y$ is compact. Let $x_0 \in X$ and $U$ be an open set in $X \times Y$ such that $x_0 \times Y \subset U$. Then there exists a neighborhood $W$ of $x_0$ such that $W \times Y \subset U$.

Proof: For each $y \in x_0 \times Y$, there exists a product neighborhood $U_{x_0, y} \times U_y$ contained in $U$. Since $Y$ is compact, there is a finite collection $y_1, \ldots, y_n$ such that $U_{y_1}, \ldots, U_{y_n}$ cover $Y$. Let $W = \bigcap_{i=1}^{n} U_{x_0, y_i}$. Now $W \times U_{y_i} \subset U_{x_0, y_i} \times U_{y_i}$ for each $i = 1, \ldots, n$, so $W \times \bigcup_{i=1}^{n} U_{y_i} = W \times Y \subset U$.

**Theorem**

Finite product of compact spaces is compact.

Proof: Note that it suffices to prove $X \times Y$ is compact if $X$ and $Y$ are compact. Let $\left\{ U_a \right\}_{a \in A}$ be an open cover of $X \times Y$. For each fixed $x \in X$, let $U_{x, 1}, \ldots, U_{x, n}$ be a finite cover of $x \times Y$. Let $U_x = \bigcup_{j=1}^{n} U_{x, j}$. $U_x$ is open, so by the tube lemma, there exists a neighborhood $W_x$ of $x$ such that $W_x \times Y \subset U_x$. Since $X$ is compact, there exists $x_1, \ldots, x_m$ such that $W_{x_1}, \ldots, W_{x_m}$ cover $X$. So $W_{x_1} \times Y, \ldots, W_{x_m} \times Y$ cover $X \times Y$. Since each tube $W_{x_i} \times Y$ is covered by $\left\{ U_{x_i, j} \right\}_{j=1}^{n}$, a finite number of open sets of $X \times Y$, $\bigcup_{j=1}^{n} \left[ U_{x_i, j} \right]_{j=1}^{n}$ is a finite subcover of $\left\{ U_a \right\}_{a \in A}$. Hence $X \times Y$ is compact.

**Theorem**

$[a, b] \subset \mathbb{R}$ is compact.

Proof: Let $\left\{ U_a \right\}_{a \in A}$ be an open cover of $[a, b]$. Let $c \in (a, b)$, where $c = \inf \left\{ x \in \mathbb{R} \left| x \in \bigcup_{a \in A} U_a \right. \right\}$. Then $c \neq a$, and there exists an open set containing $b$.

**Theorem**

$\prod_{j=1}^{n} [a_j, b_j] \subset \mathbb{R}^n$ is compact.

**Theorem: Heine-Borel Theorem**

$X \subset \mathbb{R}$ is compact if and only if $X$ is closed and bounded.
**Local Compactness**

**Definition: Local Compactness**

*\( X \) is locally compact at \( x \) if there is a compact subset \( C \) of \( X \) which contains a neighborhood of \( x \).

*\( X \) is locally compact if it is locally compact at every \( x \in X \).

**Example**

\( \mathbb{R} \) is locally compact.

**Definition: One-Point Compactification**

Let \( Y \) be compact and Hausdorff. Suppose \( X \subseteq Y \) such that \( Y - X = \{ \infty \} \) is one point and that \( \overline{X} = Y \). Then we call \( Y \) a one-point compactification of \( X \).

**Theorem**

Let \( X \) be locally compact and Hausdorff, but \( X \) is not compact. Then \( X \) has a one-point compactification. Moreover, if \( Y_1 \) and \( Y_2 \) are both one-point compactification of \( X \), then \( \text{Id}_X \) extends to a homeomorphism \( h: Y_1 \to Y_2 \) which takes \( \infty_{Y_1} \) to \( \infty_{Y_2} \).

**Proof:**

Let \( Y = X \cup \{ \infty \}, \infty \notin X \). Define the topology on \( Y \) as follows: \( U \subseteq X \) is open iff

- \( U \subseteq X \) and \( U \) is open in \( X \), or
- \( U = A' \) where \( A \subseteq X \) is compact.

\( Y \) is Hausdorff: Take two points \( x_1 \) and \( x_2 \) in \( Y \). If \( x_1, x_2 \in X \), done. Otherwise, let \( x \in X \) and \( \infty \in Y \). There exists \( U_x \) open and \( C \) such that \( x \in U_x \subseteq C \subseteq X \) (since \( X \) be locally compact). \( \infty \in C \) which is open. So \( U_x \cap C \neq \emptyset \).

\( Y \) is compact: Let \( \{ U_{\alpha} \}_{\alpha \in A} \) be an open cover of \( Y \). \( \infty \) is in some \( U_{\alpha} \) and \( U_{\alpha}' \) is compact. Note that \( \{ \infty \} \) is closed (since \( Y \) Hausdorff). \( X \) is open, so \( X \cap U_{\alpha} \) is open for all \( \alpha \in A \). Then \( \{ X \cap U_{\alpha} \}_{\alpha \in A} \) is an open cover of \( X \), and hence of \( U_{\alpha}' \). So a finite collection \( U_{\alpha_1}, \ldots, U_{\alpha_n} \) covers \( U_{\alpha}' \), and \( U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n} \) covers \( Y \).

\( \overline{X} = Y \), i.e. \( \infty \in \overline{X} \): Let \( U_{\infty} \) be an open set containing \( \infty \). Then \( U_{\infty} = A' \), where \( A \subseteq X \) is compact and \( A \neq X \), so \( U_{\infty} \cap X \neq \emptyset \).

**Corollary**

Let \( X \) be Hausdorff. Then \( X \) is locally compact if and only if given \( x \in X \) and neighborhood \( U_x \) of \( x \), there exists a neighborhood \( V_x \) of \( x \) such that \( V_x \) is compact and \( V_x \subseteq U_x \).

**Proof:**

(\( \rightarrow \)) \( X \) is locally compact and Hausdorff, so \( X \subseteq Y \) where \( Y \) is compact and Hausdorff by one-point compactification. Given \( U_x \), \( U_x'' = \overline{U_x} \) is closed in \( Y \) and thus so compact. So there exists \( V_x \) and \( N(\overline{U_x}) \) neighborhoods of \( x \) and \( U_x \) such that \( V_x \cap N(\overline{U_x}) = \emptyset \). So \( V_x \subseteq U_x \) and \( \emptyset \notin \overline{V_x} \). Hence \( \overline{V_x} \) is compact and \( \overline{V_x} \subseteq U_x \).

(\( \leftarrow \)) Take \( C = \overline{V_x} \) compact. Then \( x \in V_x \subseteq U_x \). This is just the definition of local compactness.

**Corollary**

If \( X \) is locally compact and Hausdorff, and \( A \subseteq X \) either open or closed, then \( A \) is locally compact and Hausdorff.

**Corollary**

\( X \) is homeomorphic to an open subspace of a compact Hausdorff space if and only if \( X \) is locally compact and Hausdorff.
LIMIT POINT COMPACTNESS

Definition: Limit Point Compact

$X$ is limit point compact if every infinite subset of $X$ has a limit point.

Theorem

If $X$ is compact, then $X$ is limit point compact.

Proof: Suppose $X$ is not limit point compact. Then there exists an infinite subset $A \subset X$ which has no limit points. So for all $x \in X$, there exists $U_x$ such that $U_x \cap A$ is at most one point, i.e. $x \notin A \Rightarrow U_x \cap A = \emptyset$ or $x \in A \Rightarrow U_x \cap A = \{x\}$. Now $(U_x)_{x \in X}$ is an open cover of $X$ but has no finite subcover since $A$ is infinite.

Definition: Sequentially Compact

$X$ is sequentially compact if every sequence $(x_i)_{i \in \mathbb{N}}$ has a convergent subsequence $(x_{n_i})_{i \in \mathbb{N}}$ such that $n_i$ is strictly increasing and $x_{n_i}$ converge.

Definition: Cauchy Sequence

In a metric space, a sequence $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence if given $\varepsilon > 0$, there exists $N > 0$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > N$.

Definition: Complete

$X$ is complete if every Cauchy sequence converges.

Lemma

If a subsequence of a Cauchy sequence converges, then so does the sequence.

Definition: Totally Bounded

A metric space is totally bounded if for all $\varepsilon > 0$, there is a finite cover of $X$ by $\varepsilon$-balls.

Definition: Lebesgue Number

Let $(U_a)_{a \in A}$ be an open cover of $X$. Then $\delta > 0$ is a Lebesgue number for $(U_a)_{a \in A}$ if given $x \in X$, $B_\delta(x) \subset U_a$ for some $a \in A$.

Lemma: Lebesgue Number Lemma

Let $X$ be complete and totally bounded. If $(U_a)_{a \in A}$ be an open cover, then there exists a Lebesgue number $\delta > 0$ for $(U_a)_{a \in A}$.

Proof: Suppose not, i.e. for all $\delta$, there exists $x_\delta$ such that $B_\delta(x_\delta) \not\subset U_a$ for any $a \in A$. Pick a sequence $\delta_n \rightarrow 0$ (e.g. $\delta_n = \frac{1}{2^n}$) and $x_{\delta_n} \in X$ as above. Since $X$ totally bounded, consider finite covers of $X$ by balls of radius $\frac{1}{2^i}$, for each $i = 1, 2, \ldots$. Inductively construct a subsequence $(x_{\delta_n})_{n \in \mathbb{N}}$ such that the tail is in one ball of radius $\frac{1}{2^i}$ for $i = 1, 2, \ldots$. The resulting sequence is Cauchy, and hence converges since $X$ complete. So $x_{\delta_n} \rightarrow x$, $x \in U_a$ for some $a \in A$, and $B_\delta(x) \subset U_a$. Now, for $i$ large enough, $\delta_n < \frac{\varepsilon}{2}$ since $\delta_n \rightarrow 0$, and that $x_{\delta_n} \in B_\delta(x)$. So $B_\delta(x_{\delta_n}) \subset B_\delta(x) \subset U_a$ by the
triangle inequality. Contradiction.

**Theorem**
If \((X, d)\) is a metric space, then the following are equivalent:
1. \(X\) is compact.
2. \(X\) is limit point compact.
3. \(X\) is sequentially compact.
4. \(X\) is complete and totally bounded.

Proof:
(1 \(\Rightarrow\) 2) Done.

(2 \(\Rightarrow\) 3) Let \(\{x_i\}_{i \in \mathbb{N}}\) be a sequence. If \(\{x_i\}_{i \in \mathbb{N}}\) is a finite set (finitely many different elements), then there is a constant subsequence which converge. So assume \(A=\{x_i| i \in \mathbb{N}\}\) is infinite and let \(x \in X\) be a limit point of \(A\). Then we can find infinite sets \(S_i, i \in \mathbb{N}\) such that \(S_i+1 \subseteq S_i\) and points \(x_i \in S_i\) with \(n_i+1 \geq n_i\) such that \(d(x_i, x) < \frac{1}{i}\) (possible since \(B_{\frac{1}{i}}(x) \cap S_i\) with \(S_i = A\) is an infinite set). Then \(x_i \to x\) since given \(\varepsilon > 0\) there exists \(M > 0\) such that \(\frac{1}{M} < \varepsilon\), and so for \(i > M\) \(d(x_i, x) < \frac{1}{n_i} < \frac{1}{M} < \varepsilon\).

(3 \(\Rightarrow\) 4) Take any Cauchy sequence \(\{x_i\}_{i \in \mathbb{N}}\). Since \(X\) is sequentially compact, there is a subsequence \(\{x_{i_n}\}_{n \in \mathbb{N}}\) which converges. So the Cauchy \(\{x_{i_n}\}_{n \in \mathbb{N}}\) converges by lemma, and hence \(X\) is complete. Now suppose \(X\) is not totally bounded. Let \(\varepsilon > 0\) be such that the \(\varepsilon\)-balls do not have a finite subcover. For \(x_1, \ldots, x_n\) such that \(d(x_i, x_j) \geq \varepsilon\ \forall i,j \leq n\), there exists \(x_{n+1}\) such that \(x_1, \ldots, x_n, x_{n+1}\) have the same property (i.e. \(d(x_i, x_j) \geq \varepsilon\ \forall i,j \leq n+1\)). So we can construct an infinite sequence \(\{x_i\}_{i \in \mathbb{N}}\) such that \(d(x_i, x_j) \geq \varepsilon\ \forall i \neq j\). This sequence has no convergent subsequence, for if it did, then given \(\frac{\varepsilon}{2}\) there exists \(M\) such that \(d(x_i, x_j) < \frac{\varepsilon}{2}\ \forall i < M\), but then \(d(x_i, x_j) > \varepsilon\ \forall i > M\), so contradiction.

(4 \(\Rightarrow\) 1) Let \(\{U_i\}_{i \in \mathbb{N}}\) be an open cover of \(X\). Let \(\delta > 0\) be a Lebesgue number for the cover. Let \(B_{\delta}(x_1), \ldots, B_{\delta}(x_k)\) be a finite covering of \(\delta\)-balls. Then for each \(i=1, \ldots, k\), \(B_{\delta}(x_i) \subseteq U_{i_\alpha}\) for some \(\alpha_i \in A\). Hence \(U_{i_\alpha}, \ldots, U_{i_\alpha}\) is a finite subcover.

**Corollary**
If \(X\) is compact metric space, then any open cover has a Lebesgue number.

**Definition: Uniform Continuity**
Let \(X\) and \(Y\) be metric spaces. \(f:X \to Y\) is uniformly continuous if given \(\varepsilon > 0\), there \(\delta > 0\) such that \(d_Y(f(x), f(y)) < \varepsilon\) whenever \(d_X(x, y) < \delta\) for all \(x, y \in X\).

**Theorem**
Let \(X\) and \(Y\) be metric spaces. If \(X\) is compact and \(f:X \to Y\) is continuous, then \(f\) is uniformly continuous.

Proof: Given \(x \in X\), there exists \(\delta_x > 0\) such that \(y \in B_{\delta_x}(x) \Rightarrow d_Y(f(x), f(y)) < \frac{\varepsilon}{2}\) since \(f\) continuous. Then \(y, z \in B_{\delta_x}(x) \Rightarrow d_Y(f(y), f(z)) < \varepsilon\) by the triangle inequality. Now the open cover \(\{B_{\delta_x}(x)\}_{x \in X}\) has a Lebesgue number \(\delta > 0\).
**The Tychonoff Theorem**

**Definition: Maximal**
A collection of sets $D$ with the finite intersection property is maximal if for any $D' \supset D$, $D' \neq D$, $D'$ does not have the finite intersection property.

**Lemma**
Given a collection $C$ of sets with the finite intersection property, there exists $D$ such that $C \subset D$ and $D$ is maximal.

Proof: Construct $D$ using Zorn's Lemma.

**Lemma**
Let $D$ be a maximal collection of sets in $X$ with the finite intersection property. Then:
1. If $A_1, \ldots, A_n \in D$, then $A_1 \cap \cdots \cap A_n \in D$.
2. If $A \cap U \neq \emptyset \ \forall U \in D$, then $A \in D$.

**Lemma**
Suppose that $D$ is a maximal collection of sets in $\prod X_{\alpha}$, where each $X_{\alpha}$ is compact. Then $\bigcap A \in D$ is nonempty.

Proof: Note that closure the projection $\pi_{\alpha}(A)$ is closed in a compact space $X_{\alpha}$, so $\bigcap A \in D$ is compact. So there exists some $x_{\alpha} \in \pi_{\alpha}(A) \ \forall A \in D$, so for any neighborhood $U_{\alpha}$ of $x_{\alpha}$, $\pi_{\alpha}(A) \neq \emptyset \ \forall A \in D$, and so $x_{\alpha} = (x_{\alpha})_{\alpha \in \alpha}$. Let $V_a = \pi_{\alpha}^{-1}(U_{\alpha}) = U_{\alpha} \times \prod_{\beta \neq \alpha} X_{\beta}$, then $V_a \in D$. Therefore finite intersections of $V_a$'s is in $D$, i.e., $V_{a_1} \cap \cdots \cap V_{a_n} = U_{a_1} \times \cdots \times U_{a_n} \times \prod_{\beta \neq a_1, \ldots, a_n} X_{\beta} \in D$, and so $x \in A$ for every $A$, so $\bigcap A \neq \emptyset$.

**Theorem: Tychonoff’s Theorem**
If $X_{\alpha}$ is compact for all $\alpha \in \mathcal{A}$, then $\prod_{\alpha \in \mathcal{A}} X_{\alpha}$ is compact.

**Countability and Separation Axioms**

**The Separation Axioms**

**Definition: Regular**
Let $X$ be a topological space where one-point sets are closed. Then $X$ is regular if a point and a disjoint closed set can be separated by open sets.

**Definition: Normal**
Let $X$ be a topological space where one-point sets are closed. Then $X$ is normal if two disjoint sets can be separated by open sets.
**Remark**
Normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff.

**Proposition**
If $X$ is regular and $U$ is a neighborhood of $x$, then there exists a neighborhood $V$ of $x$ such that $\overline{V} \subset U$.

Proof: $U^c$ is closed. So there exist open sets $V_1$ and $V_2$ such that $x \in V_1$, $U^c \subset V_2$, $V_1 \cap V_2 = \emptyset$. So $x \in \overline{V_1} \subset U$.

**Proposition**
If $X$ is normal and $U$ is a neighborhood of a closed set $A$, then there exists a neighborhood $V$ of $A$ such that $\overline{V} \subset U$.

Proof: $U^c$ is closed. So there exist open sets $V_1$ and $V_2$ such that $A \subset V_1$, $U^c \subset V_2$, $V_1 \cap V_2 = \emptyset$. So $A \subset \overline{V_1} \subset U$.

**The Urysohn Lemma**

**Theorem: Urysohn Lemma**
Let $X$ be normal, $A$ and $B$ closed such that $A \cap B = \emptyset$. Let $[a, b] \subset \mathbb{R}$. Then there exists a continuous function $f : X \to [a, b]$ such that $f(A) = a$ and $f(B) = b$.

Proof: It is sufficient to take $[a, b] = [0, 1]$ (since they are homeomorphic). Now for every rational $q \in (0, 1)$, construct open sets $U_q \subset X$ such that if $0 < p < q < 1$ then $\overline{U_p} \subset U_q$, and that $A \subset U_0$, $A \subset \bigcup_{p=1}^{q} U_p$, $B \cap U_p = \emptyset$. Let $U_1 = X - B \supset A$ and $U_0$ be an open set containing $A$ with $U_0 \subset U_1$. Let $\phi : \mathbb{N} \to \mathbb{Q} \cap [1, 2]$ where $\phi(1) = 0$, $\phi(2) = 1$. Suppose $U_{\phi(i)}$, $\ldots$, $U_{\phi(n)}$ are constructed. $\phi(n+1)$ is between two closest neighbors, $\phi(i)$ and $\phi(j)$, on the list. Since $\phi(i) < \phi(j)$, $\overline{U_{\phi(i)}} \subset U_{\phi(j)}$, so can pick $U_{\phi(n+1)}$ to be a neighborhood of $\overline{U_{\phi(j)}}$ (that is $\overline{U_{\phi(n+1)}} \subset U_{\phi(j)}$). Hence by induction, define $U_q \subset \mathbb{R}$ for all rationals $q \in (0, 1)$.

Now define $f(x) = \inf_{q \in \mathbb{Q} \cap [0, 1]} \{ x \in U_q \}$. Then $f(A) = 0$ and $f(B) = 1$. It remains to prove the continuity of $f$. First note that $x \in U_q \Rightarrow f(x) \leq r$ and $x \notin U_q \Rightarrow f(x) \geq r$. Now let $U$ be an open neighborhood of $f(x) \in [0, 1]$. Then there exists open interval $(q_1, q_2)$ such that $f(x) \in (q_1, q_2) \subset U$. Now $f^{-1}((q_1, q_2)) \supset U_{q_1} \supset U_q$, which is open and contains $x$, and that $f(U_{q_1} \cap \overline{U_q}) \subset (q_1, q_2)$. Therefore $f$ is continuous.

**The Tietze Extension Theorem**

**Theorem: Tietze Extension Theorem**
Let $X$ be normal, $A \subset X$ be closed, and $f : A \to [a, b]$ or $f : A \to \mathbb{R}$ be continuous. Then $f$ may be extended to a continuous map defined on all of $X$.

Proof:
For $f : A \to [a, b]$, it is sufficient to show for $[a, b] = [-1, 1]$. Build a collection $g_1, g_2, \ldots$ of approximates to $f$ such that $f - g_1 - g_2 - \cdots$ converges to 0 uniformly. That is $g = \sum g_n \to f$ on $X$.

- Let $B_1 = f^{-1}([-1, -\frac{1}{2}])$, $C_1 = f^{-1}([-\frac{1}{2}, \frac{1}{2}])$, $D_1 = f^{-1}((\frac{1}{2}, 1])$. Then $B_1$ and $D_1$ are closed and disjoint. By the Urysohn lemma, we can find $g_1 : X \to [-\frac{1}{2}, \frac{1}{2}]$ such that $g(B_1) = -\frac{1}{2}$, $g(D_1) = \frac{1}{2}$, and that $|f(x) - g_1(x)| < \frac{1}{4}$ $\forall x \in A$. We get obtain $f - g_1 : A \to [-\frac{1}{4}, \frac{1}{4}]$.

- Now by letting $B_2 = (f - g_1)(([-1, -\frac{1}{2}])$, $C_2 = (f - g_1)^{-1}([-\frac{1}{2}, \frac{1}{2}])$, $D_2 = (f - g_1)^{-1}((\frac{1}{2}, 1])$, we can get $g_2 : X \to \frac{1}{2}([-\frac{1}{2}, \frac{1}{2}])$ by Urysohn lemma, with $g(B_2) = \frac{1}{4}(-\frac{1}{4})$, $g(D_2) = \frac{1}{4}(\frac{1}{4})$, and
\[ f(x) - g_i(x) - g_j(x) < \frac{1}{i^2} \quad \forall x \in A. \]

- By induction, let \( B_x := (f - g_1 - \cdots - g_{n-1})^{-1}((\frac{1}{3})^{n-1}[-1, -\frac{1}{3}]) \), \( C_x := (f - g_1 - \cdots - g_{n-1})^{-1}((\frac{1}{3})^{n-1}[\frac{1}{3}, 1]) \), and \( D_x := (f - g_1 - \cdots - g_{n-1})^{-1}((\frac{1}{3})^{n-1}[\frac{1}{3}, 1]) \). Then we can get \( g_x : X \to ((\frac{1}{3})^{n-1}[-\frac{1}{3}, \frac{1}{3}]) \) by Urysohn lemma, with \( g(B_x) = (\frac{1}{3})^{n-1}(\frac{1}{3}) \), \( g(C_x) = (\frac{1}{3})^{n-1}(\frac{1}{3}) \), and \( |f(x) - g_i(x) - \cdots - g_j(x)| < \frac{1}{i^2} \quad \forall x \in A. \)

- Now take \( g = \sum g_i \). It is the extension we are looking for.

For \( f : A \to \mathbb{R} \), it is sufficient to show for \( f : A \to (-1, 1) \). Now, there exists \( g : X \to [-1, 1] \) such that \( g = f \) on \( A \) by the first part of the proof. Let \( B = g^{-1}((-1 \cup 1)) \). Then \( A \) and \( B \) are disjoint closed sets. By Urysohn lemma, there exists \( \phi : X \to [0, 1] \) such that \( \phi(A) = 1 \) and \( \phi(B) = 0 \), hence \( \phi f = f \) on \( A \), and \( \phi g = 0 \) on \( B \). Then \( \phi g : X \to (-1, 1) \) is the extension we are looking for.

**Definition: Separates Points**

Suppose \( \{f_\alpha\}_{\alpha \in A} \) is a collection of functions such that \( f_\alpha : X \to [0, 1] \). Let \( f : X \to \prod_{A} [0, 1] \) be defined by \( (f(x))_\alpha = f_\alpha(x) \). Suppose that for \( x \neq y \), there exists an \( \alpha \) such that \( f_{\alpha}(x) \neq f_{\alpha}(y) \) (hence \( f \) is 1-1). We say \( f \) separates points.

**Theorem**

Suppose \( X \) is normal and has a countable basis. Then there exists a countable collection of continuous functions \( f_i : X \to [0, 1] \) such that, given \( x_0 \in X \) and a neighborhood \( U_{x_0} \) of \( x_0 \), there exists \( i \) such that \( f_i(x_0) = 1 \) and \( f_i \equiv 0 \) outside \( U_{x_0} \).

**Remark**

If we have such functions, then \( f \) separates points since given \( x, y \in X \), there exists \( U_x \) such that \( y \notin U_x \) by normality.

**The Urysohn Metrization Theorem**

**Definition: Completely Regular**

A space \( X \) is completely regular if one-point sets are closed (\( T_1 \)), and if given a point \( p \in X \) and a closed set \( A \subset X \) such that \( p \notin A \), there exists a continuous function \( f : X \to [0, 1] \) such that \( f(p) = 1 \) and \( f(A) = 0 \).

**Theorem: Urysohn Metrization Theorem**

If \( X \) is completely regular (or normal) and second countable (countable basis), then \( X \) is metrizable.

Proof: Let \( B_a \) and \( B_m \) be basis elements such that \( \overline{B_a} \subset B_m \). There exists a continuous function \( f_{n,m} : X \to [0, 1] \) such that \( f_{n,m}(\overline{B_a}) = 0 \) and \( f_{n,m}(B_m) = 1 \) by Urysohn lemma. In a regular space, given a point \( x \in X \) and a neighborhood \( U \) of \( x \), there exists another neighborhood \( V \) of \( x \) such that \( x \in \overline{V} \subset U \). Hence given \( x, y \in X \) there exists \( B_a \) and \( B_m \) such that \( x \in B_m \subset B_a \), \( y \notin B_a \), and so \( f_{n,m} \) separates points. Now \( \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) is 1-1 and onto, so \( \prod f_{n,m} : X \to [0, 1]^{\mathbb{N}} \) is 1-1. \( [0, 1]^{\mathbb{N}} \) is metrizable.

**Remark**

If \( X \) is normal but not second countable, take \( \prod f : X \to [0, 1]^{\mathbb{N}} \) where \( f \in C^0(X, I) \) (continuous function from \( X \) to \( I \)). If \( X \) is compact Hausdorff (hence normal), this is a homeomorphism.

**The Stone-Čech Compactification**
Theorem
Let \( X \) be completely regular (or normal). Then there exists a compactification of \( Y \) of \( X \) (i.e. \( Y \) is compact Hausdorff and \( X = Y \)) with the property that any bounded continuous function \( f : X \to \mathbb{R} \) extends uniquely to a function \( g : Y \to \mathbb{R} \). \( Y \) is called the Stone-Čech Compactification.

Proof: Let \( B(X, \mathbb{R}) \) be the set of all bounded continuous functions \( f : X \to \mathbb{R} \). For each \( f \in B \) let \( I_f = [-\alpha_f, \alpha_f] \) contain the image of \( f \). Let \( Z = \prod_{f \in B(X, \mathbb{R})} I_f \), which is compact Hausdorff by Tychonoff theorem. Define \( h : X \to \prod_{f \in B(X, \mathbb{R})} I_f \) by \( (h(x))_f = f(x) \). Let \( Y = h(X) \) be the compactification. Now let \( i : Y \to Z \) be the inclusion, \( i|X = h \), then \( \pi_f \circ i \) extends \( f \) uniquely.

Corollary
Let \( X \) be completely regular and \( W \) be compact Hausdorff. If \( \phi : X \to W \) is a continuous function, then \( \phi \) extends to Stone-Čech.

Proof: \( W \to I^d \) is an imbedding, so \( X \to W \to I^d \) is continuous. Hence each coordinate extends.

Metrization Theorems and Paracompactness

Definition: Refine
A collection \( B \) of subsets of \( X \) is said to refine \( A \) if \( \bigcup_{U \in B} U = \bigcup_{U \in A} U = X \) and for each \( U \in B \) there exists \( U \in A \) such that \( B \subseteq A \).

Definition: Local Finiteness
An open cover \( A \) of \( X \) is called locally finite if for any \( x \in X \), \( x \in U \) for finitely many \( U \in A \).

Definition: Paracompactness
\( X \) is paracompact if every open cover \( A \) of \( X \) has a locally finite open refinement.

Theorem
Every metrizable space is paracompact.

Definition: Partition of Unity
Given a locally finite cover \( A \), a partition of unity is a collection of continuous functions \( \phi_U : X \to [0,1] \) such that \( \phi \neq 0 \) on \( U \) and \( \sum \phi_U(x) = 1 \) for all \( x \in X \).

Complete Metric Spaces and Function Spaces

COMPLETE METRIC SPACES

Definition: Cauchy Sequence
Let \( (Y, d) \) be a metric space. A sequence \( (y_n)_{n \in \mathbb{N}} \) in \( (Y, d) \) is Cauchy if given \( \varepsilon > 0 \) there is an \( N > 0 \) such that
Definition: Complete
A metric space $(Y, d)$ is complete if every Cauchy sequence converges.

Definition: Standard Bounded Metric
The standard bounded metric associated to $d$ is $\tilde{d}(x, y) \equiv \min\{d(x, y), 1\}$.

Definition
If $Y$ is metric and $A$ is a set, $Y^A$ is the set of functions from $A$ to $Y$.

Definition: Bounded
Let $Y$ be metric. A function $\phi: A \to Y$ is bounded if $\text{diam}(\phi(A))$ is finite. Let $B(A, Y)$ denote the set of bounded functions from $A$ to $Y$.

Note: If $Y$ has a bounded metric, then all functions are bounded.

Definition: Sup Metric
Let $(Y, d)$ be a metric space and $\phi_1, \phi_2 \in B(A, Y)$. The sup metric on $Y^A$ is $\rho(\phi_1, \phi_2) = \sup \{d(\phi_1(a), \phi_2(a)) \mid \forall a \in A\}$.

Definition: Uniform Metric
Let $(Y, d)$ be a metric space and let $\tilde{d}$ be the standard bounded metric. The uniform metric on $Y^A$ is $\bar{\rho}(\phi_1, \phi_2) = \sup \{\tilde{d}(\phi_1(a), \phi_2(a)) \mid \forall a \in A\}$.

Remark
$\rho(\phi_1, \phi_2) < 1 \Leftrightarrow \bar{\rho}(\phi_1, \phi_2) < 1$, in which case $\rho = \bar{\rho}$.

Proposition
Let $f_n \in B(A, Y)$. Then $f_n \to f$ in the uniform metric if and only if $f_n \to f$ uniformly.

Theorem
If $(Y, d)$ is complete, then $(Y^A, \bar{\rho})$ is complete.

Proof: Let $(f_n)_{n \in \mathbb{N}}$ be Cauchy in $Y^A$. Then given $\varepsilon > 0$, $\bar{\rho}(f_n, f_m) = \sup \{\tilde{d}(f_n(a), f_m(a)) \mid \forall a \in A\} < \frac{\varepsilon}{2}$ for all $n, m > N_0$. So $(f_n(a))_{n \in \mathbb{N}}$ is Cauchy for all $a \in A$, and hence $f_n(a) \to f(a)$. So $\bar{\rho}(f_n, f) = \sup \{\tilde{d}(f_n(a), f(a)) \mid \forall a \in A\} < \frac{\varepsilon}{2} < \varepsilon$ for all $n > N_0$, so $f_n \to f$.

Proposition
Suppose $A$ is a topological space and $Y$ a metric space. Let $C(A, Y)$ be the set of continuous functions from from $A$ to $Y$. Then $C(A, Y)$ is closed in $B(A, Y)$ with the uniform metric.

Corollary
If \( Y \) is complete, then \( C(A,Y) \) is also complete with the uniform metric. If \( Y \) is complete and \( A \) is compact, then \( C(A,Y) \) is complete in the sup metric.

**Definition: Completion**

Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. Let \(i:X \to Y\) be an isometric embedding, that is \(d_X(x_1,x_2)=d_Y(i(x_1),i(x_2))\). \(Y\) is the completion of \(X\) if \(\overline{i(X)}=Y\) and \(Y\) is complete.

**Theorem**

Every metric space has a completion.

Proof: Embed \((X,d_X)\) into \(C(X,\mathbb{R})\) (bounded) with the sup topology as follows. Fix \(x_0 \in X\); let \(a \in X\). Define \(\phi_a(x)=d(x,a)-d(x,x_0)\). Then \(\phi_a(x)\leq d(a,x_0)\) be the triangle inequality (hence bounded). Let \(i:X \to C(X,\mathbb{R})\) be given by \(i(a)=\phi_a\). Now, \(\rho(i(a),i(b))=\sup_{x \in X}|\phi_a(x)-\phi_b(x)|=\sup_{x \in X}|d(x,a)-d(x,b)+d(x,x_0)|=\sup_{x \in X}|d(x,a)-d(x,b)|\leq d(a,b)\) by triangle inequality. However, taking \(x=a\), \(|d(a,a)-d(a,b)|=d(a,b)\). Hence \(\rho(i(a),i(b))=d(a,b)\), so \(i\) is an isometry. Let \(Y=\bar{i(X)}\).

**Peano Space-Filling Curve**

Corollary

There exists a continuous and onto map \(\phi:I \to I \times I\).

**Compactness in Metric Spaces**

**Theorem**

A metric space is compact if and only if it is complete and totally bounded.

**Definition: Equicontinuous**

Let \(X\) be a topological space, \(Y\) a metric space. The set of functions \(F \subseteq C(X,Y)\) is equicontinuous at \(x_0 \in X\) if given \(\varepsilon > 0\) there exists a neighborhood \(U_{x_0}\) of \(x_0\) such that \(d(f(x), f(y)) < \varepsilon\) for all \(f \in F\) and for all \(x,y \in U_{x_0}\).

\(F\) is equicontinuous if it is equicontinuous at each \(x_0 \in X\).

**Examples**

- Suppose that \(F \subseteq C^1(I,\mathbb{R})\) and \(|f'(x)| < 1\) \(\forall f \in F, x \in I\), then \(F\) is equicontinuous.
- If \(d(f(x), f(y)) < C(d(x,y) \alpha)\) for some fixed \(C\) and fixed \(\alpha\), then \(F\) is equicontinuous.

**Proposition**

If \(F \subseteq C(X,Y)\) is totally bounded in the uniform metric, then \(F\) is equicontinuous.

Note: \(d\) itself may or may not be bounded.

Proof: Let \(\varepsilon > 0\) be given. Assume \(\varepsilon < 1\). There exists \(f_1,\ldots,f_k\) such that \(B_{\varepsilon/3}(f_i)\) cover \(F\). Since each \(f_i\) is continuous, there exists a neighborhood \(U_{x_0}\) of \(x_0\) such that \(d(f_i(x), f_i(y)) < \varepsilon/3\) for all \(x,y \in U_{x_0}\). Given \(f \in F\), \(f \in B_{\varepsilon/3}(f_i)\) for some \(i\). Now \(d(f(x), f(y)) \leq d(f(x),f_i(x))+d(f_i(x),f_i(y))+d(f_i(y),f(y))\), and since
Let Corollary bounded, be given. Given any Proof: Let bounded. \( f \in B_{\text{sup}}(f) \Rightarrow d(f(y), f(y)) \leq \frac{\varepsilon}{3} \), so \( d(f(x), f(y)) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \). So \( F \) is equicontinuous.

**Proposition**

If \( X \) and \( Y \) are compact and \( F \subset C(X,Y) \) is equicontinuous, then \( F \) is totally bounded.

**Proof:** Let \( 0 < \varepsilon < 1 \) be given. Since \( F \) equicontinuous, let \( U_a \) be neighborhoods of \( a \in X \) such that \( d(f(x), f(y)) \leq \frac{\varepsilon}{3} \) for all \( x, y \in U_a \). \( X \) is compact, so let \( \{U_{a_i}\}_{i=1}^n \) cover \( X \). Let \( \{V_{\text{ii}}(y_i)\}_{i=1}^n \) be a finite cover of \( Y \) of \( \frac{\varepsilon}{3} \) - balls centered at \( y_i \). Now consider the set of functions \( \alpha \) where \( \alpha : \{1, \ldots, k\} \to \{1, \ldots, l\} \). If there is \( f \in F \) such that \( f(x) \in V_{\text{ii}}(y_{\alpha(i)}) \) for each \( i = 1, \ldots, k \), choose one label it \( f_a \). Then we get a finite collection of \( \varepsilon \) - balls \( \{B_{\varepsilon}(f_a)\} \). Now let \( f \in F \). For each \( i = 1, \ldots, k \), choose \( \alpha(i) \) such that \( f(x) \in V_{\text{ii}}(y_{\alpha(i)}) \). Let \( x \in X \), then \( x \in U_{a_i} \) for some \( a_i \). So \( d(f(x), f_{\alpha(i)}(x)) \leq d(f(x), f_{\alpha(i)}) + d(f_{\alpha(i)}, f_{\alpha(i)}(x)) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \). Hence \( \{B_{\varepsilon}(f_a)\} \) cover \( F \), so it is totally bounded.

**Definition: Pointwise Bounded**

\( F \in Y^X \) is pointwise bounded if \( \{f(x)\}_{f \in F} \) is a bounded set in \( Y \) for each \( x \in X \).

**Theorem: Ascoli’s Theorem, Classical Version**

Let \( X \) be compact. Let \( F \subset C(X,\mathbb{R}^n) \). \( F \) has a compact closure if and only if \( F \) is equicontinuous and pointwise bounded.

**Proof:** Let \( G \) denote the closure of \( F \).

- \( F \subset G \) and \( F \) is equicontinuous and pointwise bounded.

- No \( G \) is closed in the complete space \( \mathbb{R}^n \) and hence is complete. Let \( \varepsilon > 0 \) and \( x_0 \in X \) be given. Since \( F \) is equicontinuous, choose \( U_{x_0} \) such that \( d(f(x), f(y)) \leq \frac{\varepsilon}{3} \) for all \( x, y \in U_{x_0} \). Given \( g \in G \), choose \( f \in F \) such that \( \rho(f, g) \leq \frac{\varepsilon}{3} \). Then \( d(g(x), g(y)) \leq \frac{\varepsilon}{3} \) for all \( x, y \in U_{x_0} \) by triangle inequality. Hence \( G \) is equicontinuous. Now let \( x_n \in X \) be given. Given any \( g_1, g_2 \in G \), choose \( f_1, f_2 \in F \) such that \( \rho(f_1, g_1) \leq 1 \) and \( \rho(f_2, g_2) \leq 1 \). Since \( F \) is pointwise bounded, \( d(f_1(x_0), f_2(x_0)) \leq M \) for all \( x_0 \in X \), and hence \( d(g_1(x_0), g_2(x_0)) \leq M + 2 \). So \( G \) is pointwise bounded. Now, for each \( a \in X \), choose \( U_a \) such that \( d(g(x), g(y)) \leq \frac{\varepsilon}{3} \) for all \( g \in G \), \( x, y \in U_a \). Since \( X \) compact, cover it with \( U_a, \ldots, U_a \). Since \( G \) is pointwise bounded, \( \bigcup_{a=1}^k \{g(a)\}_{g \in G} \) is bounded, so suppose it lies in \( B_{\varepsilon}(0) \subset \mathbb{R}^n \). Then \( g(X) \subset B_{\varepsilon}(0) \forall g \in G \). Let \( Y = \overline{B_{\varepsilon}(0)} \), which is compact. Then \( G \subset C(X,Y) \) is totally bounded under \( \rho \).

- \( G \) is compact since it is complete and totally bounded.

**Corollary**

Let \( X \) be compact. Let \( F \subset C(X,\mathbb{R}^n) \). \( F \) is compact if and only if \( F \) is closed and bounded under the sup metric, and equicontinuous.

- If \( F \) is compact, it is closed and bounded. Since \( \overline{F} = F \), it is equicontinuous.

- If \( F \) is closed, so \( \overline{F} = F \). \( F \) is bounded under the sup metric, so it is pointwise bounded. Also, \( F \) is equicontinuous.
So \( F \) has a compact closure. But \( F \) is closed, so \( \bar{F} = F \) is compact.

**Compact-Open Topology**

**Definition: Compact-Open Topology**
Let \( X \) and \( Y \) be topological spaces. Describe a basis for \( C(X, Y) \) as follows. \( S(K, U) \equiv \{ f \in C(X, Y) | f(K) \subseteq U \} \) is open if \( U \subseteq Y \) is open and \( K \subseteq X \) is compact.

**Definition: Evaluation Map**
The map \( ev : C(X, Y) \times X \to Y \) defined by \( ev((f, x)) = f(x) \) is called the evaluation map.

**Theorem**
If \( C(X, Y) \) has the compact-open topology and \( X \) is locally compact Hausdorff, then \( ev : C(X, Y) \times X \to Y \) is continuous.

Proof: Let \( (f, x) \in C(X, Y) \times X \) and a neighborhood \( V \subseteq Y \) of \( ev((f, x)) = f(x) \) be given. By the continuity of \( f \), \( f^{-1}(V) \) is open and contains \( x \). Since \( X \) is locally compact Hausdorff, there exists a neighborhood \( U \) of \( x \) such that its compact closure \( \bar{U} \subseteq f^{-1}(V) \), and hence \( f(\bar{U}) \subseteq V \). Let \( K = \bar{U} \). Then \( (f, x) \in S(K, V) \times U \) is open, and \( ev(S(K, V), U) \subset V \).

**Definition**
Given a function \( f : Z \times X \to Y \), it gives rise to a function \( F : Z \to Y^X \) defined by \( F(z)(x) = f(z, x) \).

Conversely, given \( F : Z \to Y^X \), there is a corresponding function \( f : Z \times X \to Y \) given by \( f(z, x) = F(z)(x) \).

\( F \) is the map induced by \( f \).

**Theorem**
Give \( C(X, Y) \) the compact-open topology. If \( f : Z \times X \to Y \) is continuous, then \( F : Z \to C(X, Y) \) is continuous.

Conversely, if \( F : Z \to C(X, Y) \) is continuous and \( X \) is locally compact Hausdorff, then \( f : Z \times X \to Y \) is continuous.

Proof:
(\( \Rightarrow \)) Suppose \( f : Z \times X \to Y \) is continuous. Let \( z \in Z \) and \( F(z) \in S(K, U) \) in \( C(X, Y) \) be given. By continuity of \( f \), \( f^{-1}(U) \subseteq Z \times X \) is open and contains \( z \times K \). Since \( K \) is compact, the tube lemma implies there is a neighborhood \( W \) of \( z \) such that \( W \times K \subseteq f^{-1}(U) \). Hence \( F(W)(K) = f(W, K) \subseteq U \), so \( F \) is continuous.

(\( \Leftarrow \)) Suppose \( F : Z \to C(X, Y) \) is continuous. Then \( j : Z \times X \to C(X, Y) \times X \) given by \( j(z, x) = (F(z), x) \) is continuous. Then \( ev : C(X, Y) \times X \to Y \) is continuous since \( C(X, Y) \) has the compact-open topology and \( X \) is locally compact Hausdorff. Therefore, \( ev \circ j : Z \times X \to Y \) given by \( (ev \circ j)(z, x) = ev(F(z), x) = F(z)(x) = f(z, x) \) is continuous.

**Baire Spaces and Dimension Theory**

**Baire Spaces**

**Definition: Baire Space**
Baire space is a space in which the intersection of a countable collection of open and dense sets is dense. That is, if \( \{ U_n \}_{n \in \mathbb{N}} \) are open and dense sets, \( \bigcap_{n \in \mathbb{N}} U_n \) is dense.
Proposition

$X$ is a Baire space if and only if the countable union of closed sets without interior has no interior, i.e. if $\{A_i\}_{i \in \mathbb{N}}$ are closed and $\operatorname{int}(A_i) = \emptyset$ $\forall i \in \mathbb{N}$ then $\operatorname{int}\left( \bigcup_{i \in \mathbb{N}} A_i \right) = \emptyset$.

Proof:

(⇒) $A_i'$ is open. If $U \neq \emptyset$ is open, then $U \not\subset A_i'$ since $\operatorname{int}(A_i) = \emptyset$. So $U \cap A_i' \neq \emptyset$, hence $A_i'$ is dense (since $\overline{T} = X$).

Since $X$ is a Baire space so $\bigcap_{i \in \mathbb{N}} A_i'$ is dense, i.e. $U \cap \bigcap_{i \in \mathbb{N}} A_i' \neq \emptyset$ for all open $U$. Therefore $U \not\subset \bigcup_{i \in \mathbb{N}} A_i$, so $\bigcup_{i \in \mathbb{N}} A_i$ has no interior.

Definition: Residual

A set $A$ in a Baire space $X$ is residual if it contains the intersection of a countable family of open and dense sets.

Proposition

If $A$ and $B$ are residual, $A \cap B$ is residual.

If $A_i$ is residual, $\bigcap_{i \in \mathbb{N}} A_i$ is residual.

Theorem: Baire Category Theorem

If $X$ is compact Hausdorff or complete metric, then $X$ is a Baire space.

Proof: Suppose $\{A_i\}_{i \in \mathbb{N}}$ is a family of closed sets with no interior. Want: $\bigcup_{i \in \mathbb{N}} A_i$ has no interior, i.e. given any open set $U$ there is a point $x \in U$ and $x \not\in \bigcup_{i \in \mathbb{N}} A_i$.

We wish to construct $U_i$ such that $U_i \subset U_{i-1}$, $U_i \cap A_i = \emptyset$, $\bigcap_{i \in \mathbb{N}} U_i \neq \emptyset$. Then let $x \in \bigcap U_i$, so $x \in U_i \forall i$ and $x \in U_0$. Then $x \not\in \bigcup_{i \in \mathbb{N}} A_i$, and hence $x \not\in \bigcup_{i \in \mathbb{N}} A_i$.

$A_i$ has empty interior, so there exists $U_0$ such that $x \in U_0$ and $x \not\in A_i$. Since $X$ is normal, there exists $U_1$ such that $x \in U_1$, $U_1 \subset U_0$, and $U_1 \cap A_i = \emptyset$. Now assume $U_i$ is constructed. $A_{i+1}$ has empty interior, so there exists $x_{i+1} \in U_i \not\subset A_{i+1}$, hence there exists $U_{i+1}$ such that $x_{i+1} \in U_{i+1}$, $U_{i+1} \subset U_i$, and $U_{i+1} \cap A_{i+1} = \emptyset$ since $X$ is normal.

If $X$ is compact Hausdorff, $\{U_i\}_{i \in \mathbb{N}}$ is a family of non-empty nested compact sets, so $\bigcap_{i \in \mathbb{N}} U_i \neq \emptyset$.

If $X$ is complete metric, add $\operatorname{diam} U_i \leq \frac{1}{i}$. Then $x_i$ is Cauchy, so $x_i \to x$. Now $x \in \bigcap U_i \forall i$, so $x \in \bigcap_{i \in \mathbb{N}} U_i \neq \emptyset$.

Fact

$\mathbb{Q} \subset \mathbb{R}$ is not residual. Given any $q \in \mathbb{Q}$, $U_q = \mathbb{R} - \{q\}$ is open and dense. But $\left( \bigcap_{q \in \mathbb{Q}} U_q \right) \cap \mathbb{Q} = \emptyset$, so $\mathbb{Q}$ does not contain the intersection of a countable family of open and dense sets.

Theorem

Consider $F \subset C(I, \mathbb{R})$ where $F = \{ f \in C(I, \mathbb{R}) \mid f$ is not differentiable at any point $x \in I \}$. Then $F$ is residual in $C(I, \mathbb{R})$.

Proof: Construct a countable family of open dense sets in $C(I, \mathbb{R})$ whose intersection is contained in $F$. To be differentiable, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists. Construct the family $\{f_k\}_{k \in \mathbb{N}}$ where the limit is larger than $k$ for $h$ small enough.
**Proposition**
Any open set in a Baire space is a Baire space.

Proof: Let $U \subset X$ be open. Let $\{A_i\}_{i \in \mathbb{N}}$ be closed subsets of $U$ with empty interior. $A_i = a_i \cap U$ where $a_i$ closed in $X$. $a_i \cap U$ has no interior, for if it did, let $v \in V \subset \text{int}(a_i \cap U)$, but $V \cap U \neq \emptyset$ and is open, and that $V \cap U \subset \text{int}(A_i)$, so contradiction. Hence $\{a_i \cap U\}_{i \in \mathbb{N}}$ are closed in $X$ with no interior. Since $X$ is a Baire space, $\bigcup (a_i \cap U)$ has no interior. Therefore, $\bigcup A_i$ have no interior in $U$.

**Theorem**
Let $X$ be a Baire space and $(Y, d)$ be a metric space. Let $f_n: X \to Y$ be a sequence of continuous functions that converge pointwise for all $x \in X$. Then $f = \lim f_n$ is continuous on a dense set of points in $X$.

Proof: Let $A_n(\varepsilon) = \{x | d(f_n(x), f_m(x)) \leq \varepsilon \quad \forall n, m > N\}$. $A_n(\varepsilon)$ is closed. $\bigcup_n A_n(\varepsilon) = X$. Now for any open $U \subset X$, $\bigcup (A_n(\varepsilon) \cap U) = U$, so at least one $A_n(\varepsilon) \cap U$ has interior. Let $U_1 = \bigcup \text{int}(A_n(\varepsilon))$ is open and dense. Let $\varepsilon = \frac{1}{n}$. Then $\Lambda = \bigcap U_{1/n}$ is residual.

Claim: $f$ is continuous at each point of $\Lambda$. Let $x \in \Lambda$ and fix $\varepsilon > 0$. Take $\frac{1}{4n} < \varepsilon$. Then $x \in U_{1/4n}$ and $x \in \text{int} \left( A_n \left( \frac{1}{4n} \right) \right)$ for some $N$ so there exists $U_s \subset A_n \left( \frac{1}{4n} \right)$. So for every $y \in U_s$ and for all $n, m > N$, $d \left( f_n(y), f_m(y) \right) < \frac{1}{4n} \Rightarrow d \left( f_n(y), f(y) \right) \leq \frac{1}{4n}$. Choose $m > N$. There exists $V_s$ such that $y \in V_s \Rightarrow d \left( f_m(x), f_m(y) \right) \leq \frac{1}{4n}$. Then for $x, y \in U_s \cap V_s$, $d \left( f(x), f(y) \right) \leq d \left( f_m(x), f(x) \right) + d \left( f_m(x), f_m(y) \right) + d \left( f_m(y), f(y) \right) \leq \frac{1}{4n} + \frac{1}{4n} + \frac{1}{4n} = \frac{3}{4n} < \varepsilon$. 29 of 29