Topological Spaces and Continuous Functions

Topological Spaces

Definition: Topology

A topology on a set X is a collection T of subsets of X, with the following properties:
1. $\emptyset, X \in T$.
2. If $u_\alpha \in T, \alpha \in A$, then $\bigcup_{\alpha \in A} u_\alpha \in T$.
3. If $u_i \in T, i = 1, \ldots, n$, then $\bigcap_{i=1}^{n} u_i \in T$.

The elements of T are called open sets.

Examples

1. $T = \{\emptyset, X\}$ is an indiscrete topology.
2. $T = 2^X =$ set of all subsets of X is a discrete topology.

Definition: Finer, Coarser

T_1 is finer than T_2 if $T_2 \subset T_1$. T_2 is coarser than T_1.

Basis For a Topology

Definition: Basis

A collection of subsets B of X is called a bais for a topology if:
1. The union of the elements of B is X.
2. If $x \in B_1 \cap B_2$, $B_1, B_2 \in B$, then there exists a B_3 of B such that $x \in B_3 \subset B_1 \cap B_2$.

Examples

1. B is the set of open intervals (a, b) in $\mathbb{R}$ with $a < b$. For each $x \in \mathbb{R}$, $x \in \left(x - \frac{1}{2}, x + \frac{1}{2}\right)$.
2. B is the set of all open intervals (a, b) in $\mathbb{R}$ where $a < b$ and a and b are rational numbers.
3. Let T be the collection of subsets of $\mathbb{R}$ which are either empty or are the complements of finite sets. Note that $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. This topology does not have a countable basis.

Claim

A basis B generates a topology T whose elements are all possible unions of elements of B. That is, the topology generated by B is the collection of arbitrary unions of the subsets of B.

Proof: Have to prove that if $u_1 \cup \cdots \cup u_n \in T$ then $u_1 \cap \cdots \cap u_n \in T$. By induction, it suffices to prove that if $u_1 \cup u_2 \in T$ then $u_1 \cap u_n \in T$. If $x \in u_1 \cap u_2$, then there exists $u_1 = \bigcup B_\alpha$, $\alpha \in A$ and $u_2 = \bigcup B_\gamma$, $\gamma \in G$. Also there exists B_{α_x} , $\alpha_x \in A$ and B_{γ_x} , $\gamma_x \in G$ such that $x \in B_{\alpha_x} \cap B_{\gamma_x}$, so $x \in B_{\alpha\gamma}$ such that $B_{\alpha\gamma} \subset B_{\alpha_x} \cap B_{\gamma_x}$. So $u_1 \cap u_2 = \bigcup_x B_{(\alpha\gamma)_x}$.

Examples

1. S, the standard topology on $\mathbb{R}$, is generated by the basis of open intervals (a, b) where $a < b$.
2. The open sets of F are complements of finite sets and $\emptyset$.

3. A basis for another topology on $\mathbb{R}$ is given by half open intervals $[a,b)$, $a<b$. It generated the lower limit topology L .
4. The open intervals (a,b) , $a<b$ with a and b rational is a countable basis. It generateds the same topology as S .

Claim: S is finer than F , and L is finer than S .

Proposition
Suppose β and β' are bases for topologies T and T' on the same space X . If they have the property that for every $B\in\beta$ and $x\in\beta$ there exists $B'\in\beta'$ such that $x\in\beta'\subset\beta$, then T' is finer than T .

Proof: If $B\in\beta$ and $x\in\beta$, there exists B_x' such that $x\in B_x'\subset B$ and $B_x'\in\beta'$. $B=\bigcup_{x\in B} B_x'\in T'$. So every $B\in\beta$ is in T' .

Definition: Sub-Base
A sub-base for a topology on X is a collection β of subsets on X satisfying $\bigcup_{B\in\beta} B=X$.

We build a basis by taking all finite intersections of the elements of β .

The Subspace Topology

Relative Topology
Given a topology T on X and a subset Y of X , T induces a topology T_Y on Y called the relative topology. $T_Y=\{t\cap Y|t\in T\}$.

Check that T_Y is a topology:
1. $\emptyset=\emptyset\cap Y$, $Y=X\cap Y$.
2. Let $S_\alpha\in T_Y$, $S_\alpha=T_\alpha\cap Y$. So $\bigcup S_\alpha=(\bigcup T_\alpha)\cap Y$.
3. $S_1\cap\cdots\cap S_n=(T_1\cap Y)\cap\cdots\cap(T_n\cap Y)=(T_1\cap\cdots\cap T_n)\cap Y$.

Closed Sets and Limit Points

Definition: Closed
A subset $A\subset X$ a topological space is closed if A^C is open.

Properties of Closed Sets
1. $\emptyset$, X are closed.
2. If A_β , $\beta\in B$ is closed, then $\bigcap_{\beta\in B} A_\beta$ is closed.

   Proof: $\left(\bigcap_{\beta\in B} A_\beta\right)^c=\bigcup_{\beta\in B} A_\beta^c$ is open.
3. If $A_{1,}\ldots,A_n$ are closed, then $\bigcup_{i=1}^{n} A_i$ is closed.

Examples
Consider the standard topology on $\mathbb{R}$.
1. Let $x\in\mathbb{R}$. $\{x\}$ is closed.

2. $I=[a,b]$ is closed.

Definition: Interior, Closure

Let X be a topological space. Let $A\subset X$. The interior of A, denoted $\mathring{A}$, is the largest open set in A. The closure $\overline{A}$ is the smallest closed set containing A.

Proposition

$x\in\mathring{A}$ if and only if there exists an open U such that $x\in U\subset A$.

Proof:

($\Rightarrow$) $x\in\mathring{A}$, take $U=\mathring{A}$.

($\Leftarrow$) If $x\in U\subset A$, U open, then $\mathring{A}\cup U=\mathring{A}$ is open and contained in A. So $U\subset\mathring{A}$ and $x\in\mathring{A}$.

Proposition

$x\in\overline{A}$ if and only if for all open U, $x\in U$, $U\cap A\neq\emptyset$.

Proof:

($\Rightarrow$) If $x\in\overline{A}$, $x\in U$ and $U\cap A=\emptyset$, then U^c is closed and contains A. So $\overline{A}\cap U^c$ is closed and contains A, but $x\notin\overline{A}\cap U^c$ which is smaller than A.

($\Leftarrow$) Now suppose $x\in U$, $U\cap A\neq\emptyset$. Consider $\overline{A}^c$, which is open. If $x\in\overline{A}^c$, then $A\cap\overline{A}^c\neq\emptyset$. Contradition. So $x\in\overline{A}$.

Definition: Limit Point

Let $A\subset X$. x is a limit point of A iff every open set U, $x\in U$, intersects A in a point different from x.

Proposition

Let A' be the set of limit points of A. Then $\overline{A}=A\cup A'$.

Proposition

$f:X\rightarrow Y$ is continuous if and only if $f(\overline{A})\subset\overline{f(A)}$ for all $A\subset X$.

Proof:

($\Rightarrow$) Suppose f is continuous and $x\in\overline{A}$. To prove $f(x)\in\overline{f(A)}$, it suffices to prove that if V is open, $f(x)\in V$, then $V\cap f(A)\neq\emptyset$. Now $f^{-1}(V)$ is open and $x\in f^{-1}(V)$ so $f^{-1}(V)\cap A\neq\emptyset$. So $V\cap f(A)\neq\emptyset$.

($\Leftarrow$) Suppose $C\in Y$ is closed. Then $f(f^{-1}(C))\subset C$, and $f(\overline{f^{-1}(C)})\subset\overline{f(f^{-1}(C))}\subset C$. So $f^{-1}(C)=\overline{f^{-1}(C)}$. Hence $f^{-1}(C)$ is closed.

Lemma

A is closed if and only if $A=\overline{A}$.

Definition: Hausdorff

A topological space X is a Hausdorff space if given any two points $x,y\in X$, $x\neq y$, there exists neighbourhoods U_x of x, U_y of y such that $U_x\cap U_y\neq\emptyset$.

Definition: T_1

A topological space X is a T_1 if given any two points $x, y \in X$, $x \neq y$, there exists neighbourhoods U_x of x such that $y \notin U_x$.

Proposition
If the topological space X is T_1 or Hausdorff, points are closed sets.

Continuous Functions

Definition: Continuity
Let X and Y be topological spaces. A function $f: X \to Y$ is continuous if $f^{-1}(U)$ is open in X for every open set U in Y.

Definition: Neighbourhood
An open set containing x is called a neighbourhood of x.

Definition: Continuity Pointwise
Let X and Y be topological spaces. A function $f: X \to Y$ is continuous at $x \in X$ iff $f^{-1}(U(f(x)))$ contains a neighbourhood of x for all neighbourhoods $U(f(x))$ of $f(x)$.

Theorem
Let X and Y be topological spaces. $f: X \to Y$ is continuous if and only if it is continuous at every $x \in X$.

Proof:
($\Rightarrow$) Let $x \in X$ and $U(f(x))$ be a neighbourhood of $f(x)$. Then $f^{-1}(U(f(x)))$ is a neighbourhood of x.
($\Leftarrow$) Let $U \subset Y$ be open. Let $x \in f^{-1}(U)$. Then $f(x) \in U$, so $f^{-1}(U)$ contains a neighbourhood V_x of x.
$\bigcup_{x \in f^{-1}(U)} V_x = f^{-1}(U)$ is open.

Example
$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ is continuous in the lower limit topology of $\mathbb{R}$, but not in the standard topology.

Identifying Some Continuous Function
X and Y topological spaces.

1. $\mathrm{Id}: X \to X$ is continuous.
   Proof: $\mathrm{Id}^{-1}(U) = U$.
2. If $y \in Y$, then $f: X \to Y$, $f(x) = y, \forall x \in X$ (a constant function) is continuous.
3. If $f: X \to Y$ is continuous and $A \subset X$ has the relative topology, then $(f|A): A \to Y$ is continuous.
   Proof: $(f|A)^{-1}(U) = f^{-1}(U) \cap A$.
4. If $A \subset X$ has the relative topology, $i: A \to X, i(a) = a$ is continuous.
5. If $f(X)$ is given the relative topology in Y and $f: X \to Y$ is continuous, then $f: X \to f(X)$ is continuous.
   Proof: If U is open in $f(X)$, there exists V open in Y such that $U = V \cap Y$ so $f^{-1}(U) = f^{-1}(V)$ is open.

Proposition
If $f: X \to Y$ is continuous and $g: Y \to Z$ is continuous, then $g \circ f: X \to Z$ is continuous.

Proof: If U is open in Z, then $(g\circ f)^{-1}(U)=f^{-1}(g^{-1}(U))$ is open.

Proposition
Let $\pi_i, i=1,2$ be the projection on the i-th factor (so $\pi_1(y_1,y_2)=y_1$ and $\pi_2(y_1,y_2)=y_2$). The π_i's are continuous.

Proof: If U is open in Y_1, $\pi_1^{-1}(U)=U\times Y_2$ which is open.

Proposition
$f:X\rightarrow Y_1\times Y_2$ is continuous if and only if $\pi_i\circ f=f_i$ is continuous for $i=1,2$.

Proof:
($\Rightarrow$) f and π_i are continuous, so $\pi_i\circ f$ is continuous.
($\Leftarrow$) $\pi_i\circ f$ is continuous means given U_i, $(\pi_i\circ f)^{-1}(U_i)$ is open, but $f^{-1}(U_1\times U_2)=(\pi_1\circ f)^{-1}(U_1)\cap(\pi_2\circ f)^{-1}(U_2)$ is open (using basis, etc.).

Proposition
The product topology is the coarsest topology with the property that $f:X\rightarrow Y_1\times Y_2$ is continuous if and only if $f_i:X\rightarrow Y_i$ is continuous.

Proof: $\mathrm{Id}:Y_1\times Y_2\rightarrow Y_1\times Y_2$ is continuous. So $\pi_1\circ\mathrm{Id}$ is continuous if U_1 is open in Y_1 and U_2 is open in Y_2. Now $U_1\times Y_2$ is open in $Y_1\times Y_2$ and $Y_1\times U_2$ is open in $Y_1\times Y_2$.

Proposition
$f:X\rightarrow Y$ is continuous if and only if $f^{-1}(C)$ is closed for all closed C in Y.

Proof: $f^{-1}(C)=(f^{-1}(C^c))^c$.

Definition: Homeomorphism
A 1-1 onto map $f:X\rightarrow Y$ whose inverse $f^{-1}:Y\rightarrow X$ is also continuous is called a homeomorphism. X and Y are said to be homeomorphic.

Proposition
Suppose X is Hausdorff or T_1, and x is a limit point of X. Then any neighbourhood U of x contains infinitely many distinct points of X.

Proof: Suppose U is a neighbourhood of x, and U has only n distinct points $x_1,\dots,x_n$ where $x_i\neq x$. Then there exists U_i neighbourhood of x such that $x_i\notin U_i$. Then $U\cap U_1\cap\cdots\cap U_n$ is a neighbourhood of x and $U\cap U_1\cap\cdots\cap U_n$ has only x in it. Contradiction.

Definition: Convergence of Sequences
The sequence x_i $,i\in\mathbb{N}$ converges to x in X iff for any neighbourhood U of x, there exists N such that $x_i\in U$ for all $i>N$. x is called a limit point of x_i $,i\in\mathbb{N}$, written as $\lim x_i=x$ or $x_i\rightarrow x$.

Proposition
If X is Hausdorff, then the limit points are unique; that is, if $x_i\rightarrow x$ and $x_i\rightarrow y$, then $x=y$.

Proof: Assume otherwise. Then there exists neighbourhoods U_x of x and U_y of y such that $U_x \cap U_y = \emptyset$. So $x_i \ , i \in \mathbb{N}$ can't converge to both x and y.

Proposition
Suppose X is Hausdorff and $x_i \rightarrow x$. Then $\{x_i \cup x\}$ is closed.

Proposition
Suppose $x \in X$ and x is not a limit point of X, then $\{x\}$ is open.

Definition: Open Cover
Let $U_\alpha, \alpha \in A$ be open sets in X. Then $U_\alpha, \alpha \in A$ is called an open cover if $X = \bigcup_\alpha U_\alpha$.

Proposition
Let $f: X \rightarrow Y$ be a function. Let $U_\alpha, \alpha \in A$ be an open over. Then f is continuous if and only if $f | U_\alpha$ is continuous for all $\alpha \in A$.

Proof: Suppose $f | U_\alpha$ is continuous for all $\alpha \in A$ and $V \subset Y$ is open. Then $f^{-1}(V) = \bigcup_{\alpha \in A} (f | U_\alpha)^{-1}(V)$. $(f | U_\alpha)^{-1}(V)$ is open in U_α, so it is open in X. Hence $f^{-1}(V)$ is open in X.

Pasting Lemma
Suppose $A, B \subset X$ are closed, $f: A \rightarrow Y$ and $g: B \rightarrow Y$, and $f = g$ on $A \cap B$. Let h be defined on $A \cup B$, $h = f$ on A and $h = g$ on B. Then $h: A \cup B \rightarrow Y$ is continuous.

Proof: Let $C \subset Y$ be closed. $g^{-1}(C)$ is closed in B and $f^{-1}(C)$ is closed in A, so $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ is closed since $A \cup B$ is closed.

Proposition
$f: X \rightarrow Y$ is continuous if and only if $f^{-1}(U)$ is open for every element U of a basis B of the topology on Y.

Proof: If U is open in Y, then $U = \bigcup_{\alpha \in A} V_\alpha$ for some collection V_α of basis elements in B. Then $f^{-1}(U) = f^{-1}(\bigcup_{\alpha \in A} V_\alpha) = \bigcup_{\alpha \in A} f^{-1}(V_\alpha)$, which is an union of open sets, so open.

The Product Topology

Examples: Product Spaces
1. X, Y sets. $X \times Y = \{(x, y) | x \in X, y \in Y\}$.
2. X_α sets indexed by $\alpha \in A$. The product $\prod_{\alpha \in A} X_\alpha$ is functions from A to X_α such that α goes into X_α.

Definition: Product Topology
Let X, Y be sets with topologies T_X and T_Y. We define a topology $T_{X \times Y}$ on $X \times Y$ called the product topology by taking as basis all sets of the form $U \times W$ where $U \in T_X$ and $W \in T_Y$.

Note: $\bigcup_{\alpha\in A} X_\alpha \times \bigcup_{\beta\in B} Y_\beta = \bigcup_{(\alpha,\beta)\in A\times B} X_\alpha \times Y_\beta$. So a basis for T_X and a basis for T_Y generate a basis for $T_{X\times Y}$.

Examples

1. Standard × standard on $\mathbb{R}^2$; basis is open rectangles.
2. Standard × standard × standard on $\mathbb{R}^3$; basis is open cubes.

Definition: Box Topology

Let $X_\alpha, \alpha\in A$ be topological spaces. A basis of open sets of a topology on $\prod_{\alpha\in A} X_\alpha$ is $\prod_{\alpha\in A} U_\alpha$ where U_α is open in X_α. The topology it generates is called the box topology.
Note: If we allow $U_\alpha \neq X_\alpha$ for finitely many $\alpha\in A$, we get the product topology.

Remarks

- Box topology is finer than the product topology.
- For fixed $a\in A$, $\pi_a : \prod X_\alpha \to X_a$ is continuous.

Proposition

The product topology is the unique topology with the property that $f : Y\to \prod_{\alpha\in A} X_\alpha$ is continuous if and only if $\pi_a \circ f : Y\to X_a$ is continuous for each $a\in A$.

Proposition

If $X_\alpha, \alpha\in A$ are Hausdorff, so is $\prod_{\alpha\in A} X_\alpha$ with the product or box topology.

Proof: If $x, y\in \prod_{\alpha\in A} X_\alpha$ and $x\neq y$, then $x_{\alpha_0} \neq y_{\alpha_0}$ for some $\alpha_0 \in A$. Let U_x and U_y be open sets in X_{α_0} such that $x_{\alpha_0} \in U_x$ and $y_{\alpha_0}\in U_y$ and $U_x \cap U_y = \emptyset$. Then $U_x \times \prod_{\alpha\neq\alpha_0} X_\alpha$ and $U_y \times \prod_{\alpha\neq\alpha_0} X_\alpha$ separate x and y.

Proposition

If $A_\alpha \subset X_\alpha$ is closed, so is $\prod A_\alpha \subset \prod X_\alpha$.

Proof: Let $x\in (\prod A_\alpha)^c$. Then $x_{\alpha_0} \in (A_{\alpha_0})^c$ for some α_0. So $x\in (A_{\alpha_0})^c \times \prod_{\alpha\neq\alpha_0} X_\alpha = (\prod A_\alpha)^c$ is an open set.

The Metric Topology

Definition: Metric

A metric on a set X is a function $d : X\times X\to \mathbb{R}_+$ such that for all $x, y, z\in X$:

1. $d(x, y) = d(y, x)$.
2. $d(x, y) = 0$ iff $x = y$.
3. $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality).

Definition: Metric Space

A metric space is a set X with a distance function d.

Definition: Open Ball

The open ball of radius $r>0$ around $x\in X$ is $B_r(X)=\{y\in X|d(x,y)<r\}$.

Lemma

If $y\in B_r(X)$, then $B_s(y)\subset B_r(x)$ for $s<r-d(x,y)$.

Proof: If $z\in B_s(y)$, then $d(x,z)\leq d(x,y)+d(y,z)<d(x,y)+s<d(x,y)+r-d(x,y)=r$.

Proposition

The open balls are a basis of a topology on X .

Proof: Let $z\in B_r(x)\cap B_s(y)$. Let $\varepsilon<\min(r-d(x,z),s-d(y,z))$. Then $B_\varepsilon(z)\subset B_r(x)\cap B_s(y)$.

Example

Let X be any non-empty set. Set $d(x,y)=1$ if $x\neq y$ for all $x,y\in X$, and $d(x,x)=0$. Then $B_{\frac{1}{2}}(x)=\{x\}$. This metric generates the discrete topology.

Definition: Metrizable

A topological X is metrizable if there exists a metric d on set X that induces the topology of X .

Proposition

A metric space is Hausdorff.

Proof: If $x\neq y$, then $d(x,y)>0$. So $B_{\frac{d(x,y)}{3}}(x)\cap B_{\frac{d(x,y)}{3}}(y)=\emptyset$, since if $z\in B_{\frac{d(x,y)}{3}}(x)\cap B_{\frac{d(x,y)}{3}}(y)$ then $d(x,z)<\frac{1}{3}d(x,y)$ and $d(y,z)<\frac{1}{3}d(x,y)$, but $d(x,y)\leq d(x,z)+d(z,y)<\frac{2}{3}d(x,y)$, so contradiction.

Examples

- If X has at least 2 points, then the indiscreet topology is not metrizable.
- The topology F (complements of finite sets on $\mathbb{R}$) is not Hausdorff and not metrizable.

Example

On $\mathbb{R}$, $d(x,y)=|x-y|$ is a metric on $\mathbb{R}$ which gives the standard topology.

Proposition

If d_1 and d_2 are metrics on X , and for each $x\in X$, $r_1,r_2\in\mathbb{R}_+$ there exists $s_1,s_2\in\mathbb{R}_+$ such that $B_{s_1}^{d_1}(x)\subset B_{r_1}^{d_2}(x)$ and $B_{s_2}^{d_2}(x)\subset B_{r_2}^{d_1}(x)$, then these two metrics generate the same topologies.

Proposition

If d_1 and d_2 are metrics on X and there exists $c_1,c_2>0$ such that $c_1d_1(x,y)\leq d_2(x,y)\leq c_2d_1(x,y)$ for all $x,y\in X$, then these two metrics generate the same topologies.

Definition: Bounded
A metric space X is bounded (or has finite diameter) if there exists $k>0$ such that $d(x,y)\leq k$ for all $x,y\in X$.

Constructing a Bounded Metric
Start with a metric d on X. We can produce a new metric with diameter 1 which gives it the same topology.
Let $\bar{d}(x,y)=\min(d(x,y),1)$. Then $B^{d}_{\min(1,r)}(x)\subseteq B^{\bar{d}}_{r}(x)$ and $B^{\bar{d}}_{\min(\frac{1}{2},r)}(x)\subseteq B^{d}_{r}(x)$, so the topologies are the same. $\bar{d}$ gives the standard bounded metric.

Example: Uniform Topology
If $X_\alpha,\alpha\in A$ are bounded metric spaces with bounded diameters (say ≤ 1), then $\prod X_\alpha$ has $d(x,y)=\sup_\alpha d(x_\alpha,y_\alpha)$.
This gives the uniform topology.
Note: Product topology is finer than the uniform topology which is finer than the box topology.

Proposition
$f:X\to Y$ is continuous at $x_0\in X$ if and only if given $\varepsilon>0$ there exists $\delta>0$ such that $d_X(y,x_0)<\delta\Rightarrow d_Y(f(y),f(x_0))<\varepsilon$.

Proof:
($\Rightarrow$) This is obvious.
($\Leftarrow$) If U is a neighborhood of $f(x_0)$, there exists $\varepsilon>0$ such that $B_\varepsilon(f(x_0))\subset U$. So there exists $\delta>0$ such that $B_\delta(x_0)\subset f^{-1}(B_\varepsilon(f(x_0)))$, and so $f(B_\delta(x_0))\subset U$.

Lemma: Sequence Lemma
Let X be a metric space. $x\in\bar{A}$ if and only if there exists $x_n\in A$ such that $x_n\to x$.

Proof:
($\Rightarrow$) If $x\in\bar{A}$, then $B_{\frac{1}{n}}(x)\cap A\neq\emptyset,\forall n\in\mathbb{N}$. Let $x_n\in B_{\frac{1}{n}}(x)\cap A$. Then $x_n\to x$.
($\Leftarrow$) If $x_n\to x$, then given a neighborhood U of x, U contains all but finitely many of the x_n's, so $U\cap\{x_n\}_{n\in\mathbb{N}}\neq\emptyset$ and so $U\cap A\neq\emptyset$.

Remark
The Sequence Lemma is true if X satisfies the First Countability Axiom: For any $x\in X$, there exists $U_n, n\in\mathbb{N}$ open neighborhoods of x such that if V is a neighborhood of x, $V\supset U_n$ for some $n\in\mathbb{N}$.

Theorem
Let X be a metric space. Then $f:X\to Y$ is continuous if and only if $f(x_n)\to f(x)$ whenever $x_n\to x$.

Proof:
($\Rightarrow$) Suppose f is continuous and U is a neighborhood of $f(x)$ and that $x_n\to x$. Then $f^{-1}(U)$ is a neighborhood of x, so there exists N such that $n>N\Rightarrow x_n\in f^{-1}(U)\Rightarrow f(x_n)\in U$.
($\Leftarrow$) To prove f is continuous, it suffices to prove $f(\bar{A})\subset\overline{f(A)},\forall A\subset X$. Let $x\in\bar{A}$, $x_n\in A$, $x_n\to x$. Then $f(x_n)\to f(x)$. So given U neighborhood of $f(x)$, there exists N such that $n>N\Rightarrow f(x_n)\in U$. Since $x_n\in A$, so $f(x_n)\in f(A)$. So $f(x)\in\overline{f(A)}$.

Definition: Uniform Convergence

Let X and Y be metric spaces. $f_n: X \to Y$ converges uniformly to $f: X \to Y$ if given $\varepsilon > 0$ there exists $N > 0$ such that for all $n > N$, $d(f_n(x), f(x)) < \varepsilon$ for all $x \in X$.

Proposition
Let X and Y be metric spaces. If $f_n: X \to Y$ converges uniformly to $f: X \to Y$ and f_n continuous, then f is continuous.

Proof: Given $x_0 \in X$ and $\varepsilon > 0$, we have to find $\delta > 0$ such that $d(f(x_0), f(y)) < \varepsilon$ whenever $d(x_0, y) < \delta$. Find $N > 0$ such that $d(f(x_1), f_n(x_1)) < \frac{\varepsilon}{3}$ for all $n > N$ and all $x_1 \in X$. Now fix $n > N$. f_n is continuous, so there exists $\delta > 0$ such that $d(x_0, y) < \delta \Rightarrow d(f_n(x_0), f_n(y)) < \frac{\varepsilon}{3}$. So
$d(f(x_0), f(y)) \leq d(f(x_0), f_n(x_0)) + d(f_n(x_0), f_n(y)) + d(f_n(y), f(y)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

The Quotient Topology

Definition: Quotient Map
Let X and Y be topological spaces and $f: X \to Y$ a continuous surjective (onto) map. Then f is a quotient map if moreover $f^{-1}(U) \subset X$ is open if and only if $U \subset Y$ is open.

Definition: Open Map
$f: X \to Y$ is open iff $f(U) \subset Y$ is open for all open $U \subset X$.

Proposition
$\pi: X \times Y \to X$ is an open map.

Proof: Let $U \subset X \times Y$ be open. Then $U = \bigcup_{i \in I} (U_i \times V_i)$, so $\pi(U) = \bigcup_{i \in I} U_i$ which is open in X.

Proposition
If $f: X \to Y$ is continuous, surjective, and open, then f is a quotient.

Proof: If $f^{-1}(U) \subset X$ is open, then $U = f(f^{-1}(U))$ is open in Y.

Definition: Equivalence Relation
A equivalence relation on a set X, $(X, \sim)$, satisfy:

1. $x \sim y \Leftrightarrow y \sim x$,
2. $x \sim x$,
3. $x \sim y,\ y \sim z \Rightarrow x \sim z$,

for all $x, y, z \in X$.
Note: X is partitioned into equivalent classes. $X/\sim$ is the set of equivalent classes.

Remark
Let $f: X \to Y$ be surjective. Let $x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2)$. Then $X/\sim$ is in 1-1 correspondence with Y.

Proposition
Given $(X,\sim)$, there is a unique topology on $X/\sim$ which makes the natural map $f:X\to X/\sim$ a quotient map and $X/\sim$ a quotient space.

Proof: $U\subset X/\sim$ is open if and only if $f^{-1}(U)$ is open in X.

Lemma
Let $\{0,1\}$ have the topology $\{\emptyset,\{0\},\{0,1\}\}$. Let $U\subset X$ be open. Then $\phi:X\to\{0,1\},\phi(x)=\{\begin{smallmatrix}0\text{ if } x\in U\\ 1\text{ if } x\in U^c\end{smallmatrix}$ is continuous.

Proposition
Let $f:X\to Y$ be surjective and continuous. Then f is a quotient map if and only if given $g:Y\to Z$, g is continuous if and only if $g\circ f$ is continuous for all g and Z.

Proof:
($\Rightarrow$) Suppose f is a quotient map. Assume $g:Y\to Z$ is continuous, so $g^{-1}(V)\subset Y$ open for all $V\in Z$, and so $(g\circ f)^{-1}(V)=f^{-1}(g^{-1}(V))\subset X$ is open since f is continuous. Assume $g\circ f$ is continuous, so $(g\circ f)^{-1}(V)=f^{-1}(g^{-1}(V))\subset X$ is open, and so $g^{-1}(V)\subset Y$ is open since f is open.

($\Leftarrow$) It remains to show that if $f^{-1}(V)$ is open in X, then V is open in Y. Let $\psi:Y\to\{0,1\},\psi(y)=\{\begin{smallmatrix}0\text{ if } y\in V\\ 1\text{ if } y\notin V\end{smallmatrix}$. ψ is continuous if and only if $\psi\circ f$ is continuous, which it is since $f^{-1}(V)$ is open.

Proposition
Let $f:X\to Y$. Let $\sim$ be the equivalence relation of X where $x_1\sim x_2\Leftrightarrow f(x_1)=f(x_2)$. Then f is continuous if and only if the natural map $\hat{f}:X/\sim\to Y$ is continuous.

Proposition
If $f:X\to Y$ is a quotient map, then $\hat{f}:X/\sim\to Y$ is a homeomorphism.

Proof: $\hat{f}$ is 1-1, onto, and continuous (since f is continuous). Let $\rho:X\to X/\sim$, which is continuous. Then $\hat{f}^{-1}\circ f=\rho$ is continuous, and $\hat{f}^{-1}$ is continuous since f is a quotient map.

Definition: Retraction
Let $A\subset X$ and $f:X\to A$. If $f|A=Id_A$ and f is continuous, then f is called a retraction.

Proposition
A retraction is a quotient map.

Proof: If U is open in A, $f^{-1}(U)$ is open in X since f is continuous. It remains to show that if $f^{-1}(U)$ is open in X, then U is open in A. Now $U=f^{-1}(U)\cap A$ since $f|A=Id_A$, so U is open in A since it is the intersection of an open set in X with A.

Connectedness and Compactness

Connected Spaces

Definition: Separation
Let X be a topological space. A separation of X is a pair of nonempty open sets, U_1 and U_2, such that $U_1 \cup U_2 = X$ and $U_1 \cap U_2 = \emptyset$.

Definition: Connected
X is connected if there is no separation of X.

Theorem
If $f: X \to Y$ is continuous and X is connected, then so is $f(X)$.

Proof: Suppose $f(X)$ is not connected. Let V_1 and V_2 open in $f(X)$ be a separation. Then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ is a separation of X.

Proposition
Suppose $Y \subset \mathbb{R}$. If there exists $a, b \in Y$ and $c \notin Y$ such that $a < c < b$, then Y is not connected.

Proof: $(-\infty, c) \cap Y$ and $(c, \infty) \cap Y$ is a separation if Y.

Corollary
If $f: X \to \mathbb{R}$ is continuous and X is connected, and $f(x_1) = a$, $f(x_2) = b$, $a < c < b$, then there exists $x_z \in X$ such that $f(x_z) = c$.

Proof: $f(X)$ is connected, so it contains c.

Theorem
The non-empty connected sets in $\mathbb{R}$ are precisely the intervals.

Proof: Suppose $Y \in \mathbb{R}$ is a connected set. Let $a = \mathrm{glb}\, Y$ and $b = \mathrm{lub}\, Y$. Y is the interval between a and b, perhaps within the endpoints. Suppose U_1 and U_2 is a separation of Y. Let $a_1 \in U_1$, $b_1 \in U_2$, $a < a_1 < b_1 < b$. Then we can find a_2 and b_2 such that $[a_{2,} b_2]$ is separated by $[a_{2,} b_2] \cap U_1$ and $[a_{2,} b_2] \cap U_2$. Let $c = \mathrm{lub}[a_{2,} b_2] \cap U_1$. Then $c \neq b_2$ since b_2 is in the open neighborhood $[a_{2,} b_2] \cap U_2$. Also $c \notin U_1$ since there exists $c + \varepsilon \in U_1$, so $c \neq \mathrm{lub}[a_{2,} b_2] \cap U_1$. Similarly $c \notin U_2$ since there exists $c - \varepsilon \in U_2$, so $c \neq \mathrm{lub}[a_{2,} b_2] \cap U_1$. Hence by the openness of U_1 and U_2, c can't be anywhere. Contradiction. So the intervals are connected.

Corollary
Let I be an interval in $\mathbb{R}$. If $f: I \to X$ is continuous, then $f(I)$ is connected.

Theorem
Suppose $X_\alpha \subset X$, $\alpha \in A$ are connected, and there exists $p \in X_\alpha$, $\forall \alpha \in A$. Then $\bigcup_{\alpha \in A} X_\alpha$ is connected.

Proof: Suppose U_1 and U_2 is a separation. Then p is in U_1 or U_2; say U_1. Then $X_\alpha \subset U_1$ $\forall \alpha \in A$, for if $U_2 \cap X_\alpha \neq \emptyset$ then $U_1 \cap X_\alpha$ and $U_2 \cap X_\alpha$ separate X_α.

Definition: Path Connected

X is path connected if given $x, y \in X$, there exists a continuous map $f : I \to X$ such that $f(0)=x$ and $f(1)=y$.

Theorem
If X is path connected, then X connected.

Theorem
If $A \subset X$ is connected and $A \subseteq B \subseteq \overline{A}$, then B is connected.

Proof: Suppose U_1 and U_2 separate B. Then $U_1 \cap A \neq \emptyset$ and $U_2 \cap A \neq \emptyset$. So $U_1 \cap A$ and $U_2 \cap A$ separate A. Contradiction.

Theorem
The product of a finite number of connected spaces is connected.

Proof: It is sufficient to prove for two. Fix a point $(a, b) \in X \times Y$. Then $\{(x, b) | x \in X\}$ is connected. So for an arbitrary point $(x_{0,} y_0)$, $\{(x, b) | x \in X\} \cup \{(x_{0,} y) | y \in Y\}$ is connected. Hence their union $X \times Y$ is connected.

Components and Local Connectedness

Definition: Component
Given X and $x \in X$, let the component of x, C_x, be the largest connected set containing x, i.e. the union of all connected sets containing x.

Proposition
For $x, y \in X$, either $C_x = C_y$ or $C_x \cap C_y = \emptyset$.

Proof: If $C_x \cap C_y \neq \emptyset$, then let $z \in C_x \cap C_y$. Now $z \in C_x \to C_z = C_x$ and $z \in C_y \to C_z = C_y$. So $C_z = C_x$ and $C_z = C_y$, so $C_x = C_y$.

Theorem
The connected components of X partition X.

Definition: Path Component
Given X and $x \in X$, let the path component of x $(PC)_x = \{y | y \text{ can be joined to } x \text{ by a path}\}$.

Theorem
The path components of X partition X. In fact, they partition the connected components of X, i.e. $(PC)_x \subseteq C_x$.

Definition: Local Connectivity
X is locally connected at x iff for each neighborhood U_x of x there is a connected neighborhood V_x of x such that $V_x \subset U_x$.
X is locally connected iff X is locally connected at each point $x \in X$.

Theorem
If a space X is locally connected, then the connected components of X are open.

Proof: Let $x \in C_x$. Then any U_x contains V_x. Now each $V_x \subset C_x$, so C_x is open.

Lemma
If a space X is locally connected, then for each open $U \subset X$, U is connected.

Theorem
A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Definition: Local Path Connectivity
X is locally path connected at x iff for each neighborhood U_x of x there is a path connected neighborhood V_x of x such that $V_x \subset U_x$.
X is locally path connected iff X is locally connected at each point $x \in X$.

Theorem
A space X is locally path connected if and only if for every open set U of X, each component of U is open in X.

Compact Spaces

Definition: Open Cover
Let $U_\alpha \subset X \ \ , \alpha \in A$ be open sets such that $\bigcup_{\alpha \in A} U_\alpha = X$. $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X.

Definition: Subcover
If $B \subseteq A$ and $\{U_\alpha\}_{\alpha \in B}$ is still a cover, then $\{U_\alpha\}_{\alpha \in B}$ is called a subcover of $\{U_\alpha\}_{\alpha \in A}$.

Definition: Compact
X is compact iff every open cover of X has a finite subcover.

Theorem
If X is compact and $f: X \rightarrow Y$ is continuous, then $f(X)$ is compact.

Proof: Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $f(X)$. Let V_α be open in Y and $V_\alpha \cap f(X) = U_\alpha$. Then $f^{-1}(V_\alpha) = f^{-1}(U_\alpha)$ is open in X. $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of X, so there exists $\alpha_{1,} \ldots, \alpha_n$ such that $f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_n}) = X$, but then $U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} = f(X)$.

Proposition
A compact set in $\mathbb{R}$ is bounded.

Proof: Cover $\mathbb{R}$ by $(z-1, z+2), z \in \mathbb{Z}$. A finite collection of these is bounded.

Proposition
A compact subset A of a Hausdorff space X is closed.

Proof: Suppose there exists $p \in \bar{A} - A$. Let $x \in A$. Let U_x and V_x be neighbourhoods of x and p respectively such

that $U_x \cap V_x = \emptyset$. $\{U_x\}_{x \in A}$ form an open cover of A . Now let $U_{x_1}, \ldots, U_{x_n}$ be a finite subcover and $U = U_{x_1} \cup \cdots \cup U_{x_n}$ and $V = V_{x_1} \cap \cdots \cap V_{x_n}$. Then $U \cap V = \emptyset$. But $p \in V$ and $V \cap A = \emptyset$. Contradiction. Hence $p \notin \bar{A} - A$ and $A = \bar{A}$ so A is closed.

Theorem
Compact sets are closed and bounded in $\mathbb{R}$, $\mathbb{R}^n$, or any metric space.

Theorem
If $f : X \to \mathbb{R}$ is continuous and X is compact, then f achieves its maximum and minimum, i.e. its lub and glb.

Proof: lub and glb are in $\overline{f(X)}$.

Definition: Finite Intersection Property
A collection of (nonempty) sets $\{B_\alpha\}_{\alpha \in A}$ has the finite intersection property if for every finite sub-collection $\{B_{\alpha_i}\}_{i=1}^n$, $B_{\alpha_1} \cap \cdots \cap B_{\alpha_n} \neq \emptyset$.

Theorem
X is compact if and only if every collection of closed subsets of X with the finite intersection property has nonempty intersection.

Proof:
($\Rightarrow$) Assume X is compact. Let $\{C_\alpha\}_{\alpha \in A}$ be a collection of closed sets, so $\{C^c_\alpha\}_{\alpha \in A}$ are open. $\{C^c_\alpha\}_{\alpha \in A}$ does not cover X if there is no finite subcover, that is $\bigcup_{\alpha \in A} C^c_\alpha \neq X \Leftrightarrow \bigcap_{\alpha \in A} C_\alpha \neq \emptyset$ if $\bigcup_{i=1}^n C^c_{\alpha_i} \neq X \Leftrightarrow \bigcap_{i=1}^n C_{\alpha_i} \neq \emptyset$.
($\Leftarrow$) Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover. Suppose $\{U_\alpha\}_{\alpha \in A}$ has no finite subcover. Then $U^c_{\alpha_1} \cap \cdots \cap U^c_{\alpha_n} \neq \emptyset$ for all finite sub-collection, so $\bigcap_{\alpha \in A} U^c_\alpha \neq \emptyset$ and hence $\bigcup_{\alpha \in A} U_\alpha \neq X$. Contradiction since $\{U_\alpha\}_{\alpha \in A}$ is an open cover.

Theorem
If X is compact and $A \subset X$ is closed, then A is also compact.

Proof: Let $\{U_\beta\}_{\beta \in B}$ be an open cover of A , $U_\beta = V_\beta \cap A$, V_β open in X . Then $\{V_\beta\}_{\beta \in B} \cup A^c$ is an open cover of X , so there exists a finite collection $\beta_1, \ldots, \beta_n$ such that $V_{\beta_1}, \ldots, V_{\beta_n}, A^c$ cover X . Hence $U_{\beta_1}, \ldots, U_{\beta_n}$ cover A .

Theorem
If X is compact and Hausdorff, and A , B are closed sets in X such that $A \cap B = \emptyset$, then there exists open sets U and V , $A \subset U$, $B \subset V$, such that $U \cap V = \emptyset$.

Proof: Let $x \in A$. There exists neighbourhoods U_x of x and V_x of B such that $U_x \cap V_x = \emptyset$. Let $U_{x_1}, \ldots, U_{x_n}$ be a finite subcover of A . Let $U = \bigcup_{i=1}^n U_{x_i}$ and $V = \bigcap_{i=1}^n V_{x_i}$. Then U and V are open and $U \cap V = \emptyset$.

Theorem
Suppose X is compact. Let $\{C_n\}_{n \in \mathbb{N}}$ be a collection of non-empty, closed, and nested sets (i.e. $C_1 \supset C_2 \supset \cdots$). Then $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$.

Proof: X is compact, so $\{C_n\}_{n\in\mathbb{N}}$ has the finite intersection property. Suppose $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$. Then there exists a finite subsequence $n_1, \ldots, n_k$ such that $\bigcap_{i=n_1}^{n_k} C_{n_i} = \emptyset$, but $\bigcap_{i=n_1}^{n_k} C_{n_i} = C_{n_k}$. Contradiction.

Theorem

Suppose X is compact and Y is Hausdorff. If $f: X \to Y$ is bijective and continuous, then f is a homeomorphism.

Proof: We need $f^{-1}: Y \to X$ to be continuous. Let $C \subset X$ be closed. Then C is compact. So $f(C) \subset Y$ is compact, and so closed. Now $(f^{-1})^{-1}(C) = f(C)$. Hence if $C \subset X$ be closed, then $(f^{-1})^{-1}(C)$ is closed. Therefore f^{-1} is continuous.

Lemma: Tube Lemma

Suppose Y is compact. Let $x_0 \in X$ and U be an open set in $X \times Y$ such that $x_0 \times Y \subset U$. Then there exists a neighborhood W of x_0 such that $W \times Y \subset U$.

Proof: For each $y \in x_0 \times Y$, there exists a product neighborhood $U_{x_0, y} \times U_y$ contained in U. Since Y is compact, there is a finite collection $y_1, \ldots, y_n$ such that $U_{y_1}, \ldots, U_{y_n}$ cover Y. Let $W = \bigcap_{i=1}^{n} U_{x_0, y_i}$. Now $W \times U_{y_i} \subset U_{x_0} \times U_{y_i}$ for each $i = 1, \ldots, n$, so $W \times \bigcup_{i=1}^{n} U_{y_i} = W \times Y \subset U$.

Theorem

Finite product of compact spaces is compact.

Proof: Note that it suffices to prove $X \times Y$ is compact if X and Y are compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $X \times Y$. For each fixed $x \in X$, let $U_{x,1}, \ldots, U_{x,n_x}$ be a finite cover of $x \times Y$. Let $U_x = \bigcup_{i=1}^{n_x} U_{x,i}$. U_x is open, so by the tube lemma, there exists a neighborhood W_x of x such that $W_x \times Y \subset U_x$. Since X is compact, there exists $x_1, \ldots, x_m$ such that $W_{x_1}, \ldots, W_{x_m}$ cover X. So $W_{x_1} \times Y, \ldots, W_{x_m} \times Y$ cover $X \times Y$. Since each tube $W_{x_i} \times Y$ is covered by $\{U_{x_i, j}\}_{j=1}^{n_{x_i}}$ a finite number of open sets of $X \times Y$, $\bigcup_{i=1}^{m} \{U_{x_i, j}\}_{j=1}^{n_{x_i}}$ is a finite subcover of $\{U_\alpha\}_{\alpha \in A}$. Hence $X \times Y$ is compact.

Theorem

$[a, b] \subset \mathbb{R}$ is compact.

Proof: Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $[a, b]$. Let $c \in [a, b]$, where $c = \text{lub}\left\{x \in \mathbb{R} \,\middle|\, [a, x] \text{ is covered by a finite subcover of } \{U_\alpha\}_{\alpha \in A}\right\}$. Then $c = b$, and there exists an open set containing b.

Theorem

$\prod_{i=1}^{n} [a_i, b_i] \subset \mathbb{R}_n$ is compact.

Theorem: Heine-Borel Theorem

$X \subset \mathbb{R}$ is compact if and only if X is closed and bounded.

Local Compactness

Definition: Local Compactness

X is locally compact at x if there is a compact subset C of X which contains a neighborhood of x.
X is locally compact if it is locally compact at every $x \in X$.

Example

$\mathbb{R}$ is locally compact.

Definition: One-Point Compactification

Let Y be compact and Hausdorff. Suppose $X \subset Y$ such that $Y - X = \{\infty\}$ is one point and that $\bar{X} = Y$. Then we call Y a one-point compactification of X.

Theorem

Let X be locally compact and Hausdorff, but X is not compact. Then X has a one-point compactification. Moreover, if Y_1 and Y_2 are both one-point compactification of X, then Id_X extends to a homeomorphism $h: Y_1 \rightarrow Y_2$ which takes ∞_{Y_1} to ∞_{Y_2}.

Proof:
Let $Y = X \cup \{\infty\}, \infty \notin X$. Define the topology on Y as follows: $U \subset Y$ is open iff

- $U \subset X$ and U is open in X, or
- $U = A^c$ where $A \subset X$ is compact.

Y is Hausdorff: Take two points x_1 and x_2 in Y. If $x_1, x_2 \in X$, done. Otherwise, let $x \in X$ and $\infty \in Y$. There exists U_x open and C compact such that $x \in U_x \subset C \subset X$ (since X be locally compact). $\infty \in C^c$ which is open. So $U_x \cap C^c = \emptyset$.
Y is compact: Let $\{U_\alpha\}_{\alpha \in A}$ be a open cover of Y. ∞ is in some U_{α_1} and $U_{\alpha_1}{}^c$ is compact. Note that $\{\infty\}$ is closed (since Y Hausdorff). X is open, so $X \cap U_\alpha$ is open for all $\alpha \in A$. Then $\{X \cap U_\alpha\}_{\alpha \in A}$ is an open cover of X, and hence of $U_{\alpha_1}{}^c$. So a finite collection $U_{\alpha_2}, \ldots, U_{\alpha_n}$ covers $U_{\alpha_1}{}^c$, and $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ covers Y.
$\bar{X} = Y$, i.e. $\infty \in \bar{X}$: Let U_∞ be an open set containing ∞. Then $U_\infty = A^c$, where $A \subset X$ is compact and $A \neq X$, so $U_\infty \cap X \neq \emptyset$.

Corollary

Let X be Hausdorff. Then X is locally compact if and only if given $x \in X$ and neighborhood U_x of x, there exists a neighborhood V_x of x such that $\overline{V_x}$ is compact and $\overline{V_x} \subset U_x$.

Proof:
($\rightarrow$) X is locally compact and Hausdorff, so $X \subset Y$ where Y is compact and Hausdorff by one-point compactification. Given U_x, $U_x{}^c = \overline{U_x{}^c}$ is closed in Y and thus so compact. So there exists V_x and $N(\overline{U_x{}^c})$ neighborhoods of x and $\overline{U_x{}^c}$ such that $V_x \cap N(\overline{U_x{}^c}) = \emptyset$. So $\overline{V_x} \subset U_x$ and $\infty \notin \overline{V_x}$. Hence $\overline{V_X}$ is compact and $\overline{V_X} \subset U_X \subset X$.
($\leftarrow$) Take $C = \overline{V_x}$ compact. Then $x \in V_x \subset U_x$. This is just the definition of local compactness.

Corollary

If X is locally compact and Hausdorff, and $A \subset X$ either open or closed, then A is locally compact and Hausdorff.

Corollary

X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact and Hausdorff.

Limit Point Compactness

Definition: Limit Point Compact

X is limit point compact if every infinite subset of X has a limit point.

Theorem

If X is compact, then X is limit point compact.

Proof: Suppose X is not limit point compact. Then there exists an infinite subset $A\subset X$ which has no limit points. So for all $x\in X$ there exists U_x such that $U_x\cap A$ is at most one point, i.e. $x\notin A\Rightarrow U_x\cap A=\emptyset$ or $x\in A\Rightarrow U_x\cap A=\{x\}$. Now $\{U_x\}_{x\in X}$ is an open cover of X but has no finite subcover since A is infinite.

Definition: Sequentially Compact

X is sequentially compact if every sequence $\{x_i\}_{i\in\mathbb{N}}$ has a convergent subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ such that n_i is strictly increasing and x_{n_i} converge.

Definition: Cauchy Sequence

In a metric space, a sequence $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence if given $\varepsilon>0$, there exists $N>0$ such that $d(x_n,x_m)<\varepsilon$ for all $n,m>N$.

Definition: Complete

X is complete if every Cauchy sequence converges.

Lemma

If a subsequence of a Cauchy sequence converges, then so does the sequence.

Definition: Totally Bounded

A metric space is totally bounded if for all $\varepsilon>0$, there is a finite cover of X by ε-balls.

Definition: Lebesgue Number

Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X. Then $\delta>0$ is a Lebesgue number for $\{U_\alpha\}_{\alpha\in A}$ if given $x\in X$, $B_\delta(x)\subset U_\alpha$ for some $\alpha\in A$.

Lemma: Lebesgue Number Lemma

Let X be complete and totally bounded. If $\{U_\alpha\}_{\alpha\in A}$ be an open cover, then there exists a Lebesgue number $\delta>0$ for $\{U_\alpha\}_{\alpha\in A}$.

Proof: Suppose not, i.e. for all δ, there exists x_δ such that $B_\delta(x_\delta)\not\subset U_\alpha$ for any $\alpha\in A$. Pick a sequence $\delta_n\to 0$ (e.g. $\delta_n=\frac{1}{2^n}$) and $x_{\delta_n}\in X$ as above. Since X totally bounded, consider finite covers of X by balls of radius $\frac{1}{2^i}$, for each $i=1,2,\ldots$. Inductively construct a subsequence $\{x_{\delta_{n_i}}\}_{i\in\mathbb{N}}$ such that the tail is in one ball of radius $\frac{1}{2^i}$ for $i=1,2,\ldots$. The resulting sequence is Cauchy, and hence converges since X complete. So $x_{\delta_{n_i}}\to x$, $x\in U_\alpha$ for some $\alpha\in A$, and $B_\varepsilon(x)\subset U_\alpha$. Now, for i large enough, $\delta_{n_i}<\frac{\varepsilon}{2}$ since $\delta_{n_i}\to 0$, and that $x_{\delta_{n_i}}\in B_{\delta_{n_i}}(x)$. So $B_{\delta_{n_i}}(x_{\delta_{n_i}})\subset B_\varepsilon(x)\subset U_\alpha$ by the

triangle inequality. Contradiction.

Theorem

If (X, d) is a metric space, then the following are equivalent:

1. X is compact.
2. X is limit point compact.
3. X is sequentially compact.
4. X is complete and totally bounded.

Proof:
($1 \Rightarrow 2$) Done.
($2 \Rightarrow 3$) Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence. If $\{x_i\}_{i \in \mathbb{N}}$ is a finite set (finitely many different elements), then there is a constant subsequence which converge. So assume $A = \{x_i | i \in \mathbb{N}\}$ is infinite and let $x \in X$ be a limit point of A. Then we can find infinite sets $S_i, i \in \mathbb{N}$ such that $S_{i+1} \subset S_i$ and points $x_{n_i} \in S_i$ with $n_{i+1} > n_i$ such that $d(x_i, x) < \frac{1}{i}$ (possible since $B_{1/i}(x) \cap S_i$ with $S_0 = A$ is an infinite set). Then $x_{n_i} \to x$ since given $\varepsilon > 0$ there exists $M > 0$ such that $\frac{1}{M} < \varepsilon$, and so for $i > M$ $d(x_{n_i}, x) < \frac{1}{n_i} < \frac{1}{M} < \varepsilon$.
($3 \Rightarrow 4$) Take any Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$. Since X is sequentially compact, there is a subsequence $\{x_{n_i}\}_{i \in \mathbb{N}}$ which converges. So the Cauchy $\{x_n\}_{n \in \mathbb{N}}$ converges by lemma, and hence X is complete. Now suppose X is not totally bounded. Let $\varepsilon > 0$ be such that the ε-balls do not have a finite subcover. For $x_1, \ldots, x_n$ such that $d(x_i, x_j) \geq \varepsilon \ \ \forall i, j \leq n$, there exists x_{n+1} such that $x_1, \ldots, x_n, x_{n+1}$ have the same property (i.e. $d(x_i, x_j) \geq \varepsilon \ \ \forall i, j \leq n+1$). So we can construct an infinite sequence $\{x_i\}_{i \in \mathbb{N}}$ such that $d(x_i, x_j) \geq \varepsilon \ \ \forall i \neq j$. This sequence has no convergent subsequence, for if it did, then given $\frac{\varepsilon}{2}$ there exists M such that $d(x_i, x) < \frac{\varepsilon}{2} \ \ \forall i > M$, but then $d(x_{n_i}, x_{n_j}) > \varepsilon \ \ i, j > M$, so contradiction.
($4 \Rightarrow 1$) Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Let $\delta > 0$ be a Lebesgue number for the cover. Let $B_\delta(x_1), \ldots, B_\delta(x_k)$ be a finite covering of δ-balls. Then for each $i = 1, \ldots, k$, $B_\delta(x_i) \subset U_{\alpha_i}$ for some $\alpha_i \in A$. Hence $U_{\alpha_1}, \ldots, U_{\alpha_k}$ is a finite subcover.

Corollary

If X is compact metric space, then any open cover has a Lebesgue number.

Definition: Uniform Continuity

Let X and Y be metric spaces. $f: X \to Y$ is uniformly continuous if given $\varepsilon > 0$, there $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$ for all $x, y \in X$.

Theorem

Let X and Y be metric spaces. If X is compact and $f: X \to Y$ is continuous, then f is uniformly continuous.

Proof: Given $x \in X$, there exists $\delta_x > 0$ such that $y \in B_{\delta_x}(x) \Rightarrow d_Y(f(x), f(y)) < \frac{\varepsilon}{2}$ since f continuous. Then $y, z \in B_{\delta_x}(x) \Rightarrow d_Y(f(y), f(z)) < \varepsilon$ by the triangle inequality. Now the open cover $\{B_{\delta_x}(x)\}_{x \in X}$ has a Lebesgue number $\delta > 0$.

The Tychonoff Theorem

Definition: Maximal
A collection of sets D with the finite intersection property is maximal if for any $D'\supset D$, $D'\neq D$, D' does not have the finite intersection property.

Lemma
Given a collection C of sets with the finite intersection property, there exists D such that $C\subset D$ and D is maximal.

Proof: Construct D using Zorn's Lemma.

Lemma
Let D be a maximal collection of sets in X with the finite intersection property. Then:

1. If $A_{1,}\dots,A_n\in D$, then $A_1\cap\cdots\cap A_n\in D$.
2. If $A\cap U\neq\emptyset\ \ \forall U\in D$, then $A\in D$.

Lemma
Suppose that D is a maximal collection of sets in $\prod X_\alpha$, where each X_α is compact. Then $\bigcap_{A\in D}\overline{A}$ is nonempty.

Proof: Note that closure the projection $\overline{\pi_\alpha(A)}$ is closed in a compact space X_α, so $\overline{\pi_\alpha(A)}$ is compact. $\bigcap_{A\in D}\overline{\pi_\alpha(A)}\neq\emptyset$ Since $\{\pi_\alpha(A)\}_{A\in D}$ has the finite intersection property and X_α is compact $\bigcap_{A\in D}\overline{\pi_\alpha(A)}\neq\emptyset$. So there exists some $x_\alpha\in\overline{\pi_\alpha(A)}\ \ \forall A\in D$, so for any neighborhood U_α of x_α $U_\alpha\cap\pi_\alpha(A)\neq\emptyset\ \ \forall A\in D$, and so $\pi_\alpha^{-1}(U_\alpha)\cap A\neq\emptyset\ \ \forall A\in D$. Let $x=(x_\alpha)_{\alpha\in A}$. Let $V_\alpha=\pi_\alpha^{-1}(U_\alpha)=U_\alpha\times\prod_{\beta\neq\alpha}X_\beta$, then $V_\alpha\in D$. Therefore finite intersections of V_α's is in D, i.e. $V_{\alpha_1}\cap\cdots\cap V_{\alpha_k}=U_{\alpha_1}\times\cdots\times U_{\alpha_k}\times\prod_{\beta\neq\alpha_1,\dots,\alpha_k}X_\beta\in D$, and so $\left(U_{\alpha_1}\times\cdots\times U_{\alpha_k}\times\prod_{\beta\neq\alpha_1,\dots,\alpha_k}X_\beta\right)\cap A\neq\emptyset$. Therefore $x\in\overline{A}$ for every A, so $\bigcap_{A\in D}\overline{A}\neq\emptyset$.

Theorem: Tychonoff's Theorem
If X_α is compact for all $\alpha\in A$, then $\prod_{\alpha\in A}X_\alpha$ is compact.

Countability and Separation Axioms

The Separation Axioms

Definition: Regular
Let X be a topological space where one-point sets are closed. Then X is regular if a point and a disjoint closed set can be separated by open sets.

Definition: Normal
Let X be a topological space where one-point sets are closed. Then X is normal if two disjoint sets can be separated by open sets.

Remark
Normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff.

Proposition
If X is regular and U is a neighborhood of x, then there exists a neighborhood V of x such that $\bar{V}\subset U$.

Proof: U^c is closed. So there exist open sets V_1 and V_2 such that $x\in V_1$, $U^c\subset V_2$, $V_1\cap V_2=\emptyset$. So $x\in\overline{V_1}\subset U$.

Proposition
If X is normal and U is a neighborhood of a closed set A, then there exists a neighborhood V of A such that $\bar{V}\subset U$.

Proof: U^c is closed. So there exist open sets V_1 and V_2 such that $A\subset V_1$, $U^c\subset V_2$, $V_1\cap V_2=\emptyset$. So $A\subset\overline{V_1}\subset U$.

The Urysohn Lemma

Theorem: Urysohn Lemma
Let X be normal, A and B closed such that $A\cap B=\emptyset$. Let $[a,b]\subset\mathbb{R}$. Then there exists a continuous function $f:X\to[a,b]$ such that $f(A)=a$ and $f(B)=b$.

Proof: It is sufficient to take $[a,b]=[0,1]$ (since they are homeomorphic).
Now for every rational $q\in[0,1]$, construct open sets $U_q\subset X$ such that if $0<p<q<1$ then $\overline{U_p}\subset U_q$, and that $A\subset U_0$, $A\subset\bigcup_{p<1}U_p$, $B\cap U_p=\emptyset$. Let $U_1=X-B\supset A$ and U_0 be an open set containing A with $\overline{U_0}\subset U_1$. Let $\phi:\mathbb{N}\to\mathbb{Q}\cap I$ where $\phi(1)=0$, $\phi(2)=1$. Suppose $U_{\phi(1)},\dots,U_{\phi(n)}$ are constructed. $\phi(n+1)$ is between two closest neighbors, $\phi(i)$ and $\phi(j)$, on the list. Since $\phi(i)<\phi(i)$, $\overline{U_{\phi(i)}}\subset U_{\phi(i)}$, so can pick $U_{\phi(n+1)}$ to be a neighborhood of $\overline{U_{\phi(i)}}$ (that is $\overline{U_{\phi(i)}}\subset U_{\phi(n+1)}$) such that $\overline{U_{\phi(n+1)}}\subset U_{\phi(i)}$. Hence by induction, define U_q for all rationals $q\in[0,1]$.
Now define $f(x)=\inf_{q\in I\cap\mathbb{Q}}\{x\in U_q\}$. Then $f(A)=0$ and $f(B)=1$. It remains to prove the continuity of f. First note that $x\in\overline{U_r}\Rightarrow f(x)\le r$ and $x\notin U_r\Rightarrow f(x)\ge r$. Now let U be an open neighborhood of $f(x)\in[0,1]$. Then there exists open interval (q_1,q_2) such that $f(x)\in(q_1,q_2)\subset U$. Now $f^{-1}((q_1,q_2))\supset U_{q_2}-\overline{U_{q_1}}$ which is open and contains x, and that $f(U_{q_2}-\overline{U_{q_1}})\subset(q_1,q_2)$. Therefore f is continuous.

The Tietze Extension Theorem

Theorem: Tietze Extension Theorem
Let X be normal, $A\subset X$ be closed, and $f:A\to[a,b]$ or $f:A\to\mathbb{R}$ be continuous. Then f may be extended to a continuous map defined on all of X.

Proof:
For $f:A\to[a,b]$, it is sufficient to show for $[a,b]=[-1,1]$. Build a collection $g_1,g_2,\cdots$ of approximates to f such that $f-g_1-g_2-\cdots$ converges to 0 uniformly. That is $g=\sum g_i\xrightarrow{\text{uni}}f$ on X.

- Let $B_1=f^{-1}([-1,-\frac{1}{3}])$, $C_1=f^{-1}([-\frac{1}{3},\frac{1}{3}])$, $D_1=f^{-1}([\frac{1}{3},1])$. Then B_1 and D_1 are closed and disjoint. By the Urysohn lemma, can find $g_1:X\to[-\frac{1}{3},\frac{1}{3}]$ such that $g(B_1)=-\frac{1}{3}$, $g(D_1)=\frac{1}{3}$, and that $|f(x)-g_1(x)|<\frac{2}{3}$ $\forall x\in A$. We get obtain $f-g_1:A\to[-\frac{2}{3},\frac{2}{3}]$.
- Now by letting $B_2=(f-g_1)^{-1}(\frac{2}{3}[-1,-\frac{1}{3}])$, $C_2=(f-g_1)^{-1}(\frac{2}{3}[-\frac{1}{3},\frac{1}{3}])$, $D_2=(f-g_1)^{-1}(\frac{2}{3}[\frac{1}{3},1])$, we can get $g_2:X\to\frac{2}{3}[-\frac{1}{3},\frac{1}{3}]$ by Urysohn lemma, with $g(B_2)=\frac{2}{3}(-\frac{1}{3})$, $g(D_2)=\frac{2}{3}(\frac{1}{3})$, and

$|f(x)-g_1(x)-g_2(x)|<(\frac{2}{3})^2 \quad \forall x\in A$.

- By induction, let $B_n=(f-g_1-\cdots g_{n-1})^{-1}((\frac{2}{3})^{n-1}[-1,-\frac{1}{3}])$, $C_n=(f-g_1-\cdots g_{n-1})^{-1}((\frac{2}{3})^{n-1}[-\frac{1}{3},\frac{1}{3}])$, $D_n=(f-g_1-\cdots g_{n-1})^{-1}((\frac{2}{3})^{n-1}[\frac{1}{3},1])$. Then we can get $g_n: X\to(\frac{2}{3})^{n-1}[-\frac{1}{3},\frac{1}{3}]$ by Urysohn lemma, with $g(B_n)=(\frac{2}{3})^{n-1}(-\frac{1}{3})$, $g(B_n)=(\frac{2}{3})^{n-1}(\frac{1}{3})$, and $|f(x)-g_1(x)-\cdots-g_n(x)|<(\frac{2}{3})^n \quad \forall x\in A$.
- Now take $g=\sum g_i$. It is the extension we are looking for.

For $f:A\to\mathbb{R}$, it is sufficient to show for $f:A\to(-1,1)$. Now, there exists $g:X\to[-1,1]$ such that $g=f$ on A by the first part of the proof. Let $B=g^{-1}(\{-1\}\cup\{1\})$. Then A and B are disjoint closed sets. By Urysohn lemma, there exists $\phi:X\to[0,1]$ such that $\phi(A)=1$ and $\phi(B)=0$, hence $\phi g=f$ on A , and $\phi g=0$ on B . Then $\phi g:X\to(-1,1)$ is the extension we are looking for.

Definition: Separates Points

Suppose $\{f_\alpha\}_{\alpha\in A}$ is a collection of functions such that $f_\alpha: X\to[0,1]$. Let $f:X\to\prod_A[0,1]$ be defined by $(f(x))_\alpha=f_\alpha(x)$. Suppose that for $x\neq y$, there exists a α such that $f_{alpha}(x)\neq f_\alpha(y)$ (hence f is 1-1). We say f separates points.

Theorem

Suppose X is normal and has a countable basis. Then there exists a countable collection of continuous functions $f_i: X\to[0,1]$ such that, given $x_0\in X$ and a neighborhood U_{x_0} of x_0 , there exists i such that $f_i(x_0)=1$ and $f_i\equiv 0$ outside U_{x_0} .

Remark

If we have such functions, then f separates points since given $x,y\in X$, there exists U_x such that $y\notin U_x$ by normality.

The Urysohn Metrization Theorem

Definition: Completely Regular

A space X is completely regular if one-point sets are closed (T_1), and if given a point $p\in X$ and a closed set $A\subset X$ such that $p\notin A$, there exists a continuous function $f:X\to[0,1]$ such that $f(p)=1$ and $f(A)=0$.

Theorem: Urysohn Metrization Theorem

If X is complete regular (or normal) and second countable (countable basis), then X is metrizable.

Proof: Let B_n and B_m be basis elements such that $\overline{B_n}\subset B_m$. There exists a continuous function $f_{n,m}:X\to[0,1]$ such that $f_{n,m}(\overline{B_n})=0$ and $f_{n,m}(B_m{}^c)=1$ by Urysohn lemma. In a regular space, given a point $x\in X$ and a neighborhood U of x , there exists another neighborhood V of x such that $x\in\overline{V}\subset U$. Hence given $x,y\in X$ there exists B_n and B_m such that $x\in\overline{B_n}\subset B_m$, $y\notin B_m$, and so $f_{n,m}$ separates points. Now $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ is 1-1 and onto, so $\prod f_{n,m}: X\to[0,1]^{\mathbb{N}}$ is 1-1. $[0,1]^{\mathbb{N}}$ is metrizable.

Remark

If X is normal but not second countable, take $\prod f:X\to[0,1]^{C^0(X,I)}$ where $f\in C^0(X,I)$ (continuous function from X to I). If X is compact Hausdorff (hence normal), this is a homeomorphism.

The Stone-Čech Compactification

Theorem
Let X be completely regular (or normal). Then there exists a compactification of Y of X (i.e. Y is compact Hausdorff and $\bar{X}=Y$) with the property that any bounded continuous function $f: X\rightarrow\mathbb{R}$ extends uniquely to a function $g: Y\rightarrow\mathbb{R}$. Y is called the Stone-Čech Compactification.

Proof: Let $B(X,\mathbb{R})$ be the set of all bounded continuous functions $f: X\rightarrow\mathbb{R}$. For each $f\in B$ let $I_f=[-\alpha_f,\alpha_f]$ contain the image of f. Let $Z=\prod_{f\in B(X,\mathbb{R})} I_f$, which is compact Hausdorff by Tychonoff theorem. Define $h: X\rightarrow\prod I_f$ by $(h(x))_f=f(x)$. Let $Y=\overline{h(X)}$ be the compactification. Now let $i: Y\rightarrow Z$ be the inclusion, $i|X=h$, then $\pi_f\circ i$ extends f uniquely.

Corollary
Let X be completely regular and W be compact Hausdorff. If $\phi: X\rightarrow W$ is a continuous function, then ϕ extends to Stone-Čech.

Proof: $W\rightarrow I^A$ is an imbedding, so $X\rightarrow W\rightarrow I^A$ is continuous. Hence each coordinate extends.

Metrization Theorems and Paracompactness

Definition: Refine
A collection B of subsets of X is said to refine A if $\bigcup_{U_B\in B} U_B=\bigcup_{U_A\in A} U_A=X$ and for each $U_B\in B$ there exists $U_A\in A$ such that $B\subset A$.

Definition: Local Finiteness
An open cover A of X is called locally finite if for any $x\in X$, $x\in U_A$ for finitely many $U_A\in A$.

Definition: Paracompactness
X is paracompact if every open cover A of X has a locally finite open refinement.

Theorem
Every metrizable space is paracompact.

Definition: Partition of Unity
Given a locally finite cover A , a partition of unity is a collection of continuous functions $\phi_{U_A}: X\rightarrow[0,1]$ such that $\phi\neq 0$ on $U_A{}^c$ and $\sum\phi_{U_A}(x)=1$ for all $x\in X$.

Complete Metric Spaces and Function Spaces

Complete Metric Spaces

Definition: Cauchy Sequence
Let (Y,d) be a metric space. A sequence $(y_n)_{n\in\mathbb{N}}$ in (Y,d) is Cauchy if given $\varepsilon>0$ there is an $N>0$ such that

$d(y_n, y_m) < \varepsilon$ whenever $n, m > N$.

Definition: Complete
A metric space (Y, d) is complete if every Cauchy sequence converges.

Definition: Standard Bounded Metric
The standard bounded metric associated to d is $\bar{d}(x, y) \stackrel{\text{def}}{=} \min\{d(x, y), 1\}$.

Definition
If Y is metric and A is a set, Y^A is the set of functions from A to Y.

Definition: Bounded
Let Y be metric. A function $\phi: A \to Y$ is bounded if $\operatorname{diam}(\phi(A))$ is finite. Let $B(A, Y)$ denote the set of bounded functions from A to Y.
Note: If Y has a bounded metric, then all functions are bounded.

Definition: Sup Metric
Let (Y, d) be a metric space and $\phi_{1,} \phi_2 \in B(A, Y)$. The sup metric on Y^A is
$\rho(\phi_{1,} \phi_2) \stackrel{\text{def}}{=} \sup\{d(\phi_1(a), \phi_2(a)) | \forall a \in A\}$.

Definition: Uniform Metric
Let (Y, d) be a metric space and let $\bar{d}$ be the standard bounded metric. The uniform metric on Y^A is
$\bar{\rho}(\phi_{1,} \phi_2) \stackrel{\text{def}}{=} \sup\{\bar{d}(\phi_1(a), \phi_2(a)) | \forall a \in A\}$.

Remark
$\rho(\phi_{1,} \phi_2) < 1 \Leftrightarrow \bar{\rho}(\phi_{1,} \phi_2) < 1$, in which case $\rho = \bar{\rho}$.

Proposition
Let $f_n \in B(A, Y)$. Then $f_n \to f$ in the uniform metric if and only if $f_n \to f$ uniformly.

Theorem
If (Y, d) is complete, then $(Y^A, \bar{\rho})$ is complete.

Proof: Let $(f_n)_{n \in \mathbb{N}}$ be Cauchy in Y^A. Then given $\varepsilon > 0$, $\bar{\rho}(f_n, f_m) = \sup\{\bar{d}(f_n(a), f_m(a)) | \forall a \in A\} < \frac{\varepsilon}{2}$ for all $n, m > N_0$. So $(f_n(a))_{n \in \mathbb{N}}$ is Cauchy for all $a \in A$, and hence $f_n(a) \to f(a)$. So $\bar{\rho}(f_n, f) = \sup\{\bar{d}(f_n(a), f(a))\} < \frac{\varepsilon}{2} < \varepsilon$ for all $n > N_0$, so $f_n \to f$.

Proposition
Suppose A is a topological space and Y a metric space. Let $C(A, Y)$ be the set of continuous functions from from A to Y. Then $C(A, Y)$ is closed in $B(A, Y)$ with the uniform metric.

Corollary

If Y is complete, then $C(A,Y)$ is also complete with the uniform metric.
If Y is complete and A is compact, then $C(A,Y)$ is complete in the sup metric.

Definition: Completion
Let (X,d_X) and (Y,d_Y) be metric spaces. Let $i: X \rightarrow Y$ be an isometric embedding, that is $d_X(x_1, x_2)=d_Y(i(x_1), i(x_2))$. Y is the completion of X if $\overline{i(X)}=Y$ and Y is complete.

Theorem
Every metric space has a completion.

Proof: Embed (X,d_X) into $C(X,\mathbb{R})$ (bounded) with the sup topology as follows. Fix $x_0 \in X$; let $a \in X$. Define $\phi_a(x)=d(x,a)-d(x,x_0)$. Then $\phi_a(x) \leq d(a,x_0)$ be the triangle inequality (hence bounded). Let $i: X \rightarrow C(X,\mathbb{R})$ be given by $i(a)=\phi_a$. Now, $\rho(i(a),i(b))=\sup_{x\in X}|\phi_a(x)-\phi_b(x)|=\sup_{x\in X}|d(x,a)-d(x,x_0)-d(x,b)+d(x,x_0)|=\sup_{x\in X}|d(x,a)-d(x,b)| \leq d(a,b)$ by the triangle inequality. However, taking $x=a$, $|d(a,a)-d(a,b)|=d(a,b)$. Hence $\rho(i(a),i(b))=d(a,b)$, so i is an isometry. Let $Y=\overline{i(X)}$.

Peano Space-Filling Curve

Cororally
There exists a continuous and onto map $\phi: I \rightarrow I \times I$.

Compactness in Metric Spaces

Theorem
A metric space is compact if and only if it is complete and totally bounded.

Definition: Equicontinuous
Let X be a topological space, Y a metric space. The set of functions $F \subset C(X,Y)$ is equicontinuous at $x_0 \in X$ if given $\varepsilon > 0$ there exists a neighborhood U_{x_0} of x_0 such that $d(f(x), f(y)) < \varepsilon$ for all $f \in F$ and for all $x, y \in U_{x_0}$.
F is equicontinuous if it is equicontinuous at each $x_0 \in X$.

Examples
- Suppose that $F \subset C^1(I,\mathbb{R})$ and $|f'(x)<1|$ $\forall f \in F, x \in I$, then F is equicontinuous.
- If $d(f(x), f(y)) < C(d(x,y))^{\alpha}$ for some fixed C and fixed α, then F is equicontinuous.

Proposition
If $F \subset C(X,Y)$ is totally bounded in the uniform metric, then F is equicontinuous.
Note: d itself may or may not be bounded.

Proof: Let $\varepsilon > 0$ be given. Assume $\varepsilon < 1$. There exists $f_1, \ldots, f_k$ such that $B^{\bar{\rho}}_{\varepsilon/3}(f_i)$ cover F. Since each f_i is continuous, there exists a neighborhood U_{x_0} of x_0 such that $d(f_i(x), f_i(y)) < \frac{\varepsilon}{3}$ for all $x, y \in U_{x_0}$. Given $f \in F$, $f \in B^{\bar{\rho}}_{\varepsilon/3}(f_i)$ for some i. Now $d(f(x), f(y)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(y)) + d(f_i(y), f(y))$, and since

$f\in B^{\bar{\rho}}_{\varepsilon/3}(f_i)\Rightarrow d(f(x),f_i(x))<\frac{\varepsilon}{3}$, $f\in B^{\bar{\rho}}_{\varepsilon/3}(f_i)\Rightarrow d(f(y),f_i(y))<\frac{\varepsilon}{3}$, $x,y\in U_{x_0}\Rightarrow d(f_i(x),f_i(y))<\frac{\varepsilon}{3}$, so $d(f(x),f(y))\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$. So F is equicontinuous.

Proposition

If X and Y are compact and $F\subset C(X,Y)$ is equicontinuous, then F is totally bounded.

Proof: Let $0<\varepsilon<1$ be given. Since F equicontinuous, let U_a be neighborhoods of $a\in X$ such that $d(f(x),f(y))<\frac{\varepsilon}{3}$ $\forall f\in F, \forall x,y\in U_a$. X is compact, so let $\{U_{a_i}\}_{i=1}^k$ cover X. Let $\{V_{\varepsilon/3}(y_i)\}_{i=1}^l$ be a finite cover of Y of $\frac{\varepsilon}{3}$ - balls centered at y_i. Now consider the set of functions $\{\alpha\}$ where $\alpha:\{1,\dots,k\}\to\{1,\dots,l\}$. If there is $f\in F$ such that $f(x_i)\in V_{\varepsilon/3}(y_{\alpha(i)})$ for each $i=1,\dots,k$, choose one label it f_α. Then we get a finite collection of ε -balls $\{B^{\rho}_{\varepsilon}(f_\alpha)\}$. Now let $f\in F$. For each $i=1,\dots,k$, choose $\alpha(i)$ such that $f(x_i)\in V_{\varepsilon/3}(y_{\alpha(i)})$. Let $x\in X$, then $x\in U_{a_i}$ for some a_i. So $d(f(x),f_\alpha(x))\leq d(f(x),f(a_i))+d(f(a_i),f_\alpha(a_i))+d(f_\alpha(a_i),f_\alpha(x))<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$. Hence $\{B^{\rho}_{\varepsilon}(f_\alpha)\}$ cover F, so it is totally bounded.

Definition: Pointwise Bounded

$F\in Y^X$ is pointwise bounded if $\{f(x)\}_{f\in F}$ is a bounded set in Y for each $x\in X$.

Theorem: Ascoli's Theorem, Classical Version

Let X be compact. Let $F\subset C(X,\mathbb{R}^n)$. F has a compact closure if and only if F is equicontinuous and pointwise bounded.

Proof: Let G denote the closure of F.
($\Rightarrow$) G compact, so G is totally bounded under the sup and uniform metric, and hence pointwise bounded. Also, G is equicontinuous. Since $F\subset G$, F is equicontinuous and pointwise bounded.
($\Leftarrow$) Note that G is closed in the complete space $\mathbb{R}^n$ and hence is complete. Let $\varepsilon>0$ and $x_0\in X$ be given. Since F is equicontinuous, choose U_{x_0} such that $d(f(x),f(y))<\frac{\varepsilon}{3}$ $\forall f\in F,\forall x,y\in U_{x_0}$. Given $g\in G$, choose $f\in F$ such that $\rho(f,g)<\frac{\varepsilon}{3}$. Then $d(g(x),g(y))<\frac{\varepsilon}{3}$ $\forall x,y\in U_{x_0}$ by triangle inequality. Hence G is equicontinuous. Now let $x_0\in X$ be given. Given any $g_1,g_2\in G$ choose $f_1,f_2\in F$ such that $\rho(f_1,g_1)<1$ and $\rho(f_2,g_2)<1$. Since F is pointwise bounded, $d(f_1(x_0),f_2(x_0))\leq M$ $\forall f_1,f_2\in F$, and hence $d(g_1(x_0),g_2(x_0))\leq M+2$. So G is pointwise bounded. Now, for each $a\in X$, choose U_a such that $d(g(x),g(y))<1$ $\forall g\in G,\forall x,y\in U_a$. Since X compact, cover it with $U_{a_1},\dots,U_{a_k}$. Since G is pointwise bounded, $\bigcup_{i=1}^{k}\{g(a_i)\}_{g\in G}$ is bounded, so suppose it lies in $B_N(0)\subset\mathbb{R}^n$. Then $g(X)\subset B_{N+1}(0)$ $\forall g\in G$. Let $Y=\overline{B_{N+1}(0)}$, which is compact. Then $G\subset C(X,Y)$ is totally bounded under ρ. Therefore, G is compact since it is complete and totally bounded.

Corollary

Let X be compact. Let $F\subset C(X,\mathbb{R}^n)$. F is compact if and only if F is closed and bounded under the sup metric, and equicontinuous.

($\Rightarrow$) If F is compact, it is closed and bounded. Since $\bar{F}=F$, it is is equicontinuous.
($\Leftarrow$) F is closed, so $\bar{F}=F$. F is bounded under the sup metric, so it is pointwise bounded. Also, F is equicontinuous.

So F has a compact closure. But F is closed, so $\bar{F}=F$ is compact.

Compact-Open Topology

Definition: Compact-Open Topology
Let X and Y be topological spaces. Describe a basis for $C(X,Y)$ as follows. $S(K,U)\overset{\text{def}}{=}\{f\in C(X,Y) | f(K)\subset U\}$ is open if $U\subset Y$ is open and $K\subset X$ is compact.

Definition: Evaluation Map
The map $ev: C(X,Y)\times X\to Y$ defined by $ev((f,x))=f(x)$ is called the evaluation map.

Theorem
If $C(X,Y)$ has the compact-open topology and X is locally compact Hausdorff, then $ev: C(X,Y)\times X\to Y$ is continuous.

Proof: Let $(f,x)\in C(X,Y)\times X$ and a neighborhood $V\subset Y$ of $ev((f,x))=f(x)$ be given. By the continuity of f, $f^{-1}(V)$ is open and contains x. Since X is locally compact Hausdorff, there exists a neighborhood U of x such that its compact closure $\bar{U}\subset f^{-1}(V)$, and hence $f(\bar{U})\subset V$. Let $K=\bar{U}$. Then $(f,x)\in S(K,V)\times U$ is open, and $ev(S(K,V),U)\subset V$.

Definition
Given a function $f: Z\times X\to Y$, it gives rise to a function $F: Z\to Y^X$ defined by $F(z)(x)=f(z,x)$.
Conversely, given $F: Z\to Y^X$, there is a corresponding function $f: Z\times X\to Y$ given by $f(z,x)=F(z)(x)$.
F is the map induced by f.

Theorem
Give $C(X,Y)$ the compact-open topology. If $f: Z\times X\to Y$ is continuous, then $F: Z\to C(X,Y)$ is continuous.
Conversely, if $F: Z\to C(X,Y)$ is continuous and X is locally compact Hausdorff, then $f: Z\times X\to Y$ is continuous.

Proof:
($\Rightarrow$) Suppose $f: Z\times X\to Y$ is continuous. Let $z\in Z$ and $F(z)\in S(K,U)$ in $C(X,Y)$ be given. By continuity of f, $f^{-1}(U)\subset Z\times X$ is open and contains $z\times K$. Since K is compact, the tube lemma implies there is a neighborhood W of z such that $W\times K\subset f^{-1}(U)$. Hence $F(W)(K)=f(W,K)\subset U$, so F is continuous.
($\Leftarrow$) Suppose $F: Z\to C(X,Y)$ is continuous. Then $j: Z\times X\to C(X,Y)\times X$ given by $j(z,x)=(F(z),x)$ is continuous. Then $ev: C(X,Y)\times X\to Y$ is continuous since $C(X,Y)$ has the compact-open topology and X is locally compact Hausdorff. Therefore, $ev\circ j: Z\times X\to Y$ given by $(ev\circ j)(z,x)=ev(F(z),x)=F(z)(x)=f(z,x)$ is continuous.

Baire Spaces and Dimension Theory

Baire Spaces

Definition: Baire Space
Baire space is a space in which the intersection of a countable collection of open and dense sets is dense. That is, if $\{U_n\}_{n\in\mathbb{N}}$ are open and dense sets, $\bigcap_{n\in\mathbb{N}} U_n$ is dense.

Proposition
X is a Baire space if and only if the countable union of closed sets without interior has no interior, i.e. if $\{A_i\}_{i\in\mathbb{N}}$ are closed and $\operatorname{int}(A_i)=\emptyset \quad \forall i\in\mathbb{N}$ then $\operatorname{int}\left(\bigcup_{i\in\mathbb{N}} A_i\right)=\emptyset$.

Proof:
($\Rightarrow$) A_i^c is open. If $U\neq\emptyset$ is open, then $U\not\subset A_i$ since $\operatorname{int}(A_i)=\emptyset$. So $U\cap A_i^c\neq\emptyset$, hence A_i^c is dense (since $\overline{A_i^c}=X$) . Since X is a Baire space so $\bigcap_{i\in\mathbb{N}} A_i^c$ is dense, i.e. $U\cap\bigcap_{i\in\mathbb{N}} A_i^c\neq\emptyset$ for all open U . Therefore $U\not\subset\bigcup_{i\in\mathbb{N}} A_i$, so $\bigcup_{i\in\mathbb{N}} A_i$ has no interior.

Definition: Residual
A set A in a Baire space X is residual if it contains the intersection of a countable family of open and dense sets.

Proposition
If A and B are residual, $A\cap B$ is residual.
If A_i is residual, $\bigcap_{i\in\mathbb{N}} A_i$ is residual.

Theorem: Baire Category Theorem
If X is compact Hausdorff or complete metric, then X is a Baire space.

Proof: Suppose $\{A_i\}_{i\in\mathbb{N}}$ is a family of closed sets with no interior. Want: $\bigcup_{i\in\mathbb{N}} A_i$ has no interior, i.e. given any open set U there is a point $x\in U$ and $x\notin\bigcup_{i\in\mathbb{N}} A_i$.
We wish to construct U_i such that $\overline{U_i}\subset U_{i-1}$, $U_i\cap A_i=\emptyset$, $\bigcap\overline{U_i}\neq\emptyset$. Then let $x\in\bigcap\overline{U_i}$, so $x\in U_i \quad \forall i$ and $x\in U_0$. Then $x\notin A_i \quad \forall i$ and hence $x\notin\bigcup_{i\in\mathbb{N}} A_i$.
A_1 has empty interior, so there exists U_0 such that $x\in U_0$ and $x\notin A_1$. Since X is normal, there exists U_1 such that $x\in U_1$, $\overline{U_1}\subset U_0$, and $\overline{U_1}\cap A_1=\emptyset$. Now assume U_i is constructed. A_{i+1} has empty interior, so there exists $x_{i+1}\in U_i-A_{i+1}$, hence there exists U_{i+1} such that $x_{i+1}\in U_{i+1}$, $\overline{U_{i+1}}\subset U_i$, and $\overline{U_{i+1}}\cap A_{i+1}=\emptyset$ since X is normal.
If X is compact Hausdorff, $\{\overline{U_i}\}_{i\in\mathbb{N}}$ is a family of non-empty nested compact sets, so $\bigcap\overline{U_i}\neq\emptyset$.
If X is complete metric, add $\operatorname{diam}\overline{U_i}\leq\frac{1}{i}$. Then x_i is Cauchy, so $x_i\to x$. Now $x\in\overline{U_i} \quad \forall i$, so $x\in\bigcap\overline{U_i}\neq\emptyset$.

Fact
$\mathbb{Q}\subset\mathbb{R}$ is not residual. Given any $q\in\mathbb{Q}$, $U_q=\mathbb{R}-\{q\}$ is open and dense. But $\left(\bigcap_{q\in\mathbb{Q}} U_q\right)\cap\mathbb{Q}=\emptyset$, so $\mathbb{Q}$ does not contain the intersection of a countable family of open and dense sets.

Theorem
Consider $F\subset C(I,\mathbb{R})$ where $F=\{f\in C(I,\mathbb{R}) | f$ is not differentiable at any point $x\in I\}$. Then F is residual in $C(I,\mathbb{R})$.

Proof: Construct a countable family of open dense sets in $C(I,\mathbb{R})$ whose intersection is contained in F . To be differentiable, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists. Construct the family $\{f_k\}_{k\in\mathbb{N}}$ where the limit is larger than k for h small enough.

Proposition
Any open set in a Baire space is a Baire space.

Proof: Let $U \subset X$ be open. Let $\{A_i\}_{i \in \mathbb{N}}$ be closed subsets of U with empty interior. $A_i = a_i \cap U$ where a_i closed in X. $a_i \cap \overline{U}$ has no interior, for if it did, let $v \in V \subset \text{int}(a_i \cap \overline{U})$, but $V \cap U \neq \emptyset$ and is open, and that $V \cap U \subset \text{int}(A_i)$, so contradiction. Hence $\{a_i \cap \overline{U}\}_{i \in \mathbb{N}}$ are closed in X with no interior. Since X is a Baire space, $\bigcup (a_i \cap \overline{U})$ has no interior. Therefore, $\bigcup A_i$ have no interior in U.

Theorem
Let X be a Baire space and (Y, d) be a metric space. Let $f_n: X \to Y$ be a sequence of continuous functions that converge pointwise for all $x \in X$. Then $f = \lim f_n$ is continuous on a dense set of points in X.

Proof: Let $A_N(\varepsilon) = \{x \mid d(f_m(x), f_n(x)) \leq \varepsilon \quad \forall n, m > N\}$. $A_N(\varepsilon)$ is closed. $\bigcup_N A_N(\varepsilon) = X$. Now for any open $U \subset X$, $\bigcup (A_N(\varepsilon) \cap U) = U$, so at least one $A_N(\varepsilon) \cap U$ has interior. Let $U_\varepsilon = \bigcup \text{int}(A_N(\varepsilon))$ is open and dense. Let $\varepsilon = \frac{1}{n}$. Then $\Lambda = \bigcap U_{1/n}$ is residual.
Claim: f is continuous at each point of Λ. Let $x \in \Lambda$ and fix $\varepsilon > 0$. Take $\frac{1}{4n} < \varepsilon$. Then $x \in U_{1/4n}$ and $x \in \text{int}\left(A_N\left(\frac{1}{4n}\right)\right)$ for some N so there exists $U_x \subset A_N\left(\frac{1}{4n}\right)$. So for every $y \in U_x$ and for all $n, m > N$, $d(f_n(y), f_m(y)) < \frac{1}{4n} \Rightarrow d(f_n(y), f(y)) \leq \frac{1}{4n}$. Choose $m > N$. There exists V_x such that $y \in V_x \Rightarrow d(f_m(x), f_m(y)) \leq \frac{1}{4n}$. Then for $x, y \in U_x \cap V_x$,
$d(f(x), f(y)) \leq d(f_m(x), f(x)) + d(f_m(x), f_m(y)) + d(f_m(y), f(y)) \leq \frac{1}{4n} + \frac{1}{4n} + \frac{1}{4n} = \frac{3}{4n} < \varepsilon$.