The Complex Plane

Motivation
Over $\mathbb{R}$, there are algebraic equations (ex: $x^2=-1$, $x^4+x^2+1=0$)with no solutions. These equations cause us to yet extend our notion of numbers to include solutions to these equations. We introduce a symbol i which we declare the solution to $x^2=-1$

Definition
$i^2=-1$.

Definition: Complex Number
A complex number is an expression of type $a+ib$, $a, b\in\mathbb{R}$.
Note: $a+ib=z\in\mathbb{C}$, $a=\mathrm{Re}(z)$ and $b=\mathrm{Im}(z)$.

Arithmetic Operations
There are arithmetic operations on $\mathbb{C}$.
- +: $(a+ib)+(c+id)=(a+c)+i(b+d)$. If $z=a+ib$ and $w=c+id$, then $z+w$ is determined by $\mathrm{Re}(z+w)=\mathrm{Re}(z)+\mathrm{Re}(w)$ and $\mathrm{Im}(z+w)=\mathrm{Im}(z)+\mathrm{Im}(w)$.
- -: $(a+ib)-(c+id)=(a-c)+i(b-d)$.
- ×: $(a+ib)(c+id)=(ac-bd)+i(ab+bc)$. If $z=a+ib$ and $w=c+id$, then zw is determined by $\mathrm{Re}(zw)=ac-bd$ and $\mathrm{Im}(zw)=ab+bc$.
- ÷: If $z=a+ib$ and $w=c+id$, then $\dfrac{z}{w}=\dfrac{ac+bd+i(bc-ad)}{c^2+d^2}$.

Theorem
Addition and multiplication are associative, communicative, and satisfy the distribution law. That is, for all z , w , v ,
- $(z+w)+v=z+(w+v)$ and $(zw)v=z(wv)$ (associative).
- $z+w=w+z$ and $zw=wz$ (communicative).
- $z(w+v)=zw+zv$ (distribution law).

As usual, $z+0=z$, $0\cdot z=0$, $1\cdot z=z$.

Polar Coordinates

Definition
If $z=a+ib$, then $z=r(\cos\theta+i\sin\theta)$, where $r=|z|=\sqrt{a^2+b^2}$, $r\geq 0$ is the normal of z , and
$\cos\theta=\dfrac{b}{\sqrt{a^2+b^2}}\Leftrightarrow\theta=\arccos\left(\dfrac{b}{\sqrt{a^2+b^2}}\right)$, $0\leq\theta<2\pi$.
Note: $\mathrm{Re}(z)=r\cos\theta$ and $\mathrm{Im}(z)=r\sin\theta$.

Theorem
1. $|zw|=|z||w|$.
2. $\mathrm{Arg}(zw)=\mathrm{Arg}(z)+\mathrm{Arg}(w) \mod 2\pi$.

De Moivre Formula
$(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$.

Complex Conjugation

Definition: Complex Conjugate

Let $z=a+ib$. Then $\bar{z}=a-ib$ is the complex conjugate to z .

Remarks

- $|z|^2=z\,\bar{z}\Leftrightarrow|z|=\sqrt{z\,\bar{z}}$.
- $|\bar{z}|=|z|$.
- $\operatorname{Arg}(\bar{z})=-\operatorname{Arg}(z)\mod 2\pi=2\pi-\operatorname{Arg}(z)$.

Quadratic Equations

Formula for solving quadratic equations naturally extends into complex quadratic equations $ax^2+bx+c=0$, $a,b,c\in\mathbb{C}$.

Every equation (except when $a=b=0$) has a solution given by $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ when $a\neq 0$.

Claim

$\pm\sqrt{z}$ makes sense, i.e. there exists $w\in\mathbb{C}$ such that $w^2=z$.

Proof: Take $|w|=\sqrt{|z|}$ and $\operatorname{Arg}(w)=\dfrac{1}{2}\operatorname{Arg}(z)$.

Triangle Inequality

For complex numbers an equivalent triangle inequality is $|z+w|\leq|z|+|w|$.
Another form of the triangle inequality is $|z-w|\geq|z|-|w|$ or $|z-w|+|w|\geq|z|$.

Equations of Lines and Circles

Equation of Lines

Let z be a variable in $\mathbb{C}$. Let $a,b\in\mathbb{C}$ be fixed complex numbers. $\operatorname{Re}(az+b)=0$ defines a line in $\mathbb{C}$. In this form, we can't get a line passing through ic , but we can write $\operatorname{Im}(z)=c=\operatorname{Re}\left(\dfrac{z}{i}\right)$.

Another form is $|z-p|=|z-q|$, $p,q\in\mathbb{C}$; these are the points in $\mathbb{C}$ which are at the same distance from p and q .

Equation of Circles

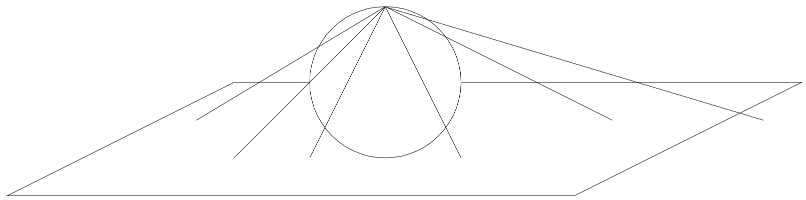
In planar geometry a circle is an equidistant set of points around a given point call the centre. $|z-c|=r$ defines a circle in $\mathbb{C}$.

Also, for any $\rho>0,\rho\neq 1$, the equation $|z-p|=\rho|z-q|$ defines a circle.

Infinity

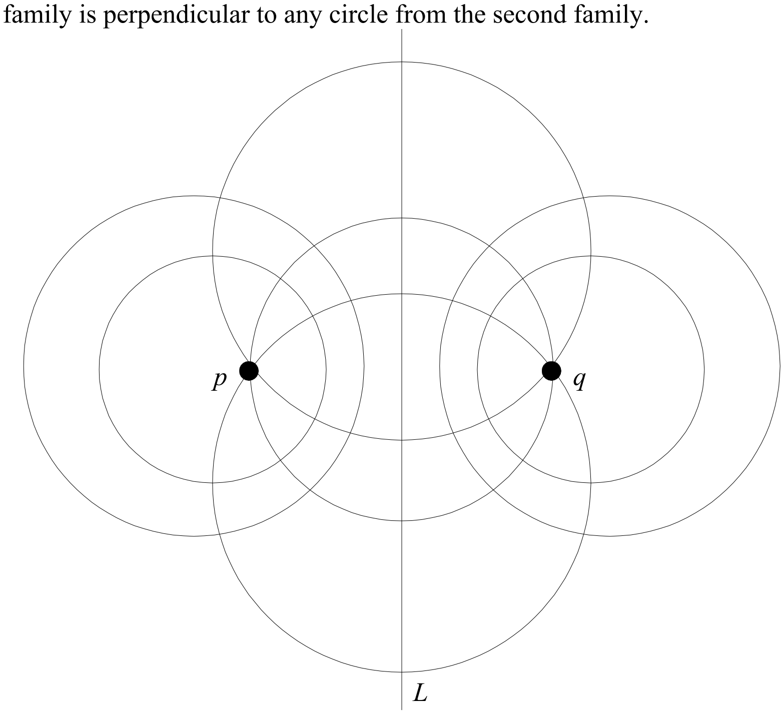
Riemann Sphere

We may insert infinity as a point to the complex plane: $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$.



Appolonis Theorem

Let first family of circles be $|z-p|=\rho|z-q|$ for all possible $\rho>0$.
Let second family of circles be perpendicular to L and passing through p and q .
Any circle from the first family is perpendicular to any circle from the second family.



Open and Closed Sets

Definition: Open Set

A set $S\subset\mathbb{C}$ is called open if for all $x\in S$ there exists $r>0$ such that $D(x;r)\subset S$.
Note: Here $D(x;r)\stackrel{\text{def}}{=}\{z\in\mathbb{C}\,|\,|z-x|<r\}$ is the (open) disk centred at x of radius r .

Definition: Boundary

Let $S\subset\mathbb{C}$. The boundary of S is the set of all points $w\in\mathbb{C}$ such that for all $r>0$, $D(x;r)\not\subset S$ but $D(x;r)\cap S\neq\emptyset$.

Definition: Closed Set

A set $S\subset\mathbb{C}$ is called closed if it contains its boundary.

Proposition

$S\subset\mathbb{C}$ is open if and only if $\partial S\cap S=\emptyset$ (no point in ∂S is contained in S).

Definition: Disjoint
A and B are disjoint if $A \cap B = \emptyset$.

Corollary
$S \subset \mathbb{C}$ is open if and only if $\mathbb{C} \setminus S$ (complement of S) is closed.

Remarks
1. $\partial S = \partial(\mathbb{C} \setminus S)$ because of the symmetry of the boundary definition.
2. It may happen that neither $\partial S \subset S$ nor $\partial S \subset \mathbb{C} \setminus S$. Then S is neither open nor closed.

Definition: Polygonal Path
A polygonal path P is the union of a finite number of straight intervals such that the end point of one interval is the starting point of the consecutive interval. The starting point of P is the start of the first interval; the end point of P is the ending point of the last interval.

Definition: Path Connected
S is called path connected if for any $x, y \in S$ there exists a polygonal path $P \subset S$ with starting point x and end point y . (P connects x and y).

Definition
$S \subset \mathbb{C}$ is said to contain infinity (more correctly, contain a neighbourhood of infiniy) if there exists $M > 0$ such that $S \supset \{ z \in \mathbb{C} \mid |z| > M \}$.

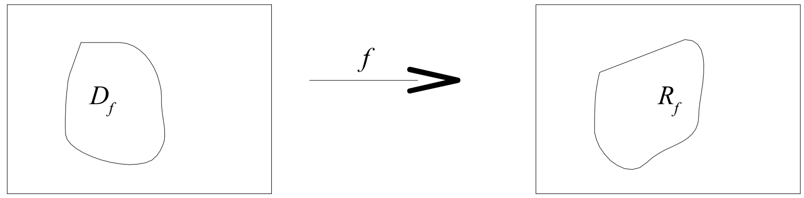
Definition: Closure
$\bar{S} = S \cup \mathrm{Bd}(S)$ is the closure of S .
Note: $\bar{S}$ is always closed.
Note: $\bar{S}$ is the smallest closed set containing S .

Functions

Definition: Function
A function is a rule which associates some (not neccessary all) complex numbers with unique complex numbers.
$w = f(z)$: z is the independent variable, w is the dependent variable.

Definitions: Domain, Range
$D_f \subset \mathbb{C}$ is the domain of definition of f . It consists of all z for which $f(z)$ is defined.
$R_f = f(D_f) \subset \mathbb{C}$ is the range f . It consists of all $w = f(z)$, i.e. all $f(z), z \in D_f$.

Geometric Representation of Complex Functions
If $f(z)$ takes on real values only (eg: $f(z) = |z|^2$), then we can draw graphs. But in general, we need a 4 dimensional space, and drawing graphs is unavailable.
Instead, we can draw two copies of $\mathbb{C}$:



Limits of Sequences

Definition: Limit

Let $\{z_n\}_{n=1}^{\infty}, z_j \in \mathbb{C}$ be a sequence of complex numbers. We say $A \in \mathbb{C}$ is the limit of $\{z_n\}_{n=1}^{\infty}$ if for all $\varepsilon > 0$ there exists $N > 0$ such that $|z_j - A| < \varepsilon$ for all $j > N$.

Remark

Consider $Z = \bigcup_{n=1}^{\infty} \{z_n\}$. If $A = \lim_{n\to\infty} z_n$, then $A \in \mathrm{Bd}(Z)$.

Proposition

Let $z_n = x_n + i\, y_n$ and $A = s + i\, t$. Then $\lim_{n\to\infty} z_n = A$ if and only if $\lim_{n\to\infty} x_n = s$ and $\lim_{n\to\infty} y_n = t$.

Proposition

Suppose $a_n, b_n \in \mathbb{R}$ and $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$. Then $a_n + i\, b_n \overset{n\to\infty}{\to} A + i\, B$.

Remarks

We have the usual properties for limits:
1. The limit of a convergent sequence is unique.
2. $\lim_{n\to\infty} (z_n + w_n) = \lim_{n\to\infty} z_n + \lim_{n\to\infty} w_n$.
3. $\lim_{n\to\infty} z_n w_n = \left(\lim_{n\to\infty} z_n\right)\left(\lim_{n\to\infty} w_n\right)$.
4. If $z_n \overset{n\to\infty}{\to} A \neq 0$ then $\frac{1}{z_n} \overset{n\to\infty}{\to} \frac{1}{A}$. (Note: $\frac{1}{z_n}$ defined for all $n > N$, not all n .)

Limits of Functions

Definition: Limit

Suppose f is a complex function defined on $S \subset \mathbb{C}$. Suppose that $x \in S$ or $x \in \mathrm{Bd}(S)$ (the obundary). Then $\lim_{z\to x} f(z) = A$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|z - x| < \delta$ $\forall\, z \neq x$ we have $|f(z) - A| < \epsilon$.
Note: Limits of f only make sense for points on S or $\mathrm{Bd}(S)$.

Remark

f and g may have different domains of definition. If S_f and S_g are the domains of definition for f and g , then the domain of definition for $f + g$, $f - g$, $f \times g$ is $S_f \cap S_g$.

Proposition

If S_f and S_g are the domains of definition for f and g, then the domain of definition for $f+g$, $f-g$, $f\times g$ is $S_f\cap S_g$. If $z_0\in\overline{S_f\cap S_g}$ and both f and g have limits at z_0, then:

- $\lim_{z\to z_0}(f(z)+g(z))=\lim_{z\to z_0}f(z)+\lim_{z\to z_0}g(z)$,
- $\lim_{z\to z_0}(f(z)-g(z))=\lim_{z\to z_0}f(z)-\lim_{z\to z_0}g(z)$,
- $\lim_{z\to z_0}(f(z)\times g(z))=\lim_{z\to z_0}f(z)\times\lim_{z\to z_0}g(z)$.

The domain of definition of $\frac{f}{g}$ is $(S_f\cap S_g)\setminus\{z\in S_g|g(z)\neq 0\}$. If $z_0\in\overline{(S_f\cap S_g)\setminus\{z\in S_g|g(z)\neq 0\}}$, and f and g have limits at z_0 and $\lim_{z\to z_0}g(z)\neq 0$, then $\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{\lim_{z\to z_0}f(z)}{\lim_{z\to z_0}g(z)}$.

Warning: For general sets $S_f, S_g\subset\mathbb{C}$, we do not always have $\overline{S_f\cap S_g}=\overline{S_f}\cap\overline{S_g}$. Generally, $\overline{S_f\cap S_g}\subset\overline{S_f}\cap\overline{S_g}$.

Definition: Continuous

f is called continouous at $z_0\in S$ if $\lim_{z\to z_0}f(z)=f(z_0)$.

Proposition

As usual, if f and g are continuous, then $f+g$, $f-g$, $f\times g$, and $f\div g$ are.

Definition

A function $f:S\to\mathbb{C}$ has limit L at ∞ ($\lim_{z\to\infty}f(z)=L$) if for all $\varepsilon>0$ there exists $M>0$ such that for all $|z|>M$ we have $|f(z)-L|<\epsilon$.

Note: S has to be unbounded, i.e. for all $M>0$ there exists $z\in S$ such that $|z|>M$.

Series

Definition: Infinite Series

Given a sequence $\{a_1, a_2, a_3, \dots\}$, define an infinite series by $\sum_{j=1}^{\infty}a_j=a_1+a_2+a_3+\cdots$.

Defintion: Parital Sum

$s_n=\sum_{j=1}^{n}a_j=a_1+\cdots+a_n$ is the partial sum.

Definition: Limit

We say that $\sum_{j=1}^{\infty}a_j=L$, $L\in\mathbb{C}$ (in other words $\sum_{j=1}^{\infty}a_j$ converges to L) if the sequence $\{s_1, s_2, s_3, \dots\}$ converges to L.

Proposition

Let $z_j=x_j+i y_j$. Then $\sum_{j=1}^{\infty}z_j=\sum_{j=1}^{\infty}x_j+i\sum_{j=1}^{\infty}y_j$, i.e. $\sum_{j=1}^{\infty}z_j$ converges if and only if $\sum_{j=1}^{\infty}x_j$ and $i\sum_{j=1}^{\infty}x_j$ converge.

Definition: Absolute Convergence

We say that $\sum_{j=1}^{\infty} z_j$ converges absolutely if $\sum_{j=1}^{\infty} |z_j|$ converges.

Proposition

If a series converges absolutely, then it converges.

Exponentia, Logarithm, and Trigonometric Functions

Definition: Exponential Function

If $z = x + iy$, then $\mathrm{e}^z \overset{\text{def}}{=} \mathrm{e}^x(\cos y + i \sin y)$, i.e. e^z is the complex number with $\mathrm{Re}(\mathrm{e}^z) = \mathrm{e}^x \cos y$ and $\mathrm{Im}(\mathrm{e}^z) = \mathrm{e}^x \sin y$.

Properties of the Exponential

- e^z is defined everywhere. Its domain is $\mathbb{C}$.
- $\mathrm{e}^z \neq 0$ because $\mathrm{e}^x \neq 0$ and $\cos y, \sin y \neq 0$ at the same time.
- $\mathrm{e}^z = \mathrm{e}^{x+iy} = \mathrm{e}^x \mathrm{e}^{iy} = \mathrm{e}^x(\cos y + i \sin y)$.
- $\mathrm{e}^{z_1+z_2} = \mathrm{e}^{z_1}\mathrm{e}^{z_2}$.
- $|\mathrm{e}^z| = \mathrm{e}^x = \mathrm{e}^{\mathrm{Re}(z)}$.
- $\mathrm{Arg}(\mathrm{e}^z) = y$. Note that $\mathrm{Arg}(\mathrm{e}^z) = \mathrm{Arg}(\mathrm{e}^{iy})$.

Remarks

1. If $z = x$ is real, then $\mathrm{e}^z = \mathrm{e}^x(\cos 0 + i \sin 0) = \mathrm{e}^x$. It agrees with the real exponential.
2. $\mathrm{e}^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ and $\mathrm{e}^z = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n$.
3. If $w \neq 0$, then there exists $z = x + iy$ such that $\mathrm{e}^z = w$. Write $\mathrm{e}^{x+iy} = w = r\mathrm{e}^{i\theta}$. Then $z = x + iy$ is given by $x = \ln r$ and $y = \theta + 2\pi k$. Note that we have infinity many solutions!

Definition: Logarithm

Define $\log(z) = \ln|z| + i\,\mathrm{Arg}(z)$ on $\mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$. It takes value in $\{z \in \mathbb{C} \mid -\pi \leq \mathrm{Im}(z) \leq \pi\}$.

Remarks

1. $\log(z)$ is continuous everywhere on $\mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$.
2. $\mathrm{e}^{\log(z)} = z = \log(\mathrm{e}^z)$.
3. $\mathrm{Arg}(z)$ is the principle value of $\arg(z)$ such that $-\pi < \mathrm{Arg}(z) \leq \pi$.
4. If $x \in \mathbb{R}$, then $\log(x) = \ln x$.

Definition: Cosine, Sine

Given the exponential, define $\cos(z) = \frac{1}{2}(\mathrm{e}^{iz} + \mathrm{e}^{-iz})$ and $\sin(z) = \frac{1}{2}(\mathrm{e}^{iz} - \mathrm{e}^{-iz})$.

Remarks

1. Both $\cos: \mathbb{C} \to \mathbb{C}$ and $\sin: \mathbb{C} \to \mathbb{C}$ coincides with the conventional cosine and sine, i.e. if $y = 0$, $\cos z = \cos x$ and $\sin z = \sin x$.
2. If $z = iy$, then sine and cosine restrict to hyperbolic trigonometric functions (up to multiplication by i); $\cosh u = \frac{\mathrm{e}^u + \mathrm{e}^{-u}}{2}$ and $\sinh u = \frac{\mathrm{e}^u - \mathrm{e}^{-u}}{2}$.
3. Laws: $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$ and $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$.

Basic Properties of Analytic Functions

Power Series

Definition: Power Series

A power series in z is defineid by $f(z)=\sum_{j=0}^{\infty} a_j (z-z_0)^j$ $z_0 \in \mathbb{C}$. $z_0 \in \mathbb{C}$ is the centre of the power series.

Definition: Radius of Convergence

The radius of convergence for a power series $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ is $R \geq 0$ or $R=\infty$ such that for any z with $|z-z_0|<R$ $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ converges absolutely and for any z with $|z-z_0|>R$ $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ diverges.
Note: $|z-z_0|=R$ is not specified.

Lemma

If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges, then it converges absolutely for all w with $|w-z_0|<|z-z_0|$.

Theorem

Any power series has a radius of convergence.

Claim

1. $\dfrac{1}{1-z}=\sum_{n=0}^{\infty} z^n$, $|z|<1$.
2. $e^z=\sum_{n=0}^{\infty} \dfrac{z^n}{n!}$.
3. $\cos(z)=\sum_{n=0}^{\infty} (-1)^n \dfrac{z^{2n}}{(2n)!}$.
4. $\sin(z)=\sum_{n=0}^{\infty} (-1)^n \dfrac{z^{2n+1}}{(2n+1)!}$.

Definition: Derivative

Let $f: D \subset \mathbb{C} \to \mathbb{C}$ and $z_0 \in \mathbb{C}$. Then $f'(z_0)=\lim_{h \in \mathbb{C} \to 0} \dfrac{f(z_0+h)-f(z_0)}{h}=\lim_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$.

Definition: Analytic

$f: D \to \mathbb{C}$ is called analytic in D if it has a derivative at every $z_0 \in D$.

Definition: Entire

f is called an entire function if it is analytic in the whole $\mathbb{C}$.

Proposition

If $f: D\to\mathbb{C}$ and $g: D\to\mathbb{C}$ are differentiable at $z_0\in D$, then so are $f+g$, $f-g$, $f\,g$, $\frac{f}{g}$ (assuming $g(z_0)\neq 0$). We have

- $(f\pm g)'(z_0)=f'(z_0)\pm g'(z_0)$.
- $(f\,g)'(z_0)=f'(z_0)g(z_0)+f(z_0)g'(z_0)$.
- $\left(\frac{f}{g}\right)'(z_0)=\frac{f'(z_0)g(z_0)-f(z_0)g'(z_0)}{(g(z_0))^2}$.

Example: Polynomials

$f(z)=z$ is an entire function, with $f'(z_0)=\lim_{h\in\mathbb{C}\to 0}\frac{z_0+h-z_0}{h}=1$. Thus any polynomial $f(z)=a_n z^n+a_{n-1}z^{n-1}+\cdots+a_1 z+a_0$ is an entire function with $f-(z)=n\,a_n z^{n-1}+(n-1)a_{n-1}z^{n-2}+\cdots+a_1$.

Theorem

If $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ has $R>0$ as its radius of convergence, then it is analytic in the disk $|z-z_0|<R$ and there we have $f'(z)=\sum_{n=1}^{\infty}n\,a_n(z-z_0)^{n-1}$.

Example: The Exponential

One way to define the exponential is via the differential equation $f(z)'=f(z)$, $f(0)=1$. $f(z)=\mathrm{e}^z$ is a solution.

Explicity, for $z\in\mathbb{C}$, $\mathrm{e}^z=\sum_{n=0}^{\infty}\frac{(z)^n}{n!}$.

Claim: $\mathrm{e}^{z_1}\mathrm{e}^{z_2}=\mathrm{e}^{z_1+z_2}$, $(\mathrm{e}^z)'=\sum_{n=1}^{\infty}n\frac{(z)^{n-1}}{n!}=\sum_{n=1}^{\infty}\frac{(z)^{n-1}}{n-1!}=\sum_{n=0}^{\infty}\frac{(z)^n}{n!}=\mathrm{e}^z$, $\mathrm{e}^0=\sum_{n=0}^{\infty}\frac{(0)^n}{n!}=1$.

Theorem

If a function $f: D\to\mathbb{C}$ admits a power series at $z_0\in D$ with radius of convergence $R>0$, then $f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$, where $f^{(n)}(z_0)$ denotes the n-th derivative at z_0.

Cauchy-Riemann Equations and Harmonic Functions

Assume $f: D\to\mathbb{C}$ is analytic. Then $f(z)=u(z)+i\,v(z)$ with $u, v: D\to\mathbb{R}$; therefore if $z=x+i\,y$ then $u(z)=u(x,y)$ and $v(z)=v(x,y)$ can be thought of as real functions.

Theorem: Cauchy Riemann Equations

Let $z_0=x_0+i\,y_0\in D$. $f: D\to\mathbb{C}$ is analytic if and only if $\frac{\partial u}{\partial x}(x_{0,}y_0)=\frac{\partial v}{\partial y}(x_{0,}y_0)$ and $\frac{\partial u}{\partial y}(x_{0,}y_0)=-\frac{\partial v}{\partial x}(x_{0,}y_0)$.

Remark

Since $u=\operatorname{Re} f$, this determins $v=\operatorname{Im} f$ up to a constant. Knowing u we can compute $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. Therefore we know

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

Theorem
If $u = \text{Re}\, f$ and f analytic, then $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Definition: Harmonic Functions
If $u : D \to \mathbb{R}$ satfisties $\Delta u = 0$, then it is harmonic.

Defintion: Harmonic Conjugate
Given harmonic functions u and v such that $f = u + iv$ is analytic, v is the harmonic conjugate (defined up to a constant).

Theorem
If $f = u + iv$ is analytic (in domain D), and if u or v or $u^2 + v^2$ is constant in D , then f is constant.

Theorem
Suppose $D \subset \mathbb{C} = \mathbb{R}^2$ is a domain, and $u, v : D \to \mathbb{R}$ are functions that are differentialbe (i.e. both partial derivatives exists) and both partial derivatives are continuous in D . If the Cauchy-Riemann Equations hold, then $u + iv : D \to \mathbb{C}$ is analytic.

Maximum Principle for Anayltic and Harmonic Functions
1. Suppose $f : D \to \mathbb{C}$ is an analytic function. If there exists $z_0 \in D$ such that $|f(z_0)| \geq |f(z)|$ $\forall z \in D$, then f is a constant function.
2. Suppose $u : D \to \mathbb{R}$ is a harmonic function. If there exists $z_0 \in D$ such that $|u(z_0)| \geq |u(z)|$ $\forall z \in D$, then u is a constant function.

Line and Contour Integral

Defintion: Path
$\gamma : [a, b] \subset \mathbb{R} \to \mathbb{C}$ is a path.

Defintion: Smooth
The path γ is smooth if $\gamma'(t)$ exists for all $a \leq t \leq b$ and is continuous.
Note: If $\gamma(t) = \sigma(t) + \tau(t)$ where $\sigma, \tau : [a, b] \to \mathbb{R}$, then $\gamma'(t) = \sigma'(t) + \tau'(t)$. At the endpoints a and b , by $\gamma'(t)$ we mean the right and left hand side derivatives respectively.

Definition: Closed
A curve γ is call closed if $\gamma(a) = \gamma(b)$.

Definition: Simple Closed
A curve γ is call simple closed if $\gamma(s) \neq \gamma(t)$ $\forall s, t \in [a, b], s \neq t$.

Definition: Piecewise-Smooth
A curve γ is called piecewise-smooth if there exists $a = a_0 < a_1 < \cdots < a_{k-1} < a_k = b$ such that $\gamma|_{[a_{l-1}, a_l]}$ is smooth.

Definition: Line Integral

Let $\gamma:[a,b]\to D$ be a smooth path. Suppose we have a function $f:D\to\mathbb{C}$. The line integral
$\int_\gamma f(z)\,dz \stackrel{\text{def}}{=} \int_a^b (f(\gamma(t)))(\gamma'(t))\,dt$.

For a piecewise-smooth path with $a=a_0<a_1<\cdots<a_{k-1}<a_k=b$, $\int_\gamma f(z)\,dz \stackrel{\text{def}}{=} \sum_{l=1}^{k}\left[\int_{\gamma_l} f(z)\,dz\right]=\sum_{l=1}^{k}\left[\int_{a_{l-1}}^{a_l}(f(\gamma(t)))(\gamma'(t))\,dt\right]$
where $\gamma_l=\gamma|_{[a_{l-1},a_l]}$.

Remark

By definition, $\int_a^b \alpha(t)+i\beta(t)\,dt=\int_a^b \alpha(t)\,dt+i\int_a^b \beta(t)\,dt$ where $\alpha(t),\beta(t):[a,b]\to\mathbb{R}$.

Property of the Line Integral

$\left|\int_\gamma f(z)\,dz\right|\le \operatorname{length}(\gamma)\cdot\max_{z\in\gamma}|f(z)|$, where $\operatorname{length}(\gamma)=\int_a^b|\gamma'(t)|\,dt$.

Jordan's Theorem

If γ is a simple closed, then $\mathbb{C}\setminus\gamma$ splits into two connected domains, one bounded (the interior) and one unbounded (the exterior).

Definition: Positively Oriented

A simple closed curve $\gamma:[a,b]\subset\mathbb{R}\to\mathbb{C}$ is called positively oriented if for a point P inside γ, $\operatorname{Arg}(\gamma(t)-P)$ increased by 2π when t goes from a to b.

Green's Theorem

Let Ω be a connected domain with piecewise smooth boundary (not necessarily connected). Orient the outside boundary positively, the inside boundaries negatively; let Γ be the resulting union of the simple closed curves with orientation. Suppose that f is defined and smooth (partial derivatives exist and continuous) in the domain $D\supset(\Omega\cup\Gamma)$. Then
$\int_\Gamma f(z)\,dz=i\iint_\Omega\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)dx\,dy$.

Note: There is a real calculus version of this theorem. Let $u,v:D\to\mathbb{R}$. Then
$\int_\Gamma u\,dx+v\,dy=\int_\Gamma u\,dx+\int_\Gamma v\,dy=i\iint_\Omega\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy$. Here if $\gamma:[a,b]\to\mathbb{R}^2$, $\gamma(t)=(\alpha(t),\beta(t))$, then
$\int_\gamma u\,dx=\int_a^b u(\gamma(t))\alpha'(t)\,dt$.

Remark

$f:D\to\mathbb{C}$ is analytic (i.e. the complex derivative $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ exists at any point of D) if and only if we have the Cauchy-Riemann equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

Note that the Cauchy-Riemann equations can be rewritten in the form of a single complex equation $\frac{\partial f}{\partial x}=-i\frac{\partial f}{\partial y}$, where

$\frac{\partial f}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$ and $\frac{\partial f}{\partial y}=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}$; the real part is the first C-R equation, the imaginery part is the second C-R equation.

Cauchy's Theorem and Cauchy's Formula

Cauchy's Theorem

Suppose $f:D\rightarrow\mathbb{C}$ is analytic, and $D\supset\Gamma,\Omega$. Then $\int_{\Gamma} f(z)dz=0$.

Proposition

If $f:D\rightarrow\mathbb{C}$ is analytic and γ is a piecewise smooth simple closed curve whose interior is contained in D , then $\int_{\gamma} f(z)dz=0$.

Definition: Simply Connected

A domain Ω is called simply connected if for any simple closed curve $\gamma\subset\Omega$ the inside of γ is connected in Ω .

Theorem

If D is a simply-connected domain and γ is any piecewise smooth closed curve, then $\int_{\gamma} f(z)dz=0$ for any analytic function in D .

Theorem

If f is analytic in a simply-connected domain, then there exists $F:D\rightarrow\mathbb{C}$ such that $F'=f$.
Note: F is unique up to a constant.

Remark

This theorem implies that if D is a simply-connected domain and f is analytic, then $\int_{\gamma} f(z)dz=0$ for any closed curve γ . Indeed for any path $\gamma:[a,b]\rightarrow\mathbb{C}$ $\int_{\gamma} f(z)dz=\int_a^b f(\gamma(t))\gamma'(t)dt=F(\gamma(t))\big|_a^b=0$.

Theorem: Cauchy Formula

$f(z)=\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{\zeta-z}d\zeta$, where $f:D\rightarrow\mathbb{C}$ is analytic, D is a simply-connected domain containing the piecewise smooth close curve γ and z is inside γ .

Example

$\frac{1}{2\pi i}\int_{\gamma}\frac{1}{\zeta}d\zeta=1$. Here $f(z)=1$ and γ is any simple closed curve around 0.

Cauchy-Goursat Theorem

If $f:D\rightarrow\mathbb{C}$ is analytic (i.e. $f'(z)=\lim_{h\rightarrow 0}\frac{f(z+h)-f(z)}{h}$ exists for all $z\in D$) and $\Omega\subset D$ is a domain with piecewise-

smooth boundary, then $\int_{\partial\Omega} f(z)\,dz=0$.

Consequences of Cauchy's Formula

Theorem

Let $f:D\to\mathbb{C}$ be an analytic function. Let $z_0\in D$ and $R>0$. Define $\Delta_{z_0}(R)=\{z \| z-z_0| <R\}\subset D$. Then there exists a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ convergent in $\Delta_{z_0}(R)$ such that $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ on $\Delta_{z_0}(R)$. Furthermore,
$a_n=\frac{1}{2\pi i}\int_{\partial\Delta_{z_0}(R)}\frac{f(z)}{(z-z_0)^n}dz$.

Note: If $n=0$, $a_0=\frac{1}{2\pi i}\int_{\partial\Delta_{z_0}(R)}\frac{f(z)}{z-z_0}dz=f(z_0)$ is just the Cauchy Formula.

Corollary

The derivative $f':D\to\mathbb{C}$ is also analytic.

Corollary

The coefficients of the power series of f are given by $a_n=\frac{f^{(n)}(z_0)}{n!}$.

Remark

$\frac{f(z)}{(z-z_0)^{n+1}}=\frac{a_0}{(z-z_0)^{n+1}}+\frac{a_1}{(z-z_0)^n}+\cdots+\frac{a_n}{z-z_0}+a_{n+1}+a_{n+2}(z-z_0)+\cdots$.

Proposition

$\int_{\partial\Delta_{z_0}(R)}\frac{1}{(z-z_0)^k}dz=0 \quad ,k\in\mathbb{N}$ unless $k=1$.

Order of Zero

Definition: Zero of a Function

Let $f:D\to\mathbb{C}$ be analytic, and $z_0\in D$. Then either $f(z_0)\neq 0$ or $f(z_0)=0$. If $f(z_0)=0$ then z_0 is a zero of f .
Note: $f(z_0)=a_0$ if $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Definition: Order of Zero

Suppose $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Let m be such that $a_m\neq 0$ but $a_1=a_2=\cdots=a_{m-1}=0$. Then m is called the order of zero of f at z_0 .
In particular, if z_0 is not a zero of f (i.e. $f(z_0)=a_0\neq 0$) then the order of zero of f at z_0 is 0.

Proposition

If m is the order of zero of f at z_0, then $g(z)=\dfrac{f(z)}{(z-z_0)^m}$ is analytic in D and $g(z_0)\neq 0$.

Conversely, if $g(z)-\dfrac{f(z)}{(z-z_0)^m}$ is analytic in D and $g(z_0)\neq 0$, then m is the order of zero of f at z_0.

Definition: Order of Zero

Note that $a_0=f(z_0)$, $a_1=f'(z_0)$, ... , $a_k=\dfrac{f^{(k)}(z_0)}{k!}$. This leads to another equivalent definition of the order of zero.

If m is the order of zero of f at z_0, then $f(z_0)=f'(z_0)=\cdots=f^{(m-1)}(z_0)=0$ and $f^{(m)}(z_0)\neq 0$.

Merera's Theorem

Suppose $f:D\rightarrow\mathbb{C}$ is a continuous function in a domain $D\subset\mathbb{C}$ such that $\int_\gamma f(z)dz=0$ for any simple closed curve γ with the interior in D. Then f is analytic.

Liouville's Theorem

Suppose $f:\mathbb{C}\rightarrow\mathbb{C}$ is analytic in the whole plane (such a fuction is called entire). If f is bounded, i.e. there exists $M\in\mathbb{R}$ such that $|f(z)|<M$ for all $z\in\mathbb{C}$, then $f\equiv\text{constant}$.

Analytic Logarithms

Assume that $f:D\rightarrow\mathbb{C}$ is an analytic function in a simply connected domain D, and $f(z)\neq 0$ $\forall z\in D$. Then there exists an analytic branch of the logarithm $\log f:D\rightarrow\mathbb{C}$, with $F=\log f$ and $F'=\dfrac{f'}{f}$.

Note: F is defined up to adding a constant.

Multiplication of Power Series

Suppose $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ and $g(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ both converge for $|z-z_0|<R$. Then $f(z)g(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n$ where $c_n=\sum_{k=0}^{n}a_k b_{n-k}$ converges in $|z-z_0|<R$.

Isolated Singularities

Definition: Isolated Singularity

If $f(z)$ is defined for all $|z-z_0|<\varepsilon$, $\varepsilon>0$ but $z\neq z_0$, then z_0 is called an isolated singularity of f.

Note: f is defined and analytic in this punctured disk centred at z_0 of radius ε.

Definition: Removable Singularity, Poles, Essential Singularity

Let $\Delta=\{z:|z-z_0|<\varepsilon\}$. Assume that $f:\Delta\setminus\{z_0\}\rightarrow\mathbb{C}$ is analytic (then z_0 is an isolated singularity).

1. z_0 is called a removable singularity if f is bounded in the neighbourhood of z_0, i.e. $\exists M, \delta>0$ such that $|f(z)|<M$ whenever $|z-z_0|<\delta, z\neq z_0$.
2. z_0 is a pole of f if $\lim_{z\rightarrow z_0}f(z)=\infty$.
3. z_0 is an essential singularity if it is neither of the above cases.

Examples

1. $f(z)=\frac{z^3-z_0^3}{z-z_0}=\frac{(z-z_0)(z^2+z_0z+z_0^2)}{z-z_0}=z^2+z_0z+z_0^2$ is an analytic function also at z_0. This is a removable singularity.
2. $f(z)=\frac{1}{(z-z_0)^2}$. This is a genuine singularity. $\lim_{z\to z_0}\frac{1}{(z-z_0)^2}=\infty$, so z_0 is a pole.
3. $f(z)=e^{\frac{1}{z-z_0}}$. If $z-z_0>0$ then $f(z)\to+\infty$. If $z-z_0<0$ then $f(z)\to 0$. If $\operatorname{Arg}(z-z_0)=\frac{\pi}{2}$ then $e^{\frac{1}{z-z_0}}=e^{ix}$ and is bounded. z_0 is an essential singularity.

Proposition
If z_0 is a removable singularity of $f:\Delta\setminus\{z_0\}\to\mathbb{C}$, then there exists an analytic function $h:\Delta\to\mathbb{C}$ such that $f(z)=h(z)$ for $z\in\Delta\setminus\{z_0\}$.

Proposition
If $f:\Delta\setminus\{z_0\}\to\mathbb{C}$ has a pole at z_0 , then there exists an analytic function $h:\Delta\to\mathbb{C}$ with $h(z_0)\neq 0$ such that $f(z)=\frac{h(z)}{(z-z_0)^m}$. m is the order of the pole.

Definition: Residue
Let $f:\Delta\setminus\{z_0\}\to\mathbb{C}$. Let $s>0$ such that $\{z:|z-z_0|=s\}\subset\Delta$. Then $\frac{1}{2\pi i}\int_{|z-z_0|=s} f(z)\,dz$ is the residue of f at z_0 , denoted $\operatorname{Res}(f,z_0)$.

Computations of Residues
1. If f is analytc in a neighbourhood of z_0 (i.e. in a domain containing z_0), then $\operatorname{Res}(f,z_0)=0$ by the Cauchy Theorem.
2. Suppose $f(z)=\frac{H(z)}{z-z_0}$ where H is analytic at z_0 (i.e. $H:U\to\mathbb{C}$ analytic, $z_0\in U$). z_0 can be pole of order 1, or removable if $H(z_0)=0$. By the Cauchy Formula, $\operatorname{Res}(f,z_0)=\frac{1}{2\pi i}\int_{|z-z_0|=s}\frac{H(z)}{z-z_0}dz=H(z_0)$.
3. More generally, suppose $f(z)=\frac{H(z)}{(z-z_0)^m}$. Since H is analytic at z_0 , $H(z)=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$, so $f(z)=\frac{a_0}{(z-z_0)^m}+\frac{a_0}{(z-z_0)^{m-1}}+\cdots+\frac{a_{m-1}}{z-z_0}+a_m+a_{m+1}(z-z_0)+\cdots$. Now $\int_{|z-z_0|=s} a_{m+k}(z-z_0)^k\,dz=0$ for all $k\neq -1$, since $a_{m+k}(z-z_0)^k$ has an antiderivative. Therfore, $\operatorname{Res}(f,z_0)=\frac{1}{2\pi i}\int_{|z-z_0|=s}\frac{a_{m-1}}{z-z_0}dz=a_{m-1}=\frac{H^{(m-1)}(z_0)}{(m-1)!}$, which is the coefficient of H at $(z-z_0)^{m-1}$.

Proposition
Suppose $f(z)=\frac{H(z)}{G(z)}$ where $F,G:U\to\mathbb{C}$ are analytic. Suppose that $G(z_0)=0$, $G'(z_0)\neq 0$, and $H(z_0)\neq 0$. Then $\operatorname{Res}(f,z_0)=\frac{H(z_0)}{G'(z_0)}$.
Note: f has a pole of order 1 at z_0 .

Theorem

Suppose that $f: A \to \mathbb{C}$ is analytic where $A = \{z \in \mathbb{C} | r < |z - z_0| < R\}$. Then there exists $a_k \in \mathbb{C}$, $k \in \mathbb{Z}$ such that $f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k = \sum_{k=-\infty}^{-1} a_k (z - z_0)^k + \sum_{k=0}^{\infty} a_k (z - z_0)^k$. Furthermore, $a_k = \frac{1}{2\pi i} \int_{|z - z_0| = \varepsilon} \frac{f(z)}{(z - z_0)^{k+1}} dz$ where $r < \varepsilon < R$.

Definition: Laurent Series

$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k$ is the Laurent series.

Note: $\operatorname{Res}(f, z_0) = a_1$.

The Residue Theorem and Its Application to the Evaluation of Definite Integrals

The Residue Theorem

Suppose γ is a simple closed curve in $\mathbb{C}$, where $\gamma = \partial \Omega$, Ω a bounded domain. Suppose $f: \Omega \setminus \{z_1, \ldots, z_n\} \to \mathbb{C}$ is an analytic function that extends to a continuous function on $\partial \Omega$ (each z_j is an isolated singularity). Then

$\int_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k)$.

Example: Rational Functions

Let $f(z) = \frac{P(z)}{Q(z)}$ where P and Q are polynomials. f is a rational function. Since both P and Q are entire functions, $\frac{P}{Q}$ is also an analytic function at z unless $Q(z) = 0$ (then we have a pole or removable singularity). We can compute the residue.

<u>Claim</u>: Suppose $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0$, $a_j \in \mathbb{C}, a_n \neq 0$ is a polynomial of degree n. If $\alpha \in \mathbb{C}$ is such that $P(\alpha) = 0$ (i.e. α is a root), then there exists a polynomial q of degree $n - 1$ such that $p(z) = (z - \alpha) q(z)$.

<u>Corollary</u>: p has no more than n roots.

<u>Claim</u>: There exists $R > 0$ such that if $|z| > R$ then $\frac{1}{2}|a_n z^n| < |p(z)| < 2|a_n z^n|$.

<u>Theorem</u>: Let P and Q be polynomials such that $\deg P \geq 2 + \deg Q$ and $Q(x) \neq 0 \;\; \forall x \in \mathbb{R}$. Then $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{k} \operatorname{Res}\left(\frac{P}{Q}, z_k\right)$ where z_k's are such that $Q(z_k) \neq 0$ and $\operatorname{Im} z_k > 0$.

Example

$\int_{-\infty}^{\infty} f(x) dx = \lim_{R \to \infty} \int_{\gamma_R} f(z) dz$, where $\gamma_R = [-R, R] \cup \{z \in \mathbb{C} | |z| = R, \operatorname{Im} z > 0\}$, holds when $\lim_{R \to \infty} \int_{\substack{|z| = R \\ \operatorname{Im} z > 0}} f(z) dz = 0$. This will hold if $f(z) = \frac{P(z)}{Q(z)}$ with $\deg P \geq \deg Q + 2$ and $Q(x) \neq 0 \;\; \forall x \in \mathbb{R}$.

There are other instances when this will hold also. Where the integral involes trigonometric functions, use $\cos x = \operatorname{Re}(e^{ix})$ and $\sin x = \operatorname{Im}(e^{ix})$, and also $\int_{\substack{|z| = R \\ \operatorname{Im} z > 0}} e^{iz} dz \leq \pi$ (Jordon's Lemma).

Jordon's Lemma

$\left| \int_{\substack{|z| = R \\ \operatorname{Im} z > 0}} e^{iz} dz \right| \leq \pi$.

Example

Consider integrals of the form $\int_0^{2\pi} f(\theta)\,d\theta$ involving trigonometric functions. Turn this integral into a complex contour integral by substituting $z = e^{i\theta}$, $dz = i\,e^{i\theta}\,d\theta \Leftrightarrow d\theta = \frac{dz}{i\,e^{i\theta}}$. Notice that $\sin z = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and $\cos z = \frac{1}{2}\left(z + \frac{1}{z}\right)$.