Chaotic Systems

- Deterministic model, SDIC (sensitive dependence on initial conditions).

Claim

If $f:\mathbb{R}\to\mathbb{R}$ is continuous and one-to-one, then f is monotonic; that is, f is either increasing ($x_1<x_2\Rightarrow f(x_1)<f(x_2)$) or decreasing ($x_1<x_2\Rightarrow f(x_1)>f(x_2)$).

Claim

Let $f:\mathbb{R}\to\mathbb{R}$ be continuous. If f has only one fixed point p, then $\text{graph}(f)\cap\text{diagonal}\neq\emptyset$; moreover, the intersection is only one point.

Claim

Let $f:\mathbb{R}\to\mathbb{R}$ be continuous, one-to-one, increasing, and have one fixed point p. Then $f(x)<p \ \ \forall x<p$ and $f(x)>p \ \ \forall x>p$. Moreover, for all $x<p$, either $f(x)<x$ or $f(x)>x$.

Claim

Let $f:\mathbb{R}\to\mathbb{R}$ be continuous, one-to-one, increasing, and have one fixed point p. Let $f^n(x)\to p$ as $n\to\infty$. Then $f(x)>x \ \ \forall x<p$ and $f(x)<x \ \ \forall x>p$.

Fixed and Periodic Points

Definition: Attracting Fixed Point

Let $f:\mathbb{R}\to\mathbb{R}$ be continuous. Let p be a fixed point of f. p is an attracting fixed point if there exists $\varepsilon>0$ such that for all $x\in(p-\varepsilon,p+\varepsilon)$, $f^n(x)\to p$ as $n\to\infty$. Equivalently, p is an attracting fixed point if there exists $\varepsilon_0>0$ such that for all $0<\varepsilon_0<\varepsilon$, $f(\overline{I_\varepsilon(p)})\subset I_\varepsilon(p)$ where $I_\varepsilon(p)=(p-\varepsilon,p+\varepsilon)$.

Claim

If $x<f(x)<-x+2p \ \ \forall x\in[p-\varepsilon,p]$ and $-x+2p<f(x)<x \ \ \forall x\in[p,p+\varepsilon]$, then p is an attracting point.

Definition: Repelling Point

Let $f:\mathbb{R}\to\mathbb{R}$ be continuous. Let p be a fixed point of f. p is a repelling fixed point if there exists $\varepsilon_0>0$ such that for all $x\in[p-\varepsilon_0,p+\varepsilon_0], x\neq p$, there exists $n>0$ such that $f^n(x)\notin[p-\varepsilon_0,p+\varepsilon_0]$. Equivalently, there exists $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$, $f(I_\varepsilon(p))\not\supseteq\overline{I_\varepsilon(p)}$.

Claim

If $f(x)>-x+2p \ \ \forall x\in[p-\varepsilon,p]$ or $f(x)<x \ \ \forall x\in[p-\varepsilon,p]$ and $f(x)>x \ \ \forall x\in[p,p+\varepsilon]$ or $f(x)<-x+2p \ \ \forall x\in[p,p+\varepsilon]$, then p is an repelling fixed point.

Definition: Global Basin of Attraction

Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and let p be a fixed point of f. The global basin of attraction of p is $\{x\in\text{Dom}(f): f^n(x)\to p\}$.

Definition: Local Basin of Attraction
Let f be continuous and let p be a fixed point of f. The local basin of attraction of p is the connected/continuous interval I such that $f^n(x) \to p \ \forall x \in I$.

Claim
Let p be a fixed point of f and let f be one-to-one locally at p. Then p is fixed for f^{-1} and f is repelling if and only if f^{-1} is attracting.

Definition: Periodic Point
p is a periodic point of f if there exists $k>0$ such that $f^k(p)=p$. k is the period of p if $f^i(p) \neq p \ \forall i<k$ and $f^k(p)=p$.

Definition: Eventually Periodic
A point q is eventually periodic if it becomes periodic for some iterate.

Observation
If q is a periodic point of period k, then q is a fixed point of f^k.

Corollary
If q is a periodic point of period k, then $f(q), f^2(q), \ldots, f^{k-1}(q)$ are also periodic points of period k. Hence there are k periodic points of period k.

Claim
Let p be a periodic point of period k. If $k \geq 3$, then f is not one-to-one.

Definition: Attracting Periodic Point
Let p be a periodic point of period k. p is an attracting periodic point if p is an attracting fixed point of f^k.

Definition: Repelling Periodic Point
Let p be a periodic point of period k. p is a repelling periodic point if p is a repelling fixed point of f^k.

Differentiable Systems

Definition: Tangent Line
Let f be differentiable. Let p be a fixed point of f. The tangent line through p is $L(p): y=f'(p)(x-p)+p$.

Claim
Let f be differentiable and p a fixed point of f. Then p a fixed point of the tangent line $L(p)$. Moreover,

- p is an attracting fixed point of $L(p)$ if and only if $|f'(p)|<1$.
- p is a repelling fixed point of $L(p)$ if and only if $|f'(p)|>1$.

Claim
Let f be differentiable and p a fixed point of f.

- If $|f'(p)|<1$, then p is an attracting fixed point of f.
- If $|f'(p)|>1$, then p is a repelling fixed point of f.

Claim
Let f be differentiable and p a periodic point of f of period k, i.e. p a fixed point of f^k. Then $f^{k\prime}(p)=\prod_{i=0}^{k-1} f'(f^i(p))$.

Doubling Function
"Model of Chaotic Systems"

Definition: Doubling Function
Let $F(x):[0,1]\to[0,1]$ be defined by $F(x)=\begin{cases}2x & x\in[0,1/2)\\ 2x-1 & x\in[1/2,1]\end{cases}$.

Note that F is equivalent to:

- $F(x)=2x_{\equiv 1}$.
- $\theta\to 2\theta$ a map on angles.
- $z\to z^2$, $z\in\mathbb{C}, |z|=1$.

Periodic Points
$\#\mathrm{Per}_n(f)=\#\mathrm{fixed}(f^n)-\sum_{m|n}\#\mathrm{Per}_m(f)$, where $\#\mathrm{fixed}(f^n)=2^n-1$.

Generalization
Let $f(x):[0,1]\to[0,1]$ be defined $f(x)=\begin{cases}f_0(x) & 0\le x<c\\ f_1(x) & c\le x\le 1\end{cases}$, where $f_0(0)=0, f_0(c)=\lim_{x\to c} f_0(x)=1$ and $f_1(c)=0, f_1(1)=1$, and $f_0'(x), f_1'(x)>\lambda$ for some $\lambda>1$.

Chaotic Systems

Definition: Expansive/Chaotic
Let $f:X\to X$ where X is metric and compact. The map/system f is expansive or chaotic if there exists $\alpha_0>0$ such that for any $x, y\in X$ there exists a positive integer n satisfying $\mathrm{dist}(f^{(x)}, f^n(y))>\alpha_0$.

Definition: "Two-Branched Expansive Map"
Let $f:[0,1]\to[0,1]$, $f(x)=\begin{cases}f_0(x) & 0\le x<c\\ f_1(x) & c\le x\le 1\end{cases}$ where $f_{0,} f_1\in C^1$ with $f_0(0)=0, f_0(c)=\lim_{x\to c} f_0(x)=1$, $f_1(c)=0, f_1(c)=1$, and $f_0'(x), f_1'(x)>\lambda>1$. f is the "two-branched expansive map".

Claim
Let f be the "two-branched expansive map". Then there exists $\alpha_1>0$ such that if $c-\alpha_1<x<c\le y<c+\alpha_1$ then $d(f(x), f(y))>\frac{1}{2}$.

Claim
Let f be the "two-branched expansive map". Then f is expansive/chaotic.

Claim
Let $g:[0,1]\to[0,1]$ be C^1 close to f, i.e. $|g(x)-f(x)|$ small and $|g'(x)-f'(x)|$ small. If f is expansive, then so is g. Therefore, f is robustly chaotic.

Definition: Conjugacy/Equivalence
$f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are conjugate/equivalent if there exists $h:\mathbb{R}\to\mathbb{R}$ and $h^{-1}:\mathbb{R}\to\mathbb{R}$ bijective such that $f=h^{-1}\circ g\circ h$.

Remark
$f=h^{-1}\circ g\circ h$ if and only if $h\circ f=g\circ h$ if and only if $h\circ f\circ h^{-1}=g$.

Lemma
$x\in\text{Fix}(f)\Leftrightarrow h(x)\in\text{Fix}(g)$.

Definition: Dense
Let $D\subseteq[0,1]$. D is dense in $[0,1]$ if for any interval $(a,b)\subseteq[0,1]$, $D\cap(a,b)\neq\emptyset$.

Claim
Let f be the "two-branched expansive map". Then $\text{Per}(f)$ is dense.

Claim
Let f be the "two-branched expansive map". Then the preorbit of f, $f^{-}([0,1])$, is dense.

Symbolic Dynamics

Definition: Shift Map
Let $\Lambda=\{0,1\}^{\mathbb{N}}$ (infinite sequences of 0 and 1, i.e. $\vec{a}\in\Lambda\Leftrightarrow a=(a_1,a_2,\ldots), a_i=0\text{ or }1$). The shift map σ is defined as $\sigma:\Lambda\to\Lambda$, $[\sigma(\vec{a})]_n=a_{n+1}$, i.e. $(a_1,a_2,a_3,\ldots)\xrightarrow{\sigma}(a_2,a_3,a_4,\ldots)$.

Periodic Points
- $\text{Fix}(\sigma)=\{\vec{0},\vec{1}\}$.
- $\text{Fix}(\sigma^2)=\{0\vec{0},0\vec{1},1\vec{0},1\vec{1}\}$.
- $\text{Fix}(\sigma^3)=\{00\vec{0},00\vec{1},01\vec{0},01\vec{1},10\vec{0},10\vec{1},11\vec{0},11\vec{1}\}$.

In general, $\text{Per}_n(\sigma)=\text{Fix}(\sigma^n)\setminus\bigcup_{m|n}\text{Per}_m(\sigma)$.

Pre-Images
$\sigma^{-1}(\vec{a})=\{(0,\vec{a}),(1,\vec{a})\}$.

Claim
Let f two-branched expansive map and σ be the shift map. The map $h:[0,1]\to\{0,1\}^{\mathbb{N}}$ defined by

$[h(x)]_n = \begin{cases} 0 & \text{if } f^n(x) \in I_0 \\ 1 & \text{if } f^n(x) \in I_1 \end{cases}$ is one-to-one, onto, and satisfies $h \circ f = \sigma \circ h$. Therefore, f and σ are conjugate.

Extensions of the Doubling Function

Example

Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = \begin{cases} 2x & \text{if } x < \frac{1}{2} \\ 2x-1 & \text{if } x \geq \frac{1}{2} \end{cases}$. Let $B(\pm\infty) = \{x : F^n(x) \xrightarrow{n} \pm\infty\}$ (basin of attraction of $\pm\infty$).

Then $B(\pm\infty) = (-\infty, 0) \cup (1, \infty)$ and $F([0,1]) = [0,1] = B(\pm\infty)^c$.

Example

Let $\varepsilon > 0$. Define $g(x) = \begin{cases} (2+\varepsilon)x & \text{if } 0 \leq x < \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases} = \begin{cases} \gamma x & \text{if } 0 \leq x < \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$ where $\gamma = 2 + \varepsilon > 2$.

- Note that $g_0(\frac{1}{2}) = \frac{\gamma}{2} > 1$, and if $\frac{1}{\gamma} < x < \frac{1}{2}$ then $g_0(x) > 1$. So $[0,1]$ is not an invariant region of g, i.e. $g([0,1]) \not\subset [0,1]$.
- Let $U_0 = (\frac{1}{\gamma}, \frac{1}{2})$. Then $B(+\infty) = (1,\infty) \cup U_0 \cup \bigcup_{m=1}^{\infty} G^{-m}(U_0)$, where $G^{-m}(U_0)$ contains 2^m connected intervals.
- g has an infinite number of periodic points.

Neutral Fixed Points and Bifurcations

Claim

Assume that $f : \mathbb{R} \to \mathbb{R}$, $f \in C^\infty$, $f(p) = p$, $f'(p) = 1$.

1. If $f''(p) > 0$, then p attracts close by points $x < p$ and repels close by points $x > p$. p is called a saddle node.
2. If $f''(p) < 0$, then p repels close by points $x < p$ and attracts close by points $x > p$. p is called a saddle node.
3. If $f''(p) = 0$ and $f'''(p) > 0$, then p repels all close by points.
4. If $f''(p) = 0$ and $f'''(p) < 0$, then p attracts all close by points.

Claim

Assume that $f : \mathbb{R} \to \mathbb{R}$, $f \in C^\infty$, $f(p) = p$, $f'(p) = -1$. Define $g = f \circ f$. Then

- $g'(p) = 1$,
- $g''(p) = 0$,
- $g^{(3)}(p) = -[2f^{(3)}(p) + 3(f^{(2)}(p))^2]$

Assuming $g^{(3)}(p) \neq 0$,

1. If $g^{(3)}(p) > 0$, p is a repelling point for g and therefore f.
2. If $g^{(3)}(p) < 0$, p is an attracting point for g and therefore f.

Note

- A pitchfork bifurcation for g produces a "period doubling bifurcation".