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Aristarchus's On the Sizes and Distances of the Sun and the Moon: *Greek and Arabic Texts*

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Introduction

In the 1920s, T. L. Heath pointed out that historians of mathematics have "given too little attention to Aristarchus" (Heath 1921, vol. 2, 1). This is still true today. The Greek text of Aristarchus's *On the Sizes and Distances of the Sun and the Moon* has received little attention; the Arabic editions virtually none.[1] For these reasons, much of what this text has to tell us about ancient and medieval mathematics and the mathematical sciences has gone unnoticed.

When one considers that many of Aristarchus's arguments are obscure and much of his mathematics cumbersome, the persistent interest in this text during the medieval and early modern periods is remarkable. It was edited and studied by Arabic scholars long after all of its mathematical methods and most of its astronomical results had become otiose. Copies of the Greek manuscripts were still being made by Latin scholars in the 17th century, well after the ascent of printed text.[2]

The work begins with a series of hypotheses that are at once crude and contradictory and yet refreshingly bold. From these, by great labor, Aristarchus derives a few precise statements about objects far outside our common purview, displaying an incisive ability with theoretical modeling. The text is a fine example of that style of Greek mathematics which produces, from seemingly intractable quagmires, results that are simple and clean. All of these features must have delighted the many generations of mathematicians who studied *On Sizes*. But perhaps they were struck by something simpler than the detailed arguments and the actual results. Perhaps they were struck by the work's fundamental, unspoken claim. *On Sizes* implies, unequivocally, that the world is mathematical; not just in a vague qualitative way, but in a precise quantitative way. It demonstrates that by starting from a few simple and readily obtainable statements one can, through the

[1] The text itself, an English translation and useful notes are provided by Heath (1913). Heath's edition is based principally on Vat. Gr. 204, the oldest Greek MS. A mathematical discussion of the treatise is given by Neugebauer (1975, 634–643), who is predominantly interested in Aristarchus's astronomical results. Wall (1975) discusses *On Sizes* in his historiographic study of Aristarchus. Newton (1977, 171–177 & 389–394) also studied the text, but it is not clear how closely he followed the details of the argument. The treatise is also discussed by Panchenko (2001).

[2] Noack (1992) has provided a thorough study of the history of the text.

methods of mathematics, produce new knowledge of things that would otherwise be beyond our grasp. This claim lies at the foundation of all the ancient exact sciences and was as exciting in the medieval and early modern periods as it was in antiquity.

Aristarchus is securely dated to the early Hellenistic period by Ptolemy who associates him with a summer solstice observation of 280 BCE (Toomer 1984, 139). This date is compatible with Archimedes' *Sand Reckoner*, which makes repeated mention of Aristarchus, discussing his methods and a number of his assumptions and results. Archimedes is assumed to have known *On Sizes*, because he credits Aristarchus with finding the size relation of *On Sizes* Prop. 9 (Heiberg 1973, vol. 2, 220). The difficulty is that two of the hypotheses that Archimedes attributes to Aristarchus are different from what we find in the treatise, but this will become less problematic when we have a better understanding of Aristarchus's approach.[3]

The first author who identifies Aristarchus as the author of *On Sizes* is Plutarch, in a work that also associates Aristarchus with the heliocentric hypothesis, written around the turn of the 2nd century CE. In *On the Face Appearing in the Circle of the Moon*, there are three discussions of Aristarchus. The first of these mentions the Stoic philosopher Cleanthes' criticisms of "Aristarchus the Samian" for holding the impious view that the earth could move (Cherniss and Helmbold 1968, 54). The second attributes to Aristarchus a treatise called *"On Sizes and Distances"* and gives a statement of the results of Prop. 7 that is close in wording to the one immediately following the hypotheses in *On Sizes* (Cherniss and Helmbold 1968, 74; cf. Heath 1913, 352). The third associates Aristarchus with the results of Prop. 17 (Cherniss and Helmbold 1968, 120). By the time Pappus was writing his *Collection* in the 4th century CE, *On Sizes* was traditionally attributed to Aristarchus and was included as a canonical text in the field of mathematical astronomy (Hultsch 1876–1878, vol. 2, 137). We, along with most who have written on Aristarchus, believe that he was the author of *On Sizes*.[4]

On Sizes fits in well with what we know of the intellectual context of the early Hellenistic period. From a philosophical perspective, it addresses questions that were being raised by the Epicureans about the validity of sense perception as a criterion of knowledge, especially with regard to the size of the sun. In terms of the current concentric sphere cosmology, in which the size of the cosmos was determined by the distance of the sun from the earth, it makes the claim that we can know the overall size of the cosmos and the sizes of the principal bodies within it. Moreover, the deductive structure of the treatise agrees with other texts in the exact sciences of this period. Indeed, the similarities between *On Sizes* and the geometric part of Archimedes' *Sand Reckoner* are

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[3] Wall (1975, 206–210) argues that all references to Aristarchus's work in *Sand Reckoner* may be to *On Sizes*. We see no reason, however, not to believe that Aristarchus wrote more than one book and that Archimedes simply takes what he needs for his own ends.

[4] Bowen and Goldstein (1994, 700, n. 20) have raised the possibility of a much later date for the text of *On Sizes*. Their primary argument is the claim that *On Sizes* fits better with the intellectual context of the first centuries around the turn of the era. They do not, however, provide the details of this argument. Rawlins (1991, 69, n. 6) also believes that *On Sizes* is a later work. He bases his opinion on the claim that Hyp. 6, discussed below, could never be held by a "serious astronomer."

striking. They address the same sorts of problems; they use the same sorts of geometric models and the same toolbox of mathematical methods to arrive at their solutions.[5]

The Greek text

The structure of the treatise

On Sizes is a work of deductive applied mathematics. It is structured like other applied mathematical texts of the early Hellenistic period, such as the two texts of Autolycus on spherical astronomy or Euclid's scientific works. Unlike these texts, however, the project of *On Sizes* is computational. Hence, the goal of the text is to develop geometric diagrams of the earth and the luminaries, and then to introduce numerical parameters to derive bounds on the sizes and distances of the sun and the moon.

The treatise begins with six assumptions. There is no caption for these given in the manuscripts but within the text they are referred to by the term "hypothesis" (ὑπόθεσις).[6] There are two types of hypotheses, arranged in two groups. The first three are geometric, in that they make assumptions about the celestial world that allow the mathematician to construct a geometric diagram. They are not physical in the ancient sense of the term. That is, they are not about the nature of the objects concerned, but treat the external aspects of these bodies in such a way as to establish the possibility of a definite geometric model. They are as follows (Heath 1913, 352):

1. That the moon receives its light from the sun.
2. That the earth has the ratio of a point and a center to the sphere of the moon.[7]
3. That, when the moon appears to us halved, the great circle dividing the dark and the bright portions of the moon points toward (νεύειν εἰς) our eye.[8]

Only Hyps. 2 & 3 are explicitly used in the course of an argument. Nevertheless, all three are implied in the geometric configuration of a number of propositions. Consider Table 1, which exhibits the deductive structure of the work. An explicit use of an hypothesis is indicated by a bullet, while an implicit use in the diagram is indicated by an *i*.

While Hyps. 1 & 3 are relatively straightforward, Hyp. 2 warrants some comment. It was usual in Greek astronomical texts to claim that the size of the earth relative to the

5 Although Erhart and Erhard Siebold [1942,1943] have questioned the authenticity of *Sand Reckoner*, following Neugebauer (1942) few scholars have taken this doubt seriously. Indeed, Archimedes is generally taken as an important source for our knowledge of Aristarchus's activity (Christianidis J., Dialetis D. and K. Gavroglu 2002).

6 Heath (1913, 353) supplies the caption.

7 Other remarks in the treatise as well as the three explicit uses of this hypothesis make it clear that "the sphere of the moon" is the sphere in which the moon moves.

8 That is, the dividing circle lies in the same plane as our eye. In the text, this hypothesis is poorly formed. Despite being supported by the Greek MSS, Pappus and the Arabic editions, the adjective μέγιστον, "greatest," modifying τὸν. . . κύχλον should be excised. Prop. 2 proves that this circle is *not* a great circle. Moreover, Prop. 5, the only proposition that uses Hyp. 3, cites it as referring to "the circle" not "the great circle."

Table 1. Logical structure of *On Sizes*. A proposition in the column headings is supported by each unit marked with a bullet in the row headings to the left. An *i* indicates that the unit is implicit in the geometric configuration of the proposition

	1_g 1_a	2_g 2_a	3_g 3_a	4_g 4_a	5_g $4c_a$	6_g 5_a	7_g 6_a	8_g $6c_a$	9_g 7_a	10_g 8_a	11_g 9_a	12_g 10_a	13_g 11_a	14_g 12_a	15_g 13_a	16_g 14_a	17_g 15_a	18_g 16_a	17_a
H1			i	i		i	i					i	i	i	i				
H2			•			i	i				i		•	•					
H3					•	i	i												
H4							•												
H5													•	•					
H6				•							•	•		•					
1_g1_a		•																	
2_g2_a			i	i								•							
3_g3_g				•								i							
4_g4_a					•														
5_g4c_a						•													
6_g5_a							•												
7_g6c_a									•						•				•
8_g6_a																			
9_g7_a										•			•				•		
10_g8_a																			
11_g9_a																			
12_g10_a													•						
13_g11_a														•	•				
14_g12_a															•				
15_g13_a																•	•		•
16_g14_a																			
17_g15_a																		•	
18_g16_a																			

sphere of the fixed stars was as a center point to a sphere; Euclid makes this claim at the beginning of his *Phenomena* (Berggren and Thomas 1996, 52–53). The assumption of Aristarchus's text, however, is much stronger.[9] Hyp. 2 requires that the size of the earth be negligible in comparison to the size of the lunar orbit. This hypothesis in used in the derivation of three propositions (Props. 3, 13 & 14), and it is implicit in the geometric configuration of a number of others; see Table 1.[10] If one accepts the view that the text is a piece of deductive mathematics, this hypothesis has a number of more or less problematic implications. It rules against observing daily lunar parallax. It denies the possibility of relating the lunar distance to the radius of the earth and establishing terrestrial distances for the luminaries. Finally, it implies that no extended terrestrial

[9] Al-Ṭūsī (1940,2), in fact, reverted to the weaker, and logically insufficient assumption relating the earth to the sphere of the zodiac. An older Arabic version, by Thābit ibn Qurra, states the same hypothesis as the Greek [124r]; see page 24 for a discussion of this text.

[10] The statement by Newton (1977, 175) that "we may doubt that Aristarchus intends for the second hypothesis to be taken literally" is not supported by a careful reading of the text.

shadow will fall on the orbit of the moon. The problems with this final consequence are serious, for it is directly contrary to Hyp. 5. Moreover, the propositions that rely on Hyp. 5 are all predicated on the assumption that the earth does, indeed, cast an extended shadow on the path of the moon (Props. 13–15).

The next three hypotheses are computational. They make assumptions about the physical world which allow the application of numerical parameters to the geometric models. These are then used to derive numerical solutions to the problems at hand. They are as follows (Heath 1913, 352):

4. That, when the moon appears to us halved, its distance from the sun is less than a quadrant by a thirtieth of a quadrant [87°].
5. That the breadth of the shadow is two moons.[11]
6. That the moon subtends a fifteenth part of a zodiacal sign [2°].

As indicated in Table 1, Hyp. 4 & 5 are used only once. Hyp. 6, on the other hand, is used four times.[12] The treatise as a whole demonstrates how relative sizes and distances can be inferred from these numerical parameters.

The direct relationship between the computational hypotheses and the primary results of the treatise is stated by Aristarchus himself. Immediately following the computational hypotheses, Aristarchus gives a short summary of the logical structure of the text, which points out how key propositions are related to each other and to the hypotheses (Heath 1913, 352–354). He frames this passage in terms of the statements of the theorems; in effect, he says that Prop. 7 results from Hyp. 4, and Prop. 9 from Prop. 7, while Prop. 15 is derived from Prop. 7 and Hyps. 5 & 6. This short summary is a rare example of a Greek mathematician discussing his results in terms of their logical requirements.

Following the summary, Heath's edition of the text proceeds with seventeen propositions presented in the deductive style that appears to have been traditional for applied mathematics texts of the early Hellenistic period. In fact, the numbering of the propositions varies in the different manuscripts.[13] On the whole, however, the MSS do not treat

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[11] The use of this hypothesis in context is more straightforward than the statement. It means the width of the earth's shadow falling on the moon's orbit appears to us as twice the angular span of the moon. The Arabic expression is clearer, "The breadth of the earth's shadow is the magnitude of two moons" (al-Ṭūsī 1940, 2).

[12] This reading of the logical dependence disagrees with Heath in three places, each for the same reason. In each instance, Heath refers the reader to a previous proposition for the repetition of a short argument which is made in the course of proving that proposition. In Prop. 11, Heath (1913, 387) refers to Prop. 4 because the step in question is also shown in Prop. 4. The step, however, is not the result of Prop. 4. Aristarchus's remark, "then it is clear from the forgoing proof . . . ," indicates that he intends us to supply the same argument as in Prop. 4, which involves a direct appeal to Hyp. 6. A similar situation occurs in Prop. 14. Here, Heath (1913, 403) appeals to Prop. 11. In fact, however, the step is not a result of Prop. 11; it too requires a direct application of Hyp. 6 and then a trigonometric lemma assumed elsewhere in the text. (For a discussion of these lemmas and Aristarchus's style of trigonometry, see page 224, below.) Finally, in Prop. 14, Heath (1913, 403) refers to Prop. 13 for an argument relying on Hyp. 5.

[13] For example, in Vat. Gr. 204, Prop. 1 is numbered 1 & 2, divided according to the two claims of the enunciation [109r]. Prop. 5 is taken as the first part of the following proposition; Prop. 8 as the last part of the proceeding. There are, hence, only 16 propositions.

Heath's Props. 5 & 8 as separate propositions; they are either corollaries or lemmas to the neighboring theorems.[14] Since neither of them have diagrams it is clear that they would not generally have been considered theorems by ancient and medieval authors. It is only in the hands of European mathematicians that these passages became separate propositions. Since Aristarchus probably did not number his text, however, such considerations are only of interest for trying to unravel the text history. The Arabic versions do, however, include one final proposition not found in the Greek sources (Prop. 17_a). The correlation of the numbers is given in Table 1; the numbers for the Arabic text are those of the printed text of al-Ṭūsī (1940); for example, $5_g 4c_a$ is the fifth Greek proposition and the corollary to the fourth Arabic proposition. We use the proposition numbers in Heath (1913) to refer to the propositions

When we consider the logical structure of the text using a table, we see that *On Sizes* exhibits structural characteristics similar to those of other systematic mathematical texts, and breaks into logical sections producing certain results. In two places, a series of several propositions act as lemmas for a final proposition (Props. 1–7, Props. 12–15). Some theorems are key results (Props. 7 & 15), while others appear to be trivial and isolated (Props. 8 & 11).

The table shows that the first six propositions all lead up to Prop. 7 and then are never used again. We may take this opening section, then, as a derivation of Prop. 7, the fundamental distance relation. As Prop. 7 shows, for the purposes of *On Sizes*, Aristarchus assumes a geocentric cosmos,[15] so that Prop. 7 states that the solar distance, D_s, is greater than 18 and less than 20 times the lunar distance, D_m; $18D_m < D_s < 20D_m$. Prop. 7 is one of the most important theorems in the book. It is used twice in the extant Greek text as well as in the final Arabic proposition (Props. 9, 15 & 17_a).

Proposition 8 is peculiar for a number of reasons. It states that during a total solar eclipse the sun and the moon are tangential to a cone, whose vertex is "at our eye" (Heath 1913, 382).[16] Its logical isolation from the rest of the text is conspicuous in Table 1, and it is justified by a direct appeal to "observation" (ἐκ τῆς τηρήσεως) making this the only place where the text mentions observation. As Neugebauer has pointed out, this proposition bears an interesting relationship to Prop. 3 (Neugebauer 1975, 635). Proposition 3 simply assumes that the cone tangent to the sun and the moon can have its vertex "at our eye" (Heath 1913, 361). Moreover, Prop. 9 makes the same assumption, and Prop. 13 asserts the equality of the angular span of the sun and moon based on the fact that the vertex of their tangent cone is "in our eye" (Heath 1913, 383 & 397–399). None of these three propositions makes any mention of eclipses or depends on the actual claim of Prop. 8, and in the case of Prop. 13 we are dealing with a situation around opposition. Moreover, they all express the geometric configuration in the same words. It seems likely that Prop. 8 is based on the same unstated assumption as Props. 3, 9 & 13

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[14] In Tabriz 3484 there is a rare case where Prop. 8 is separately numbered as the first of two sevens (Chavoshī 2005, 177).

[15] As we know from Archimedes, in another treatise Aristarchus assumed a heliocentric cosmos (Heiberg 1973, vol. 2, 218). Nevertheless, both the figure and text of Prop. 7 make it clear that the cosmos of *On Sizes* is geocentric (Heath 1913, 376–378).

[16] As Heath (1913, 383, n. 2) points out, this rules against the possibility of an annular eclipse.

concerning the angular span of the luminaries. Perhaps, Prop. 8 is meant to be a claim about the implications of this assumption for the physical phenomena of solar eclipses. In the ancient and medieval sources, this proposition is not separately numbered. There is a real possibility that this proposition was interpolated into the Greek text by an editor who wished to discuss certain aspects of solar eclipses.

Propsositions 9 & 10 are results of Prop. 7. Prop. 9, based on similar triangles, simply asserts that the inequalities relating the radii of the luminaries are the same as those of the distances.[17] Propsosition 10 states bounds on the ratio between the volumes, based on the inequalities between the diameters. Prop. 10 is an important result, but it is not very useful. Propsosition 9, which is used three times, is one of the more fruitful theorems (Props. 10, 13 & 17).

In the current state of the text, Prop. 11 is another oddity. It relates the lunar diameter, d_m, to the lunar distance, D_m. It shows that $2/45 D_m > d_m > 1/30 D_m$. Since this does not seem like an inherently exciting result, and because it is never used again in the text, it is possible that Prop. 11 is a later addition by a mathematically able commentator. On the other hand, it is also possible that this theorem served some role in Aristarchus's project. It is the only theorem that relates a size to a distance, and it may have been meant to provide a metric link between these two features of the cosmos. It should also be pointed out that this proposition is inconsistent with Hyp. 2 and Prop. 17. Hyp. 2 states that the size of the earth is immeasurably small relative to the size of the lunar orbit, while Prop. 17 shows that the moon is considerably smaller than the earth. Hence the moon must be immeasurably small relative to its orbit, yet Prop. 11 proves a definite numerical relationship between the moon and its sphere.

Thus far, the treatise has begun with a tight sequence of theorems (Props. 1–7) followed by another sequence (Props. 8–11), which harvests the consequences of Prop. 7 but also contains two structurally isolated theorems (Props. 8 & 11). These are followed by Props. 12–15, another continuous run of theorems. Proposition 15, the goal of this series, is the final logically important proposition in the treatise. It relates the solar diameter, d_s, to the terrestrial diameter, d_e, by proving that $19:3 < d_s : d_e < 43:6$. Proposition 15 is used twice in the Greek text and also in the final Arabic proposition (Props. 16, 17 & 17_a). It is also an intrinsically important result because it relates the size of the sun to something more accessible, the size of the earth. The demonstration of this result is not simple and involves a number of auxiliary theorems that introduce interesting theoretical objects and unspecified assumptions.

Although Prop. 12 implicitly relies on two of the early propositions in the exposition of the geometric diagram, it is logically a direct result of Hyp. 6. It introduces a numerical approximation for the size of the diameter of the circle that divides the light and dark portions of the moon. This circle, implied by Hyp. 1 and established by Prop. 2, becomes a locally important theoretical object in the derivation of Prop. 15. We call it the *dividing circle*, its diameter, in the plane of the moon's orbit, the *dividing line*.[18] For

[17] While the enunciation is about diameters the proof itself is about radii. The mathematical equivalence was too obvious to warrant notice.

[18] This terminology follows the Arabic, الدائرة الفاصلة (al-Ṭūsī 1940, 12). Neugebauer (1975, 639) calls the dividing line the *terminator*.

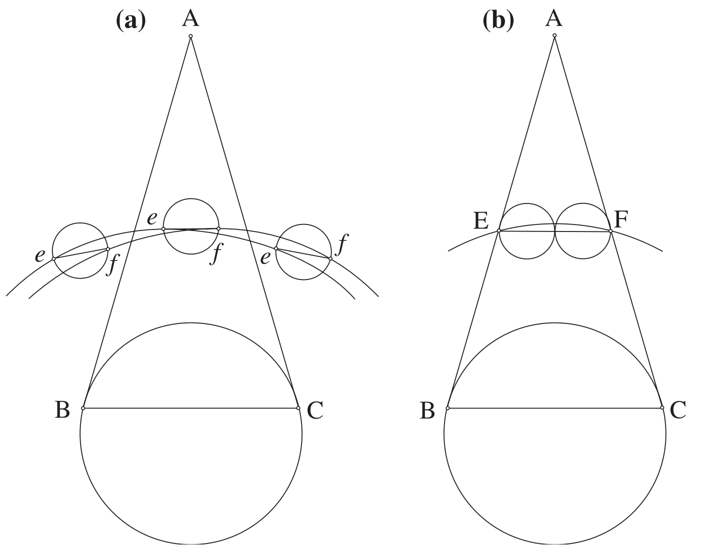

Fig. 1. Diagrams for the dividing circle, endpoint circle, and endpoint chord

the purposes of Props. 13 & 14, the endpoints of the dividing line are considered to move on a single circle within the earth's shadow (τοῦ κύκλου, καθ' οὗ φέρεται τὰ ἄκρα τῆς διαμέτρου τοῦ διορίζοντος ἐν τῇ σελήνῃ τό τε σκιερὸν καὶ τὸ λαμπρόν), which intersects the edges of the shadow (Heath 1913, 392). This is a simplifying assumption. In fact, the two endpoints do not move on the same curve and because both the sun and the moon move, and at different speeds, the determination of the shape of the curves described by the endpoints is nontrivial. The only situation in which the endpoints of the dividing line would actually describe a single circle is if the sun and the moon were diametrically opposite and moved in the same direction at the same angular velocity. In that case, however, the moon and the earth's shadow would not move relative to one another and the endpoints of the dividing line would never cross the earth's shadow.

In order for the endpoints to cross the shadow, the moon must move rapidly relative to the sun, as is observed. Consider Fig. 1(a), in which ABC is the shadow cone of the earth. There are various possibilities that could explain Aristarchus's simplifying assumptions. For example, if we set the moon moving faster than the sun in the same direction, then the dividing line, ef, will not remain parallel to itself, but its endpoints will describe two curves that are close to each other in the vicinity of the earth's shadow, ABC. Perhaps these two curves are approximated by a single circle. Another possibility is that Aristarchus is momentarily holding the sun still while the moon moves through the terrestrial shadow. In this case, the endpoints of the dividing line, ef, will form two curves that will approach nonconcentric circles as the rays of light falling on the moon approach parallel; that is, as the sun is considered to be infinitely far away from the moon. Perhaps Aristarchus is approximating these curves by a single circle.

Although we cannot be certain of his actual assumptions, something like these considerations must lie behind his simple claim that there is such a circle and that it intersects the earth's shadow. We will call this circle the *endpoint circle*, because it is supposed to be formed by the motion of the endpoints of the dividing line. Consider Fig. 1(b). Aristarchus's endpoint circle intersects the shadow of the earth in two points, which are joined by the chord EF. We call this chord the *endpoint chord*. The relationship of the endpoint chord to the lunar diameter is the subject of Prop. 13, and this chord also plays an important role in Prop. 14. Hence, this series of propositions introduces two purely theoretical geometric objects, the endpoint circle and the endpoint chord, whose only functions are to facilitate the derivation of the size relation between the sun and the earth.

Prop. 16, an immediate result of Prop. 15, states the volume relationship between the sun and the earth. Props. 17 & 18 complete the picture with the two size relationships between the earth and the moon.

The Arabic editions contain a further proposition, Prop. 17_a, which relates the lunar distance, D_m, to the distance between the moon and the vertex of the earth's shadow, D_{m_v}.[19] It proves that $71{:}37 < D_m{:}D_{m\_v} < 3{:}1$, a result that, like Prop. 11, appears to be of no particular interest on its own. There are at least three ways of interpreting this additional theorem; it was included in the Greek text from which the Arabic scholars worked and goes back to Aristarchus, it was a scholium in a Greek MS and was brought into the text by later scholars, or it was an original result added by an Arabic mathematician. These possibilities will be discussed further below.

It may be helpful for readers to have a summary listing of the primary results. The numerical results are stated in terms of a relation between two magnitudes; this relation may be expressed either as a pair of inequalities (Prop. 7, 9 & 11) or ratio inequalities (Prop. 10, 15–18 & 17_a). Nevertheless, the functional similarity of these expressions was understood in practice. Indeed as we will see, transformations between inequalities of simple parts and ratio inequalities were part of Aristarchus's basic computational methods. Hence, every relation in *On Sizes* may be expressed as a pair of numerical ratios bounding a ratio of magnitudes, $n_1{:}n_2 < A{:}B < n_3{:}n_4$, as we will do below.

Consider Fig. 2. Where D_s is the solar distance, D_m the lunar distance, D_{m_v} the distance between the center of the moon and the vertex of the terrestrial shadow at mid-eclipse, d_s the solar diameter, d_e the terrestrial diameter, and d_m the lunar diameter, *On Sizes* establishes the following relations:

Prop. 7: $18:1 < D_s : D_m < 20:1$
Prop. 9: $18:1 < d_s : d_m < 20:1$
Prop. 10: $5832:1 < {d_s}^3 : {d_m}^3 < 8000:1$
Prop. 11: $1:30 < d_m : D_m < 2:45$
Prop. 15: $19:3 < d_s : d_e < 43:6$
Prop. 16: $6859:27 < {d_s}^3 : {d_m}^3 < 79507:216$
Prop. 17: $108:43 < d_e : d_m < 60:19$
Prop. 18: $1259712:79507 < {d_e}^3 : {d_m}^3 < 216000:6859$
Prop. $\mathbf{17_a}$: $71:37 < D_m : D_{m\_v} < 3:1$

A number of historians have claimed that Aristarchus failed to give the distances of the luminaries, despite the title of the treatise (Panchenko 2001, 24; Neugebauer 1975,

[19] The text and translation of al-Ṭūsī's version of this proposition is given in Appendix A.

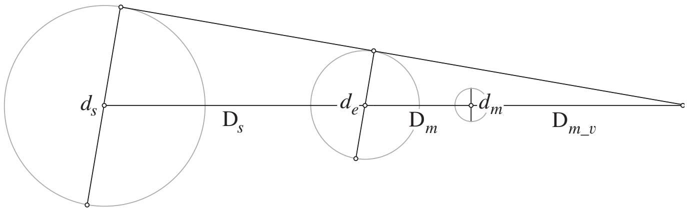

Fig. 2. Diagram of the relations determined by *On Sizes*

636; Tannery 1912, 386). The tacit assumption must be that the only proper way to do so would be to relate these distances to earth radii, since Aristarchus does, in fact, state the ratios of these distances in terms of one another.

Panchenko (2001, 24–25) attempts to show that the distances, in terms of earth radii, can be easily deduced from the treatise as it stands. His procedure, however, pays no attention to Aristarchus's mathematical methods; he simply asserts mean figures where Aristarchus works with upper and lower bounds and he makes a distinction between "our eye" and the center of the earth which is almost entirely absent from the text and is ruled out by Hyp. 2.[20] These considerations allow Panchenko (2001, 25) to give nice round numbers for the distances and to assert that Aristarchus is engaged in a "kind of 'Pythagorean' play" involving "the number which marks the famous Metonic luni-solar cycle." A comparison of the mathematical methods in these two texts, however, leads one to the belief that it is Panchenko, not Aristarchus, who is involved in number play. If indeed any simple numerical results underly Aristarchus's approach, they are hidden. The focus of Aristarchus's text is on quantitative bounds and their logical relation to numerical and geometric assumptions about the cosmos.

Nevertheless, it is possible to use the approach Panchenko suggests and Aristarchus's mathematical methods to derive numeric ratios bounding the ratios of the distances of the luminaries and the earth's diameter. We proceed as follows.

We apply the equality of terms operation[21] to the results of Props. 11 & 17 to yield

$$57:8 < D_m : d_e < 215:18^{22}. \tag{1}$$

We then apply the same operation to (1) and Prop. 7 to yield

$$513:4 < D_s : d_e < 2150:9. \tag{2}$$

Arguments like this could also help explain the presence in the text of Prop. 11, which otherwise might seem unmotivated. Similar considerations can be used to understand

[20] In fact, Aristarchus also introduces this difference in Props. 13 & 14; however, this distinction is problematic and deserves a separate discussion; see page 232.

[21] The operations on ratios, and their usage in Aristarchus's computations, are discussed below; see page 225.

[22] Panchenko (2001, 24) wants $D_m : d_e$ to be a simple 9 : 1, but it is clear that the bounds are too crude to secure this mean value, given Aristarchus's methods.

Prop. 17_a. Ratio operations on Props. 17_a & (1) can be used to get bounds on the ratio $d_e:(D_m+D_{m\_v})$. Similarity of triangles would then give bounds on $d_s:(D_s+D_m+D_{m\_v})$.

In this way, all the significant relations between the labeled lines in Fig. 2 would be determined, so that all the metric characteristics of the terrestrial shadow would be known. It is, however, not immediately obvious why this would be an important goal. Moreover, any claim about the incompleteness of the extant work must contend with the fact that the text itself gives a summary of its results which includes nothing beyond the current content of the Greek text.[23] While it is certainly possible to generate results beyond those in the text using ancient methods, and while some of these results may seem more satisfying to us than the relations Aristarchus proves, we must admit the real possibility that *On Sizes* served a purpose within its own context, which was fully satisfied by the theorems established in it.

In order to try to understand this context, it will be helpful to draw on another important ancient source for information about Aristarchus. In the *Sand Reckoner*, Archimedes tells us that the sphere in which the sun moves is called the cosmos "by most astronomers" (ὑπὸ... τῶν πλείστων ἀστρολόγων) (Heiberg 1973, vol. 2, 218). Since Archimedes was certainly in position to know such things, we should accept that the majority of astronomers in Aristarchus's time were working within a concentric sphere model of the cosmos. Moreover, in this model, the distances between any spheres beyond that of the sun were so insignificant that the sphere of the sun could effectively be taken as that of the whole cosmos. In this framework, *On Sizes* may be taken as addressing the concerns and issues of an ongoing cosmological tradition.

Aristarchus can be read as arguing that the scale and scope of the cosmos is determined on the basis of a few simple observations through the methods of mathematics. In particular, he shows that, although the sphere of the sun may be insensibly distant from that of the fixed stars, the sphere of the moon must be much closer to us. Moreover, this reading makes some sense of Aristarchus's interest in the terrestrial shadow. In a cosmos the size of the solar distance, the terrestrial shadow becomes an important feature. In particular, one may be interested to know how far its vertex extends, so as to know the lower limit of the distance of those celestial bodies that are never obscured by it.

Mathematical methods

On Sizes presents us with a number of aspects of Greek mathematics that we would otherwise know little about, and assumes a background knowledge of lemmas and operations that are not demonstrated in surviving texts. In other words, *On Sizes* gives us access to an entire tradition of Greek mathematics so well-established that a working mathematician could simply assume a knowledge of its foundation on the part of his readership.

[23] Neugebauer (1975, 636, n. 4) himself pointed out this difficulty.

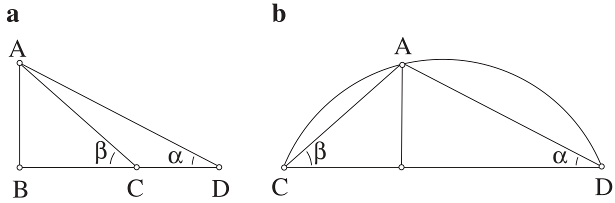

Fig. 3. Diagrams for the trigonometric lemmas used by the Hellenistic astronomers

Trigonometry

For Greek mathematicians, trigonometry was always concerned with the actual mensuration of triangles. In its mature form, which was in use at least as early as the mid-second century BCE, it was based on the use of chord tables and the functional relations they expressed. Before the development of these methods, however, some Hellenistic mathematicians approached trigonometric problems by using a group of lemmas to produce ratio inequalities that relate angle ratios to side ratios in right triangles.[24] This tradition of approximating trigonometric solutions is attested by both Aristarchus and Archimedes.

The first two lemmas are given a general enunciation by Archimedes. In the *Sand Reckoner*, in the course of a proof that the apparent diameter of the sun is greater than the side of a regular 1000-gon inscribed in a celestial great circle, Archimedes asserts a pair of ratio inequalities relating angles and sides in right triangles under the same height (Heiberg 1973, vol. 2, 232). Consider Fig. 3(a). He states, in effect, that if $BD > BC$, then

$$\beta : \alpha < BD : BC, \tag{T.L. 1}$$

and

$$\beta : \alpha > AD : AC. \tag{T.L. 2}$$

The earliest text we have that demonstrates the first lemma is Euclid's *Optics* (Heiberg 1895, 164–166).[25] The second lemma is first proved, in a trivial variant, T.L. 2a, by Ptolemy in his treatment of the chord table, *Alm.* I 10. Consider Fig. 3(b). Ptolemy shows that if $AD > AC$, then[26]

$$\overset{\frown}{AD} : \overset{\frown}{AC} > AD : AC. \tag{T.L. 2a}$$

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[24] Knorr (1985) collected and studied all of the variant proofs of two of these lemmas that are extant in the Greek mathematical corpus.

[25] When Knorr (1985, 370) studied these lemmas he still believed that the version attributed by Heiberg to Euclid was the earlier text. Following Jones, however, he later came to view the version attributed by Heiberg to Theon of Alexandria as earlier (Knorr 1991, 195, n. 7). Jones (1994) and Knorr (1994) present the case for this position.

[26] $\overset{\frown}{AD} : \overset{\frown}{AC} = \beta : \alpha$ so that T.L. 2a is an immediate consequence of T.L. 2. Ptolemy, however, does not base his proof of this lemma on a previous proof of T.L. 2 (Toomer 1984, 54–55).

Aristarchus also uses a third trigonometric lemma, T.L. 3. Consider Fig. 3(a). The lemma Aristarchus requires amounts to the statement that if $BD > BC$, then

$$BD : BC > (90^\circ - \alpha) : (90^\circ - \beta). \qquad \text{(T.L. 3)}$$

We are not aware of any ancient proof of this lemma. Heath gives a proof based on the ancient demonstrations of the other lemmas (Heath 1913, 377, n. 1).[27]

The trigonometric lemmas are used in five propositions (Props. 4 [T.L. 1 & 2], 7 [T.L. 3 & 2a], 11 [T.L. 1 & 2a], 12 [T.L. 2a] & 14 [T.L. 1]). Aristarchus never makes any general reference to the enunciation of the lemmas, he simply asserts their results based on his current diagram, assuming their application as part of his mathematical toolbox. Given the practice of Greek mathematicians, this is a strong argument that there was a tradition of using these theorems which was well enough known that Aristarchus could simply assume a readership with knowledge of their validity. The existence of this tradition is also attested by Archimedes' *Sand Reckoner*, which employs a similar style of trigonometry. We will see an example of Aristarchus's trigonometry below.

Operations on ratios and inequalities

Although Greek geometers appear to have had little interest in the arithmetical operations that formed the core of medieval Arabic and early modern Latin algebraic methods, they nevertheless employed a group of operations that feature in almost every bit of interesting mathematics of the Hellenistic period. These operations were generally performed on ratios; that is, proportions and ratio inequalities. As *On Sizes* shows, however, Greek mathematicians did convert between proportions and ratio inequalities, on the one hand, and equalities and inequalities of magnitudes, on the other. Because they had a strong tendency to work with unit fractions, however, these transformations are only found in *On Sizes* when simple parts are involved.

The most complete surviving foundation for the theory of ratios of general magnitudes is *Elem.* V, while a number of propositions relating to ratio and proportion are proved independently for whole numbers in *Elem.* VII.[28] The Euclidean text of *Elem.* V defines six of the operations for ratios and proves four of them for proportions. This ratio theory was almost certainly meant to provide a foundation for a set of current practices, not to create new practices or enumerate all existing ones. It is certainly the case that the operational practice of the Hellenistic geometers was broader than that covered in the *Elements*.

Some of the more interesting uses of ratio manipulation, which are not justified in the *Elements*, are practiced by mathematicians who probably worked around the time of, or fairly shortly after, the composition of that book and were, in all likelihood, not much under its influence. Aristarchus and Archimedes are key witnesses to this early tradition.

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[27] A further lemma, attributed to Apollonius by Ptolemy in *Alm.* XII 1, should perhaps be included among these (Toomer 1984, 558–559). Although there is no evidence that it was ever used in trigonometric calculation, its subject matter and proof structure indicate that it belongs in the same tradition.

[28] For recent studies of the foundations of ratio theory in *Elem.* V see Acerbi (2003), Vitrac (1990–2001, vol. 2, 35–68) and Mueller (1981, 118–151).

In practice, the operations on ratios were invoked by set expressions that marked them as *operations*. They were consistently referred to by a specific word or phrase, grammatically either a dative of means, an adverb or a prepositional phrase. For our purposes it is not necessary to give an extensive account of these operations.[29] We may simply list them as follows:

1. Inversion (*Elem.* V def. 13):[30]

$$A : B \gtreqless C : D \Longrightarrow B : A \lesseqgtr D : C.^{31}$$

2. Alternation (*Elem.* V def. 12 & *Elem.* V 16):

$$A : B \gtreqless C : D \Longrightarrow A : C \gtreqless B : D.$$

3. Composition (*Elem.* V def. 14 & *Elem.* V 18):

$$A : B \gtreqless C : D \Longrightarrow (A + B) : B \gtreqless (C + D) : D.$$

4. Separation (*Elem.* V def. 15 & *Elem.* V 17):

$$A > B \text{ and } A : B \gtreqless C : D \Longrightarrow (A - B) : B \gtreqless (C - D) : D.$$

5. Conversion (*Elem.* V def. 16):

$$A > B \text{ and } A : B \gtreqless C : D \Longrightarrow A : (A - B) \lesseqgtr C : (C - D).^{32}$$

6. Equality of Terms[33] (*Elem.* V def. 16 and *Elem.* V 22 & 23):

$$\begin{aligned} A_1 : A_2 \quad &\gtreqless \quad B_1 : B_2 \quad \text{and } \ldots \\ A_{n-1} : A_n &\gtreqless B_{n-1} : B_n \Longrightarrow \\ A_1 : A_n \quad &\gtreqless \quad B_1 : B_n, \end{aligned}$$

[29] There are a number of summaries of the six operations on ratios. See for example Heath (1926, vol. 2, 134–136 & 164–184), Dijksterhuis (1987, 52–54), Netz (1999a, 139–140), Vitrac (1990–2001, vol. 2, 50–56 & 61–65) and Taisbak (2003, 44–45).

[30] Although the *Elements* only discusses the operations in terms of proportions, they were used in practice for ratio inequalities as well.

[31] We are aware of no ancient proof that the inequality reversed under inversion, but this was well understood in practice.

[32] Conversion is defined but not proved in the *Elements*. It was probably understood by ancient authors as a shorthand for successive applications of separation, inversion and composition. This would explain why changing the inequality required no comment.

[33] The literal expression is "ratio through an equal," (δι' ἴσου λόγος). Heath (1926, 136) takes the expression to refer to the equal number of intervening terms. Vitrac (1990–2001, vol. 2, 52) intends something similar by the phrase "rapport à égalité de rang." We use the expression *equality of terms* to avoid any confusion that might arise from the more literal *through equality* or the traditional Latin *ex æquali*.

and

$$A_1 : A_2 \gtreqless B_1 : B_2 \text{ and } A_2 : A_3 \gtreqless B_3 : B_1 \Longrightarrow A_1 : A_3 \gtreqless B_3 : B_2.$$

Aristarchus uses these operations in a number of interesting ways. Due to the nature of the mathematical problems, he applies the operations to ratio inequalities whereas the *Elements* only discusses them in terms of proportions. In Prop 4, the first numerical proposition, the mathematics is simple enough that he can work with inequalities the entire time. In Prop. 7, he transforms the inequalities to ratio inequalities in order to carry out manipulations, then is able to transform these back into inequalities involving simple parts. In Prop. 11, he stays with simple parts but has to double one, giving the only real common fraction in the text, 2/45 (δύο με′) (Heath 1913, 386). In most of the later propositions, the numeral relationships can no longer be expressed in simple parts, so Aristarchus works entirely with ratio inequalities.

The basic problem-solving strategy is to use a combination of geometric construction and ratio manipulation to derive numerical bounds. In this sense, Aristarchus uses the operations on ratio to achieve the same kinds of results that a medieval or early modern mathematician would obtain through arithmetic operations.

Aristarchus's language shows that he was fully aware of the structural similarity between inequalities and ratio inequalities. In a number of places, he performs a ratio manipulation at the same time as a transformation, so that it is impossible to tell whether he intends the operation to occur on the inequality or the ratio inequality. In at least one case, he performs a ratio manipulation directly on an inequality.[34]

This text also makes it clear that Greek mathematicians knew that the operation of equality of terms is the same as taking the ratio of the product of the antecedents to the product of the consequents. In fact, Aristarchus uses this feature of the equality of terms operation as an important computational technique. The first time this occurs, in Prop. 13, Aristarchus explicitly says that the final ratio is the ratio of one product (συνηγμένος) to another (Heath 1913, 398).[35] Following this, in Props. 14, 15, 17 & 17_a, he simply asserts the operation and carries out the multiplication.

This text shows that the strict lines that are sometimes said to have been drawn between ratios and simple parts, between ratio manipulation and arithmetic, were not always as strict in practice as a work like the *Elements* might lead us to believe.

On Sizes *Prop. 4*

In order to work through a concrete example of Aristarchus's mathematical practices, we give an account of his demonstration of Prop. 4. In this proposition, Aristarchus argues that the dividing circle is not perceptibly different from a great circle. Because the sun is much larger than the moon, it is shown, in Prop. 2, that the light side of the moon is greater than a hemisphere. Proposition 3 proves that maximum difference between the dividing circle and a great circle occurs when the visual cone which contains both the

[34] See page 228 below.
[35] See page 246 for further discussion of this expression.

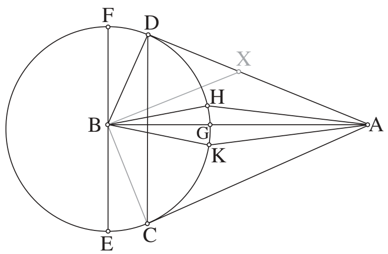

Fig. 4. Diagram for Prop. 4 (Heath 1913, 364). There is no X or BX in any MS. The line BC appears in the Arabic MSS but not in the Greek

sun and the moon has its vertex at "our eye." Hence, for Prop. 4, we only need a figure containing our eye and the moon.

Consider Fig. 4. Let our eye be A and the center of the moon B.[36] Let a plane have been drawn through A and B, cutting the moon in the great circle $ECDF$ and the visual cone in lines AC, AD and the dividing circle in DC. Then the circle with diameter DC and perpendicular to AB is the dividing circle. Let $FE \parallel DC$, and let $\overset{\frown}{HG} = \overset{\frown}{GK} = 1/2\ \overset{\frown}{FD}$. Let KB, BH, KA, AH and BD have been joined. Since, by the Hyp. 6, the moon subtends $1/15$ of a zodiacal sign, $\angle CAD = 1/15\mathbf{Z}_{sign}$ [$= 2°$].[37] But $1/15\mathbf{Z}_{sign} = 1/180\mathbf{C}$, hence $\angle CAD = 1/180\ 4\mathbf{R} = 1/45\mathbf{R}$. And $\angle BAD = 1/2\angle CAD$, hence $\angle BAD = 1/45(1/2\mathbf{R})$ [$= 1°$]. Now since $\angle ADB$ is right, $\angle BAD : 1/2\mathbf{R} > BD : DA$ [T.L. 1].[38] Hence, $BD < 1/45\ DA$. So, $BG \ll 1/45\ BA$ [$BG = BD$ and $DA < BA$], and, by separation, $BG < 1/44\ GA$.

Since $BG < 1/44\ GA$, then $BH \ll 1/44\ AH$ [$BH = BG$ and $AH > AG$]. But $BH : AH > \angle BAH : \angle ABH$ [T.L. 2], so $\angle BAH < 1/44\angle ABH$. Moreover, $\angle KAH = 2\angle BAH$ and $\angle KBH = 2\angle ABH$, therefore $\angle KAH < 1/44\angle KBH$. But $\angle KBH = \angle DBF = \angle CDB = \angle BAD$ [by construction, *Elem.* I 29, *Elem.* VI 8], therefore $\angle KAH < 1/44\angle BAD$. But $\angle BAD = 1/45(1/2\mathbf{R})$; hence, $\angle KAH < 1/3960\mathbf{R}$ [$= 1/44°$, $1/2 \times 1/45 \times 1/44 = 1/3960$]. But a magnitude seen under such an angle is imperceptible

[36] See Heath (1913, 364–371) for text and translation.

[37] Aristarchus uses three units of angular measure: the circle, the right angle and a zodiacal sign. We denote these by $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{Z}_{sign}$. An estimate of 2° for the angular size of the moon is large. In this proposition, however, an overestimate for the apparent size of the moon only underscores the results of the proof. It should be noted that Aristarchus is reported to have had another figure for the apparent size of the sun which is much closer to our current value. In the *Sand Reckoner*, Archimedes reports that Aristarchus found that the apparent diameter of the sun was $1/720$ of a circle, that is $1/2°$ (Heiberg 1973, vol. 2, 248).

[38] This is Aristarchus's first use of a trigonometric lemmas and it is not immediately obvious. Cut off $DX = DB$. Then since $DA > DX$, $DA : DX > \angle DXB : \angle DAB$ [T.L. 1]. But $DX = BD$ and $\angle DXB = 1/2\mathbf{R}$, therefore, by inversion, $BD : DA < \angle BAD : 1/2\mathbf{R}$.

to our eye.[39] And $\overset{\frown}{KH} = \overset{\frown}{DF}$. Moreover, $\overset{\frown}{KH}$ is viewed straight on, whereas $\overset{\frown}{DF}$ will generally be viewed obliquely and, hence, appear smaller still.[40]

The mathematical methods of On Sizes

Proposition 4 shows the two most interesting features of Aristarchus's computational approach. The first is the use of ratio manipulation to produce new numeric relations. The second is the application of the trigonometric lemmas to convert a relation involving angles to one involving segments and conversely.

Most of the mathematical manipulation in this proposition is done on simple parts. The majority of these are basic conversions between the various units of measure; the circle, right angle and zodiacal sign. The first ratio manipulation is an inversion and is glossed over by Aristarchus the first time he uses a trigonometric lemma (see note 38). The second ratio operation is quite interesting. Here, the operation is applied to an inequality; or, perhaps, the inequality, because it involves simple parts, is being treated as a ratio inequality. That is, since $BG < 1/45\ BA$, we assert $BA : BG > 45 : 1$, so that, by separation, $GA : BG > 44 : 1$, from which we claim $BG < 1/44\ GA$. Whether Aristarchus thought of this operation as performed on the ratio inequality or directly on the inequality is immaterial. What is important is that he worked in a tradition that understood inequalities and ratio inequalities as interchangeable.[41] Because ratio operations were an important tool, these were transfered to inequalities as well. Nevertheless, the transformation between ratios and parts almost always occurs when multiples or simple parts are involved.[42]

Trigonometric calculation shows that $\angle KAH \approx 0.0178°$, while Aristarchus gives an upper bound of $\angle KAH < 0.0227°$, expressed as $1/3960\mathbf{R}$. Ostensibly, the only piece of numerical information that Aristarchus uses in this derivation is Hyp. 6 on the assumed angular size of the moon, $\angle DAC = 2°$. In fact, however, the usage of T.L. 1 is also significant. Because of the need to derive a ratio inequality involving BD, Aristarchus

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39 The angular span would be $\approx 0.0227°$.

40 Neugebauer (1975, 639–640) considers the final part of Prop. 4 to be "slightly garbled." He finds it strange that arc DF has been not been laid off to one side of line BA, as it would appear at half moon. Aristarchus, however, probably chooses this arrangement because he wants to show that arc DF is still "imperceptible" when it is seen straight on, under its greatest possible angular span. The argument Aristarchus gives about arc DF appearing under a smaller angle from A than arc HK is rather odd because, as Neugebauer points out, arc DF cannot be seen at all from A. The intent, however, is clear and our loose summary of the conclusion captures the general sense of the argument.

41 The remarks by Fowler (1987, 246–248) on Aristarchus's *On Sizes* are useful but he goes too far in his claim that the techniques for manipulating ratio inequalities are completely distinct from those for manipulating simple parts. In particular, the statement, "when the language of ratios is in use, it is not mixed with the language of multiples or parts, even in the most obvious cases," is misleading (Fowler 1987, 247).

42 One exception to this tendency is found in Archimedes' *Sand Reckoner*, in which he infers from $1/200\mathrm{R} : \alpha < 100 : 99$ that $\alpha > 99/20{,}000\mathrm{R}$ (Heiberg 1973, vol. 2, 232).

is compelled to introduce a 45° angle. This angle then furnishes the primary numerical element in the comparison of $\angle KAH$ and $\angle BAD$. An angle closer to 1° would have given a better approximation but would not have produced a relation involving BD.[43] Since Aristarchus thinks his upper bound is sufficiently small, a better approximation is unnecessary in this case.

Although this approximation is satisfactory for Prop. 4, where it makes the case stronger, the same procedure for handling Hyp. 6 is used again in Props. 11, 12 & 14. In these theorems, however, the requirement of working with 45° and a relation involving line BD highlights some of the drawbacks to Aristarchus's geometric method of approximation. Trigonometric calculation shows that these propositions should be sensitive to small changes in the value of the lunar disk, so that bounding a 1° angle with a 45° angle is too crude to give accurate results. In fact, however, Hyp. 6 is only used for computing upper bounds. Hence, even in these later propositions, Aristarchus probably has no need to use a better approximation than 45° because this would only cause his upper bound to get larger while his lower bound stayed the same.[44]

In order to derive quantitative information involving $\angle KAH$ from the fact that $\angle BAD = 1°$, Aristarchus has to transform a quantitative relation involving $\angle BAD$ and another given angle into a relation involving sides. He manipulates this relation involving sides from one triangle to another and then transforms it back into a relation involving angles, both of these transformations being made with the trigonometric lemmas. Hence, the trigonometric lemmas serve a function in Aristarchus's trigonometry similar to that of the chord table in later Greek trigonometry; they allow the geometer to transform statements relating angles to statements relating sides. The trigonometric lemmas, however, are ratio inequalities; hence, each time one is used some accuracy is lost.

Although we have referred some of the steps in Prop. 4 to propositions in Euclid's *Elements*, we should not think that Aristarchus himself thought of these steps as supported by specific theorems. The *Elements* was probably composed either during or slightly before Aristarchus's time. In all likelihood, Aristarchus's toolbox is a loosely defined body of geometric knowledge, including an understanding of how proportions and ratio inequalities can be manipulated. Aristarchus also assumes the reader has the mathematical background to follow steps that have no justification in any theoretical text that we possess. In particular, the trigonometric lemmas and the basic operations on inequalities and ratio inequalities are the elementary mathematics that a reader of his text can be expected to know.

Four other theorems in *On Sizes* (Props. 7, 11, 12 & 14), along with a theorem in the *Sand Reckoner*, fill out the rest of our evidence for trigonometric calculation before the development of chord tables (Heath 1913, 376–380, 386–391, 402; Heiberg 1973, 232). With regard to trigonometric procedures, these four theorems offer little we have not already seen. Again, we encounter the same basic mathematical tools: use of the trigonometric theorems to transform between given angle and side relations, manipulations of proportions and ratio inequalities, and the transformation of ratio inequalities

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[43] See note 38, above.

[44] These issues are further discussed in Appendix B.

into simple inequalities. We find that angles are expressed in various units whereas sides, being given no units, are simply compared.

All of this work in early Hellenistic trigonometry takes place in the context of mathematical astronomy, as in fact does nearly all known Greek trigonometry. An important difference should, however, be noted. Whereas Hipparchus and Ptolemy use mathematics to serve the needs of an astronomy that is, at least ostensibly, based on careful observation and geared toward practical as well as theoretical concerns, Aristarchus and Archimedes use astronomical problems as a domain in which to demonstrate the power of mathematics to address questions about the physical world.

Neugebauer (1975, 643) has pointed out the striking similarity between the approaches of *On Sizes* and the *Sand Reckoner*. These texts are meant to display both the mathematical skills of their authors and the power of mathematics generally to analyze complex problems with some precision. They are committed to mathematical precision before astronomical accuracy. From our perspective, looking back after the development of trigonometric functions, the methods of the Hellenistic mathematicians appear to lack elegance. Yet these methods were, in fact, a significant advance for applied mathematics. By using the fundamental properties of the right triangle, mathematicians were able to assert relations between angles and sides. These approximations could, at least in principle, be made to fit the given value more or less tightly at the geometer's discretion.

The role of hypotheses

One of the most striking features of *On Sizes* is the use of simple hypotheses to derive numerical results. As well as the six hypotheses stated at the beginning of the text, there are a number of implicit assumptions. From the perspective of the intellectual context of the text, the most important of these are the fundamental unstated assumptions about the structure of the cosmos and the celestial bodies. Aristarchus can take it for granted that the sun and moon are spheres moving on concentric, spherical orbits about a central, spherical earth. This was presumably the common opinion among mathematical authors in his time.

There are other unstated assumptions which have more bearing on the development of the argument. In Props. 3, 9 & 13. Aristarchus assumes that the cone tangent to the sun and the moon can have its vertex at our eye. Presumably, this is based on the observational claim that they both appear under the same angular span. Also in Prop. 4, we are told that an angle which is $1/3960$ of a right angle is imperceptible to us. This assumption is quite reasonable since this angle is $\approx 0;1,21,48^\circ = 0.02272^\circ$; nevertheless, it is based on an unstated assumption about the limits of our visual abilities.

These two assumptions, along with the six explicit hypotheses, are related in some way to observation. The nature of this observational basis, however, is sometimes problematic. The distance relation of Prop. 7 is based on the claim that the angular distance between the luminaries is 87° at quadrature (Hyp. 4). Indeed, the derived distances are quite sensitive to small changes in this angle. In fact, however, given a more current value for this angular separation ($\approx 89;50^\circ$) and the margin of error for ancient observational practices, it should not have been possible to distinguish between the actual angular separation and a right angle. Aristarchus is probably taking what he considers to be the

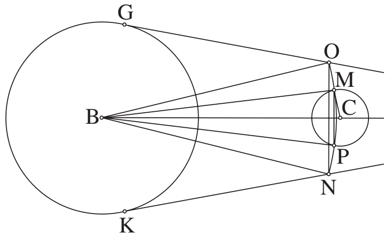

Fig. 5. Simplified diagram of Prop. 14

greatest possible value for the angular separation and showing that under this limiting assumption, the distances and sizes are still very large.

The assumed angular diameter of the moon (Hyp. 6), and hence of the sun, presents a somewhat different problem. Being called upon in four propositions, this is ostensibly the most important numerical parameter in the text. The value which is chosen, however, is much too large. Moreover, according to Archimedes, a better value, one quarter of that assumed in *On Sizes*, was known to Aristarchus (Heiberg 1973, vol.2, 248). The usual explanation is that Aristarchus found, or adopted, the superior value later in his career. Although this reading is certainly possible, it is not necessary. Considerations internal to the treatise itself may provide other explanations. In fact, changing the size of the lunar disk produces little effect in the final result, and this only in the upper bound of the ratio $D_s : D_m$.[45] The numbers involved, however, are more manageable throughout the entire calculation when one uses 2° as opposed to $1/2^\circ$. This means, as Neugebauer (1975, 643) believed, that 2° could well have been chosen simply as a convenient numerical parameter. This fact further supports the view that the treatise is intended less as a contribution to technical astronomy than as a cosmological demonstration of the power of mathematics.

We have already mentioned that there is a contradiction between Hyps. 2 & 5, which relate the sizes of the earth and its shadow individually to the lunar orbit. Nevertheless, these issues deserve further reflection. Hyp. 2 states that the size of the earth is negligible compared to the sphere of the lunar orbit. This hypothesis functions in two different ways in the text. On the one hand, it is a rather innocent assumption, allowing us to ignore lunar parallax; it is so used in Prop. 3. On the other hand, in Prop. 13 & 14, it is used as part of the computational apparatus to apply the assumed angular span of the moon to the angle at the center of the earth.

In Prop. 13, the primary numerical consideration is Hyp. 5; namely, the fact that the angular span of the moon is taken as subtending half the earth's shadow. The angle, however, is seen from the center of the earth. This means that the surface of the earth, which casts the shadow, and the position of the observer, at the center of the earth, are now found together in the same figure. The implication of Hyp. 2, however, is that there can be no geometric distinction between "our eye" and the center of the earth.

The situation becomes more pronounced in Prop. 14. Consider Fig. 5. Where B is the center of the earth, MP the moon lying within the earth's shadow, and C the center of the moon, Aristarchus asserts that $BC : CM > 45 : 1$. This statement, following from

[45] See Appendix B.

both Hyps. 2 & 6, is one of the theorem's two numerical arguments.[46] The second is the claim that $ON < 2MP$, which is a direct result of Hyp. 5. For both of these numerical conditions to hold as expressed, we must at the same time believe that the earth is as a point to the sphere of the moon and that this point somehow casts an extended shadow on that sphere. In other words, Aristarchus was willing to maintain two contradictory hypotheses simultaneously, in order to obtain numerical results.

We have already noted that Props. 13 & 14 contain a number of simplifying assumptions; in particular the presupposition of two purely theoretical objects, the *endpoint circle* and the *endpoint chord*. These objects are invoked in order to produce a geometric configuration that is susceptible to trigonometric computation. These are another sort of presupposition, although not hypotheses in the strict sense. Moreover, they are theoretical as opposed to observational. Here again, Aristarchus is willing to make some suppositions, which cannot be strictly accurate, in order to obtain his desired result. This means, as we stated earlier, that Aristarchus is doing something much more mathematical than astronomical. He is advancing hypotheses for the sake of the argument.

These realizations about Aristarchus's use of hypotheses are of considerable historical importance, especially as they relate to Archimedes' discussion of the way Aristarchus worked with hypotheses. Archimedes' *Sand Reckoner* is a work cut from the same cloth as *On Sizes*. It begins with a number of hypotheses, two of which are explicitly based on the preceding work of Aristarchus. From these assumptions, Archimedes proceeds to develop an upper bound for the size of a greatly expanded cosmos, to fill this cosmos with sand and then to exhibit a number, exceeding the number of these grains of sand, stated in his new system of numeration, specifically designed to handle such large numbers. Because he wants a cosmos that is as large as possible, he introduces a heliocentric hypothesis, which he attributes to Aristarchus.

Archimedes tells us that "Aristarchus brought out writings of certain hypotheses, in which it results from the suppositions that the cosmos is many times" larger than usual (Heiberg 1973, vol. 2, 218). These hypotheses are (a) that the earth revolves around a stationary sun and (b) that the size of the earth's orbit is as a point to the sphere of the fixed stars. Archimedes is not quite satisfied with Aristarchus's expression for the relationship of the terrestrial orbit to the cosmos because he wants to use it for computational purposes; nevertheless, an examination of the role of Hyp. 2 in *On Sizes* makes it clear that this is just the sort of hypothesis that Aristarchus would use for establishing the geometric characteristics of his diagrams. It is clear, then, that these other writings explored the structural implications of making certain nonstandard assumptions about the geometric configuration of the cosmos. On the whole, in terms of method and approach this work was probably quite similar to *On Sizes*.

The evidence of the *Sand Reckoner* supports this and Archimedes almost certainly intends the text as a nod to Aristarchus, whom he mentions by name no less than ten times. Moreover, *On Sizes* and *Sand Reckoner* are closer to each other than they are to any other text in the surviving mathematical corpus. In all likelihood, Archimedes wrote *Sand Reckoner* as a sort of tribute to a predecessor he admired, using the same general approach and mathematical methods. Although *Sand Reckoner* has a playful air, there

[46] The proportion is implied by Hyp. 6 and T.L.1, in the same way as in Prop. 4. See page 228.

are also serious elements, such as the discussion of an instrument for measuring the solar disk (Heiberg 1973, vol. 2, 222–226). Even the goal of *Sand Reckoner*, computing an upper bound for the number of grains of sand that would fill a vast cosmos, can broadly be construed as part of the same program as *On Sizes*. Although the number itself is inconsequential, the very fact that it is calculated is a testament to the idea the cosmos can be known through computation.

This is the sort of context in which we should situate Aristarchus's heliocentric hypothesis. It seems that, at the beginning of the Hellenistic period, Aristarchus specialized in a form of mathematical cosmology that drew specific numeric results from a small set of assumptions. The assumptions could be more or less true; nevertheless, the aim of the project was to show that different sets of hypotheses had definite implications for the structure of the cosmos.

The Arabic Text

Historical remarks on the Arabic versions

The survival of the Greek text of *On Sizes* is likely due to its role in the curriculum used to train astronomers in Alexandria in the 4th century of our era. It was one of a body of works known as the *Little Astronomy*, which mostly dealt with the mathematics of the celestial sphere and the consequences of the motion of that sphere for a central, spherical earth.[47] As such, it was the subject of a brief commentary by Pappus around the beginning of the 4th century (Hultsch 1876–1878, 554–560; Heath 1913, 412–414). Together these texts were intended to bring the student of mathematical sciences from a knowledge of Euclid's *Elements* to the point where he could begin to study Ptolemy's *Almagest*. The whole of this corpus was translated into Arabic, by various individuals, mostly in the 9th century. These works formed the core of a somewhat variable canon of works which, because of its position in the curriculum, was known to Arabic authors as the *Middle Books*, and by at least the 13th century included some original Arabic works.[48] Among those who made translations, revisions and additions to this corpus were Isḥāq ibn Ḥunayn, al-Kindī, Qusṭā ibn Lūqā, the Banū Mūsā, Thābit ibn Qurra and Naṣīr al-Dīn al-Ṭūsī.

At least two Arabic versions of *On Sizes* are known. The most common is the edition (تحرير) made by al-Ṭūsī sometime in the 13th century. This was done as part of his larger project to produce new editions of the canonical works of Greek mathematics and

[47] This collection included the surviving works of Autolycus, Euclid's *Phaenomena*, the two *Spherics* of Theodosius and Menelaus, and Hypsikles' adaptation of Babylonian numerical methods for the calculation of rising times.

[48] The claim by Mogenet (1950, 166) that the formation of the Arabic collection was the work of al-Ṭūsī is refuted by al-Samawʾ al who refers to a lost 9th century commentary on this collection by Qusṭā ibn Lūqā, *On the Middle [Books] which Must be Read Before the Almagest* (Sezgin 1978, 66). Moreover, Ṭūsī himself, in the introduction to his edition of the Archimedean' *Lemmas*, quotes a passage by the 10th century scholar al-Nasawī referring to the *Middle Books* (al-Ṭūsī 1940, 2; Schoy 1926, 32, n. 1).

astronomy, probably in the 1240s during his Ismāʿīlī period in Alamūt (Ragep 1993, 12–13). Sezgin (1978, 75) lists thirteen manuscripts of this edition in European and Middle Eastern libraries and it is the only version that has been printed (al-Ṭūsī 1940).[49] Although for a number of other texts in his edition of the *Middle Books* Ṭūsī names the translator and states what he knows of the text history, in the case of *On Sizes* he is silent on these matters.

The other version of the text is known from a single, privately owned manuscript, usually called the Kraus MS, because it was sold by the bookseller H. P. Kraus (Lorch 2001, 28; Kheirandish 1999, vol. 1, xxvii; Kraus 1974, 45, no. 18).[50] The colophon of this text states that it is a revision (اصلاح) by Thābit ibn Qurra, and hence complied in the late 9th century [f. 133r]. A comparison of the two versions shows that this is an earlier edition of the treatise, one that is, in places, textually different from that of al-Ṭūsī.

We do not know who translated *On Sizes* from Greek into Arabic, but the secondary literature, nonetheless, agrees that the translator was Qusṭā ibn Lūqā.[51] The earliest attribution to Qusṭā in these sources is that by Uri (1787, 208) in his catalog of the Bodleian collection of Eastern MSS. This assertion has the advantage over almost all others in being quite clear about its manuscript basis; that is, Arch. Selden. A. 45, f. 142v–150r (= Uri no. 875). We have consulted the MS in question, however, and like the other Ṭūsī MSS we have seen, it contains no information about the translator. The Kraus MS, likewise, is silent on the question of the translator. Qusṭā is indeed a likely candidate for the translator, and al-Ṭūsī tells us explicitly that Qusṭā's translations were the basis for his editions of a number of other treatises in the *Middle Books*.[52] Moreover, as just noted, al-Samawʾal tells us that Qusṭā wrote a commentary on some version of the whole collection (Sezgin 1978, 182). Nevertheless, we should reserve judgment on this question until we have a better knowledge of the manuscript evidence.

Remarks on Thābit's revision

As the colophon suggests, this text is meant to be a mathematical improvement over the source translation. Since we do not know what, if any, translation Thābit used as he composed his text, it is impossible to state whether the differences between this text and the Greek are due to Thābit or to an earlier Arabic scholar. Since Thābit's text was to be

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[49] Chavoshī (2005) has also published a facsimile of one of the MS of this version.

[50] The MS contains ten treatises of the *Middle Books*. The text of *On Sizes* is on 124r–133r, of which the penultimate folio is missing, although the foliation is continuous. 132v breaks off in the middle of an alternate proof to Prop. 15, also found in Ṭūsī, while 133r begins with Prop. 17_a in progress.

[51] Heath (1913, 320) simply states as a fact that Qusṭā was the translator; Sezgin (1978, 75), more prudently says "perhaps" he was the translator. Noack (1992, 37–38, n. 6) again takes Qusṭā as the translator and supports this with a note cataloging a long list of scholars who asserted this to be true and largely left the matter at that.

[52] The Latin tradition contains some references to other possible Arabic translators but these are obscure and inconclusive (Noack 1992, 40–41).

understood as a correction, we will assume that the substantial changes are his, while admitting that this may not hold in every case. Since the oldest Greek manuscripts were copied from one or two prototypes in Byzantium in the 9th and 10th centuries, we should admit the possibility that the Baghdad mathematicians had access to a somewhat different version of the Greek text, which was attested in at least one Greek MS in Baghdad by the end of the 9th century.

The general tendency of Thābit's revision is to flesh out the text in a number of ways. For example, the astronomical significance of a geometric object may be added in the course of an argument, or certain mathematical details may be adduced as justification. Proof structure is also reorganized in a number of places, especially when a diagram has been redrawn. All these changes were presumably made for mathematical, and didactic, reasons. The text is also made lengthier by the fact that all numbers are written out longhand, whereas the Greek often uses numerals.

There are three places where Thābit's text includes substantial additional material. One of these is based on a scholium found in a number of the Greek MSS, a second is a complete reworking of a scholiast's argument to explain the reduction of a ratio inequality involving large numbers to one with more manageable numbers, and the third is the addition of a final theorem, not found in our Greek sources.

The first of these occurs in Prop. 3, where Aristarchus assumes, without proof, that if pairs of tangents to a circle are drawn from two external points, the chord joining pairwise points of tangency will be less when the intersection of the pair of tangents is closer to the circle. A scholium attempting to prove this statement is found in Vat. Gr. 204, f. 110r (Fortia d'Urban 1810, vol. 1, 114–118).[53] Thābit's edition includes both the argument and the diagram of the scholium [f. 125v], the scholium diagram having been included within the diagram for the proposition.

The second occurs at the end of Prop. 15, where Thābit appends an alternate argument for the reduction of a ratio inequality [f. 132v]. A rambling justification of this reduction is given in a scholium in Vat. Gr. 204, on a page packed with notes [117r; Fortia d'Urban 1810, 190–193]. Thābit's approach is nicer than that of the scholium, doing more in fewer steps.

Finally, Thābit's text ends with a theorem not found in the extant Greek MSS, Prop. 17_a. The theorem has its own figure and relies on the two most important results of the text, Props. 7 & 15. Although this theorem could have been added by an Arabic scholar, it is at least as likely that it was found in the Greek sources, either as an additional theorem or an interesting scholium.

The text in the Kraus MS itself contains a fair number of copyist's errors, particularly in the numbers and geometric letter names. The diagrams, on the other hand, are as good as any we have seen in the manuscripts of *On Sizes*, Arabic or Greek. They are clearly the work of someone with a sound grasp of the mathematics involved.[54] It may be significant that these diagrams were drawn as the text was being copied, as can be seen from the

[53] Heath (1913, 362) brackets the phrase "according to the lemma," which is likely a reference to the scholium.

[54] Kheirandish (1999, xxvii) is also of the opinion that this MS was prepared by an expert.

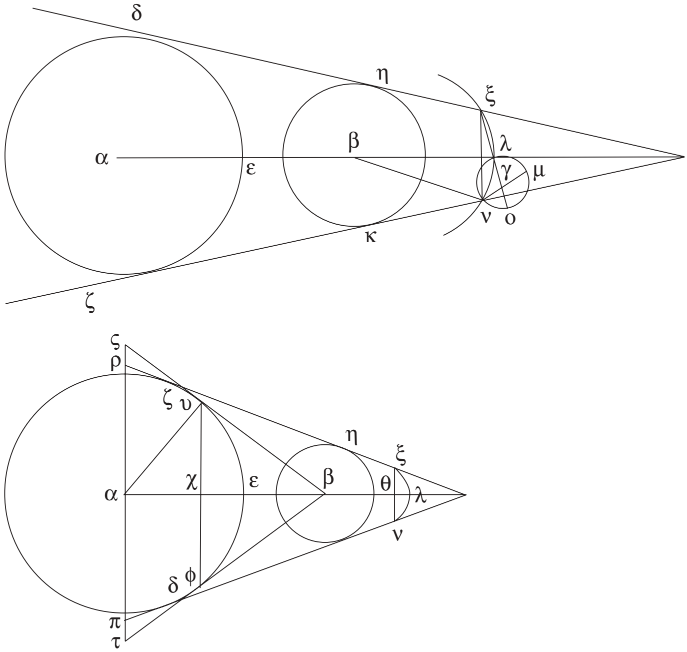

Fig. 6. Vat. gr. 204 diagrams for Prop. 13 [f. 115r–115v]

fact that text wraps around them closely.[55] The fact that the geometric letter names, in Arabic, are not derived by a strict system of transliteration from the Greek suggests that Thābit reworked the diagrams, and a comparison of the diagrams for Prop. 13 makes a strong argument that this is, indeed, what happened.

The diagrams for Prop. 13 present a real mathematical difficulty. As well as the issues we have already raised with regards to Prop. 13, the geometric requirements for Aristarchus's diagram contains a further simplifying assumption, which cannot strictly be true and yet is maintained for the sake of the argument. Consider Fig. 1 (page 220) and Fig. 6. Proposition 13 requires that the moon be entirely within the shadow of the earth. The geometry of the diagram, however, also stipulates that the dividing line be on the lunar hemisphere facing the earth, such that a tangent drawn from the center of the earth to the moon will intersect the endpoint chord where it intersects the cone of the terrestrial shadow.[56] Clearly, it is not possible to draw a diagram satisfying both these requirements.

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[55] This can be contrasted with cases where we know the diagrams were drawn after the text was complete, because a number of the diagrams do not fit into the boxes left for them. This practice is confirmed, for example, by a translation of Ptolemy's *Flattening the Surface of the Sphere* in which empty boxes have been left by the copyist (Anagnostakis 1984).

[56] It fact, however, since the moon is much smaller than the earth (Prop. 9), the dividing line must be on the hemisphere which faces away from the earth (Prop. 2). On the other hand, following Prop. 4, we may take it to be perceptibly equivalent to a great circle.

There are at least two ways to approach this complication, neither of which is fully satisfactory. Either, (1) the moon may be drawn so that it sticks out of the terrestrial shadow, or (2) the moon may be made tangent to the surface of the shadow cone, so that the intersection of the endpoint line and the tangent drawn from the center of the earth will not fall on the surface of the shadow. Either of these configurations violates one of the conditions of the proposition.

The MSS of the Greek tradition take the first route, as best seen in Vat. Gr. 204; see Fig. 6.[57] The Greek MSS use two diagrams for Prop. 13. In many of the extant diagrams the quality of the diagrams is quite poor and it is clear the copyists had some difficulty with the mathematics involved. Even in Vat. Gr. 204, which is the most competently drafted, the lines in the moon do not appear to have been drawn with due consideration for the mathematical argument. It was only with the publication of Commandino's Latin translation that European scholars were furnished with a composite, mathematically sound diagram, and they have used it ever since. (Commandino 1572, 22v ff.).[58]

Thābit took the other course. He gives a single figure and makes the moon tangent to the cone of the terrestrial shadow; see Fig. 7. Since the diagram is well drawn, the intersection of the tangent drawn from the center of the earth to the moon cannot fall on the line of the terrestrial shadow. Indeed, the entire endpoint line falls inside the shadow. The MSS of the Ṭūsī tradition, however, do not preserve the precision of Thābit's drawing. For example, in Arch. Selden. A. 45, the moon again protrudes from the terrestrial shadow [f. 147r], while in Tabriz 3484, it floats freely, entirely within the shadow (Chavoshī 2005, 179).

It is likely that Thābit's Greek sources, as ours, preserved figures that were unsatisfactory from a mathematical perspective. Hence, he set out to correct them on the basis of the geometric requirements of the text. In the case of Prop. 13, he made a single combined diagram, changed the lines in the moon, and reordered the letter names; all of which, in turn, led to minor differences in the proof of the theorem.

This brief discussion sheds some light on what it meant for Thābit to make an *improvement* of the text. We will examine three propositions in more detail below, to give a sense of Thābit's practices in the preservation of this text.

Remarks on Ṭūsī's edition

There are a number of reasons for believing that al-Ṭūsī made his edition using the Thābit revision, or some text closely related to it. Whereas Ṭūsī's general tendency is to make his edition more concise, all substantial additions found in Thābit's text are also in Ṭūsī's. Also, a number of Thābit's rearrangements to proof structure are reproduced by Ṭūsī. Moreover, and most telling, the letter names agree in every case between the

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[57] Noack (1992, Taf. VI A, XI B, XII B, XIII A, XIV B & XXV) reproduces a number of the Greek diagrams. A nice color image of the first diagram in Vat. Gr. 204 can be seen online at http://www.ibiblio.org/expo/vatican.exhibit/exhibit/d-mathematics/Greek_math2.html.

[58] The figure given by Heath (1913, 394) follows Wallis (1688, fig. 23), who in turn uses Commandino's diagram.

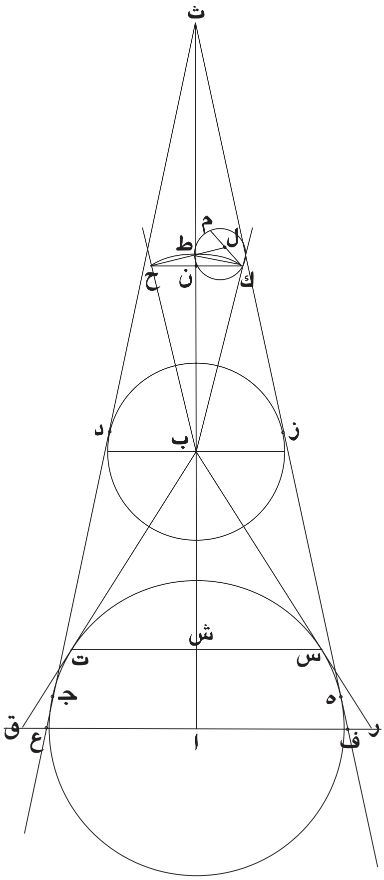

Fig. 7. Krause MS diagram for Prop. 13 [f. 131r]

two Arabic versions even though they have not been systematically transliterated from the Greek.

On the whole, Ṭūsī's text differs less from Thābit's than Thābit's from the Greek. Ṭūsī has made very few structural changes and added very little to the text. In the few cases where he adds material, this is because Thābit's version is deficient from a mathematical perspective. On the whole, Ṭūsī's object seems to have been to make a less prolix version of the treatise. He has trimmed the text using a number of different means; stylistically he uses considerable ellipsis, he condenses mathematical argument wherever possible and he eliminates repetition in proof structure. On the level of convention, he represents many numbers with numerals. These observations agree with those of Rashed (1996, 9, 12–27) regarding Ṭūsī's edition of the Banū Mūsā's *Treatise on Measuring Plane and Spherical Figures*, a work also found in his *Middle Books*.

It will be useful to make a few detailed comparisons of the three versions of the treatise – the Greek, Thābit and Ṭūsī. This will provide instantiations of the generalizations made above.

Comparison of the three versions

The first difference appears in the titles of the treatises. The Greek title is *On the Magnitudes and Distances of the Sun and the Moon*, while Thābit has *On the Volumes and Distances of the Two Luminaries, the Sun and the Moon*. Ṭūsī, on the other hand, has omitted the names as obvious, using only *On the Volumes and Distances of the Two Luminaries*. The striking feature of the Arabic titles is the word "volume" (*jirm*, جرم), which is repeated throughout the treatise. The word *jirm* (*body*, *mass*, or *bulk*) is not a common translation for the Greek version's "magnitude" (μέγεϑος), which can be used for a geometric quantity of any dimension, whereas *jirm* generally denotes a three-dimensional measure. In any case, the phrase "magnitudes of the sun and the moon" only appears in the title of the Greek work, while *jirm* appears frequently in the Arabic (Props. 10, 16 & 18).

Structural differences

The structural difference that appears almost immediately is that of the formal divisions of the propositions. The six traditional divisions of a Greek proposition were first articulated by Proclus in late antiquity in his commentary to *Elements* I (Friedlein 1873, 203–207; Netz 1999b). The *enunciation* states the proposition in general terms. It is followed by the *exposition*, which sets out some specific lettered objects satisfying the conditions of the enunciation. The *specification* then asserts the proposition for these specific lettered objects. The bulk of the proposition is given over to any auxiliary *construction* and the actual *proof*, which establishes that the assertion is true for the specific objects set out. Finally, the *conclusion* reasserts the general claim of the proposition.

In general, we can identify these categories of exposition in Aristarchus but they are more loosely differentiated and less regularly ordered than we find in the *Elements*. For example, Aristarchus will give some of the construction intermixed with the exposition, state the specification after the construction or introduce construction steps in the proof as needed. Moreover, he rarely gives a conclusion.

Nevertheless, we may use the framework of this structure to compare the three versions. In the following, we give some examples of differences in the exposition and specification that are typical.

Prop. 7 The Greek includes the construction in the exposition but omits the specification of the first part, although the second specification is given. Thābit has added a specification for the first part of the theorem and is followed in this by Ṭūsī.

Prop. 11 The Greek gives the exposition followed by a brief specification, "I say that it is according to the enunciation" (λέγω ὅτι γίγνεται τὰ διὰ τῆς προτάσεως) (Heath 1913, 386). Thābit has fleshed this out to a full specification and is followed in this by Ṭūsī.

Prop. 13 This complicated theorem is in three parts; the first two demonstrate upper and lower bounds for two ratio inequalities, the third gives just a lower bound. In the Greek text, the construction is included in the exposition and followed immediately by the proof of the first part. The second part, however, is introduced by its own specification. There is again no specification for the third part. In the Thābit text, the proposition begins abruptly, the proceeding theorem having no statement of proof (QED). Following the enunciation, there is a general specification for the whole proposition (فاقول ان الامر يكون على ما قلنا فيما تقدم) [130r]. There are three more specifications but there is some disorganization. For example, there is no specification for the first half of part two, but there is for the second. Ṭūsī basically follows Thābit; however, he gives the statement of proof for the forgoing proposition and drops the general specification.

Prop. 14 In the Greek, the exposition has been abridged to "let it be the same diagram as before" with no specification (Heath 1913, 398). Thābit sets out "the things in the figure" (الاشياء في الشكل), gives one statement of exposition and then a full specification [131r]. Ṭūsī follows Thābit.

Prop. 15 The proposition is in two parts. The Greek gives a full exposition in the beginning but only states the specification for the second part. Thābit sets out the "things in the previous figure" and then gives a bit of exposition [131v]. This is followed by a full specification for both parts; the second specification is given again to introduce the second part of the proof. Ṭūsī follows Thābit but his statement of the first specification is more concise.

These comparisons again show that Ṭūsī made his edition on the basis of the Thābit revision. There are no significant textual additions in the Ṭūsī text that are not based on the Thābit text. All differences between the Arabic versions can be explained by Ṭūsī's tendency to abridge the text.

While the Arabic versions show some divergence from the Greek with regard to expositions and specifications, the three texts are very close in the expression of the conclusions. The one significant difference is in the statements of proof at the end of a theorem. Aristarchus never gives the final assertion of proof that became canonical in later Greek mathematical writing, "which was to be shown" (ὅπερ ἔδει δεῖξαι). Thābit, on the other hand, always gives this assertion as "and that is what we wanted to prove" (وذلك ما اردنا بيانه) and Ṭūsī shortens it to "and that is what we wanted" (وذلك ما اردناه).

These data support two conclusions. The first is that of Netz (1999b), who argues that the traditional division of Greek propositions was likely a framework developed by Proclus to describe canonical texts such as the *Elements*; it was not an absolute structure, which working mathematicians strove to maintain. The second is that al-Ṭūsī's practice of condensing his text tended to eliminate the repetitious elements of Greek proof structure.

Specific differences

The prefatory material, consisting of the hypotheses and the brief sketch of the structure of the treatise, is the section where we find the closest agreement between the three versions. The one conspicuous exception is Hyp. 2, which Ṭūsī reformulated to read, "The measure (قدر) of the earth with respect to the sphere of the signs (فلك البروج) is the measure of the center and point" (al-Ṭūsī 1940, 2). Since Thābit, following the Greek, expresses this in terms of the size of the earth compared to the lunar orbit, Ṭūsī probably thought the error was a simple slip and corrected it to the more commonly held position. He apparently did not notice that the hypothesis concerning the lunar orbit is actually required in three places; see Table 1.

We turn now to a discussion of a few specific propositions which may be taken as illustrative of some of the points we have made about the differences between the three versions.

To facilitate comparison, where possible we change the Arabic letter names to those found in Heath (1913). As we have said, however, most of the lettering has been reorganized from what one would expect if Thābit were using the standard transliteration of Greek letter names. For the three propositions discussed below (Props. 4, 7 & 13), Thābit has relabeled his figure in such a way that the letter names introduced in his enunciations follow the Arabic, abjad order.

Prop. 4

We have already discussed this theorem at length (see page 228). It shows that the dividing circle in the moon is virtually identical with a great circle. A comparison of the three different versions of this theorem, will exhibit some of the typical differences in the texts: changes in the argument between the Greek and Thābit and simplification in mathematical expression between Thābit and Ṭūsī.

The details of the proof vary between the Greek and Thābit. Consider Fig. 4. The Greek argument runs as follows: by hypothesis (Hyp. 6), the moon stands on $^1/_{15}\mathbf{Z}$, so $\angle CAD = {}^1/_{15}\mathbf{Z}$. But $^1/_{15}\mathbf{Z}$ is $^1/_{180}\mathbf{C}$, so $\angle CAD = {}^1/_{180}\mathbf{C}$. Hence, $\angle CAD = {}^1/_{180}\, 4\mathbf{R} = {}^1/_{45}\mathbf{R}$ (Heath 1913, 367).

Thābit, on the other hand, with no reference to the hypothesis, states that the moon subtends $^1/_{15}\mathbf{Z} = {}^1/_{45}\, 3\mathbf{Z}$ so $\angle CAD = {}^1/_{45}\mathbf{R}$. The argument is then garbled by claiming that, since $\angle BCA = \mathbf{R}$, then $\angle CAB = {}^1/_{45}({}^1/_2\mathbf{R})$ [126r]. In fact, the size of $\angle CAB$ depends only on $\angle CAD$ and the fact that $\angle BCA$ is right should be asserted at the beginning of the following section, as it is in the Greek. Moreover, the line BC, on which this argument depends, does not appear in the Greek diagrams or text. Ṭūsī, on the other hand, follows Thābit through this argument. It seems probable that Thābit drew a new diagram for this theorem and then wrote a slightly different proof, which used a line in his diagram that does not appear in the Greek.

A step of the construction can be taken as an example of the way al-Ṭūsī renders Thābit's text more concise. Where Thābit says "Let us join line DC so that the circle, whose diameter is *DC standing on line AB at right angles* (القائمة على خط اب على زوايا قائمة), is less...", Ṭūsī simplifies with "We draw DC, and the circle, whose diameter is DC, *with AB perpendicular to it* (و اب عمود عليه), is less..." (al-Ṭūsī 1940, 6; 126r).

Prop. 7

This key proposition provides bounds for the solar distance in terms of multiples of the lunar distance. There are differences between Thābit and the Greek in terms of both structure and argument. The diagram in the earliest Greek manuscripts is somewhat poorly drawn and it is likely that Thābit reworked it [*Vat. Gr.* 204, f. 113v]. The letter names he uses follow the abjad order of his exposition with the result that not single letter corresponds to what we would expect from strict transliteration. Ṭūsī follows Thābit in these structural changes. We will get a sense for the structural differences by looking at the exposition and construction.

[Greek:] Let A be the center of the sun, B that of the earth. Let AB be joined and produced. Let C be the center of the moon when halved; let a plane be extended through AB and C, and let the section made by it in the sphere on which the center of the sun moves be the great circle ADE. Let AC, CB be joined and BC produced to D. Then because the point C is the center of the moon when halved, the angle ACB will be right. Let BE be drawn from B at right angles to BA. Then ... (Heath 1913, 377)

[Thābit:] Let the point of our eye be point B, and let the center of the sun be point A. We join BA and let us produce the plane passing through line AB and the center of the moon, when the moon is halved in light. Hence, the section, which is produced by it in the sphere of the sun, is a great circle; let it be circle AXE.[59] And let line XBE pass through point B,[60] and let BA stand on it at right angles. Hence, the center of the moon, when it is halved in light, is located between the lines AB, BE, EA.[61] Let it be point C, and let us join lines BC, CA. I say ... [127r]

The differences between the two texts are numerous. As seen in Fig. 8, the diagram has been redrawn and the argument rewritten. The "center of the earth" has become "our eye." The Greek labels the center of the moon when it is introduced; Thābit waits until the end of the exposition. Thābit gives the great circle of the sun's orbit a different name, based on a point that is not found in the Greek diagram or used in the Greek text. The Greek draws the lines BC, CA in the middle of the passage and asserts that they are perpendicular to one another. Thābit draws them at the end and makes no claims about them.

Ṭūsī follows Thābit with subtle differences. "Our eye" becomes "the eye." He omits words like "point" and "line" where the object is mathematically obvious. In one place, he adds the word "arc" to distinguish the object in question from lines.[62] The jussive of geometric operations tends to become the imperfect. There are, however, no substantial differences in this passage between Ṭūsī and Thābit.

This close correspondence between the Kraus MS and Ṭūsī's edition is not maintained in all places. In some cases, al-Ṭūsī supplies steps in the mathematical argument that have gone awry in the earlier text. The following example shows a case of textual corruption in Thābit that is revealed by a comparison with Ṭūsī's text.

Toward the end of the first part of the theorem, the text of Thābit's revision loses its way. Two of the numbers are wrong and some of the argument has gone missing.

> **[Thābit:]** So, the ratio of line FG to line GE is greater than the ratio of *nine* to five. Hence, if we compose, line FG to line GE will be greater than the ratio of twelve to five,

---

[59] Point X is not labeled in the Greek diagram.

[60] Where the MS has بنقطة اجط جـ اد, we read بنقطة آ خط جـ اد.

[61] EA is not actually a line. Ṭūsī corrects this by writing "arc EA" (al-Ṭūsī 1940, 8).

[62] See note 61.

> which is equal (مثل) to the ratio of thirty six to *two*, which is equal to the ratio of eighteen to one. [127v]

The two italicized numbers are incorrect, although correctable on the basis of the mathematics involved. By a simple slip "nine" (التسعة) was written in place of "seven" (السبعة). In fact, the MS has التسعة with a caret under the ت, indicating that the error may have been caught.

It is a little less easy to explain the "two" which should read "fifteen." The written forms of these numbers are quite different and the mathematical statement it makes is clearly false. The situation, however, becomes clear when we see how al-Ṭūsī handles this text.

> **[Ṭūsī:]** So, the ratio of FG to GE is greater than the ratio of seven to five, and by composition the ratio of FE to GE is greater than twelve to five, that is the ratio of thirty six to fifteen. And the ratio of GE to EH is greater than the ratio of fifteen to two, so by equality [of terms] the ratio of FE to EH is greater than the ratio of thirty six to two, that is the ratio of eighteen to one. (al-Ṭūsī 1940, 8)

Ratios are twice asserted between 36 and another number, 15 and 2. A copyist has simply dropped the text between the two occurrences of 36. This indicates that Ṭūsī made his edition on the basis of an MS that had not suffered this parablepsy, and he may well have relied on various sources. Perhaps the Thābit revision was standard in the compilations of *Middle Books* which circulated prior to al-Ṭūsī's edition. It is worth noting that in both places in Prop. 7 where Ṭūsī supplies arguments that are missing in the Kraus MS, the missing steps are found in the Greek.

Prop. 13

This lengthy and technically important proposition relates the endpoint chord to three other lines. The first part of the theorem compares it to the diameter of the moon, the second to the diameter of the sun and the third to the line that is the sun's diameter produced to the extended cone of the terrestrial shadow (line RQ in Fig. 8).

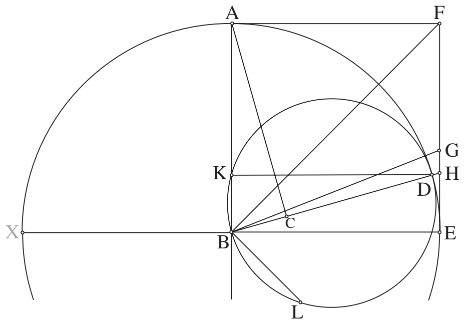

Fig. 8. A modified diagram for Prop. 7. In all MSS, the orbit of the sun, AE, is complete. Point X, on the left, does not appear in the Greek MSS, nor is it used in the Greek proof. It is, however, used in the proof of both Arabic versions, although it is not labeled in the Krause MS

As already noted the diagram in the Arabic versions has been redrawn and, because the lines in the moon are different, this has produced minor changes in the proof; see Fig. 9. As with Prop. 7, it is clear that other aspects of the theorem have been somewhat rewritten on the basis of the new figure; there is some reordering of the exposition and demonstration and the details of a number of mathematical arguments are different. We will look a three examples, all of which show Thābit's tendency to flesh out mathematical arguments and Ṭūsī's tendency to make these arguments more concise.

In the first part of Prop. 13, Aristarchus simply asserts that $7921 : 4050 > 88 : 45$ (Heath 1913, 397). It is usual for modern commentators to explain this step with recourse to continued fractions (Heath 1913, 397, n. 1; Tannery 1912, 385; Fortia d'Urban 1810, vol. 2, 86), but it appears that no late ancient or medieval scholar understood the step in this way.[63] There is a loquacious scholium in a number of the Greek MSS which points out that $7921 : 4050 = 88^1/_{90} : 45$, so that $7921 : 4050 > 88 : 45$ (Fortia d'Urban 1810, vol. 1, 166–169). Thābit, however, is much more to the point.

> **[Thābit:]** The ratio of seven thousand nine hundred twenty-one to four thousand fifty is greater than the ratio of eighty-eight to forty-five; and that is because when we make the ratio of eighty-eight to forty-five like seven thousand nine hundred twenty to some number, that number will be greater than four thousand fifty. [130r–v]

The ratio $7921 : 4050$ has been derived through the equality of terms operation. It is then reduced by an approximation which is slightly smaller. The ratio $88 : 45$ is selected because two geometrically related lines have just been shown to have a greater ratio than $89 : 45$. Since, $89 : 45 > 7921 : 4050$, Aristarchus takes a ratio slightly smaller by diminishing the first term by one. He can then check that this is a lower bound, as Thābit describes. Whether or not Aristarchus actually proceeded in this manner is irrelevant. What matters is that it is possible to make sense of the numbers in the text without recourse to continued fractions. Thābit, and al-Ṭūsī following him, apparently thought these numbers were obvious on the basis of geometric considerations and simple arithmetic.[64]

The next example is fairly typical of the types of differences we find between the three texts. Consider Fig. 9. In the third part of Prop. 13, Aristarchus makes an argument which relies on the geometry of certain lines in the sun.

> **[Greek:]** ... WU has to UA a greater ratio than that which 89 has to 90. But, as WU is to UA, so is UA to SA, because SA, UW are parallel. (Heath 1913, 399)

Thābit fleshes this out with a full geometric argument.

> **[Thābit:]** The ratio of line UW to UA is greater than the ratio of eighty nine to ninety. And the ratio of UW to UA is as the ratio of line UA to line AS, because the angles of

[63] It is worth noting that in order to argue that Aristarchus proceeded by means of continued fractions one would have to accept his having used a slightly different algorithm in the two cases where this type of step is found; see Appendix B.

[64] For the large-number ratio which is approximated in Prop. 15 (Heath 1913, 407), Thābit has a different explanation [132v]. It involves ratio operations and arithmetic, but again no continued fractions. He is followed in this by Ṭūsī (al-Ṭūsī 1940, 18).

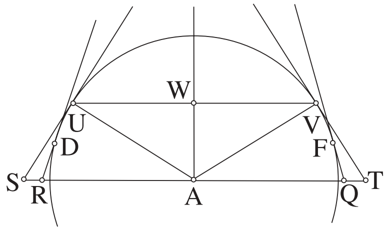

Fig. 9. A detail of the diagram for Prop. 13

> triangle AWU are equal to the angles of triangle SAU – and that is because line SA is parallel to line UW and the angles SUA, UWA are right.

Ṭūsī then takes Thābit's argument and simplifies it considerably.

> **[Ṭūsī:]** The ratio of UW to UA is greater than the ratio of 89 to 90. And the ratio UW to UA is as the ratio of UA to AS, since the triangles AUW, SUA are similar. (al-Ṭūsī 1940, 15)

This series of passages shows quite well the sorts of changes that the medieval editors tended to make in their texts. Thābit generally expanded the mathematical argumentation to give more justification than he found in the Greek. In this case, he did so on the basis of a reconstructed figure. He probably decided what the Greek diagram should have looked like on the basis of mathematical considerations of the Greek text. In fact, neither line AU nor AV are found in the Kraus MS, although they appear the Tūsī MSS that we have seen.

The final example we will look at involves a multiplicative use of the equality of terms operation, which Aristarchus introduces in Prop. 13 and uses a number of times in the remainder of the treatise.

Consider Fig. 9. At the very end of Prop. 13, Aristarchus shows that $d_s : QR > 89:90$, and that *endpoint chord*: $d_s > 22:225$. He then uses equality of terms to argue that the ratio *endpoint chord* : QR is much greater than

> **[Greek:]** the result from (ὁ συνηγμένος ἔκ) 22 and 89 to that from 90 and 225; that is 1958 to 20,250. (Heath 1913, 398)

The term used for "result" is not an established technical term for the product of a multiplication. The expression is probably ellipsis for "the number that results" (ὁ συνηγμένος ἀριθμός).[65] It should be noted that Aristarchus later uses the Euclidean technical term for product, where, toward the end of Prop. 15, he performs a similar equality of terms operation and expresses the product as "the number comprised by ... " (ὁ περιεχόμενος ἀριθμὸς ὑπο ...) (Heath 1913, 406).

Thābit was apparently struck by this lack of technical vocabulary and decided to specify the operation in some detail; first giving a vague description in the terminology

[65] Wallis (1688, 75), for example, translates it as such.

of ratios and then introducing the Arabic technical term for product. His version of the passage states that the ratio *endpoint chord* : QR is much greater than

> **[Thābit:]** that which is from the antecedents, one of them to (إلى) the other – they are twenty-two and eighty-nine – to that which is from the consequents, one of them to the other – they are two hundred twenty-five and ninety.[66] That which is from the product (ضرب) of the antecedents, one of them by (في) the other, is one thousand nine hundred fifty-eight; and that which is from the product of the consequents, one of them by the other, is twenty thousand two hundred fifty. [130v]

As usual, al-Ṭūsī simplified the passage, writing in a more straightforward technical idiom. He asserts that the ratio *endpoint chord* : QR is much greater than

> **[Ṭūsī:]** the ratio of the result from the product of one of the consequents by the other – that is 22 by 89, which is 1958 – to the result of the product of one of the antecedents by the other – that is 225 by 90, which is 20,250. (al-Ṭūsī 1940, 15)

The examples drawn from these three propositions may be taken as substantiations of the general claims made above about the differences between the three treatises.

Conclusion

On Sizes has played a variety of roles in quite different contexts throughout its long history. Probably the context we know least about is that of its composition. In fact, *On Sizes* itself should be taken as an important source for our understanding of the mathematical sciences of the early Hellenistic period. As do certain works of Archimedes, it shows a usage of hypotheses assumed simply for the sake of the argument, not because they are absolutely held to be true. Indeed, *On Sizes* exhibits an adroit use of assumptions, some of which are even contradictory, in order to derive computational results. Moreover, along with Archimedes' *Sand Reckoner*, it attests to ratio operations that are much more arithmetical than we would expect from the *Elements* and to a form of trigonometry that predated the construction of chord tables. This helps to give a more rounded picture of mathematical practices roughly contemporary with, or shortly after, the composition of the *Elements*.

At some point in Greek antiquity, *On Sizes* was collected together with other elementary works of mathematical astronomy and, after the *Almagest* had become canonical, served as part of a course of instruction intermediate between the *Elements* and the *Almagest*. Although the extent to which these subjects were actually so taught, for whom and by whom, remains largely unknown, in Pappus' *Collection* we find *On Sizes* included in a variable group of texts which were taught in the "field of astronomy" (τὸν ἀστρονομούμενον τόπον (Hultsch 1876–1878, 474.))

[66] The MS has "two hundred twenty." We read this with وخمسة where mathematically required.

This corpus of texts in the exact sciences was incorporated into Arabic scientific culture in the 9th century by the Baghdad mathematicians. *On Sizes* itself was made into a serviceable Arabic text by one of the most remarkable of these men, Thābit ibn Qurra. Apparently, Thābit thought the Greek text, and particularly the diagrams, required mathematical revision and he undertook this work in what was, at this time, a common genre for the Arabic adaptation of a Greek treatise. In fact, assuming that the main features of the Byzantine version we know were shared by the version Thābit used, *On Sizes* becomes a good source for understanding the genre of *revision* as it was practiced in ᶜAbbāsid Baghdad.

In the 13th century, Naṣīr al-Dīn al-Ṭūsī produced a new edition of the *Middle Books*, which now played an important role in mathematical education in the Islamic world. Although the canon was still based on Greek classics, it had for some time now also included original works by the Baghdad mathematicians. Ṭūsī sought to bring this corpus into better conformity with the mathematical practice of his time, applying the standards of contemporary mathematical language and correcting any errors that had entered the manuscript tradition. Again, because we posses the text of Thābit's revision, *On Sizes* is an important source for our knowledge of Ṭūsī's practices in making this collection. Indeed, *On Sizes* can be taken as an example of the technical aspect of the acquisition and assimilation of Greek scientific writings into an Islamic context (Sabra 1987).

It was in these collections of mathematical astronomy, both Arabic and Greek, that European scholars first encountered *On Sizes*. Little survives of these early transmission efforts and the first printing of a Latin translation included our text tucked away in an eclectic volume by Georgio Valla of 24 works in philosophy, medicine, music theory, and mathematics (Noack 1992, 47–52). The first significant Latin text is the translation and commentary made by Commandino (1572) near the end of this life (Noack 1992, 53–65). The work appears as an independent treatise based around redrawn and mathematically coherent diagrams and accompanied by commentary fleshing out the mathematical argument. This publication has shaped the way the treatise has been read ever since. In fact, Wallis (1688) included Commandino's translation, diagrams, and comments in his edition of the Greek text. The interest with which *On Sizes* was still read in the 16th and 17th centuries is underlined by the fact that the first critical edition, made on the basis of a number of important manuscripts, was established by Wallis, himself an important contributer to the new mathematics of the 17th century. In this way, we find an author of current developments in mathematical techniques using the latest methods of textual criticism to establish a text that, by that time, could only have been of historical interest.

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Appendix A: *On Sizes* 17_a

Al-Ṭūsī's Text

We edit the Ṭūsī text of this theorem because a key folio is missing from the Kraus manuscript. The text is established on the basis of two MSS:

B: Bodleian Library, Arch. Selden. A. 45, ff. 149v–150r
T: Tabriz National Library 3484, 184.

(يز) نسبة بعد رأس مخروط الظل عن مركز القمر، إذا كان القمر على سهم المخروط المحيط بالشمس والارض، الى بعد مركز القمر عن مركز الارض اعظم من نسبة ٧١ الى ٣٧ واصغر من نسبة ٣ الى الواحد· فليكن مركز الشمس ا، ومركز الارض ب· ونصل ا ب · وليمر به سطح· فيحدث في الشمس عظيمة ه د، وفي الارض عظيمة ز ح ، وفي المخروط خطا جـ د، جـ ه· وليكن مركز القمر ط· ونصل د ا، ز ب، ونخرجهما الى ك ، ل · فلان نسبة دك الى ز ل اقل من نسبة ٤٣ الى ٦، تكون نسبة ا جـ الى جـ ب كذلك وبالخلاف، نسبة ب جـ الى جـ ا اغظم من نسبة ٦ الى ٤٣· وبالتفصيل، نسبة جـ ب الى ب ا اغظم من نسبة ٦ الى ٣٧· وقد مر ان نسبة ا ب الى ب ط اغظم من نسبة ١٨ الى الواحد· فبالمساواة، نسبة جـ ب الى ب ط اغظم من نسبة ضرب ٦ في ١٨، وهو ١٠٨، الى ضرب ٣٧ في الواحد· وبالتفصيل، نسبة جـ ط الى ب ط اغظم من نسبة ٧١ الى ٣٧· وايضا فنسبة د ك الى ز ل كانت اعظم من نسبة ١٩ الى ثلثة· فنسبة ا جـ الى جـ ب كذلك · وبالخلاف، نسبة ب جـ الى جـ ا اصغر من نسبة ٣ الى تسعة عشر، وبالتفصيل، نسبة جـ ب الى ب ا اصغر من نسبة ٣ الى ١٦· ونسبة ا ب الى ب ط ايضا اصغر من نسبة ٢٠ الى الواحد· فبالمساواه، نسبة جـ ب الى ب ط اصغر من نسبة ٦٠ الى ١٦، اعنى من نسبة ١٥ ال ٤· وبالتفصيل، نسبة جـ ط الى ط ب اصغر من نسبة ١٢ الى ٤، اعنى من نسبة ٣ الى الواحد· وذلك ما اردناه· ثمّ كان·

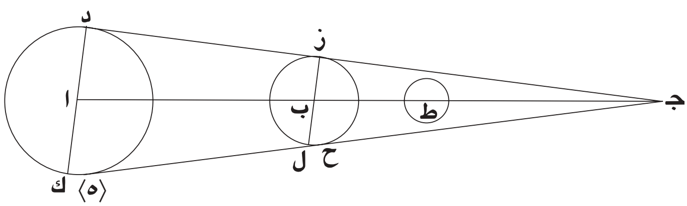

Fig. 10. B, fol. 150r. Point ح is not found in B, but is in T. Point ه appears in neither MS.

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2 ٧١] ٧٩ **BT**. 5 خطا] خطي **T**. 6 كذلك] لذلك **BT**, cor. al-Ṭūsī (1940). 12 كذلك] لذلك **BT**, cor. al-Ṭūsī (1940). 14 ايضا] ايضا ايضا **T**. 15 ٦٠] ٢٠ **BT**. 16 ٣] ١٣ **BT**.

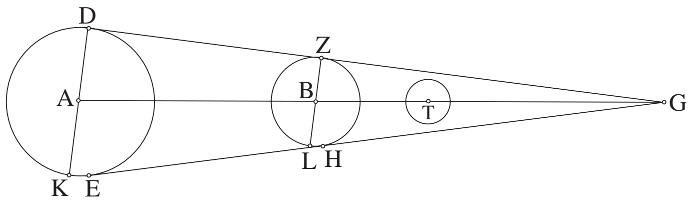

Fig. 11. Diagram for *On Sizes* 17_a

English Translation

(17) The ratio of the distance of the vertex of the shadow cone from the center of the moon, when the moon is on the axis (*sahm*) of the cone containing the earth and the sun, to the distance of the center of the moon from the center of the earth is greater than the ratio 71 to 37 and less than the ratio 3 to one.

Let the center of the sun be A and the center of the earth B. We join AB and let a plane pass through it; hence there results in the sun the great [circle] ED, and in the earth the great [circle] ZH, and in the cone the lines GD, GE. And let the center of the moon be T. We join DA, ZB, and we extend them to K, L. And since the ratio of DK to ZL is less than the ratio of 43 to 6 [Prop. 15], the ratio of AG to GB is like that. And by inversion, the ratio of BG to GA is greater than the ratio of 6 to 43. And by separation, the ratio of GB to BA is greater than the ratio of 6 to 37. And it happened that the ratio of AB to BT was greater than the ratio of 18 to one [Prop. 7]. And through equality, the ratio of GB to BT is greater than the ratio of the product of 6 by 18, which is 108, to the product of 37 by one. And by separation, the ratio of GT to BT is greater than 71 to 37.

And again, the ratio of DK to ZL is greater than the ratio of 19 to three [Prop. 15]. So the ratio of AG to GB is like that. By inversion, the ratio of BG to GA is less than the ratio of 3 to nineteen. And by separation, the ratio of GB to BA is less than the ratio of 3 to 16. And the ratio of AB to BT is again less than 20 to one [Prop. 7]. So through equality, the ratio of GB to BT is less than 60 to 16, that is than 15 to 4. And by separation GT to TB is less than the ratio of 12 to 4, that is than 3 to one, which is what we wanted.

Appendix B: The size of the lunar disk

Hyp. 6, which asserts the size of the lunar disk, primarily affects the numerical results in Props. 12–15.[67] We have used a computer to test the effect of varying the size of the lunar disk while carrying through the computations using the same mathematical steps as Aristarchus. This has shown that, in fact, the results of Prop. 15 have little dependence on the size of the lunar disk. In particular, the lower bound, 15_a, is independent of the lunar disk. The upper bound, 15_b, however, is also fairly stable. For instance, if we set

[67] Although the lunar disk is also used in Prop. 4, this has no computational effect on the theorems that follow.

the lunar disk to 10° we get 8 : 1, and if to 1/10° we get 897 : 128 ($\approx$ 7); whereas 2° gives 43 : 6 ($\approx$ 7.16).

To see why the lower bound is independent of the size of the solar disk it may be helpful to look at a chart of the internal dependencies of Props. 12–15. Props. 12 & 14 only assert one numerical statement in each case, a lower bound. Proposition 13 asserts five numerical statements: two pairs of bounds and one lower bound. Proposition 15 asserts a pair of bounds. We number the parts of this sequence as follows: 12, 13_a, 13_b, 13_c, 13_d, 13_e, 14, 15_a and 15_b.

	12	13_a	$\mathbf{13_b}$	13_c	$\mathbf{13_d}$	$\mathbf{13_e}$	**14**	15_a	$\mathbf{15_b}$
H6	•						•		
7								○	○
9				○					
12			•			•			
13_a				○			○		
13_b					•				
13_c								○	
13_d						•			
13_e									•
14									•

In the table, a bullet shows that a result in the column headings depends, computationally, on a result in the row headings. Hence, the table shows that the third part of Prop. 13 depends on Prop. 9 and the first part of Prop. 13. A computation that is affected by the size of the lunar disk is indicated by solid bullet, •. An empty bullet, ○, represents a computation that is independent of the size of the lunar disk. Computations that are a direct result of the geometry of the figure are ignored. As can be seen, 15_a, the lower bound of Prop. 15, is independent of Hyp. 6.

To understand why the upper bound is also stable, it will be necessary to look at some of the details. Since we know that Aristarchus elsewhere postulated a solar disk of 1/2°, we will use this as an example. We carry through all calculations using rational numbers. There are two steps where we cannot claim to know Aristarchus's computational algorithm. These are the reductions of ratios in large numbers to more manageable numbers in Props. 13 & 15, which modern commentators have explained with continued fractions (Heath 1913, 397 & 407).[68] Although we are unconvinced of the usefulness of continued fractions to explain ancient and medieval mathematical practice, we have employed them for the sake of this exercise. To get the exact numbers Aristarchus obtains, we have to handle the fractions slightly differently in the two cases. In Prop. 13, we derive four terms of the continued fraction expansion and round to the third, whereas in Prop. 15 we take three terms and round to the second. For the calculations using a lunar disk of 1/2°, we have taken three terms and simply dropped the rest. We summarize the differences with the following table.

[68] See the specific comparisons of the three versions, Prop. 13.

	2° ratio	2° value	1/2° ratio	1/2° value
12	89 : 90	0.99	359 : 360	0.10
13_a	2 : 1	2.00	2 : 1	2.00
13_b	88 : 45	1.96	179 : 90	1.99
13_c	1 : 9	0.11	1 : 9	0.11
13_d	22 : 225	0.10	179 : 1800	0.10
13_e	979 : 10125	0.10	64261 : 648000	0.10
14	675 : 1	675.00	10800 : 1	10800.00
15_a	19 : 3	6.33	19 : 3	6.33
15_b	43 : 6	7.17	176 : 25	7.04

The *ratio* columns give whole number ratios as found following Aristarchus's methods. The *value* columns give decimal equivalents for ready comparison. It can be seen that the only result that significantly differs between lunar disks of 2° and 1/2° is Prop. 14, which being different by two orders of magnitude should seriously effect the final upper bound.

The reason that it does not lies in the use of the operation of *conversion*. When the result of Prop. 14 is used in Prop. 15 it is first subjected to conversion. Since, this ratio is a multiple, $n : 1$, conversion gives a ratio of the form $n : (n - 1)$. Hence, where n is sufficiently large, as far as its effects on future computations are concerned, this ratio will equal one. For the numbers involved in these computations, this ratio is always sufficiently close to unity to have a negligible effect on the final result. In fact, Prop. 13_e, which varies much less than Prop. 14 has as much effect on the final upper bound of Prop. 15_b.

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