GREEK MATHEMATICS

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This overview discusses the tradition of theoretical mathematics that formed a part of Greek literary culture. This was not the only kind of mathematics that existed in the ancient Greek world. There were also traditions of elementary school mathematics, and the subscientific traditions of mathematics that were handed down by various professionals who used mathematics in their work, such as tradesmen, builders, accountants and astrologers.\(^1\) In fact, in the ancient Mediterranean, these subliterary traditions almost certainly formed the vast majority of all the mathematics studied and practiced, while literary, theoretical mathematics was practiced by only a privileged few.\(^2\) Nevertheless, the elite, literary status of the theoretical mathematicians, along with the brilliance of their work, insured that much of their project was preserved for posterity, while the predominantly oral, subscientific tradition survives only in scattered, material fragments and a few collections passed down through the manuscript tradition, attributed, probably erroneously, to Heron (c. mid-1\(^{st}\)-3\(^{rd}\)).

In the Classical period, when the forms of literary mathematical texts were being established, elite, theoretical mathematicians appear to have done as much as they could to separate themselves from professionals who used mathematics, just as they strove to distinguish themselves from other elites, such as sophists and philosophers who did not engage in mathematical activity. Theoretical mathematics was originally not a professional, institutionalized activity. During the Hellenistic period, when this, now more institutionalized, theoretical mathematics was applied to serious problems in natural science and engineering, there were a number of attempts to wed the theoretical and practical traditions of mathematics but the social context and institutional settings of the two remained distinct. Finally, in the Imperial and Late Ancient periods, although creative mathematics was less practiced, mathematical scholarship was thoroughly institutionalized in the philosophical schools and mathematics and philosophy were finally united, although for many mathematical scholars of the late period, philosophy was accorded a superior position.

The range of ideas and activities that were designated by the word *mathēmatikē* are not identical to those denoted by our word *mathematics*.\(^3\) From the earliest times, *mathēmatikē* was connected with any branch of learning, but came to denote the mathematical sciences centered around arithmetic, geometry, astronomy and harmonics. From the time of the Pythagoreans to that of Ptolemy (mid-2\(^{nd}\) CE), astronomy and harmonics were not regarded as applications of mathematics, but as core areas of the enterprise. *Mathēmatikē* denoted those literary disciplines that used mathematical techniques or investigated mathematical objects, whether actual or ideal, and which included fields such as optics, sphere-making, or astrology along with abstract investigations such as the theories of whole numbers or conic sections.

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\(^1\) The term "subscientific" is due to J. Høyrup. See, for example, J. Høyrup, Sub-scientific Mathematics: Observations on a Pre-Modern Phenomena," *History of Science*, 27 (1990), 63–87.


Nevertheless, ancient thinkers had a fairly clear idea of what constituted a mathematical work and who was a mathematician. They described different arrangements of the mathematical sciences and set out various relationships between mathematical fields and other branches of theoretical knowledge. Ancient discussions of mathematicians revolve around a core group of frequently repeated names. We find debates about the legitimacy of certain mathematical arguments in philosophical works, but rarely in the surviving mathematical texts. Mathematicians defined mathematics by the nature of the texts they wrote. From around the middle of the 5th century BCE, mathematical texts were highly structured, involving explicit arguments, often centered around diagrams and letter names or other technical apparatus, and mathematicians were those people who could produce such texts. They developed particular ways of speaking, or rather of writing, and these further served to reinforce the exclusive tendencies of this small literary group. This chapter explores the little we know about the social contexts and mathematical practices of these authors, and the rather more we know about their surviving works.

DIRECT AND INDIRECT TRADITIONS

Our evidence for ancient Greek mathematical activity comes, almost exclusively, from texts that were passed down through the medieval period in contexts that were generally not devoted to mathematical activity and by individuals who often did not themselves produce original mathematics. Although this is true for all of the ancient Greek theoretical sciences, the situation is perhaps more pronounced in the case of mathematics.

On the one hand, Greek mathematicians produced texts, not only to transmit their results to posterity, but also to teach others how to engage in a range of mathematical activities, of which the production of written texts was only a small part. On the other hand, with regard to theoretical mathematics, we have almost no material evidence, few non-mathematical accounts and even fewer direct sources. Hence, we are in the awkward position of having to use only a small fraction of the total texts that were once produced to try to reconstitute the contexts of mathematical knowledge and activities in which these texts were produced and read. A sense that the texts that we now read do not teach us everything we need to know in order to understand how Greek mathematicians actually did mathematics has been felt by scholars since at least as far back as the early modern period, and this has led to a number of transitions in the way the primary texts are read. Indeed, the majority of the texts we still posses were probably produced to be vehicles of transmission, and hence meant to satisfy a different set of criteria than texts that might have been written to train mathematicians, or to bear more directly on mathematical activity. Nevertheless, if we wish to understand what Greek mathematicians actually did, and how, then we have no real alternative but to study the texts that have been transmitted through the manuscript tradition.

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4 Netz identified 144 mathematicians in ancient sources, many of which are only known to us by name. See, R. Netz, “Classical Mathematicians in the Classical Mediterranean,” Mediterranean Historical Review, 12 (1997), 1–24.

5 Exceptions to this general pattern are Archimedes’ Method, Apollonius’ Cutting off a Ratio, and, perhaps, some books in the Heronian corpus.
The manuscript sources are generally divided into direct and indirect traditions. The direct tradition are manuscripts of source texts, in Greek, while the indirect traditions are commentaries and summaries in Greek along with translations and their commentaries, largely in Arabic and Latin. For understanding Greek mathematics, the most important indirect traditions are the Arabic translations that derive from the 8th to 10th century translation movement in Baghdad, and the 12th to 13th century Latin translations, from either Greek or Arabic. From this description, it might seem that the direct tradition could be treated as the principle source, so that the indirect traditions could be neglected except in cases where the direct tradition was deficient.

The difficulty with this assumption, however, is that even in the direct tradition the mathematical texts were subjected to numerous revisions over the centuries, the details of which are now mostly lost to us. In the case of religious and literary texts, the actual words of the original author were considered sacrosanct and the ancient and medieval editors conceived of their role as the preservation of these words themselves. In the case of the exact sciences, however, the texts were often edited by scholars who were themselves practitioners or teachers of the fields that the texts transmitted. These scholars often took the scope of their role to include a correction of the words of text based on their own understanding of the ideas that the words conveyed. Hence, the Greek mathematical texts must be understood as canonical in the sense that the canon was somewhat flexible and subject to repeated reinterpretation. Both the selection of texts that we are now able to read, and the specific words in which we read them, are the result of this repeated reworking and reexamination of the canon. For these reasons, in order to determine how Greek mathematics was actually practiced, we are often in the position of having to reconstruct a lost context of mathematical activity on the basis of both the direct and indirect traditions. In order to get a sense for some of the vagaries of the transmission, we look at two examples.

The work of Archimedes (c. 280s–212) will furnish our first example. We know of this corpus through a number of early modern copies of a lost Byzantine manuscript (Heiberg’s A), a 13th century Latin translation by William of Moerbeke (c. 1215–1286) made on the basis of A and another lost Byzantine manuscript (Heiberg’s B), and a third Byzantine manuscript (Heiberg’s C) that was made into the famous palimpsest in the 12th century. Neither the Arabic nor the pre-Moerbeke Latin

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7 See D. Gutas, Greek Thought, Arabic Culture (London: Routledge, 1999).
8 There is no comprehensive treatment of the sources of Greek mathematics. For the Greek manuscript tradition, see F. Acerbi, Il silenzio delle sirene (Rome: Carocci, 2010), pp. 269–375. For the Arabic tradition, see F. Sezgin, Geschichte des Arabischen Schriftums, Bände V–VII (Leiden: Brill, 1974–1979).
10 The manuscript copies are Laurent. 28.4, Marc. gr. 305, Paris gr. 2360 and 2361, Vat. gr. Pii II nr. 16, Monac. 492.
11 Moerbeke’s autograph is Ottob. lat. 1850. See M. Clagett, Archimedes in the Middle Ages, Volume II: Translations from the Greek by William of Moerbeke, 2 parts (Philadelphia: The American Philosophical Society, 1976).
traditions are crucial in our assessment of Archimedes’ writings. Although the Arabic tradition is important for some of the minor works,\textsuperscript{13} it appears that \textit{On the Sphere and Cylinder} was the only one of Archimedes substantial treatises that was translated into Arabic. Hence, our knowledge of Archimedes is based on three, presumably, independent Byzantine manuscripts that were probably produced as part of the Byzantine revival of the 9\textsuperscript{th} century and one or two other Greek manuscripts that were in Baghdad around this same time. In fact, compared to other major Greek mathematical sources, such as the works of Apollonius (c. late-3\textsuperscript{rd} BCE) or Pappus (c. early-4\textsuperscript{th} CE), this is a fairly rich basis. One thing that we notice immediately, however, is that a number of treatises – including those on which Eutocius (c. early-6\textsuperscript{th} CE) wrote commentaries – are written in Koine, whereas other treatises are partially written in Archimedes’ native Doric.\textsuperscript{14} Since Eutocius himself, and others in his milieu, edited the works they studied, we may presume that these changes in dialect were introduced by such editorial work. We cannot now know what other changes were introduced in this process. Nevertheless, it is clear that late ancient and medieval editors felt that they were fully justified in making fairly extensive changes without comment. We cannot be certain that the texts were not already edited before the late ancient period and we also do not know what changes were introduced around the 9\textsuperscript{th} century in Constantinople when the three Byzantine manuscripts for which we now have any direct evidence were produced.

The next example that we will look at is that of Apollonius.\textsuperscript{15} We know the Greek version of Apollonius’ \textit{Conics} through a single Byzantine manuscript,\textsuperscript{16} of which all other extant manuscripts are copies.\textsuperscript{17} What we find in this manuscript, however, is not an original work by Apollonius, but an edition of the first four books of the original eight made by Eutocius, over six centuries later, as part of his project to expound classical works of advanced Hellenistic geometry. A second important Greek source for Apollonius’ activity comes from another single Byzantine manuscript containing Pappus’ \textit{Collection},\textsuperscript{18} a loose grouping of writings on various mathematical topics. From this text we learn about aspects of Apollonius’ work for which we would not otherwise have any evidence, such as his interest in systems of large numbers, or his adherence to Euclid’s organization of geometry into those fields that can be han-

\textsuperscript{13} For example, Knorr has argued at length for the importance of the medieval tradition of \textit{Dimension of the Circle}. See, W. Knorr, \textit{Textual Studies in Ancient and Medieval Geometry} (Boston: Birkhäuser, 1989), pp. 375–816.

\textsuperscript{14} The question of Archimedes’ dialect is made difficult by the fact that much of the Doric in the received text was produced by the editor J.L. Heiberg, in response to the fact that the manuscripts contain a strange mixture of Koine and Doric. See J.L. Heiberg, \textit{Archimedes}, vol. 2, pp. x–xviii (note ??). See also, R. Netz, “Archimedes’ writings: through Heiberg’s veil,” in K. Chemla, ed., \textit{The History of Mathematical Proof in Ancient Traditions}, pp. 188–191.


\textsuperscript{16} Vat. gr. 206.


\textsuperscript{18} The manuscript containing Pappus’ \textit{Collection} is Vat. gr. 218.
dled with elementary constructions (straight edge and compass), with conic sections (parabola, hyperbola and ellipse) and those that require more involved curves (spirals, loci and so forth).

For our understanding of Apollonius’ mathematics, however, the Arabic tradition is as important a source as the Greek. In the 9th century, the group of scholars around the Banū Mūsā acquired a copy of the Conics in a version which had not been modified by Eutocius, but from which the eighth book had already gone missing. Through the mathematical work of al-Ḥasan ibn Mūsā (mid-9th CE), the chance discovery of a copy of the Eutocius version in Damascus, and the philological and mathematical expertise of Thābit ibn Qurra (c. 830–901) and others, an Arabic version of the seven remaining books was eventually completed. When we compare this version with the Greek, there are a number of differences but it is not clear which one is closer to whatever Apollonius wrote. Indeed, we no longer possess the Conics that Apollonius wrote. We have the descendent of an edition made by Eutocius, in Greek, and another of that made by the scholars in the circle of the Banū Mūsā, in Arabic. The Arabic tradition has also preserved On Cutting off a Ratio, a text in what Pappus calls the “field of analysis,” otherwise only known from a discussion in Pappus’ Collection VII. Hence, in order to try to evaluate Apollonius’ mathematics, it is necessary to read a variety of texts, none of which he actually wrote, and some of which are not even translations or summaries of his work.

As these two examples serve to show, the significance of the manuscript tradition for interpreting the received text has to be evaluated independently in each case. Nevertheless, it is clear that the texts we are working with have been modified over the centuries. This is even more pronounced in the case of the texts that were more often read, such as the Elements or the treatises of the so-called Little Astronomy. The Greek text of the Elements is preserved in two main versions, that in most of the manuscripts is called “the edition of Theon” although there is some disagreement among the principle sources, while another, possibly non-Theonian version is extent in one manuscript. At the end of the 9th century, there were at least three Arabic versions – a translation by al-Ḥajjāj (c. late-8th–early-9th) with a revision, as well as a translation by Ishaq ibn ʿ Hunayn (830–910) that was revised by Thābit ibn Qurra – of which only Thābit’s correction remains, but not without substantial incorporation of the older versions. All of the various Arabic texts, however, are different, in
places, from the Greek, and it is not clear that the Greek versions have not undergone as much change since the Baghdad translations were made as the Arabic versions. This means that the Arabic versions should also be used to assess the original source, but this is made difficult by the numerous variants in the Arabic tradition and the fact that only parts of the text have been published.

A similarly complicated assortment of variants can be found in the sources for the group of texts known as the Little Astronomy, in the late ancient period, or the Middle Books, during the medieval period. By the late ancient period, these texts were grouped together by teachers like Pappus and described as the texts to mastered between Euclid’s Elements and Ptolemy’s Almagest. Hence, these treatises, like the Elements, were often studied, and thus often edited. For example, there are two substantially different Greek versions of Euclid’s Optics and Phenomena, while there are at least three early Arabic versions of the Spheres by Theodosius (c. early-2nd–mid 1st) and at least two of On the Sizes and Distances of the Sun and the Moon by Aristarchus (c. early-3rd BCE). Once again, there are differences between the Greek and Arabic traditions, such as extra propositions in the Arabic On the Sizes and Distances or in the Greek Spheres. Moreover, it is often difficult to decide, in any objective way, which variant should be ascribed to the older source.

Because the texts of Greek mathematics have been subjected to repeated editorial work, we must regard them as historically contingent objects, in some ways created by the process of transmission itself. The texts, as we find them, are the products of a literary culture, produced by literary practices and made for literary consumption. Nevertheless, the mathematics that they contain was originally produced in a context of activity, now mostly lost to us, of which the production and consumption of literary texts formed only a part.

**THE SOCIAL CONTEXT**

Although the scanty nature of our evidence makes it difficult to describe in detail the social circumstances in which Greek mathematicians worked, we can, nevertheless, paint a picture in broad strokes. In order to assess the social setting, we read discussions of mathematics and mathematicians in literary authors and philosophers.

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make inferences based on the few places where Greek mathematicians make personal comments in their writings, and make some guesses on the role of higher mathematics in education based on the writings of mathematicians and philosophers. It should be obvious that this discussion will be speculative.30

It is clear that, in Greco-Roman culture, in distinction from professionals who used mathematics in their work, theoretical mathematicians did not form a professional group that had been educated in a standardized way and earned their living through developing and teaching their mathematical skills, although some mathematicians apparently did earn a living through teaching and writing mathematics. Rather, we find a broad array of backgrounds: Archytas (c. mid-5th–mid-4th) was a statesman and a general,31 Hippocrates (c. mid-5th BCE) was a wealthy merchant,32 Eudoxus (c. mid-4th BCE) was a respected legislator and a philosopher with many students,33 Eratosthenes (c. late-3rd BCE) was the head of the library of Alexandria,34 Archimedes was associated with the royal court of Syracuse,35 Hypatia (c. late-4th–early-5th) was the daughter of a mathematical scholar and herself taught philosophy to youths of the Alexandrian social and political elite.36 The one thing that these individuals all share is that they had privileged lives and were participants in the type of high culture that revolved around literary and philosophical pursuits. Indeed, the esteem accorded to theoretical mathematics by most philosophers and the appearance of mathematicians in dialogs and other literary works, indicates that mathematicians had a respected place in this high culture, even if they had little or no institutionalized place in society.

Although it is uncertain what role theoretical mathematics had in education, it seems that this role increased throughout the ancient period. In the Classical and early Hellenistic periods, most schools of higher education were centered on a specific philosophical tradition and mathematics education would have depended on the importance of mathematics within the school’s thought. Of the mathematical curriculum of the most famous school of antiquity, Plato’s Academy, we know almost nothing.37 Whereas one could probably study mathematics, and even the history of mathematics at the Lyceum, it is unlikely that much, if any, mathematics was taught at Epicurus’ Garden, despite the close relationship between the thought of Epicurus (341–270) and that of Democritus (c. mid-5th–mid-4th). Throughout the Hellenistic and Imperial periods, the teaching of higher mathematics appears to have become more established, at least in certain times and places. We are told by Pappus, for example, that Apollonius studied under the pupils of Euclid at Alexandria,38 Apollo-
nius says that Conics V will be useful for the “student” of analysis, and he appears to have organized whole treatises to be of use in teaching techniques in geometrical analysis. The fact that some people did study treatises of theoretical mathematics is made evident by two papyri from the Imperial period that contain enunciations and unlettered diagrams from Euclid’s Elements, P. Oxy. I 29 (Elements II 4 & c. 5) and P. Berol. 17469 (Elements I 8–10). These appear to have been lists of enunciations and symbolic diagrams compiled so as to be of use for study and memorization. By the late ancient period, teachers, like Pappus and Theon (c. mid-4th CE) of Alexandria, or Eutocius of Ascalon, were organizing treatises into canons for study, producing new editions of the classics, writing commentaries on important works and producing text-based studies of specialized fields. Nevertheless, although it is clear that mathematics was occasionally taught, it is not clear if there were any general patterns to the teacher-student relationship. Although in some schools, most students may have listened to lectures on elementary geometry and spherics, it is unlikely that many progressed on to more advanced topics, such as geometrical analysis or arithmetics. Most working mathematicians would have been lucky to have one or two advanced students in their lifetime, and in many cases, the “students” referred to in our texts may have been rhetorical students that the author hoped posterity would furnish.

Throughout the course of the ancient period, it seems likely that Greek mathematicians generally worked alone and not in research groups or schools. Of course, there are some clear exceptions to this. In Athens, during the Classical period there were small groups of mathematicians who worked together, or at least on the same set of problems. Some of these, such as Eudoxus, then returned to their homes along the Eastern Mediterranean and founded schools of mathematical and philosophical instruction. During this period, Athens was the main center of mathematical activity, but there were also peripheral nexuses, of which a striking example was the group at Cyzicus. Another exception is that of Alexandria during the Hellenistic and Imperial periods. Starting from the time of Euclid (c. early 3rd BCE), there were almost continuously a few mathematicians working in Alexandria, perhaps associated with the Museum and Library. Archimedes publicized most of his work by sending it to mathematicians working in Alexandria, but it does not seem that there were more than two or three working there at any time for whom he had much respect. As mentioned, Apollonius was said to have studied with Euclid’s students in Alexandria, one of whom may have been Archimedes’ correspondent Conon (c. mid-3rd BCE) who we are told, by Apollonius, had a debate with a certain Nicoteles about the use-
fulness of the content of *Conics IV for dionysis*. Whether this debate took place in Alexandria or by correspondence is unclear. We also learn from Apollonius’ introductions that small groups of geometers occasionally met in other places to discuss mathematics. Hypsicles (c. late-2nd BCE) tells us that when a certain Basilides of Tyre was in Alexandria, he and Hypsicles’ father spent much of their time discussing a mathematical work by Apollonius. Thus, in some cases, Athens and then Alexandria acted as centers that attracted talented mathematicians from the peripheries, while in other cases, they were cultural nexuses where people studied and disseminated the works of important mathematicians, such as Eudoxus and Archimedes, who chose to live in peripheral locations.

Our general impression is that mathematics was practiced by a small group of individuals from the privileged classes. Although this activity was never popular, it was respected, due to the special interest that most philosophers held in mathematical knowledge. As neopythagorean and neoplatonic thought became more established in the Imperial and Late Ancient periods, the special relationship between the mathematical sciences and philosophy also became more institutionalized.

**PRACTICES AND METHODS**

Although almost none of the surviving documents tell us how Greek mathematicians actually taught and produced mathematics, we can make some conjectures about this based on what the sources do say, and the types of mathematics that are preserved. In the following, we will examine three primary nexuses of mathematical activity: oral practices, material practices and literary practices. In our sources, we can perceive a gradual transition from a more oral tradition, based around public arguments made about diagrams and instruments, to a more literary tradition that involved reading and writing texts containing elaborate arguments, tables, and special symbols that would have been difficult for anyone to follow without engaging the written works as material objects.

Of the formative, primarily oral, period of Greek mathematics, we know very little. It is now generally accepted that the Greeks produced little or no deductive mathematics before the mid-5th century, when Hippocrates was active. It was also around this time that Greek mathematicians began writing down their results. Nevertheless, it is clear that the practice of mathematics at this time was still highly oral. John Philoponus (c. mid-6th CE) tells us that Hippocrates learned mathematics during his time in Athens, by associating with philosophers, while he was waiting for the resolu-

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46 See J.L. Heiberg, *Apollonius*, vol. 2, p. 4 (note ??). Chronologically, however, Apollonius could not have studied.

47 See J.L. Heiberg, *Apollonius*, vol. 1, p. 192 (note ??).


49 We have no reliable source containing a text of deductive mathematics prior to that attributed to Hippocrates by Simplicius in his *Commentary on Aristotle’s Physics A 2* (185a 14). Netz makes the argument that Hippocrates was one of the first to write down mathematical texts. See, R. Neta, “Eudemus of Rhodes, Hippocrates of Chios and the Earliest Form of a Greek Mathematical Text,” *Centaurus*, 46 (2004), 243–286.
tion of his lawsuit against certain pirates who had plundered his cargo. From Plato's writings, we have the images of Socrates teaching Meno's slave boy mathematics by discussing diagrams in a public square, and Theaetetus (c. early 4th BCE) and Socrates working through a question pertaining to commensurability, which was presumably meant to be reminiscent of the way Theaetetus studied mathematics under Theodorus (c. late-5th BCE). When we reflect on the fact that deductive mathematics arose during the period of the sophists, when Greek, and particularly Athenian culture, put a premium on the ability to convince others of one's position in public forums, it is clear that mathematical practice also originally involved the oral presentation of arguments in public spaces.

Moreover, throughout the ancient period, the most common institutional location for mathematical activities was in schools that were predominantly devoted to teaching philosophy and rhetoric. Since there were no schools of higher mathematics, we must assume that the bulk of the higher education of mathematicians, like other intellectuals, took place in schools of philosophy, where they studied the skills of winning others to their position through oral disputation and rational argument. We still find considerable evidence for such oral practices in the elementary texts, such as Euclid's *Elements* or Theodosius' *Spheric*. The format of the propositions and the repetitive language lends itself to oral presentation and memorization, and the fact that earlier propositions are cited by a synopsis of the enunciation indicates that the listener was expected to memorize the enunciation.

As well as drawing diagrams and making arguments about them, Greek mathematicians engaged in a range of material practices involving specialized instruments, of which we now have only indirect evidence. It has long been recognized that the constructive methods of Euclid's *Elements* are a sort of abstraction of procedures that can actually be carried out with a straightedge and compass. More recently, it has been recognized that the constructions of Theodosius' *Spheric* are also meant to be applicable to actual globes. The construction of mechanical globes was brought to a high level by the most mechanical of all the ancient mathematicians: Archimedes. We are told that the Roman consul Marcus Claudius Marcellus brought back to Rome two devices built by Archimedes for modeling the heavens, and, according to Pappus, Archimedes wrote a book on *Sphere-Making*. Hence, as with oral practices, we find that the material practices have left their mark in the preserved texts.

In a number of places, mathematical authors explicitly describe the sorts of instruments that they used in the course of their research. Eutocius attributes to Plato, somewhat dubiously, a sort of mechanical sliding square, which could be used to find

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50 Philoponus, *Commentary on Aristotle’s Physics* A 2 (185a 16).
51 Plato, *Meno* 82a–85b, and *Theaetetus* 147d–148b.
52 Netz discusses the formulaic nature of Greek mathematical prose in R. Netz, *The Shaping of Deduction in Greek Mathematics* (Cambridge: Cambridge University Press, 1999), pp. 127–167. Although Netz focuses on the cognitive roles of formulae, it is also clear that they would have facilitated memorization and oral presentation.
53 This is also supported by the format of two of the papyri containing material from the *Elements*: *P. Oxy.* I 29 (*Elements* II 4 & 5) and *P. Berol.* 17469 (*Elements* I 8–10). See note 52.
56 Pappus, *Collection* VIII 3.
two mean proportionals between two given lines. Diocles (c. early-2nd BCE), in On Burning Mirrors, describes how we can use a flexible ruler, made of horn, to draw an accurate parabola. Nicomedes (c. mid-3rd BCE) is said to have built a mechanical device for inserting a line of a given length between two given objects, known as a neusis construction. These, and many other passages make it clear that Greek mathematicians were engaged in a range of material practices that involved the accurate reproduction of the objects that they studied.

Interest in the mathematizable properties of instruments is evidenced from texts like Pappus’ Collection VIII, which shows how to carry out geometric constructions with a straightedge and a compass set at a fixed opening. Moreover, a number of fields of applied, or mixed, mathematics were based around the set of operations that could be carried out with specific instruments. Ancient gnomonic, the study of sundials, was based on constructions that could be carried out with a set-square and a compass. The methods developed for projecting the objects on a sphere onto a plane were closely related to the practices involved in drawing star maps in the plane. Finally, the analemma methods of spherical astronomy involved the use of analog calculations that were carried out by performing physical manipulations on a prepared plate, and in some cases a hemisphere.

Whereas these activities were mostly employed in research and teaching, there were also material practices that involved the production and use of literary texts. One important area of this activity involved the production of literary diagrams. Whereas we have descriptive evidence that Greek mathematicians were concerned with the visual accuracy of their drawings, the figures that we find preserved in our manuscript sources are so far from such accuracy that it seems there must have been special principles operative in the production of these literary diagrams. In the manuscript sources, we find, for example, a square representing any rectangle, a regular pentagon representing any polygon, circular arcs representing conic sections, straight lines representing curved lines, curved lines representing straight lines, and so forth.

The two most consistent features of these diagrams are the use of an unnecessarily regular object to represent a general class of objects, and a basic disregard for visual accuracy in favor of the representation of key mathematical relationships. It seems that ancient authors developed a schematic type of diagram which would be easy to copy and which could be used, in conjunction with the text, to produce a more accurate diagram when the need arose. The literary diagram was an object of communication that served to mediate between the readers and the mathematical objects.

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58 Diocles, On Burning Mirrors, Prop. 4.
63 The best evidence for this is Ptolemy’s Analemma and Heron’s Dioptra 35. See also N. Sidoli, “Heron’s Dioptra 35 and analemma methods: An astronomical determination of the distance between two cities.” Centaurus, 47 (2005), 236–258.
under investigation. As Greek mathematics became literary, the diagram secured a central place in the production of mathematical texts so that we find diagrams even in places like Elements VII–IX, on number theory, where they do not convey essential information and appear merely as a trope.

All of our preserved texts, however, come from this literary period, and hence we see little change in the use of diagrams in our sources. This can be contrasted with the use of tables. Whereas it would be possible to follow the mathematical details in Euclid’s Elements in an oral presentation, in order to verify even a simple calculation in Ptolemy’s Almagest one needs to have access to a copy of the chord table. While the proto-trigonometry of Aristarchus’ On the Sizes and Distances of the Sun and Moon could be followed in detail with just a working knowledge of geometry, the accurate trigonometry developed by Hipparchus (c. mid-2nd BCE) and others in the late Hellenistic period was a literary practice involving the consultation and manipulation of written sources.

The literary practices of Greek mathematicians naturally extended to the literary production of the texts themselves. Greek mathematicians were members of a small group of individuals in Greco-Roman society who produced works of high literature and they took pains to secure this social position. It has been argued that there are parallels between both the language and the structure of Greek mathematical works and other types of literary production. It is also clear that Greek mathematicians, like other ancient intellectuals, engaged in various editorial and pedagogical projects to revise their works and to make them more accessible to students and less specialized readers.

This was true not only for the structure and overall presentation of their works, but for the language itself. Although individual authors had their own personal style, Greek mathematicians developed a distinctive mathematical style that can be recognized in all mathematical texts. This style becomes especially conspicuous when we see it mishandled by a non–mathematician, such as in an argument by Theon of Smyrna (c. early-2nd CE) that if the parameters of Hipparchus’ solar model are given, the position of the sun is determined. It is not clear to what extent the homogeneity of mathematical style was due to the attention of the original authors or to the care of their later editors, nevertheless, already by the middle of the Hellenistic period the production of mathematics had become a fully literary activity, closely involved with the careful study of written books. This must have formed yet another barrier to entry into the small group of individuals who produced original mathematics. While a fair number of people may have studied the mathematical sciences by attending lectures and sessions at various schools, only a small number of these could have advanced to the study of written mathematical texts, either by being wealthy enough to buy their

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66 Some of the editorial practices of the late ancient period are discussed by A. Cameron, “Isidore of Miletus and Hypatia: On the Editing of Mathematical Texts,” Greek, Roman and Byzantine Studies, 31 (1990), pp. 103–127. For a treatment of the pedagogical purposes of Greek mathematical authors see J. Mansfeld, Prolegomena Mathematica From Apollonius of Perga to the Late Neoplatonists (Leiden: Brill, 1998).

own books or by being bright enough to receive their teacher’s permission to use the school’s books.

CONSTRUCTIONS

One of the most distinctive methodological features of Greek mathematics is the use of constructions. Construction, or construction-based thinking, is found not only in geometry but also in number theory, arithmetics, the exact sciences and in general investigations of what, and how, mathematical objects are given.

Constructive techniques are conspicuous in elementary geometry. There has recently been much attention paid to the role of diagrams in Greek mathematical thought, however, it is only through the mediating process of construction that the diagram has any deductive force. Constructions are used in very nearly every proposition in order to introduce new objects whose properties are then used as starting points in chains of deductive inference. In the Elements, construction postulates are introduced to justify the construction procedures that are used in the propositions that the ancients called problems, but not necessarily those used in what they called theorems. In other geometrical texts a variety of constructive processes are used that are never explicitly postulated, such as setting a line of a given length between two objects or passing a plane through a solid object. More elaborate construction procedures are set out and then justified in problems. Problem solving, which was a key area of geometric activity, always involved the construction of some specific object.

In geometric texts, many different verbs were used to denote various types of constructions, depending on what the geometer intended to do. Lines could be produced (agō) between two given points, circles drawn (graphō) with a given center and distance, solid objects cut (temnō) by a passing plane, diameters in spheres set out (ektithēmi) on plane surfaces, parallelograms erected (sunistēmi) on given lines, and so forth. In the enunciation of problems, these operations were stated in the infinitive, whereas in the construction of either type of proposition they were stated in the perfect imperative passive. Despite the fact that the construction is explicitly stated as complete, it is clear that it represents the most active part of mathematical practice. Moreover, the construction is often the most creative part of a mathematical argument, since it introduces new objects into the domain of discourse, which are entirely at the mathematician’s discretion.

Although construction is generally associated with geometry, constructivist thinking permeated other branches of Greek mathematics as well. All of the problems in Euclid’s number theory, Elements VII–IX, show us how to find (euriskō) numbers,

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which are hence assumed to exist from the beginning.\footnote{See I. Mueller, Philosophy of Mathematics and Deductive Structure in Euclid’s Elements (Cambridge, MA: MIT Press, 1981), p. 60.} The problems in Euclid’s number theory, however, are the active component of the theory and provide the algorithms that are used in the rest of the theory. In geometric texts, as well, objects that are assumed to exist are sometimes found by construction, such as the center of a circle, in Euclid’s Elements III 1, or of sphere, in Theodosius’ Spherics I 2, so that these problems are analogous to the algorithms in the number theory. Furthermore, in works of more advanced arithmetics, such as the Arithmetics of Diophantus (c. 3rd CE), a number of different constructive operations are invoked such as to find (euriskō), to separate (diaireō), to add (prostithēmi) and to make (poieō). Although these expressions assume the existence of the numbers involved, because rational numbers are expected, limits to the possibility of solution must sometimes be invoked. One of Diophantus’ problems, such as Arithmetics II 8, “To separate a proposed square into two squares,”\footnote{P. Tannery, Diophantus Alexandrini Opera Omnia (Leipzig: Teubner, 1893), vol. I, p. 90.} can later function as an algorithm used in further problems, in much the same way as a problem in the geometric texts. One difference is that for Diophantus the constructive procedures are themselves joined together making a series of conditions for the solution to the problem, such as Arithmetics I 7, “To take away two given numbers from the same number and to make the remainders have a given ratio to one another,”\footnote{P. Tannery, Diophantus, vol. I, p. 24 (note 3).} whereas in the geometric texts only one construction is stipulated and all the conditions are expressed as modifications of the objects, such as Theodosius’ Spherics II 15, “If a lesser circle in a sphere and some point on the surface of the sphere, which is between it and the circle equal and parallel to it, be given, to draw through the point a great circle tangent to the given circle.”\footnote{C. Czinczenheim, Édition, traduction et commentaire des Sphériques de Théodore, These, Paris IV, 2000, p. 102.} This difference occurs both at the grammatical level, and also in terms of the procedures of solution. In geometric problems, there is only one verb, in the infinitive, and a single geometric object is constructed satisfying all of the conditions, which are stated as modifications of the nouns. In Diophantus, the conditions are stated as a series of verbs and they are satisfied individually as the problem proceeds.\footnote{For Arithmetics I 7, the two conditions are rather simple, but they are still handled sequentially. For more involved problems the conditions are solved individually. For example, in Arithmetics III 1, after satisfying two of the conditions, Diophantus says “two of the conditions (epigmata) are now solved.” See P. Tannery, Diophantus, p. 138 (note 3).}

The notion of construction was so fundamental for Greek mathematicians that they developed a sort of conceptual framework to handle constructive processes as a theory of givens, formalized in Euclid’s Data.\footnote{See C. M. Taisbak, Euclid’s Data (Copenhagen: Museum Tusculanum Press, 2003).} Given objects are those that are found at the start of the discourse or can be deduced from these, or objects that are constructed at the discretion of the mathematician. Given objects exist in a definite and usually unique way and their properties are known and manageable.\footnote{See F. Acerbi, “The Language of the “Givens”: Its Form and its Use as a Deductive Tool in Greek Mathematics,” Archive for History of the Exact Sciences, 65 (2011), 119–153.} The neoplatonic commentator Marinus of Neapolis (late-5th BCE) reports a number of definitions of
the concept of given, which he attributes to various mathematicians. After discussing various ways that we can understand the notion of given, Marinus settles on the concepts of known and provided (porimon), claiming that what is provided is that which we are able to make or to construct, for example drawing a circle or finding three expressible lines that are only commensurable in square. This agrees with Euclid’s definition of given in the Data: definition 1 reads, “Given is said of figures, lines and angles of which we are able to provide an equal.” The notion of provision is an attempt to formalize the productive processes through which the mathematician gained mastery of the subject. In the Data, the notion of given, and hence of provision, is slowly expanded to include computations and deductive inferences, and in later authors, such as Heron and Ptolemy, we find chains of givens being used to justify computational techniques.

In these ways, construction fulfilled a number of important roles for Greek mathematicians. On a practical level, construction formalized and abstracted various active procedures that were necessary in actually doing mathematics. On a more theoretical level, it allowed mathematicians to introduce new objects whose properties could then be used to prove theorems or solve problems. On a fundamental level, it provided concrete instantiations of specific objects to be provided for mathematical discourse. Since all Greek mathematics was functionally based on specific objects, constructions performed an essential function.

OPERATIONS

Although in a general sense we can regard constructions as operations, in this section we focus on those operations that can be performed on a statement, expression, or number. While there is little operational mathematics in elementary geometrical treatises, such as the early books of the Elements, or Theodosius’ Spheric, as soon as we begin to read higher geometry, number theory, arithmetics, or the exact sciences, we encounter long passages of deductive reasoning in the form of chains of mathematical operations.

From both a theoretical and practical perspective, Greek mathematicians privileged operations on ratios and proportions over the arithmetic operations. A theoretical justification for many of the common ratio manipulations that were in practice was provided by Elements V, thought to have been formulated by Eudoxus. Almost all of the theorems of the second half of this book deal with manipulations that can be carried out on proportions. For example, the operation of separation, justified in Elements V 17, is going from a proportion of the form \( a : b :: c : d \) to one of the form \( a - b : b :: c - d : d \), where \( a > b \). The operation of combination, justified in Elements V 18, is the converse. Although in the Elements these operations are only justified for proportions, Greek mathematicians also applied them to ratio equalities and, occasionally, equations. This gives the impression that Greek mathematicians

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79 H. Menge, Euclid, vol. 6, pp. 250, 240 (note ?).
80 H. Menge, Euclid, vol. 6, p. 2 (note ?).
81 See, for example, T. Heath, Aristarchus of Samos (Oxford: Claredon Press, 1913), and J.L. Heiberg, Archimedes, vol. 2 pp. 216–258 (note ?).
sharply distinguished between proportions and equations, and there is some truth to
this. Equations were taken to be statements about different things that were equal
in quantity, whereas proportions were claims that two ratios were the same. Nev-
evertheless, despite this distinction, Greek mathematicians were aware that equations
and proportions could be interchanged, and occasionally subjected equations to ratio
manipulations, or proportions to arithmetic operations. Of course, all of the ratio
manipulations could be rewritten as arithmetic operations, but Greek geometers
apparently had no interest in doing this. In fact, even in places where one might expect
to find only arithmetic operations, such as in the calculation of the size of a length or
an angle, we still encounter the use of ratio manipulations.

Greek mathematicians, of course, could also perform arithmetic operations, how-
ever, they did not spend much effort attempting to formalize or justify these oper-
ations. More difficult operations, such as taking roots, were not explicitly discussed
in much detail before the late ancient period. Arithmetic operations were per-
formed on individual terms, whole proportions and ratio inequalities, and equations.
The three ancient and medieval algebraic operations were probably regarded as special
cases of such arithmetic operations, applicable under certain specified conditions. In
the introduction to his Arithmetics, Diophantus describes the two primary algebraic
operations that can be performed on an equation to solve for an unknown number.
He says that if "a kind becomes equal to the same kind but not of the same quantity,
it is necessary to take away the similar from the similar on each of the sides, in order
that that a kind should be equal to a kind." In other words, if we have an equation in
which numbers, some number of unknowns, or higher terms are found on both sides
of the equation, we must subtract the common term from both sides, so as to bring
it to the other side, as we would say. The second operation is "to add a kind missing
from either of the sides, in order that an extent kind will come to be for each of the
sides." That is, as we would say, we make all our terms positive. These operations
may be repeated as necessary until only one of a number, some number of unknowns,
or higher terms are found on each side of the equation. The third operation is not
stated until it is needed at the beginning of the Arabic Book IV, which follows the
Greek Book III. The text says that, if after the other two operations have been per-
formed we have a statement equating unknowns of higher degree, then, "we divide
the whole by a unit of the lesser in degree of the two sides, until there results for us
one kind equal to a number." In other words, we reduce the equation to the lowest
degree possible. There is no attempt in the text to formally relate these operations
to operations of arithmetic or to extend to further operations of this type on analogy.

See, for examples, Aristarchus’ On the Sizes and Distances of the Sun and Moon 4, Apollonius’ Conics
III 6, or Ptolemy’ Almagest I 10.

The extraction of square roots are described by a scholiast to Elements X, and Theon in his Com-
mentary to the Almagest. Heron gives an example of taking a cube root, but does not give his method in
detail. See T. Heath, A History of Greek Mathematics (Cambridge: Cambridge University Press, 1921),
pp. 60–63.

In Diophantus’ terminology a kind (eidos) is mathematically related to what we would call a term
of a polynomial, although he does not appear to have conceived of a polynomial as a series of terms
combined by operations.

P. Tannery, Diophantus, p. 14 (note 2?).

P. Tannery, Diophantus, p. 14 (note 2?).

with the other arithmetic operations, such as multiplication or division. Hence, these operations appear to be isolated as an operation for eliminating missing terms, an operation for grouping like terms on one side of the equation, and an operation for reducing certain equations of higher degree.

Series of operations were also grouped into algorithms. In *Elements* VII there are a number of problems that involve algorithms, for example *Elements* VII 2, to find the greatest common measure of two numbers, and VII 34, to find the least common multiple of two numbers. The only actual operations involved in these problems, however, are arithmetical and they are not postulated, but simply assumed as obvious. Following a presentation of the algorithm, there is a proof that the algorithm accomplishes its goal. In other authors, we find various types of algorithms that involve a series of operations and are often unjustified, or justified by a numerical example. For example, Heron, in *Measuring* I 8, gives a general algorithm for finding the area of a triangle given its sides, which is followed not by a proof, but a worked example. An interesting type of algorithm is found in trigonometric texts, such as Ptolemy’s *Analemma* or *Almagest*. Here we find general statements of the algorithm using the *givens* terminology, where the steps of the calculation involve arithmetical operations, ratio manipulations, use of a chord table and so forth. Although, worked examples are often given there is no general attempt to justify the procedure. Hence, although algorithms were functionally recognized, there was little attempt to treat them formally.

**STRUCTURES**

A striking feature of Greek mathematical texts is their organization. Like other literary texts, Greek mathematical works were divided into books. The books often varied in length, which depended on the mathematical content they developed. In the case of the more elementary texts, such as the *Elements* and the treatises of the *Little Astronomy*, which would have often been used in schools, these books began immediately with mathematical content. More advanced works, however, often began with an introduction, for example in the form of an epistle to a colleague or student, in the Hellenistic period, or, in the Imperial period, more commonly a short address to a student or patron. These epistles provide introductory material meant to be useful for understanding the goal of the theory developed in the text and the tools used to develop it. The mathematics itself is then divided up into clear sections: an introduction, which often includes definitions or axioms, followed by various units of text. In the medieval manuscripts of most ancient mathematical texts these units are numbered as propositions, however, even in unnumbered texts, such as Ptolemy’s *Planisphere*, the sections are clearly distinguished. Propositions, as a type of textual unit, are then grouped together into theories, which are only differentiated on the basis of mathematical content. For example, *Elements* I begins with a theory of triangles and their congruency, followed by a theory of parallelism and a theory of area, which are then used to prove the so-called Pythagorean theorem. The interweaving of, sometimes, obscure individual units to form an overall theory produced an element of

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narrative that Greek mathematicians had great skill in exploiting.\textsuperscript{89}

The most common types of unit are the two types of propositions that we have already discussed: theorems and problems. These are what we find taking up the majority of theoretical treatises. There are, however, other types of units, such as analyzed propositions, computations, tables, algorithms, and descriptions. Not all of these types of texts are found in all works and some of them, such as tables, are rarely found outside the exact sciences. For example, a description is a discussion of a mathematical figure or model that explains the properties of the objects but contains little or no argument. While these are common in the exact sciences they are rare in pure mathematics – an exception is Archimedes’ \textit{Sphere and Cylinder} I 2.\textsuperscript{90}

It was recognized already by Proclus (c. 5\textsuperscript{th} CE) that a Euclidean proposition is carefully structured.\textsuperscript{91} He names the following parts: \textit{enunciation} (protasis), \textit{exposition} (ektthesis), \textit{specification} (diorismos), \textit{construction} (katakeue), \textit{demonstration} (apodeixis) and \textit{conclusion} (sumperasma). This division, however, is limited to the theorems and problems of \textit{Elements} I. In more involved problems, such as \textit{Elements} III 1, to find the center of a circle, or \textit{Spherics} II 15, to draw a great circle through a point tangent to a lesser circle, there are two further parts: first there is a specification of the problem followed by a construction that solves the problem, then there is a specification of the demonstration followed by a second construction for the sake of the demonstration. Moreover, other elements can also be recognized. M. Federspiel has noted that in many cases the beginning of the demonstration makes explicit reference to statements that are made possible by the construction, in a section he calls the \textit{anaphor}.\textsuperscript{92} Like the other parts of a proposition, the anaphora is flexible and in some case, as \textit{Elements} III 1, blends seamlessly into the demonstration.

The flexibility of these divisions must be emphasized. Outside of pedagogical texts such as the \textit{Elements} or Theodosius’ \textit{Spherics}, these parts did not seem to exercise much constraint and we find Archimedes and Apollonius mixing them up or omitting them all together. Nevertheless, the realization that Greek mathematical units are structured has led to useful insights and a number of scholars have put forward structures for various units. H. Hankle observed that there are four parts in a standard problematic analyzed proposition: \textit{transformation}, \textit{resolution}, \textit{construction} and \textit{demonstration}.\textsuperscript{93} K. Saito and N. Sidoli have added \textit{diorism} to these four and noted that in the case of theoretic analyzed propositions the division is somewhat different: \textit{construction}, \textit{deduction}, \textit{verification}, \textit{inverse deduction}.\textsuperscript{94} In the case of Diophantus’ problems, we can again recognize various parts although they are not always clearly

\textsuperscript{89} For a discussion of narrative structure in Greek mathematics, see R. Netz, \textit{Ludic Proof} (Cambridge: Cambridge University Press, 2009), pp. 66–114.

\textsuperscript{90} This unit is unnumbered in the manuscripts.


distinguished: enunciation, instantiation, treatment, solution and test. Although all of these structures seem to have functioned more as guidelines than as dictates to good practice, and although their existence may owe much to later editorial efforts, they are still useful to us in reading and understanding Greek mathematics.

The next level of structure that we notice is in the arguments. The most simple argument type is direct. After the given objects have been stated and any necessary constructions performed, some claims can then be asserted about these objects. In simple cases, these claims can lead directly, through a chain of implications and operations, to the conclusion. More often, however, other starting points must be introduced: either through a new appeal to the construction, introducing a new construction, appealing to a previous theorem in the same work, or to theorems assumed to be known to the reader, known as the toolbox. References to previous theorems, and often to the toolbox, are often invoked by a brief summary of the enunciation of the theorem, or simply by a claim of the mathematical fact. Another common form is the indirect argument, in which a claim contrary to what the mathematician wishes to prove is assumed, followed by constructions and arguments leading to a contraction with the hypotheses or with previously established results. Although indirect arguments are very common in our sources, not all Greek mathematicians found them satisfactory; Menelaus (c. turn of 2nd CE) tells us in the introduction to his Spherics that he will avoid them. Another common argument structure is the proof by cases, which might be combined with indirect argument, as in the double-indirect argument, often used by Archimedes. Using this argument structure, one shows that two objects are equal by showing that one is neither greater nor less than the other. Occasionally, we find an argument structure that extends beyond a single unit of text. The most common case of this is found in simple converses. In order to prove that \( A \) holds if and only if \( B \), Greek mathematicians would first show that \( A \) implies \( B \) and then that \( B \) implies \( A \), in two separate units. Another example of this was a two-stage method of showing that four magnitudes are proportional, \( a : b :: c : d \), which involved first showing that the proportion holds when \( a \) and \( b \) are assumed to be commensurable, and then, in a second theorem using this and a double-indirect argument, showing that the proportion also holds when \( a \) and \( b \) are incommensurable.

This structure was used in Archimedes’ Equilibrium of Planes I 6 and 7, Theodosius’ Spherics II 9 and 10 and Pappus’ Collection V 12 and VI 7–9.

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98 Whether or not these units were originally separate theorems, or have been made so by later editors is an open question.

99 This two-step argument structure is discussed by W. Knorr, “Archimedes and the Pre-Euclidean Proportion Theory,” Archives internationals d’histoire des sciences, 28 (1978), 183–244, and H. Mendell, “Two Traces of Two-Step Eudoxan Proportion Theory in Aristotle,” Archive of History of Exact Sciences, (2007), 3–37. J.L. Beggren doubts that Equilibrium of Planes I 6 and 7 were due to Archimedes due to
The final level of structure is that of individual verbal expressions. Greek mathematicians, of course, developed a system of technical idioms to handle their discipline. As well as a nomenclature, which became fairly standardized over time, they developed formulaic expressions that allowed them to condense their texts and help their readers keep mindful of the mathematical objects themselves. Since Greek mathematics was still essentially rhetorical, the use of operations and constructions was facilitated by a highly abbreviated diction that relied on various features of the Greek language in order to function. Ancient Greek is a gendered, inflected language with a definite article which allowed Greek authors of all genres to condense their terminology through various types of ellipsis that would not be possible in a language like English, or Latin. In mathematics, particularly, these expressions needed to be regular in order to still be intelligible. This process led to the use of formulaic expressions involving particularities of the language such as prepositions, and number and gender agreement between definite articles and nouns. In this way, \( \text{hē hupo tōn ABG} \) (the between the ABGs) means \( \text{angle ABG} \), whereas \( \text{to hupo tōn ABG} \) (the between the ABGs) means \( \text{rectangle AB \times BG} \). These processes allowed Greek mathematician to make involved statements without unnecessary verbiage so as to focus on the objects and their relations. These expressions, however, cannot be literally translated into English. For example, Archimedes uses a sentence that would be literally translated as “the of the on the AH to the on the HB having gained the of the AB to the HB is the on the AH to the between the GHBs” to convey \( (AH^2 : HB^2) \times (AH : HB) :: AH^2 : (GH \times HB) \). It is clear, nevertheless, that these expressions could be used to express patterns of abstract thought, even in the absence of symbolism.

**OBSCURE ORIGINS**

The origins of Greek mathematics either in Greek philosophy or as an independent discipline used to be a favorite topic for historians of Greek mathematics. The difficulty is that we have little certain evidence about the details of these origins. From Pappus, in the beginning of the 4th century CE, we have a discussion of Theaetetus’ work on irrational lines in the 4th century BCE. From Proclus, in the 5th century, we have accounts of the mathematical work of Thales (c. early-6th BCE), Pythagoras (c. turn of 5th BCE) and a short history of Greek mathematics leading up to Euclid. From Eutocius, at the end of the 5th century CE, we have treatments of the contributions of Hippocrates and Archytas, in the 5th and 4th centuries BCE. From Simplicius,
in the 6th century CE, we have the most substantial fragment of this early work, a near quotation of Hippocrates’ writing on his treatment of lunes. One of our most important sources for this early period is the so-called catalog of geometers, which appears as a passage in Proclus’ *Commentary on Elements Book I*. Parts of this passage are thought to go back to a now lost *History of Astronomy* written by Aristotle’s pupil and colleague, Eudemus of Rhodes (c. late-4th BCE). Proclus, however, does not explicitly attribute this passage to Eudemus, in contrast to other borrowings, and it has clearly been modified by other authors over the years. Moreover, this passage does not provide much description of the mathematical activities of the individuals it names and, hence, serves us mostly as a relative chronology. The earlier evidence we have for the origins of Greek mathematics, such as in the writings of Plato and Aristotle, is often vague, not attributing work to individual mathematicians, or presenting incomplete arguments. It should be clear from this that our most expansive sources are far removed in time from the persons and events they recount and must be treated with due caution.

The images of Thales and Pythagoras writing detailed mathematical proofs now seem to be the stuff of legends. When we consider how late our sources for this activity are, the other writings that remain from this early period, by these thinkers or their contemporaries, and the tendency of Greek doxographers to produce rationalized histories by associating well-known results with well-known thinkers, it appears increasingly unlikely that the early presocratics or Pythagoreans produced writings containing deductive mathematics. Although Thales and Pythagoras may well have made various correct assertions about elementary geometry and some sort of argument for why these were true, the proofs attributed to him by Proclus are now generally thought to have been reconstructed by ancient authors. Although Pythagoras almost certainly had a profound interest in numbers, there is no early evidence that he or the early Pythagoreans produced deductive mathematics. The earliest Pythagoreans of whom we can develop a clear picture are Philolaus of Croton (c. turn of 4th BCE) and Archytas of Tarentum; and the mathematical work of Philolaus is rather meager, while Archytas is already a contemporary of Plato’s and his mathematics does not appear to be uniquely Pythagorean.

Although much of the early history of Greek mathematics was probably reconstructed by Eudemus and other authors on the basis of vague attributions and the guid-

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106 Proclus, in his *Commentary to Elements Book I*, assigns Elements I 15, 26 to Thales and Elements I 32 and 42 to the Pythagoreans.

107 W. Burkert argued that there is no early evidence for mathematical activity by Pythagoras himself and that there was nothing uniquely Pythagorean about the mathematics practiced by later members of the school. See, W. Burkert, *Lore and Science in Ancient Pythagoreanism* (Cambridge, MA: Harvard University Press, 1972). A case that the early Pythagoreans worked in deductive mathematics was remade by L. Zhmud, *Wissenschaft, Philosophie und Religion im frühen Pythagoreismus* (Berlin: Bücher, 1997), but has not gained much acceptance.

ing belief that mathematics must develop in a rational way, in the case of Hippocrates of Chios, Eudemus seems to have had some written sources. Indeed, Simplicus, in his *Commentary on Aristotle’s Physics*, tells us that Eudemus regarded Hippocrates as one of the earliest mathematicians.\(^\text{109}\) We are told that Hippocrates worked on the problem of doubling the cube and reduced this to the more general problem of constructing two mean proportionals between two given lines,\(^\text{110}\) and was the first to write up the principles of geometry in an *Elements*.\(^\text{111}\) Most importantly, however, we have a long fragment of his writing preserved through Eudemus’ *History of Geometry*. In this passage, he shows how to square three different types of lunes, in an attempt to use these figures to reduce the problem of squaring a circle to something more manageable.\(^\text{112}\) From these writings we learn that at this time there were certain problems that were considered worthy of solution and that Hippocrates approached these using a general strategy of reducing them to problems that were soluble by constructions made on lettered diagrams. From the details of his arguments, we see that the types of constructions allowed and the starting points of the argument, were still taken at the geometer’s discretion.

Another important development in the 5\(^{\text{th}}\) century was the discovery of incommensurability – that is, the realization that there are cases where two given magnitudes have no common measure.\(^\text{113}\) For example, the combination of Pythagorean interests in number theory and various geometric studies came together in deductive arguments that the diagonal of a square is incommensurable with its side.\(^\text{114}\) This, in turn, lead to considerable interest in incommensurability and excited the activity of some of the greatest mathematicians of the 5\(^{\text{th}}\) and 4\(^{\text{th}}\) centuries: Theodorus, Eudoxus and Theaetetus. What we think we know about the work of Theodorus is the product of repeated reconstructions by modern scholars based on a few tantalizing hints found mostly in Plato’s writings.\(^\text{115}\) In the case of Eudoxus and Theaetetus, however, some of their work is considered to have formed the substantial basis of *Elements* V and X respectively and hence to form the foundations of ratio theory and to provide the most complete surviving study of incommensurability.

In the 4\(^{\text{th}}\) century, there were a number of important mathematicians who were traditionally associated with Plato’s Academy. Although it is now clear that the image of Plato as an organizer of science is more a product of the scholars of the early Academy than a reflection of reality,\(^\text{116}\) Plato’s writings are full of mathematical references and claims about the benefits of a mathematical education. Nevertheless, there is no reason to believe that Plato had more influence on his mathematical colleagues than they had on him. On the contrary, it seems clear that Plato used the ideas that


\(^{110}\) For example, Eutocius, *Commentary on Archimedes’ Sphere and Cylinder*, As Eratosthenes. See J.L. Heiberg, *Archimedes*, vol. 3, p. 88 (note ??).

\(^{111}\) See xxx


\(^{113}\) Knorr dates this discovery to 430–410 BCE. See W. Knorr, *Evolution*, p. 40 (note ??).

\(^{114}\) See W. Knorr, *Evolution*, pp. 21–28 (note ??).

\(^{115}\) See W. Knorr, *Evolution*, pp. 62–130 (note ??).

he learned from Theodorus, Archytas and Theaetetus to develop his own philosophy.

Whatever the case, this period of Athenian cultural influence acted as catalyst for important mathematical work. Solutions to the problem of duplicating the cube, in Hippocrates' reduction to the problem of finding two mean proportionals, were put forward by Archytas, Eudoxus and Menelaus.\(^{117}\) Archytas produced theorems in number theory that may form the basis of *Elements* VIII and the Euclidean *Section of the Canon*.\(^{118}\) A theory of conic sections was developed by Eudoxus, Menelaus and others and applied to the solution of various problems. It is difficult to appraise how general or complete this theory was.

Eudoxus was one of the most brilliant mathematicians of this period. He developed a ratio theory that was closely related to the ratio theory we now read in *Elements* V, but which may have relied on a two-stage argument, showing first the case of commensurable magnitudes and then that of incommensurable magnitudes.\(^{119}\) Although this work has generally been read as an attempt to make a general theory of ratio, it is not a complete foundation for contemporary or later mathematical practice and hence can be interpreted as a collection of theorems useful for geometry.\(^{120}\) Eudoxus also wrote a number of theorems concerning the mensuration of objects that were praised by Archimedes and which probably involved double indirect arguments. In astronomy, he put forward the two-sphere model of the cosmos with a spherical earth in the center of a spherical firmament, and a model of homocentric spheres to account for some set of celestial phenomena, unfortunately the details of these works are a matter of speculation.\(^{121}\)

By the time Alexander began his conquests, Greek mathematics had undergone considerable development. Of the three great problems of antiquity – the quadrature of a circle, the duplication of a cube and the trisection of an angle – the first two had been clearly articulated and addressed and the duplication of the cube had been solved. Although the details of their content are matters of reconstruction and speculation, general theories of number, ratio and incommensurable magnitudes had been developed and texts treating the elements of geometry were in circulation. We do not know the precise relationship between these early works and Euclid's *Elements*, but there is no direct indication in the sources that early geometers were interested in restricting the set of constructions that could be carried out to those that are abstractions from a straightedge and compass.

\(^{117}\) See W. Knorr, *Ancient Tradition*, pp. 50–76 (note ?).

\(^{118}\) See W. Knorr, *Evolution*, pp. 211–225 (note ?). Note, however, that the attribution to Archytas of *Elements* VIII and *Section of the Canon* is rather tenuous. See C. Huffman, *Archytas*, pp. 451–470 (note ??).


\(^{120}\) See K. Saito, “Phantom Theories of pre-Eudoxean Proportion,” *Science in Context*, 16 (2003), 331–347.

GEOMETRY IN THE HELLENISTIC PERIOD

Around the beginning of the Hellenistic period there were a number of projects to consolidate and reformulate the considerable body of mathematical knowledge that Greek scholars had produced. At the early Lyceum, Eudemus of Rhodes wrote his histories of geometry, arithmetic and astronomy. Although it is not clear how many written sources he had for the earliest periods, his work became a major source for the later estimation of these fields. Around the turn of the century, or shortly after, Euclid, the most widely read mathematician of the ancient period, undertook the project of giving a solid foundation to, and a clear articulation of, nearly all branches of the exact sciences.

We know nothing of Euclid’s life. It is often assumed that he worked at Alexandria, due to the circulation of a legend that he told King Ptolemy that there is no royal road to geometrical knowledge, and to Pappus’ statement that Apollonius worked under his students in Alexandria. Claims that he worked at the Museum, or the Library, are simply embellishments of these two hints. What we do know is that he produced a considerable body of mathematical work, the influence of which increased throughout the Hellenistic period and which had become canonical by the Imperial period.

Euclid worked widely in nearly every area of mathematics and the exact sciences. Most of these works were concerned with geometry or the application of geometry to natural science. Although Euclid will always be known as the author of the Elements, he also wrote works on conic sections, solid loci, porisms and other specialized works related to geometrical analysis. He was regarded by Pappus as one of the three primary authors of the field of geometrical analysis, along with Aristaeus (c. mid-4th–mid-3rd) and Apollonius. In the exact sciences, he wrote works on spherical astronomy, mechanics, optics, catoptrics and harmonic theory, although the authenticity of the last three has been questioned. Because of the fame of the Elements, and because his more advanced works have been lost, modern scholars have often seen Euclid as more of a compiler and textbook writer than as a productive mathematician. This was not, however, the assessment of ancient mathematicians, such as Apollonius and Pappus. Nevertheless, it is clear that Euclid had a great interest in codifying mathematical knowledge and setting it on a secure foundation.

We see this initially in his division of geometry into different domains based on the types of techniques that can be used and, more explicitly, in his arrangement of the Elements. The Elements begins with a series of definitions, postulates and common notions, some of which may not have been authentic, but most of which almost

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122 For a full study of Eudemus' work, see L. Zhmud, The Origin of the History of Science in Classical Antiquity (Berlin: Walter de Gruyter, 2006).
123 There are two versions of this legend, both from late authors: one involves Euclid and King Ptolemy, while the other involves Menaechmus and Alexander the Great. See Proclus, Commentary on Elements I, chap. 4, and Stobaeus, Anthology II 31.115.
124 Pappus, Collection VII 35.
125 Pappus, Collection VII 1.
126 These questions of authenticity are highly subjective, often relying on dubious arguments such as the claim that an author who produced the Elements, which has so many good arguments in it, could not also have produced the Section of the Canon, which contains some sloppy arguments.
127 For a discussion of the authenticity of the early definitions, see L. Russo, “The Definitions of
certainly were. In particular, the postulates, which deal with construction, angle measurement and parallelism, are logically required by the development of the book. The straightedge and compass construction techniques and the parallel postulate, along with the logical structure developed on the basis of these, may well have been Euclid’s original contribution to the foundations of geometry. It is clear from what Aristotle says that the theory of parallel lines was still plagued with logical issues when he was writing, and we find no mention of the construction postulates in authors before Euclid and no interest in a restriction to these constructions in authors who lived around his time, such as Autolycus (c. early 3rd BCE), Aristarchus or Archimedes. Hence, the Elements can be read as a treatise which develops mathematics along the Aristotelian principle of showing as much as possible on a limited set of starting points.

The early books of the Elements, I-VI, treat plane geometry, however, this includes Elements V on theory of ratios between magnitudes, where magnitude is understood as the abstraction of a feature of a geometric object – the size of an angle, the area of a figure, the length of a line, and so forth. The next three books, VII–IX, treat number theory and Elements X deals with incommensurability, which arises as a subject when we try to apply the theory of numbers to magnitudes, such as lines and areas. The final three books develop a theory of solid geometry.

It has often been remarked that Elements V and X do not fit very well into this overall plan, and hence must have been taken essentially unaltered from previous works. It is also possible, however, to read these books as being in their proper place in the overall deductive structure. If we read the text as having a single architecture, Euclid’s strategy appears to have been to introduce an idea or theory as late as possible, so as to show how much could be done without it. Once it became necessary to introduce new concepts and methods, however, Euclid does not treat them as purely instrumental, but presents them in the broader context of an articulated theory. This strategy can be exemplified by the first book. The structure of Elements I shows that congruency of triangles can be shown independently of the parallel postulate while equality of areas cannot. The so-called Pythagorean theorem, Elements I 47, is a culmination of the book, but we should not understand it as the goal. The book has various goals, such as the production of constructive methods for geometric problem-solving, the articulation of a theory of congruency that is only loosely related to constructive geometry, the foundation of a theory of parallelism and its development into a theory of area, which results in Elements I 47 and in the theory of the application of areas of Elements II.

Another major area of Euclid’s mathematical endeavor was in what came to be known as the “field of analysis.” Although most of this work has been lost, it
formed an important foundation for advanced research in mathematics and contributed
to Euclid’s high reputation as a mathematician in antiquity. Along with the *Data*,
which has been discussed, he produced a *Conics*, which was later superseded by that
of Apollonius, a short work on the *Division of Figures*, a treatise on loci that form
surfaces, and a treatise called *Porisms*, treating problem-like propositions that make
assertions about what objects or relations will given if certain conditions are stated
about given objects. While the *Data* was probably written more as a repository of
useful results than as a treatise to be read from beginning to end, the *Solid Loci*
and *Porisms* may have been written to teach students useful habits of thought in the
analytical approach to problems. Whatever the case, by the Imperial period, these
works were taken by Pappus, and other teachers of advanced mathematics, as part of
the canon for the study of analysis.

In contrast to Euclid, Archimedes is the ancient mathematician that we know the
most about. From his own writings we learn that his father was an astronomer,\(^{132}\)
that he was associated with the court of Hieron II of Syracuse, and that he kept in
regular contact with his mathematical colleagues working in Alexandria. Because of
his iconic status in antiquity, many legends circulated about Archimedes, most of
which are probably apocryphal.\(^{133}\) Many of these tales concern his death, and many
of them are likely to have been exaggerated as well. Nevertheless, it seems clear that
he was an old man when he was killed by Roman soldiers during the sack of Syracuse
under Marcus Claudius Marcellus, in 212 BCE.\(^{134}\) It is also likely that the siege of
the city was prolonged due to engines of war that Archimedes built, although none
of these are likely to have been burning mirrors. Moreover, from the prefaces that he
wrote to his works, we are able to get a sense of his personality. He comes off as having
been justly proud of his abilities and aware of the fact that there were very few who
could really appreciate what he was doing. He also, however, had a playful side and
sometimes teased the Alexandrian mathematicians by sending them false propositions
and encouraging them to find the proofs.\(^{135}\) This playfulness also extended to his own
work as we see from his analysis of an ancient game, the *Stomachion*, and his work the
*Sand Reckoner*, in which he shows that his system of large numbers, which cannot
be represented in the normal Greek number system, can handle counting the number
of grains of sand that it would take to fill even Aristarchus’ absurdly large universe in
which the sun is assumed to be at the center of the earth’s orbit.\(^{136}\)

The relationship between Archimedes and Euclid is difficult to untangle. Archi-
medes was probably Euclid’s younger contemporary, but since Euclid’s dates are only
vaguely known, it is also possible that Euclid passed away some years before Archime-
des was born. Euclid’s direct influence on Archimedes is also difficult to detect. On
the one hand, Archimedes appears to have been unimpressed by Euclid’s architectural

\(^{132}\) H. Menge, *Euclid’s Data* (note ??), and J. Hogendijk, “The Arabic version of Euclid’s *On Divisions*,” in

\(^{133}\) This is based on a plausible conjecture for a meaningless passage in our received manuscripts. See


\(^{135}\) See J.L. Heiberg, *Archimedes*, vol. 2, pp. 2–4 (note ??).

style, showed no interest in a restriction to straightedge and compass constructions, and occasionally proved things using different techniques from those we find in the Elements. On the other hand, he does not repeat material we find in Euclid’s works and it may be that, like a number of other great mathematicians, he came to appreciate Euclid’s project more as he got older and was increasingly more influenced by the Elements throughout the course of his long career.\(^\text{137}\) Proclus’ claim that Archimedes mentions Euclid used to be explained away by pointing out that a reference to Elements I 2 in Sphere and Cylinder I 2 is an obvious interpolation; however, in the palimpsest there is another reference to Euclid elsewhere in Sphere and Cylinder that may be authentic.\(^\text{138}\)

Archimedes’ works can be divided into three types: geometrical, mechanical and computational. In all of three of these, however, we can see certain features that are characteristic of Archimedes’ personal style. They tend to be short, never more than two books. They cover distinct problems or areas and start with axioms and construction techniques suitable to the task at hand, with little interest in reducing these to more elementary concepts or methods. They show an abiding interest in mensuration, often in the form of a numerical comparison between the properties of a well-known object and those of a less tractable one. On the whole, Archimedes wrote advanced mathematical texts for mathematicians and, with the notable exception of his Method, seemed to have had little interest in questions of pedagogy or foundations.

Archimedes wrote no elementary geometric treatises. He wrote a text on On Spiral Lines, which defined the curves by moving points, set out some of their basic properties, based on arithmetic progressions and constructions involving setting a given length between given objects, and concluded by finding a number of significant areas related to the curves. In On Conoids and Spheroids, he opens with a series of definitions and a discussion of the problems to be addressed, proves a number of theorems on arithmetic series and the areas of ellipses and then enters the main body of the work, in which he treats the volumes of various conics of revolution and their sections. One of his most striking works is On the Sphere and Cylinder. The first book is a systematic treatment of the relationships between various objects like a circle and polygons that inscribe and circumscribe it, a cone and pyramids that inscribe and circumscribe it, a sphere and the series of sections of a cone that inscribe and circumscribe it, and so on, which leads to finding key relationships between a sphere and a cone, or cylinders, which inscribe or circumscribe it. The second book then uses this material to solve a number of difficult problems and to show theorems related to the magnitude of the volume and area of segments of a sphere.

Archimedes’ surviving works in mechanics have to do with statics and the equilib-rium of floating bodies. In On the Equilibria of Planes, he demonstrates the principle of the balance, and this appears to have played a major role in his research. In the so-called Method, he explains to his colleague Eratosthenes how he used the idea of a


virtual balance to derive many of his important mensuration results, found in works like
*Quadradure of the Parabola* and *Conoids and Spheroids*, using the notion of sus-
pending what we could call infinitesimals so as to maintain equilibrium on a balance.\(^{139}\)
Using as an example a hoof-shaped object that he had not treated previously,\(^ {140}\) in
*Method* 12–15, Archimedes shows how one can first investigate the solid heuristically
using the virtual balance, then develop a proof strategy using indivisibles, and finally
write a fully rigorous proof using a double indirect argument. Although the *Method*
was "lost" until 1906 and appears to have exerted no influence on the development of
mathematics, it is an invaluable glimpse into the working habits of antiquity’s greatest
mathematician.

Probably the most widely read of Archimedes’ computational works was the *Mea-
surement of the Circle*. Although our current version of the text appears to be an
abridged and highly edited epitome of Archimedes’ original work, it provides us with
at least part of his general approach to the classical problem of squaring the circle.\(^ {141}\)
It begins by showing that the area of a circle is equal to a triangle whose base is the
circumference of the circle and whose height is the radius of the circle and it concludes
with a general, although slow, iterative method for approximating the value of \(\pi\) by
inscribing and circumscribing a circle with a polygon. In *Measurement of the Circle*,
this process ends with the 96-gon; however, Heron, in his *Measurements*, states that
Archimedes produced an even more precise set of bounds, but there are sufficient dif-
ficulties with the numbers in the received manuscripts to have generated a good deal
of discussion but little agreement.\(^ {142}\) As well as his interest in large numbers, shown
in *Sand Reckoner* and the *Cattle Problem*, Archimedes may have done some work in
combinatorics, although the evidence for this is not certain.

Archimedes, unlike Euclid, did not write for beginners, nor did he make grand
efforts towards systemization. Instead, he wrote monographs addressing specific sets
of problems or areas of theory that were susceptible to mensuration but were difficult
enough to demonstrate his considerable abilities. When the necessity for a rigorous
presentation required that he include trivial material, we can sense his boredom and
haste, but when he ventures into unchartered waters he is careful to present a thorough
proof, occasionally giving two arguments for the same result.\(^ {143}\)

The most successful mathematician of the next generation was Apollonius; how-
ever, he appears to have followed more closely in the tradition of Euclid than that of
Archimedes. The only indication that we have of Apollonius continuing an Archi-
medean project is Pappus’ description in *Collection* II of Apollonius’ system of large
numbers. On the whole, however, Pappus’ statement that Apollonius studied with the

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\(^{139}\) See E. Hayashi and K. Saito, *Tenbin no Majutsushi: Arukimedesu no Sūgaku* (Tokyo: Kyoritsu
Shuppan, 2011).

\(^{140}\) A section of a right cylinder formed by a plane passing through the diameter of the base.

\(^{141}\) For the history of the text, see W. Knorr, *Textual Studies in Ancient and Medieval Geometry* (Boston:

\(^{142}\) There are as many reconstructions for these values and their derivation as there are scholars who
have studied the matter. W. Knorr provides a summary of previous work, followed by his own recon-
struction. See W. Knorr, “Archimedes and the Measurement of the Circle: A New Interpretation,”
*Archive for History of Exact Sciences*, 15 (1975), 115–140.

\(^{143}\) Examples of the former are *Sphere and Cylinder* I 28–32 and *Spheroids and Conoids* 11, and of the
latter are *Sphere and Cylinder* II 8.
students of Euclid in Alexandria seems to fit well with his mathematical style. His works are both systematic and comprehensive. He had an interest in laying the foundation of the areas in which he worked and in providing elementary texts for more advanced mathematics, such as the theory of conic sections and the field of analysis. Like Euclid, he divided his texts according to the constructive approaches that were used: straightedge and compass, conic sections, and general curves.

Apollonius’ most significant work was his *Conics*, the first four books of which provided “a training in the elements” of conic sections, and the latter four exploring more advanced topics that were of use in geometric analysis. The final book of the treatise, in which Apollonius showed how his theory could be used to solve interesting problems, has been lost since ancient times. The entire book was clearly written to be of use in analysis: in the introductions to the individual books, Apollonius explains how the material he covers will serve the student of analysis, and Pappus includes the *Conics* as the last work in his treatment of the field of analysis. The elementary books were apparently based on the work of Euclid, although Apollonius redefined the conic sections, derived the principle properties (symptōmata) from a general conic construction, gave a more general treatment, and furnished a number of theorems that were not known to Euclid. The more advanced books are said to be an addition of the work of various predecessors, but again Apollonius points out how he clarified, simplified and extended their contributions.

The treatise begins by showing the analogies that exist between various types of conic sections such as lines, triangles, circles, parabolas, ellipses and hyperbolas, and how the fundamental properties of the three final conic sections can be derived from relationships between certain straight lines related to the original cone. After developing the basic theory of tangents, *Conics* I ends by providing constructions of the conic sections given certain straight lines that involve finding the cone, of which the sought conic is a section. *Conics* II then treats diameters and asymptotes and ends with a series of problems involving tangents. Book III provides a sort of metrical theory of conic sections and their tangents that shows the invariance of certain areas constructed with parallels to the tangents, and which Apollonius regarded as essential for a complete solution to the three and four line locus problems. The final books deal with topics like the intersections of various conics sections, minimum and maximum lines, equal and similar sections, and the diameters of conic sections and the figures that contain them. In each case, we see that the material is related to certain types of problems in the analytic corpus, and Apollonius is careful to point how these books will be useful to various aspects of analysis. The goal of the treatise, then, was to furnish theorems useful to analysis in such a way that the theory of conic sections

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144 It is also possible, however, that the idea that Apollonius studied with Euclid's students was simply an inference by Pappus, or someone else, based on the similarity of their mathematical style.

145 See Pappus, *Collection* IV 36, for a discussion of the division into three classes of problems, or domains of geometry.

146 J.L. Heiberg, *Apollonius*, vol. 1, pp. 2–4 (note ??).


148 Greek mathematicians used the term “hyperbola” to refer to just one branch of the modern curve and referred to a pair of branches as “opposite sections.” Nevertheless, Apollonius realized there was an important relationship between the two and in many ways treated them under a unified approach.
would be laid on a solid foundation, analogous to that of elementary geometry, and so that the topics which were raised should be handled with a certain completeness. Apollonius’ tendency to treat things exhaustively is also seen in the other area to which he made extensive contributions, the field of analysis. Besides the *Conics*, Apollonius wrote six of the treatises that Pappus mentions as belonging to the “field of analysis.” Of these, *Cutting off a Ratio* survives in an Arabic translation, and the others can be plausibly reconstructed based on Pappus’ description, at least as far as the general structure and the methods used. In contrast to Archimedes’ short works, the topics covered in these works do not seem to be inherently interesting. For example, *Cutting off a Ratio* exhaustively solves the problem of producing a line through a given point cutting two given lines such that the segments cut off between the intersections and two given points on the lines have a given ratio. The treatment of this mundane problem is then carried out in 21 cases for the arrangements of the original given objects (dispositions), and 87 cases for the arrangements of the line that solves the problem (occurrences). The entire treatment is analytical, giving analyzed propositions for both problems and theorems, and stating the total number of possible solutions and specifying their limits. Although it is possible that some of this material could have been of use in studying conic sections, recalling that Apollonius claims that *Conics* itself was to be of use in analysis, it seems likely that his short works were meant to be training texts in analytical methods. By reading these works, one could develop a strategy for formulating an analytical approach as well as considerable experience in understanding the structure of analytical arguments.

When we look at the scope of his work, it seems that Apollonius viewed himself as Euclid’s successor. He produced a body of texts that could be studied as a course in analysis, dividing the subject matter of these treatises up according the types of constructions involved and crowned by what he rightly believed would become the elements of conic theory. Nevertheless, Apollonius, naturally, brought his own personal interests to his work. In contrast to Archimedes, he did not spend much effort on questions of mensuration but focused on relative placement and arrangement: treating at considerable length topics such as the number and location of the intersections of curved lines, the disposition of maximal and minimal lines, the placement and arrangement of tangents, and so forth. Moreover, he took Euclid’s inclination to completeness to new heights and treated a number of topics so exhaustively as strike many modern readers as excessive.

The Hellenistic period was the most active for researches in pure geometry. As well as the three authors we have surveyed, a number of other important mathematicians were at work during this period, most of whom are now known to us only by name. Nevertheless, in the works of these others which do survive in some form, such as Diocles’ *Burning Mirrors*, in Arabic, or Hypsicles’ treatment of a dodecahedron and the icosahedron inscribed in the same sphere, modified to make an *Éléments XIV*, we find explicit mention of the work of Euclid, Archimedes and Apollonius.

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149 R. Rashed and H. Bellosta, *Apollonius de Perge, La section des droites selon des rapports* (note ?).
152 See B. Vitrac and A. Djebbar, “Le Livre XIV des Éléments d’Euclid : versions grecques et arabe
and a clear indication of respect. Despite everything that we have lost, it seems clear, from the regard in which they were held by their contemporaries and successors, that these three mathematicians were the most active geometers of the Hellenistic period and we are able to make a fair appraisal of their efforts.

**ARITHMETICS AND ALGEBRAIC THINKING**

There has been much debate as to the role that algebra and algebraic thinking have played in Greek mathematics. Although few scholars still argue for interpreting the Greek theory of the application of areas as “geometric algebra,” there is still considerable evidence for algebraic modes of thought in arithmetical problem solving and the theories devoted to this activity. Nevertheless, depending on how we define our terms, it is still possible to argue that Greek activity falls short of algebra. If, on the one hand, we regard algebra as an explicit study of equations and their methods of solution as reduced to the arithmetic operations, then it is possible to argue that Greek work in this area was proto-algebraic. If, on the other hand, we regard algebra as use of an explicit unknown, the application of various stated operations to equations, and the exposition, through examples, of methods that have various applications, then Greek mathematicians did, indeed, do algebra. Whatever the case, for this style of mathematics, we can use the term *arithmetics*, which is not far from the Greek usage.

Reflecting the various subscientific traditions of practical, and recreational, mathematics, some examples of problems in practical arithmetics, of the kind that would have been taught in schools, have survived on fragments of papyri and excerpted in compilations, especially in the Heronian corpus. In the case of elementary arithmetics, which are clearly the products of an oral tradition, no justification of the method of solution is given and no operational, or algebraic, reasoning in explicitly invoked. Understanding and reapplying the methods for which these sources are evidence would have required verbal explanations provided by a teacher. Although most of the problems in these traditions are of the first degree, there are a number of second-degree problems as well that appear to have been solved by the application of a set of identities that must have formed part of the oral instruction. Learning to solve these sorts of problems was probably part of the education of professionals such as builders and accountants, and may have been part of the general education of the literate. Because second-degree equations were not used for practical applications before the early modern period, their presence in our early sources indicates that, even in this practical tradition, the development of general problem-solving skills was a goal of mathematical practice and education.


Concurrent with this practical tradition, Greek mathematicians produced an advanced, theoretical form of arithmetics that went well beyond the practical needs of schoolteachers and engineers. Our knowledge of higher Greek arithmetics is due entirely to the *Arithmetics* of Diophantus. Although there were probably other works in this tradition, and Diophantus himself refers to another of his own works, the *Porisms*, none of these have survived. Hence, our only knowledge of this tradition comes through the rather idiosyncratic work of Diophantus, who, like the Hellenistic geometers, was more interested in solving problems than in giving a general exposition of the methods he employed.

All that we may say with certainty about Diophantus is that he was associated with Alexandria and lived sometime between the middle of the 2nd century BCE and the middle of the 4th CE. These dates derive from the fact that he mentions Hypsicles, in his *On Polygonal Numbers*, and is mentioned by Theon of Alexandria. The testimonials that are used to date Diophantus more precisely, to the middle of 3rd century, date from 11th century. They can be, and have been, called into question.

Our knowledge of Diophantus’ most famous work, the *Arithmetics*, is based on both the Greek and Arabic traditions. The Greek tradition, in six books, survives in a number of late-medieval and early-modern manuscripts, while the Arabic translation, of four books, is known in only a single manuscript. What is curious is that the Arabic books, which are numbered IV through VII, indeed follow the first three Greek books so that the final Greek books, which are numbered IV through VI, must actually follow the Arabic books, probably as VIII, IX and X. Since there is no trace of the other books in either tradition, they must have been lost early in the transmission. There are a number of differences of presentation between the Greek and Arabic books. While the Greek text employs symbolic abbreviation that are explained in the text, the Arabic version is fully rhetorical, conforming to the practice in other early medieval algebraic texts in Arabic. Furthermore, the Arabic text contains far more elementary details, going through the resolution of simple equations, giving full computations and verifying that the numbers so found solve the original problem. Hence, it has generally believed that the Arabic books are derived from the *Commentary to the Arithmetics* produced by Hypatia.

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158 For example, see W. Knorr, “Arithmētikē stoicheiōsis: On Diophantus and Hero of Alexandria,” *Historia Mathematica*, 20 (1993), 180–192, who also argues that Diophantus wrote a work on the elements of arithmetics.


161 This commentary is attributed to Hypatia in the *Souda*, see P. Tannery, *Diophantus*, vol. 2, p. 36 (note ??).
For the historian of mathematics, the *Arithmetics* is a difficult text because it contains considerable mathematics but few general discussions of method. Hence, it is susceptible to a broad range of interpretations. Nevertheless, the discovery of the Arabic books has made it clear that Diophantus teaches his methodology in much the same way as Apollonius in *Cutting off a Ratio*, by repeated exposure to the application of specific methods. The *Arithmetics* provides a general framework for approaching problems and then sets out many different types of problems that can be handled by these methods.\(^{162}\)

The *Arithmetics* is structured as a series of problems, where *problem* has the sense that it generally does in Greek mathematics. The focus of the treatment is more on the production of certain numbers that meet the conditions of the problem and less on the resolution of the equation that finds these numbers, which is often alluded to only briefly. Most of the text of a problem in the *Arithmetics* is devoted to setting out the problem in the terms of the “arithmetical theory,” while the solution of the equation that results from this is relatively short and often referred to with a cryptic reference to the operations involved, “Let the common wanting [terms] be added and like [terms taken] from like.”\(^{163}\) This constructive aspect of the project is expressed by the refrain that concludes many of the propositions, in which Diophantus points out that the numbers “make the problem” (*poiousi to problēma*). As in geometry, the goal is to produce a specific object.

There are sometimes necessary conditions that must be stipulated to make a rational solution possible. The only times Diophantus explicitly refers to such a condition he uses the term *prosdiorismos*, but this may be because in these particular cases he has actually introduced a further condition, so that the word should be understood to mean “further specification.” In a number of cases he uses *diorizesthai*, a verbalization of the standard *diorismos*, for initial specifications.\(^{164}\) Diophantus is only concerned to give one solution, despite the fact that he is aware that there are often more possible solutions.

In the introduction to the work, Diophantus explains the basics of his method of applying symbolic abbreviations and the use of an unknown number. He gives a symbol for squares, \(\delta^2\), cubes, \(\kappa^3\), units, \(\mu^0\), and states that a number that is some multitude of units is “called an unknown (alos) number, and its sign is \(\zeta\).”\(^{165}\) There is also a sign introducing wanting terms, \(\mathcal{A}\), which always follows the extent terms.\(^{166}\)

With these symbols, what we call a polynomial can be written as a series of signs and numerals. For example, \(\zeta \tau \mu^0 \pi \aleph \delta^0 \kappa \beta\) can be anachronistically transcribed as \(5x + 1 - 23x^2\). Diophantus, however, did not have the concept of a polynomial as made

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\(^{165}\) A. Allard, *Diophante*, p. 375 (note ??).

\(^{166}\) Wanting terms are mathematically equivalent to negative terms, however, ancient and medieval mathematicians did not seem to think of them as operational terms.
up of a number of terms related by operations, but rather expressed a collection of
various numbers of different kinds (είδος) of things, some of which might be in deficit,
where certain stated operations could be carried out on equations involving these sets
of things. Perhaps we should think of \( \zeta \pi \mu \varpi \lambda \delta \kappa \beta \) as expressing something like:

\[ 5x, 1, \text{less } 23x^2. \]

Using these these symbols, in \textit{Arithmetics} I, as a means of demonstrating the
utility of his new methods, Diophantus treats the sort of simple problems whose
solutions would have been taught in many schools and would have been well known
to professionals who used arithmetic methods. He shows how to use the conditions
of the problem to produce an equation involving an unknown number, which can then be
solved to produce the sought numbers, where all of the numbers involved are assumed
to be rational. In \textit{Arithmetics} II and III, Diophantus begins to treat indeterminate
problems that are handled by finding a equation involving the unknown number,
which must be set equal to a square. A number of the problems solved in \textit{Arithmetics}
II, namely 8–11 and 19, compose a toolbox used extensively in rest of the treatise. In
the Arabic books, IV–VII, Diophantus continues to show how the methods set out in
\textit{Arithmetics} II can be applied to more difficult problems, now involving higher powers
of the unknown. The introduction to \textit{Arithmetics} IV introduces a third operation that
allows us to reduce an equation containing two powers of the unknown to one with
a power equal to a number.\textsuperscript{167} This allows the treatment of equations with higher
powers of the unknown. In the last three Greek books, the problems become yet
more difficult, often involving the construction of an auxiliary problem because the
solution to the original problem requires special conditions to be met in order
to make a rational solution possible. The final Greek book treats problems involving the
determination of metrical properties of right triangles, given various conditions.

Greek arithmetics exhibits a number of features that we regard as essential to
algebra, such as the use of an unknown and the application of algebraic operations.
Nevertheless, a number of other important features of algebra are not present. For
example, the solutions and various types of equations, as such, does not seem to have
been a subject of direct study. Hence, arithmetics appears to have formed a group of
problem solving methods that were used as a techniques for the production of sought
numbers but which did not focus on the equation as a subject of study.

COMBINATORICS

Although it used to be believed that the Greeks did no substantial work in com-
binatorics, it is now clear that this assessment was simply due to a loss of the primary
sources. The evidence that Greek mathematicians worked in combinatorics is slight,
either due to the vagaries of our transmission or to a relatively narrow range of their
activity. Although we find a number of recursive arguments and proofs in various
mathematical works, and some clear combinatorial statements in later authors, such
as Pappus, it is not clear that any works devoted to combinatorial mathematics were
written. What seems much more likely is that combinatorial methods were developed

in the context of the technical works on logic and then disappeared with the loss of these texts.

It has been argued that Archimedes was involved in combinatorial reasoning in his Stomachion, a study of an ancient puzzle game, which was similar to the modern tangram, although more complicated. Unfortunately, our evidence for this text is only fragmentary and there is no explicit use of combinatorial reasoning. Moreover, the rest of the ancient evidence for this game also gives little indication of combinatorial methods.

We are on much better footing with the evidence from ancient logic. The most certain testimony comes from Plutarch, who tells us that all “the arithmeticians” and particularly Hipparchus, contradict the claim made by Chrysippus that the number of conjunctions produced through ten assertibles is greater than a million. In fact, Hipparchus calculated that the number of conjoined assertibles for affirmation is 103,049 while that for negation is 310,954. It has been shown that these numbers are the 10th Schröder number and half the sum of the 10th and the 11th Schröder numbers, which are the correct solution to a well-defined problem involving bracketing ten assertions, and their negations. Since the calculation of these numbers could almost certainly not have been carried out by brute force, it is clear that Hipparchus must have had at his disposal a body of combinatorial techniques that he could draw on in formulating and solving this problem. Moreover, a survey of the ancient literature on logic reveals a concern with combinatorial thinking, which goes back at least as far as Aristotle. It seems likely that combinatorial methods were developed in the technical study of logic, which naturally gave rise to many combinatorial assertions and problems.

THE EXACT SCIENCES

The Greek exact sciences were produced by people who regarded themselves, and were regarded by their contemporaries, as mathematicians. There was no special institutional setting for people who did optics or astronomy. What divided optics from geometry, or observational astronomy from spherics, was the different traditions of texts in which these fields were transmitted. Nevertheless, ancient thinkers divided up the disciplinary space of the mathematical sciences in various ways, which probably

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169 The fragment of the Stomachion that survives in Arabic is different from that which survives in Greek. See, H. Suter, “Der Loculus Archimedes oder das Systemachion des Archimedes,” Abhandlungen zur Geschichte der Mathematik, 9 (1899), 491–499. For the Greek text, see the previous note.
171 Plutarch, On the Contradictions of the Stoics, 1047c–e.
172 The manuscripts actually have 310,952 for the second number, but this appears to be an error.
reflected individual interests and tastes as much as institutional or educational realities. Plato famously justified the Pythagorean division of the mathematical sciences into arithmetic, geometry, music and astronomy, which would later be known as the quadrivium, but a look at the work of Archytas, for example, makes it clear that an evaluation of the contents and methods of these early sciences must not be made on the basis of later categories. Aristotle, placed the mathematical sciences, as one branch of the theoretical sciences, between theology and physics; and within the mathematical sciences asserted various subordinate relationships, such as optics to geometry, or harmonics to arithmetic. Geminus (c. 1st BCE), in some general work on mathematics, provides an extensive classification of the mathematical sciences that reflects the genres of the texts that we still possess. The first division was into pure and applied mathematics, of which the applied branches form what we can call the exact sciences. The branches of pure mathematics were arithmetic and geometry, while those of applied mathematics were mechanics, astronomy, optics, geodesy, harmonics (kanonike) and calculation (logistikē). Ptolemy, who considered himself a mathematician, seems to agree with Aristotle’s division of the theoretical sciences, but he took his point of departure by asserting that only mathematics is capable of producing knowledge and placing it above the other two. Moreover, in the Harmonics, he makes it clear that he regards mathematics as the study of beautiful things, of which the highest branches are astronomy and harmonics. In this way, geometry and arithmetic are reduced to the position of “indisputable instruments.” As these examples make clear, there was no generally accepted classification of the mathematical sciences and the organization and importance of the different fields was highly influenced by the author’s own interests.

An important area of mathematics that was necessary in the exact sciences was the application of numerical methods to problems of mensuration. Although these techniques were developed by a number of mathematicians in the Hellenistic and Imperial periods, most of the surviving texts of this type are preserved under the authorship of Heron. While the Heronian texts also preserve many problems in the subscientific traditions that would have formed part of the education of professionals, such as engineers and accountants who used mathematics in their work, one of Heron’s primary goals appears to have been to produce academic mechanics and theories of mensuration following the pattern of some of the more established exact sciences such as optics or astronomy. Heron blends the methods of the Hellenistic geometers with the interests and approaches of mechanics and other practical fields, and, unsurprisingly, he mentions Archimedes more than any other single author. In Heron’s work, we find a number of practices that typify the Greek exact sciences. He uses the structures

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175 See Plato, Republic, VII 520a–532c, and Archytas, frag. 1 and 3.
176 For example, Aristotle, Metaphysics, 6.1, 1026a and 11.7 1064a–b.
179 Ptolemy, Almagest I 1.
180 Ptolemy, Harmonics, III 3.
181 I. Düring, Die Harmonielehre des Klaudios Ptolemaios (Elanders: Göteborg, 1930) p. 94.
of mathematical propositions, such as theorems and problems, analysis and synthesis, although their content and actual meaning is altered to accommodate new subject matter. He models objects in the physical world using geometric diagrams and then focuses his discussion on these diagrams.\textsuperscript{183} He blends metrical, computational approaches with geometric, constructive ones, and uses arguments involving given\textsuperscript{s} to provide a theoretical justification for computations.\textsuperscript{184} Although Heron is one of the first authors we have who uses these techniques and his field was mechanics and the mathematics of engineers and architects, the fact that similar techniques are found in the astronomical writings of Ptolemy indicate that they were probably general methods of the exact sciences developed in the middle or end of the Hellenistic period.

In terms of the development of Greek mathematics, the most significant exact science was astronomy. For the purposes of handling problems that arose in astronomy, Greek mathematicians developed a number of mathematical techniques that are only found in astronomical texts but became an essential part of the canonical education of mathematicians in the late ancient and medieval periods.

For the purposes of ancient astronomy, timekeeping and cosmology, Greek mathematicians developed a branch of applied geometry known as spherics. Spherics was the development of a theoretical geometry of the sphere and its application to problems of spherical astronomy, that is, the study of the motion of the fixed stars and the sun, regarded as located at some point on the ecliptic. Some works in spherics were probably produced in the Classical period, by Eudoxus and others, however, as usual, the first texts in this genre that survive are from the beginning of the Hellenistic. We have works in pure spherical astronomy by Autolycus and Euclid, but it is clear from the way these works are presented that there was also a body of knowledge of spherical geometry on which they could draw.\textsuperscript{185}

Towards the end of the Hellenistic period, Theodosius produced a new edition of elementary spherical geometry, his Spherics, which was so successful that previous versions of this material only maintained historical value and were eventually lost.\textsuperscript{186} Theodosius' Spherics is in three books, the first of which is purely geometrical and the second two of which deal with topics applicable to spherical geometry, but still expressed in an almost purely geometrical idiom. The first book treats the properties of lesser circles and great circles of a sphere that are analogous with of the properties of the chords and diameters of a circle in Elements III. The second book explores those properties of lesser circles and great circles of a sphere that are analogous with those of circles and lines in Elements III, which leads to theory of tangency, and theorems dealing with the relationships between great circles and sets of parallel lesser circles. Although still expressed in almost purely geometrical terms, the book ends with a number of theorems of largely astronomical interest, having to do with circles that can model the horizon, the equator and the always visible, and always invisible, circles. The third book deals with what we would call the transformation of coordinates, or the projection of points of one great circle onto another, and concludes with theorems

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\textsuperscript{183} See, for example, Heron, Dioptra 35.

\textsuperscript{184} See, for example, Heron, Measurements, I 7 & 8; and Measurements, I 10, or Measurements III.


that can be interpreted as concerning the rising and setting times of arcs of the ecliptic, again without naming these objects explicitly; as Euclid had, over 200 years earlier, in his *Phenomena*.

Around the end of the first century CE, Menelaus took a new approach to spherical geometry that focused on the geometry of great circles on a sphere and incorporated Hellenistic developments in trigonometry, enabling him to produce the theoretical basis for an elegant, metrical spherical astronomy. Only fragments of the Greek text of Menelaus’ *Sphēris* survive, but we have a number of Arabic and Latin versions. Our knowledge of this work is somewhat tentative because only late, heavily modified, versions of the text have been edited or studied in depth; nevertheless, we may sketch out the general approach and trajectory of the treatise. Book I develops a theory of the congruency of spherical triangles modeled on the first part of *Elements* I, but including the congruency of pairs of triangles having three equal angles. This is then followed by a run of theorems that involve inequalities that can be asserted in a given spherical triangle in which the sum of two sides is either equal to, greater than, or less than, a semicircle. Book II, then, gives a theory of bundles of great circles having a given angle to a given great circle acting as an analogy with the theory of parallels in *Elements* I. The versatility of this approach is then demonstrated with three theorems on the intersections of great circles with sets of parallel lesser circles, which elegantly do the work of six long theorems in Theodosius’ *Spherics* III. Book III begins with the theorem passed down under Menelaus’ name, but which may have been already well-known in his time, which gives a compound ratio between the chords that subtend various parts of a convex quadrilateral made up of great circles. This is then combined with Menelaus’ theory of triangles and his theory of bundles of equiangular great circles to provide tools for developing a fully metrical spherical astronomy. In this way, Menelaus combined the Hellenistic trigonometric methods with his new theory of the spherical triangle to produce a new spherical trigonometry.

A major area of mathematical development in astronomical texts was trigonometry, which in antiquity always literally involved the mensuration of elements of a triangle. We may talk of Greek trigonometry in three main stages. The first stage, which is proto-trigonometric, is attested in the astronomical writings of Aristarchus.

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190 Not even the book divisions are consistent in the various versions of the text; see M. Krause, *Die Spärik von Menelaos*, p. 8–9. The following numbers are those in Krause’s edition.


and Archimedes, *On the Sizes and Distances of the Sun and the Moon* and the *Sand Reckoner*. In these texts, the authors implicitly rely on a pair of ratio inequalities that hold between a ratio of corresponding sides and a ratio of corresponding angles of a pair of triangles. Although both Aristarchus and Archimedes seem to assume these inequalities as a well-known lemma, we have a number of proofs of their validity in later authors.\footnote{See W. Knorr, “Ancient Versions of Two Trigonometric Lemmas,” *Classical Quarterly*, 35 (1985), 361–391.} Using these inequalities, Greek mathematicians were able to derive fairly precise upper and lower bounds on sought angles or lengths. There were, however, a number of difficulties involved in this method. The computations were quite cumbersome, and the more lengthy the computation, the more accuracy would be lost in regard to the term to be bounded. Furthermore, these inequalities are only sufficiently precise for small angles, and if large angles were involved they would produce substantially inaccurate results. Although these methods were sufficient for the laborious calculation of a few chosen values it is difficult to imaging that an accurate, predictive astronomy could have been established using such techniques.

The next stage in the development of Greek trigonometry, although not well documented in our sources, appears to have been directly spurred by the desire to produce a predictive astronomy based on geometric models. During the Hellenistic period, Greek astronomers came into possession of Babylonian sources that showed them that it was possible to produce a highly accurate, predictive astronomy and provided them with observation reports and numerical parameters to put this project into effect.\footnote{See A. Jones, “The Adaptation of Babylonian Methods in Greek Numerical Astronomy,” *Iris*, 82 (1991), 441–453, and A. Jones, “On Babylonian Astronomy and its Greek Metamorphoses,” in F.J. Ragep and S. Ragep, eds., *Tradition, Transmission, Transformation* (Leiden: Brill, 1996) pp. 139–155.} By at least the time of Hipparchus, in the 2nd century BCE, the goal was to produce a numerically predictive astronomy on the basis of geometric models. In order to meet this goal, Greek mathematicians developed tabular trigonometry, which consisted of a group of theorems about right triangles and tables that allowed a length to be determined as a function of a given angle, and conversely. Tabular trigonometry allowed them to use observed angles and period relations to calculate the numerical parameters of assumed geometric models. Although it is not certain that Hipparchus was the first to produce a chord table, there is evidence that Hipparchus tabulated the chords subtending angles at $\frac{1}{2}^\circ$ intervals, from $0^\circ$ to $180^\circ$, for a circle with $d = 6.875$.\footnote{The argument for the details of Hipparchus’ chord table is given in G. Toomer, “The Chord Table of Hipparchus and the Early History of Greek Trigonometry,” *Centaurus*, 18 (1973), 6–28, and D. Duke, “Hipparchus’ Eclipse Trios and Early Trigonometry,” *Archive for History of Exact Science*, 56 (2002), 427–433.} Using such chord tables, mathematicians, as Hipparchus, Diodorus (c. mid-2nd BCE–3rd CE) and Menelaus, were able to solve computational problems in plane and spherical trigonometry by treating them with forms that we would write using the sine function, since $\text{Crd}(2\alpha) = 2 \sin \alpha$.

The final stage of Greek trigonometry is found in the writings of Claudius Ptolemy. In his *Almagest* I 10–13, Ptolemy sets out the mathematics necessary for the trigonometric methods he will use. He begins by showing how a chord table of $\frac{1}{2}^\circ$ intervals, on a circle with $d = 120$, could be calculated using a number of geometrically derived formulas and an approximation of the chord of $\frac{1}{2}^\circ$ using a lemma similar to
that at the basis of the proto-trigonometric tradition. This serves as a justification for
his more precise chord table, although Ptolemy does not actually say that he used these
methods to derive it.\textsuperscript{197} This more precise chord table was the computational tool at
the foundation of Ptolemy’s plane and spherical trigonometry, as found in works such
as the \textit{Almagest}, the \textit{Planisphere}, and the \textit{Analemma}.\textsuperscript{198} For spherical trigonometry,
Ptolemy restricted himself to the theorem based on a convex quadrilateral of great
circle arcs, known as the Menelaus theorem, avoiding any use of the more powerful
spherical-trigonometric methods that Menelaus had developed.

A final mathematical tool of the exact sciences should be mentioned: tables. Ta-
bles presumably entered Greek mathematical practice along with predictive astronomy
from Babylonian sources during the Hellenistic period. Although mathematicians
such as Hipparchus, Diodorus and Menelaus must have used tables, again, our earliest
texts that contain tables are by Ptolemy, particularly the \textit{Almagest} and the \textit{Analemma}.
Despite the fact that tables were used long before the concept of function became
explicit, they exhibit relationships that are similar to certain modern conceptions of a
function.\textsuperscript{199} For example, tables were treated as general relations between members
of two, or more, sets of numbers. Moreover, the algorithms that describe how to use
the tables make it clear that they describe a computational rule that maps a single
member of the domain to a single member of the codomain. These proto-functions
were used in a number of ways to handle the metrical aspects of geometrical objects
and moving components of a geometrical model.

In Ptolemy’s writings, tables are sets of numerical values that correspond to lengths
and arcs in the geometric models from which they are derived. At least in principle,
they are produced by direct derivation from geometric objects with assumed numeric
values. We can understand the tables themselves as a numerical representation of the
underlying model, which is geometric. The tables are then used, either by Ptolemy
or by the reader, to provide an evaluation of specific numerical values that represent
the underlying model.

In the \textit{Almagest}, mathematical tables are a component of Ptolemy’s goal of produc-
ing a deductively organized description of the cosmos, presented in an essentially single
argument. Indeed, Ptolemy makes a number of explicit assertions that the structure
of the tables in the \textit{Almagest} should exhibit both the true nature of the phenomena
in question and have a suitable correspondence with the mathematical models.\textsuperscript{200} For
Ptolemy, a table, like a mathematical theorem, is both a presentation of acquired
mathematical knowledge and a tool for producing new mathematical results.

\section*{EXPOSITORS AND COMMENTATORS}

\textsuperscript{197} For an argument that Ptolemy actually used interpolation methods in constructing his chord
table see G. Van Brummelen, \textit{Mathematical Tables in Ptolemy’s Almagest}. Ph.D. Thesis, Simon Fraser
University, Mathematics and Statistics, 1993, pp. 46–71.
\textsuperscript{198} See N. Sidoli and J.L. Berggren, “The Arabic Version of Ptolemy’s \textit{Planisphere},” \textit{SCIAMVS}, 8
\textsuperscript{199} This argument was made by O. Pedersen, “Logistics and the Theory of Functions,” \textit{Archives Inter-
\textsuperscript{200} See, for example, J.L. Heiberg, \textit{Claudii Ptolemaei opera omnia}, vol. 1 (Leipzig: Teubner, 1898–
1903), part 1, pp. 208, 251.
Our knowledge of the substantial texts of Greek mathematics comes through the filter of the scholarship of the mathematicians of late antiquity, most of whom were associated with schools of philosophy and regarded mathematics as an important part of a broader cultural and educational project. We have seen that the texts of the earlier periods were edited and commented upon by these mathematical scholars, and this process acted as an informal process of selection, in so far as texts which did not receive attention had a dramatically reduced chance of being passed down.

These late-ancient scholars were primarily responsible for creating the image of theoretical mathematics that was transmitted to the various cultures around the Mediterranean in the medieval and early modern periods. Through their teaching and scholarship, they established various canons of the great works of the past, arranged courses of study through select topics, reinforced a sound and lasting architecture by shoring up arguments and making justifications explicit, and, finally, they secured their place in this tradition by intermingling their work with that of their predecessors and situating the whole project in contemporary modes of philosophic discourse.

One of the most impressive of these scholars was Pappus of Alexandria, who was a competent mathematician, a gifted teacher and made important strides to associate mathematics with areas of interest in philosophy by constantly arguing for the relevance of mathematics to other aspects of intellectual life. Pappus worked in many areas of the exact sciences, wrote commentaries on canonical works, such as the Elements and the Almagest, and produced a series of short studies that were later gathered together into the Mathematical Collection. It is clear from Pappus’ writing, that he was part of an extended community of mathematicians and students who had regard for his work and interest in his teaching.

Pappus’ Collection, although incomplete, is indispensible for our understanding of both the early history of Greek mathematics and the main trends of mathematical thought in late antiquity. Book II, which is fragmentary, discusses a system of large numbers attributed to Apollonius. Books III and IV present a number of topics in advanced geometry framed in philosophical discussions that show, among other things, how mathematical argument can be of relevance to philosophical issues. Here we find discussions of the difference between theorems and problems, the division of geometry into linear, solid and curvilinear methods, treatments of mechanical and geometrical curves, examples of problem solving through analysis and synthesis, discussions of geometrical paradoxes, and examples of solutions to three classical problems: squaring a circle, duplicating a cube and trisecting a angle. Pappus’ style is to situate his own work in a wealth of material drawn from historically famous mathematicians, to mix geometrical and mechanical approaches and to combine numerical

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203 For a study of Pappus in his intellectual context see S. Cuomo, Pappus of Alexandria and the Mathematics of Late Antiquity (Cambridge: Cambridge University Press, 2000).
examples with pure geometry. Book V deals with isoperimetric and isovolumetric figures, again situating the discussion in the context of past work, for example, that of Archimedes, Zenodorus (c. 3rd–2nd), and Theodosius.\textsuperscript{205} The next two books were clearly produced as part of Pappus’ teaching activities and they are mainly made up of lemmas to canonical works. Book VI deals with what Pappus calls “the field of astronomy,” which came to be known as the \textit{Little Astronomy}.\textsuperscript{206} It organizes and discusses the treatises that students should master before studying Ptolemy’s \textit{Almagest}.\textsuperscript{207} Book VII covers “the field of analysis,” which consisted of a group of works by authors such as Aristaeus, Euclid, and Apollonius.\textsuperscript{208} It begins with a general account of analysis, discusses the overall content of each work, and provides numerous lemmas for individual propositions, many of which give justifications for common practices of the Hellenistic geometers, such as operations on ratios and the use of compound ratios.\textsuperscript{209} Book VIII treats theoretical mechanics, in much the same vein as Heron, whom Pappus refers to a number of times. It models various machines geometrically and sets out the mathematical theory of certain practical constructions.\textsuperscript{210}

The other mathematical scholars of the late ancient period were also involved in teaching and expounding the classics, and hence mostly worked through the medium of commentaries. Theon of Alexandria, in the 4\textsuperscript{th} century, edited works by Euclid and wrote commentaries to Ptolemy’s \textit{Almagest} and \textit{Handy Tables}. Hypatia, his daughter, collaborated with her father on various projects and wrote commentaries to Apollonius and Diophantus.\textsuperscript{211} In the following century, Proclus of Athens, wrote a commentary on the \textit{Elements}. Eutocius of Ascalon, in the 6\textsuperscript{th} century, edited works by Archimedes and Apollonius, and wrote commentaries to them. This work was a continuation of a tradition of commentating and editing that began in the Imperial period. The scholars of this period paid particular attention to issues of logical completeness, formal structure, and readability. They produced fuller texts with more explicit arguments, wrote auxiliary lemmas, introduced internal references to other parts of the canon, restructured the treatises and individual elements of the text, added introductions and conclusions, advocated explicit classifications, rewrote theories from new perspectives, and summarized long works for the purposes of study.\textsuperscript{212}

All of this was part of a broad trend, begun in the Imperial period by authors such as Geminus, Heron and Ptolemy, to incorporate the mathematical sciences into the


\textsuperscript{206} F. Hultsch, \textit{Pappus}, p. 474 (note ??).


\textsuperscript{208} A. Jones, \textit{Book 7 of the Collection}, p. 83 (note ??).

\textsuperscript{209} For essays on the lost treatises of the field, see A. Jones, \textit{Book 7 of the Collection}, pp. 510–619 (note ??).

\textsuperscript{210} The Arabic version of Book VIII, which has not yet been thoroughly studied, contains a number of topics not found in the Greek version. For an example, see D.E.P Jackson, “Towards a resolution of the problem of τὰ ἑνὶ διαστήματι γραφόμενα in Pappus’ Collection Book VIII” (note ??).

\textsuperscript{211} For Hypatia’s life and work, see M. Dzielska, \textit{Hypatia of Alexandria} (note ??).

philosophical tradition. Although in the Classical and early Hellenistic periods, philosophers showed interest in mathematical approaches, there is little indication that mathematicians had a similar regard for philosophy. The mathematicians of the late ancient period, however, were concerned that mathematics be part of an education in philosophy and rhetoric. Their texts show a combination of modes of thought from the traditions of pure mathematics with those from the various exact sciences, and a mixture of philosophical concerns with mathematical issues. Their project, situated as it was in the philosophical schools, argued both explicitly and implicitly for the value of mathematics to philosophy. The final stages of Greek mathematical practice furnished the image of Greek mathematics that remains with us to this day: that of a fully explicit, interconnected, literary product.

CONCLUSION

Theoretical Greek mathematics underwent considerable change and development throughout the centuries. What began as a leisure activity for elite scholars was then applied in solving interesting problems in the natural sciences and became of practical use to a wide range of scholars and professionals. The resulting combination of the theoretical and practical traditions was then institutionalized in courses in mathematical sciences at the philosophical schools. Throughout this long transition, however, Greek mathematics, as a type of intellectual activity, was primarily defined by the cohesion of certain traditions of practices and texts, not by the social position, or institutional setting, of the practitioners themselves.
