An application of concepts from statics to geometrical proofs

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Abstract: This paper describes an application of statics to geometrical proofs in the classroom. The aim of the study was to find out whether the use of concepts and arguments from statics can help students understand and produce proofs of geometrical theorems. The theorem studied was that the medians in a triangle meet at a single point, which is the centre of gravity of the triangle. The classroom experiment showed that most students were successful in using arguments from statics in their proofs, and that they gained a better understanding of the theorems. These findings lend support to the claim that the introduction of statics helps students produce proofs and grasp their meaning.

Introduction

This paper investigates a novel approach to the effective teaching of proof: the use of concepts and models from physics. In this approach, proving is embedded in the context of physics. Ideas from physics that are already familiar to students, such as the concepts of the centre of gravity and of balance, serve as tools for proving geometrical theorems, and students are prompted to create proofs based upon physical considerations. That is, students are encouraged to build a mathematical proof by taking as given one or more principles of physics. This can be made clear by the following example. Let us take as given the following three principles of statics:

P1: The uniqueness of the centre of gravity (each system of masses has one and only one centre of gravity).

P2: The lever principle (the centre of gravity of any two masses lies on the straight line joining the masses, and its distances from the masses are inversely proportional to them).

P3: The principle of substitution (if any two individual masses are replaced by a single mass equal to the sum of the two masses and positioned at the centre of gravity of the two masses, then the location of the centre of gravity of the total system of masses remains unchanged).

These three principles can then be used in proving the following geometrical theorem known as the Triangles Median Theorem:

The medians of a triangle intersect at a single point, and on each median this point is located two-thirds of the way from the vertex. The point of intersection is known as the centre of gravity of the triangle, or the centroid.

To produce a rather simple proof using arguments from physics, one would consider the vertices of the triangle (Figure 1) as loaded with equal masses of unit weight (at points A, B and C) and connected by rigid weightless rods. The midpoint of any side is its centre of gravity (by P1 and P2). Therefore the unit masses at points A and C can be replaced by a mass of weight 2 at E, the midpoint of AC (by P3). If we then connect this midpoint with the third vertex, B, to form a median, we can conclude that the centre of gravity of the whole triangle must lie on this median, and by P2 the median must be divided in the ratio 2:1. Since this construction can be repeated with the other two sides, the three medians must meet in one and the same point, the centre of gravity. (There are several geometric proofs of this interesting theorem, which is a corollary of the more general Ceva theorem.)

A mathematical proof, by definition, can take a set of explicit givens and use them, applying the principles of logic, as the basis for a deductive argument. A deductive argument need be no less compelling if the givens happen to come from the science of statics, as they do in the above example. In fact the physical context, appealing as it does to physical intuition, has the advantage of making the plausibility of the conclusion more readily apparent.

The role of context in learning to prove

Previous studies have shown that helping students construct proofs and justify their thinking depends on providing a context meaningful to the students. Such a context must offer well-chosen tasks, hands-on experiences, opportunities for exploration, and discussions with peers and teachers (Balacheff, 1991; Lampert, 1990; Maher and Martino, 1996; Yackel and Cobb, 1996). Current research on proving has therefore centred on the creation of rich contexts that provide a variety of opportunities for students to explore ideas, investigate, experiment, and make and test mathematical hypotheses. Indeed, it has been shown that environments in which opportunities for exploration have been provided by dynamic geometry software are quite effective in helping students learn how to prove (Hadas, Hershkowitz and Schwarz, 2000; Jones, 2000; Marrades and Gutiérrez, 2000; Mariotti, 2000).

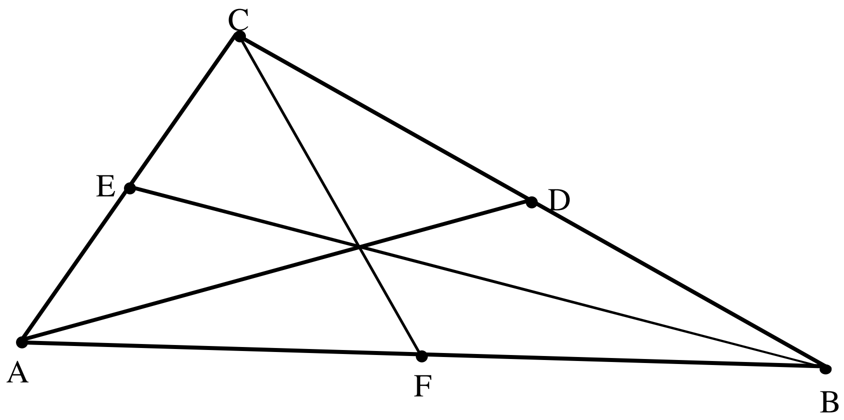

Figure 1. Centre of gravity of a triangle ABC

The study

In this study we explored the potential of using concepts from physics in creating an effective context for helping students learn proof. The study was carried out in an Ontario mixed-grade 11 and 12 class during the second semester of the school year in the year 2001. There were 21 students in the class, 13 female and 8 male. The youngest student was 16, the oldest 18; the average age was 16.8. Fourteen of the 21 students were in their eleventh year of schooling (third year of high school). Of the 21 students attending the course, 19 agreed to participate in the study after signing assent forms and obtaining the consent of their parents.

The majority of the students in the class were considered academically high achievers (though not necessarily in mathematics) and had been placed in a program of studies designated as "enhanced". The teacher, in consultation with the researchers, designed a teaching unit consisting of four 75-minute periods. The first period introduced the lever principle, the concept of the centre of gravity as the location of the fulcrum or balancing point, and the substitution principle. The students answered questions on these new concepts by completing two worksheets and working through tasks related to levers, including balancing objects with actual levers.

During period 2 the class first discussed the concepts of the existence and the uniqueness of the centre of gravity, as well as their responses to the worksheet questions on the material taught during period 1. Periods 3 and 4 were devoted to discussing the remaining worksheets and wrapping up the unit. At the end of the fourth period the students were asked to complete two proofs of the triangle median theorem, one based upon arguments from statics and the other on geometrical arguments.

Additional data for this study was collected in a number of ways. Two of the researchers observed the classroom, making notes on the teachers' lectures and recording the activities of the student workgroups. All of the students' work was done on six separate worksheets, which were handed in to the teacher to mark. After the unit was complete, each student was asked to fill out a questionnaire on various aspects of the material and its interpretation. Each student was then interviewed by one of two researchers to clarify answers given and to elicit answers to questions left unanswered. These interviews were recorded and transcribed.

Findings and Discussion

The classroom teacher rated the students' performance on the two proofs, statics and geometric, on a scale of 0 to 3.The rating scheme was as follows: very little of the proof is correct (0); the proof is deficient and incomplete (1), the proof contains a number of minor errors but otherwise is reasonably organised (2), and the proof is thorough and well organised, with at most very minor omissions or errors (3). The results are summarized in Table 1.

Table 1. Students' performance on proofs of the Triangles Median Theorem (N = 19)

	Ratings					
Question	**0**	**1**	**2**	**3**	**n/a**	**mean**
1. Statics Proof (Triangle)	4	9	4	1	1	1.1
2. Geometric Proof (Triangle)	1	10	4	3	1	1.5

Some examples will serve to illustrate the students' work and the teacher's thinking in the assignment of rank.

For the first proof, the statics proof of the Triangles Median Theorem, the majority of students (13 out of 19 respondents) supplied only an incomplete argument (ratings 0 or 1). Student 17, who was given a rating of 1 by the classroom teacher, can be taken as typical. Student 17 sometimes applies the Substitution and Lever Principles correctly, but does not refer to these principles by name. Further, she shows the replacement of masses at C and B with 2 kg at D on one diagram, but makes no clear reference to this in the text of her solution. The conclusion is incorrectly derived in any case, because her argument states that the Uniqueness of the Centre of Gravity is a consequence of the division of each of the three medians in a 2:1 ratio. In addition, the fact that the medians should have a common point of intersection is not specifically stated.

In the geometric proof of the Triangles Median Theorem, the majority of students (11 out of 19 respondents) were again unable to complete the proof. Student 6, with a rank of 1 assigned by the classroom teacher, was typical. He answered parts a) i) ii) almost correctly, omitting only the statement that that EF||CB by the Triangle Side Splitting Theorem. However, he erroneously concluded that this proves that all three medians intersect at a single point. Moreover, he stated that G and G' are the same point in part c) because "all 3 lines have to intersect at the same point". In so stating, he assumed what he was setting out to prove, perhaps influenced by a seemingly obvious physical relationship and importing it at the wrong point in the argument.

The highest rank received for this problem was 3. We can again take Student 17 as representative. She provided excellent solutions to all parts, being one of only two students to realize that parts a) i) and ii) without part b) does not prove that all three medians are concurrent. Minor mistakes are made in part i) when writing "(SS~)" for the Side-Angle-Side Similar Triangle Theorem instead of (SAS~); and neglecting to state the reason for the statement "EG/GB = FG/GC = 2/1" in part a) ii).

Students' responses to the questionnaires

The students responded to a questionnaire put together based on the researchers' experience with a similar teaching experiment in the previous year (Hanna et. al., 2001).

Table 2. Students' responses to the questionnaires (N = 19)

Question	**Responses**		
			Don't know
Are the physical arguments convincing?	Yes (16)	No (0)	(3)
Which proof was clearer?	Physical (7)	Geometric (8)	(4)
Which proof was easier to remember?	Physical (8)	Geometric (6)	(5)

	Yes	No	Don't know
Does everyday experience make the arguments from physics clearer?	10	3	6
Does everyday experience help you understand the Lever Principle?	11	2	6
Do the proofs from physics help you understand the nature of proof?	11	1	7
Do the physics proofs help you understand *why* the theorems are true?	12	2	5
Do you find the proofs from physics elegant?	6	5	8

On the whole the students seemed to find the physical arguments based on the three principles of statics interesting, convincing and explanatory. Their responses, summarised in the Table 2, also tell us something about what they think about proof in general.

The students' responses to the written questionnaire were followed up with recorded interviews that were later transcribed. For the most part, the interviewers focused on questions that had remained unanswered in the questionnaires. In some cases, though, the interviewers questioned a student further about a questionnaire answer that was particularly interesting or unclear. The interviews were recorded and transcribed, and the students' comments were later grouped around themes where possible. Limitations of space permit us to report only the comments on the first question, but these comments do give us considerable insight into what constitutes conviction in the minds of the students.

Are the arguments from physics convincing?

Almost all the students said they found the arguments from physics convincing. What is interesting is the wide variety of reasons they gave. The students' reasons fall broadly into three types: they saw the physical arguments as convincing because they were 1) easy to understand, 2) physically coherent, or 3) consistent with other known results or otherwise verifiable.

Physically Coherent: One student found the theorem convincing simply because it applies the laws of physics. Students 5, 15 and 16 were convinced by the cohesion of the physical principles. Student 15 stated that she was persuaded because she "Understood the concept of transferring weights and determining the centre of gravity." This is similar to Student 16, who claims that the theorem is "very convincing… because the idea of locating CGs [centres of gravity] and moving weight masses around and whatnot makes sense." Student 5 sums up with the crux of the theorem itself: "We know about the UCG [uniqueness of the centre of gravity] so they have to meet at one point." All these students found the theorem convincing because it makes use of principles that agree with their physical intuition.

Easy to Understand: Student 9 states that "It was convenient and much easier to use the physics principles to prove the theorem." Student 4 agrees with this so long as "Your concept is clear." Student 6 says that the arguments are convincing because "They are not too abstract, so they're easier to understand." This is perhaps because, as Student 7 points out, "[They] are logical and obvious." Student 18 claims that the Triangle Median Theorem is convincing because "It proved the theory and is easy to understand, even for people who don't really understand math." The drift of these answers seems to be that the arguments from physics carry conviction because they are transparent.

Consistent with Known Results: A final grouping of students found the Triangle Theorem convincing because it agrees with either geometry or the results of experimentation. On the one hand, some students felt that the arguments from physics were convincing because they made the same claims as the purely geometrical arguments. Students 1 and 4 state that the theorem is persuasive because it is "Consistent with the mathematical proofs" and because one arrives at "The same answer in both methods." Student 13 seems to be expressing the same idea, if a little obscurely, with the statement that "It is the same way to prove it only different methods."
Some students, on the other hand, thought that the arguments were convincing because it would be possible to verify them physically. Student 2 thought that the theorem was credible because, "It works with statics, and a more hands-on application could be used to understand the theorem." Student 16 was similarly convinced by the possibility of physical verification, stating that "Also it can be proven using weights and whatnot so that helps further understanding." The notion of external verification also seems to underlie Student 17's claim that the physical arguments are convincing because "They lead to a valid answer." It should be noted that the students never actually did a physical verification of the Triangle Theorem; they did not actually see a triangle balanced on its centre of gravity. But a number of students still put forward the mere possibility of such a physical verification as grounds for conviction.

Conclusion

In this study we investigated a novel application of statics to geometrical proof in the classroom, seeking to determine if and how it might help students understand and produce a proof of a geometrical theorem. Our results show that in constructing their proofs the students found the use of arguments from statics quite convincing, and also that we were successful in conveying to the students the broader idea that concepts from statics can be used in proving mathematical theorems. These results strengthen our belief that teaching geometrical proofs using concepts and principles from physics can make a significant contribution to creating the sort of rich context in which students can best learn proving. It appears to be a promising approach worthy of further exploration.

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