Teaching proof in the context of physics

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Abstract: This paper describes an application of statics to geometrical proofs in the classroom. The aim of the study was to find out whether the use of concepts and arguments from statics can help students understand and produce proofs of geometrical theorems. The two theorems studied were: (1) that the medians in a triangle meet at a single point which is the centre of gravity of the triangle, and (2) the Varignon theorem, that the lines joining the midpoints of successive sides of a quadrilateral form a parallelogram. The classroom experiment showed that most students were successful in using arguments from statics in their proofs, and that they gained a better understanding of the theorems. These findings lend support to the claim that the introduction of statics helps students produce proofs and grasp their meaning.

Kurzreferat: Im folgenden Beitrag wird eine Studie beschrieben, in der die Verwendung von Begriffen und Argumenten aus dem Bereich der Statik zur Förderung des Verständnis von geometrischen Beweisen untersucht wurde. Dabei geht es einerseits um die Aussage, dass sich die Seitenhalbierenden eines Dreiecks in einem Punkt schneiden, welcher gerade der Schwerpunkt des jeweiligen Dreiecks ist, und andererseits um das Varignon Theorem, nach dem man ein Parallelogramm erhält, wenn man die Mittelpunkte der Seiten eines Vierecks verbindet. Das Unterrichtsexperiment zeigte, dass die meisten Schüler bei der Anwendung der Argumente aus der Statik erfolgreich waren und ein besseres Verständnis bezüglich der geometrischen Aussagen aufbauten. Die Ergebnisse stützen die Vermutung, dass die Einführung von Statik für Schüler beim Beweisen hilfreich ist.

ZDM-Classification: E53, G43, M10, M50

1. The Context

A mathematical proof, by definition, can take a set of explicit givens (such as, axioms, accepted principles or previously proven results), and use them, applying the principles of logic, to create a valid deductive argument. The deductive argument is no less compelling if the givens happen to come from the science of statics, as they do in the example below. In fact, the physical context, appealing as it does to physical intuition, has the advantage of making the plausibility of the conclusion more readily apparent (Hanna and Jahnke, 2002; Uspinskii, 1961).

This paper investigates a novel approach to the effective teaching of proof: the use of concepts and models from physics. In this approach, proving is embedded in the context of physics. Ideas from physics that are already familiar to students, such as the concepts of balancing objects and of the centre of gravity, serve as tools for proving geometrical theorems, and students are prompted to create proofs based upon physical considerations. That is, students are encouraged to build a mathematical proof by taking as given one or more principles of physics.

This idea can be made clear by the following example. Let us take as given the following three principles of statics:

P1: The uniqueness of the centre of gravity (each system of masses has one and only one centre of gravity).

P2: The lever principle (the centre of gravity of any two masses lies on the straight line joining the masses, and its distances from the masses are inversely proportional to them).

P3: The principle of substitution (if any two individual masses are replaced by a single mass equal to the sum of the two masses and positioned at the centre of gravity of the two masses, then the location of the centre of gravity of the total system of masses remains unchanged).

These three principles can then be used in proving the following geometrical theorem:

The medians of a triangle intersect at a single point and on each median this point is located two-thirds of the way from the vertex. The point of intersection is known as the centre of gravity of the triangle, or the *centroid*.

To produce a rather simple proof using arguments from physics, one would consider the vertices of the triangle as loaded with equal masses of unit weight and connected by rigid weightless rods. The centre of gravity of any side is the midpoint of that side (by P1 and P2). Therefore the unit masses at its ends can be replaced by a mass of weight 2 at its midpoint (by P3) without altering the centre of gravity of the entire triangle. If we then connect this midpoint with the third vertex to form a median, we can conclude that the centre of gravity of the whole triangle must lie on this median, and by P2 the median must be divided in the ratio 2:1 (Figures 1 and 2). Since this construction can be repeated with the other two sides, the three medians must meet in one and the same point, the centre of gravity. (There are several geometric proofs of this interesting theorem, which is a corollary of the more general Ceva theorem).

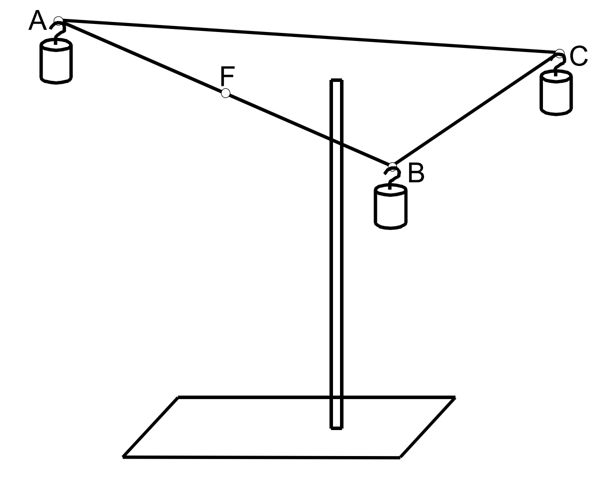

Figure 1. Triangle with equal masses at the vertices A, B and C.

2. Previous research on teaching proof

Several studies have suggested that success in helping students construct proofs and justify their thinking depend upon both the cognitive development of the students and the provision of a context meaningful to them. A meaningful learning context includes well-chosen tasks, hands-on experiences, and opportunities for exploration and discussion with peers and teachers. Several researchers have investigated the learning of proof through the provision of heuristic examples (Reiss and Renkl, 2002), or within environments that provide many opportunities for students to explore ideas, investigate, experiment, and make and test mathematical hypotheses (Balacheff, 1991; Ball, 1991; Lampert, 1990; Maher and Martino, 1996; Yackel and Cobb, 1996). These researchers have shown that such environments are quite effective in helping students learn how to prove. More recent studies have also shown the effectiveness of the opportunities for exploration provided by dynamic geometry software in particular (Goldenberg and Cuoco, 1998; Hadas, Hershkowitz and Schwarz, 2000; Jones, 2000; Marrades and Gutiérrez, 2000; Mariotti, 2000).

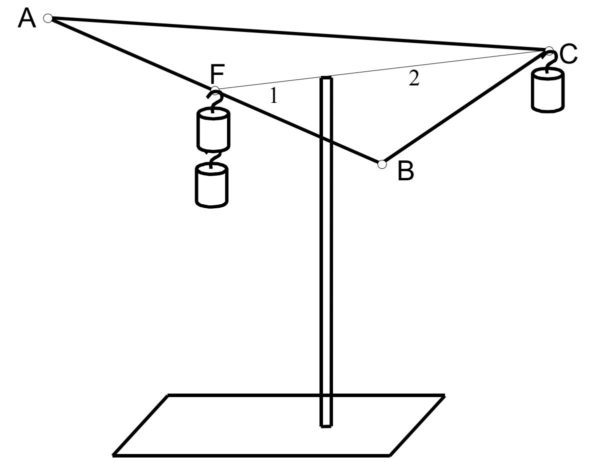

Figure 2. Masses at A and B are moved to the midpoint F of AB.

Tall (1999) recommends using different types of proof to address different stages of cognitive development. The type he calls "physical enactive proof" entails carrying out a physical action to demonstrate the truth of something, thought experiments, imagining prototypical instances, and formal proof for university students at an advanced stage of cognitive development. He observes that the simpler representations of proof such as the use of physical demonstrations may be appropriate for almost any stage of cognitive development.

Building upon this previous research, the present study seeks to determine the effectiveness of giving students opportunities for exploration in another fashion, through the application to proof of principles from statics, coupled with related hands-on experiments. In other words, it explores the use of principles from statics as an effective context for learning deductive proof.

3. Research Design

Following an earlier study on the use of ideas from physics in mathematical proof (Hanna, Jahnke, de Bruyn & Lomas, 2001), this second study was carried out in an Ontario mixed-grade 11 and 12 class during the second semester of the school year 2000-2001. There were 21 students in the class, 13 female and 8 male. The youngest student was 16, the eldest 18; the average age was 16.8. Fourteen of the 21 students were in their eleventh year of schooling (third year of high school), having completed the grade 11 mathematics course in the first semester of the same school year. Of the 21 students attending the course, 19 (7 boys and 12 girls) agreed to participate in the study after signing assent forms and obtaining the consent of their parents.

The majority of the students in the class were considered academically high achievers (though not necessarily in mathematics) and had been placed in a program of studies designated as "enhanced". The other students were chosen for this class on the basis of their high achievement in previous mathematics courses. Because the students in this class were expected to study topics that were not part of the regular curriculum, or to investigate in greater depth some of the topics of the curriculum, the new unit was seen as course work which all were expected to complete.

The teacher, in consultation with the researchers, designed a teaching unit consisting of four 75-minute periods. The first period introduced the lever principle, the concept of the centre of gravity as the location of the fulcrum or balancing point, and the substitution principle. The students answered questions on these new concepts by completing two worksheets, and worked through balancing objects with actual levers and other tasks related to levers.

In the course of the lessons the teacher handed out six worksheets for the students to complete. The first two worksheets introduced the students to the three principles of statics mentioned above (the uniqueness of the Centre of Gravity, the Lever Principle, and the Substitution Principle). These worksheets helped the students develop an intuitive understanding of these fundamental concepts by asking them to describe fully systems of two, three, and four weights attached to a single rod about a fulcrum. During period 1 the students completed these first two worksheets working individually. These exercises were entirely numerical and did not involve any work with proofs. Responses to these first two worksheets are not included in our results.

During period 2 the class first discussed the concepts of the existence and the uniqueness of the centre of gravity, as well as their responses to the work-sheet questions on the material taught during period 1. The students worked in groups of two or three to prove the concurrency and trisection of the medians in a triangle and then discussed their findings with the entire class.

Towards the end of the second period students started working on a proof of the Varignon theorem (the midpoints of the sides of any quadrilateral are vertices of a parallelogram), and on completing additional worksheets. During period 3, pairs of students completed their proof of the Varignon theorem and discussed it with the entire class. The students also answered a few questions on the use of arguments from statics in proving both the medians-in-a-triangle theorem and the Varignon theorem.

Period 4 was devoted to discussing the remaining worksheets and wrapping up the unit. At the end of this period the students also answered a questionnaire and were interviewed by the researchers.

4. Data sources

The data for this study was collected in a number of ways. Two of the researchers observed the classroom, making notes on the teachers' lectures and recording the activities of the student workgroups. All of the students' work was done on six separate worksheets, which were handed in to the teacher to mark. These worksheets provided the researchers with data that could be used to assess the students' understanding of the material. After the unit was complete, all the students were asked to fill out a questionnaire with 11 questions on various aspects of the material and its interpretation. Each student was then interviewed by one of two researchers to clarify answers given on the questionnaires and fill in unanswered questions. These interviews were recorded and transcribed.

For the purpose of this study only two sources of data proved essential:

1) The last four students' worksheets, 3, 4, 5, and 6, which asked the students to first prove that the medians of a triangle are concurrent, once using the principles of statics developed in the class, and again using purely geometric considerations. The students were then asked to supply a geometrical proof that a Varignon quadrilateral is a parallelogram, followed by a statics-based proof of the same property. (These worksheets are reproduced in the appendix.)
2) The questionnaires supplemented by the recordings of the interviews that provided minor changes in the answers to the questionnaires, and in a few places gave additional insight into the students' thinking.

5. Findings and Discussion

5.1 Tasks

Worksheets 3 and 4, reproduced in the appendix, asked the students to provide two proofs of the Triangle Medians Theorem: "The three medians of a triangle are concurrent (intersect in a single point). The point of intersection divides each median in a ratio 2 : 1." One proof was to be based on arguments from statics (as shown in figures 1 and 2), whereas the other was to be a traditional geometric proof (see appendix).

Worksheet 5, also reproduced in the appendix, asked the students to provide two proofs, one based on statics and one geometric, of the theorem known as the Varignon Quadrilateral: "The midpoints of the sides of any quadrilateral are vertices of a parallelogram."

5.2 Student Performance

Students' performance varied from proof to proof. Across all four proofs, students' responses were rated on a scale of 0 to 3. The rating scheme was as follows:

0 – very little of the proof is correct; contains isolated facts but shows little in the way of understanding;
1 – the proof is deficient and incomplete, while the proof may be complete in parts, there are serious or numerous errors;
2 – the proof contains a number of minor errors or omissions but otherwise is reasonably organised and indicates a good understanding of the problem;
3 – the proof is thorough and well organised, with at most very minor omissions or errors.

Table 1 displays the number of students who obtained ratings from 0 to 3 as well as those who did not respond or were absent (n/a). Inspection of Table 1 shows that very few students attained the maximum rating of 3 for the Triangle proof, and that more than half the class produced proofs that were judged to contain serious errors (ratings 0 or 1). Performance was better on the quadrilateral Varignon theorem, where more students (9 and 11 students in the case of the statics and the geometric proofs respectively) produced proofs that were acceptable or contained only minor errors (ratings 2 or 3).

Table 1: Number of students who obtained ratings of 0 to 3, and mean performance by question (N=19)

Question	Ratings					
	0	1	2	3	mean	n/a
Triangle: Statics Proof	4	9	4	1	1.1	1
Triangle: Geometric Proof	1	10	4	3	1.5	1
Quadrilateral: Statics Proof	2	4	8	1	1.5	4
Quadrilateral: Geometric Proof	-	4	5	6	2.1	4
n/a no answer (missing information)						

For both the Triangle Medians theorem and the quadrilateral Varignon theorem, students' performance was slightly better on the geometric proof. Although the difference was insubstantial, this would seem to indicate that the shift from a traditional geometric to a statics context did not make it easier for the students to build a deductive proof. This might not be surprising, given at least two factors that reduced the likelihood of such an effect: 1) the students were already familiar with the geometric context, and 2) the teacher was more experienced teaching the geometric proof. It should be stressed that in both contexts, that of geometry and that of statics, the students were engaged in providing deductive arguments, not empirical ones.

Table 1 also indicates that the students were more successful on the second theorem, the Varignon quadrilateral, on both the geometric and the statics proofs. This is probably due to a learning effect, as the students gained experience and understanding from their first proof and from the subsequent class discussion of the Triangle Medians theorem.

5.3 Examples

Some examples will serve to illustrate the students' work and the teacher's thinking in the assignment of rank.

For the first proof, the statics proof of the Triangles Median Theorem, the majority of students (13 out of 18 respondents) supplied only an incomplete argument (were rated 0 or 1). Student 17, who was given a rank of 1 by the classroom teacher, can be taken as typical. Student 17 sometimes applies the Substitution and Lever Principles correctly, but does not refer to these principles directly. Indeed, she shows the replacement of masses at A and B with 2 kg at F (Figures 1 and 2) on one diagram, but makes no clear reference to this in the text of the solution. The conclusion is incorrectly derived, in any case, because the argument states that the Uniqueness of the Centre of Gravity is a consequence of the division of each of the three medians in a 2:1 ratio. Also, the fact that the medians should have a common point of intersection is not specifically stated.

In the geometric proof of the Triangles Median Theorem, the majority of students (11 out of 18 respondents) were again unable to complete the proof. Student 6, with a rank of 1 assigned by the classroom teacher, was typical. He answered parts a) i) ii) (see worksheet 4 in the Appendix) almost correctly, only omitting the statement that EF||CB by the Triangle Side Splitting Theorem. However, he erroneously concluded that this proves that all three medians intersect at a single point. Moreover, he stated that G and G' are the same point in part c) because "all 3 lines have to intersect at the same point," in other words, assuming concurrence or importing physical considerations.

The highest rank received for this problem was 3. We can again take Student 17 as representative. She provided excellent solutions to all parts, being one of only two students to realize that parts a) i) and ii) without part b) does not prove that all three medians are concurrent. Minor mistakes are made in part i) when writing "(SS~)" for the Side-Angle-Side Similar Triangle Theorem instead of (SAS~); and neglecting to state the reason for the statement "EG/GB = FG/GC = 2/1" in part a) ii).

The performance of the students on the geometric proof of the Varginon Quadrilateral was quite good overall, and better than that on any other worksheet. A few students may be noted by way of example. Student 4 received a rank of 2. In his solution he demonstrated a clear understanding of the problem, with only a few small errors and omissions. Student 4 correctly demonstrated four pairs of similar triangles but omitted the relevant angles to prove that the line segments parallel. For example, after stating "tri.AHE ~ tri.ADB," he should have stated that ang.AHE = ang.ADB by the similarity of these triangles and that HE||DB by the parallel theorem (see worksheet 5).

Student 5 provides an example of the rating 3. Student 5 was the only student who made use of the Triangle Side Splitting Theorem. This student provided a good solution, though the proof would have been clearer if he had stated which triangles were involved in each application of the Side Splitting Theorem. Moreover, the conditions used in each application of the theorem to prove line segments parallel, were not exactly the same as the conditions given in earlier classroom work, and student 5 failed to make these precise conditions explicit.

The students performed better on the statics proof of the Varginon Quadrilateral than they did on the statics proof of the Triangles Median Theorem. Student 15, with a rank of 2, provides an example of the sort of answers the students gave on this problem. Student 15 correctly shows, in part i), that the Centre of Gravity lies on GE and also on HF, but fails to demonstrate that the centre of gravity is at the midpoint of these segments, which is what is required for the conclusion that she draws in part ii). Although she uses the principles of statics correctly, they are not explicitly named.

5.4 Trends in Student Performance

In general, the students did better on the Quadrilateral Theorem than on the Triangle Theorem as they got back into the swing of writing proofs. There was some improvement on both the statics-based proof and the geometric proof. In the case of the statics-based proof this should not be surprising, as none of the students had ever written a statics-based proof before; it is also very unlikely that any of them had ever read one.

Moreover, although the principles of statics introduced in this unit are intuitively obvious, they can also, by the same token, be an obstacle to learning. Fischbein (1999, p. 22) has observed that:

> ... the intuitiveness of a certain property tends to obscure in the student's mind the mathematical importance of it. An apparently trivial property seems to discard the necessity and utility of mentioning it explicitly, of proving it or defining it.

In addition, it does take some experience to see how certain principles of statics can be made to function in a mathematical argument. Our intuition of the principles themselves only develops as we see how they function and what they entail. Much like a mathematical axiom, these principles of statics are neither very helpful nor very informative outside the context of what they explain and imply. But, as discussed by de Villiers (1998) and Koedinger (1998), one would expect their importance and usefulness to become recognised as students acquire more experience in employing them as an integral part of proving.

5.5 Students' responses to the questionnaires

The students responded to a questionnaire put together based on the researchers' experience with a similar teaching experiment in the previous year (Hanna et. al., 2001). On the whole the students seemed to find the physical arguments based on the three principles of statics interesting, convincing and explanatory. Their responses also tell us something about what they think about proof in general. The results, taken from the written questionnaires, are summarised in Table 2 on the following page.

The students' written responses to the questionnaire were followed up with a recorded interview that was later transcribed. For the most part, the interviewers only focused on questions that had remained unanswered in the questionnaires. In some cases, though, the interviewer probed a student further about a questionnaire answer that was particularly interesting or unclear.

Comments were recorded question by question and grouped around themes where possible. Most comments have simply been noted, and only interesting or divergent comments have been quoted. Almost all of the comments cited here are taken from the questionnaires. A few exceptions, taken from transcripts of the interviews, have been noted as such.

The first set of questions give us some insight into what constitutes conviction in the minds of the students and what makes a proof clear and memorable for them.

Table 2: Responses to selected questions of the questionnaire (N=19)

Question	Responses		
1. Are the physical arguments convincing in the case of:			
Triangle Theorem?	Yes (16)	No (0)	?(3)
Quadrilateral Theorem?	Yes (14)	No (2)	?(3)
2. Which proof was clearer in the case of:			
Triangle Theorem?	Physical (7)	Geometric (8)	?(4)
Quadrilateral Theorem?	Physical (1)	Geometric (13)	?(5)
3. Which proof was easier to remember in the case of:			
Triangle Theorem?	Physical (8)	Geometric (6)	?(5)
Quadrilateral Theorem?	Physical (5)	Geometric (9)	?(5)
7. Do the proofs from physics help you understand the nature of proof?			
	Yes (11)	No (1)	?(7)
8. Do the physics proofs help you understand why the theorems are true?			
	Yes (12)	No (2)	?(5)
"?": no answer or don't know.			

Question 1: Are the arguments from physics convincing? If so, why?

Almost all the students report that they found the arguments from physics convincing in the case of Triangle Theorem. What is interesting is the wide variety of reasons given. The students' reasons fall broadly into three types; they saw the physical arguments as convincing because they were 1) easy to understand, 2) physically coherent, or 3) consistent with other known results or otherwise verifiable.

Easy to Understand

Student 9 states that, "It was convenient and much easier to use the physics principles to prove the theorem." Student 4 agrees with this so long as "Your concept is clear." Student 6 says that the arguments are convincing because "They are not too abstract, so they're easier to understand." This is perhaps because, as Student 7 points out, "[They] are logical and obvious." Student 18 claims that the Triangle Theorem in the physics context is convincing because, "It proved the theory and is easy to understand, even for people who don't really understand math." The drift of these answers seems to be that the arguments from physics carry conviction because they are transparent.

Physically Coherent

Student 12 found the Triangle Theorem convincing because it applies the laws of Physics. Students 5, 15 and 16 were convinced by the cohesion of the physical principles. Student 15 stated that she was persuaded because she "Understood the concept of transferring weights and determining the centre of gravity." This is similar to Student 16, who claims that the theorem is "very convincing… because the idea of locating CGs [centres of gravity] and moving weight masses around and whatnot makes sense." Student 5 sums up with the crux of the theorem itself: "We know about the UCG [uniqueness of the centre of gravity] so they have to meet at one point." These students find the Triangles Theorem convincing because it makes use of principles that agree with their physical intuition.

Consistent with Know Results

A final grouping of students found the Triangle Theorem convincing because it agrees with either geometry or the results of experimentation. On the one hand, some students felt that the arguments from physics were convincing because they made the same claims as the purely geometrical arguments. Students 1 and 4 state that the theorem is persuasive because it is, "Consistent with the mathematical proofs," and one arrives at "The same answer in both methods." Student 13 seems to be expressing the same idea, if a little obscurely, with the statement that "It is the same way to prove it only different methods."

On the other hand, other students felt that the arguments were convincing because it would be possible to verify them physically. Student 2 thought that the theorem was credible because, "It works with statics, and a more hands on application could be used to understand the theorem." Student 16 was similarly convinced by the possibility of physical verification stating that, "Also it can be proven using weights and whatnot so that helps further understanding." Some similar notion of external verification must underlie Student 17's claim that the physical arguments are convincing because "They lead to a valid answer." It should be noted that the students never actually did a physical verification of the Triangle Theorem. They did not actually see a triangle balanced on its centre of gravity, because this demonstration was not done in class, but had only a mental image of the centre of gravity as the point at which the triangle could rest on a fulcrum and remain balanced. But a number of students still put forward the mere possibility of such a physical verification as grounds for conviction.

A few students found the physical arguments in the Quadrilateral Theorem sufficiently confusing to be left uncertain or skeptical of the result, but again the overwhelming majority found the arguments convincing. Most of the comments focused on how much more difficult Theorem 2 was when compared to Theorem 1. The comments again point out that conviction arises from coherency and consistency. Student 19 states that the theorem was convincing "Because it relies on the first theorem," or, as Student 1, because it is "Provable through the triangle medians theorem." Student 1 finds the theorem convincing because it is "Provable… using geometrical proofs." Student 12 was persuaded by the theorem's completeness, saying that "Everything seemed balanced and precise. Nothing was left out."

Questions 2 and 3: Which proof, the geometric one or the one based on arguments from physics was clearer? Which was easier to remember?

Clear

The eight students, who thought that the geometric proof was clearer in the Triangle Median Theorem, said that this was so because they had encountered such theorems before and had developed a certain degree of familiarity with geometric proofs. Seven students found the arguments from physics clearer. A couple of these gave interesting reasons. Student 1 said that “The proof using the arguments from physics was simpler and it seemed more relevant than using geometric methods because it requires less ‘outside’ (non-formal) explaining.” In other words, the proof based on physical intuition was more to the point. Student 12 claimed that the physical argument was clearer because it was “Much more in depth and follows a certain path.” The interviewer followed up on this response and asked the student to expand upon it. The Student 12’s response was that “Geometry is equal in one side but physics is using laws of physics pertaining to the actual things we do in life. It’s more in-depth mathematics. It is just equal to two things and it seems equal.” The point here seems to be that the arguments from physics are more convincing because the analogy is deeper. The student’s claim appears to be that we have geometry on “one side” and “the actual things we do in life” on the other side. In the mind of this student, the arguments based on physical principles were related to both abstract geometry and physical reality and thus carried more weight and were more persuasive. The above mentioned students’ claims seem to confirm observations made by Tall (1991) and Goldenberg (2001) that students tend to base their reasoning on holistic visual representations.

Almost all of the students felt that the geometrical arguments were clearer in the case of the Quadrilateral Theorem, but here they gave more reasons. Again, most found the geometric argument clearer because they had more familiarity, but some found it more visually intuitive and concisely argued. Student 15 found it clearer because she could “Visually understand it.” Student 19 claimed that the geometric proof was clearer because “It made more sense, it was visible and evident - You could write it in words.” Student 2 found it “More precise in terms of reasoning.”

A couple of students spoke up for the physical argument, however. Although Student 3 found the geometrical argument clearer based on familiarity, he acknowledged that “The quadrilaterals were more easily proved with the physics stuff.” Student 12 stated that the “Physics arguments were clearer since they were much easier to prove and required little knowledge.”

Memorable

Slightly more students found it easier to remember the physical arguments than the geometrical arguments in the case of the Triangle Median Theorem. This was because the physical arguments were either 1) more recent, 2) shorter or 3) more intuitively obvious.

Student 9 says that he found the physical arguments easier to remember: “I think they were more fresh in my mind because it was new to me.” Students 2 and 5 found the physical arguments easier to remember because there was “Less… to remember” and they were less “Involved.” Student 15 found the physical argument easier “Because it was more or less one single formula.” A few students seemed to find that the physical principles made the proof intuitively obvious. Student 1 stated that “The physics proof was easier because things can be assumed without proving. Using the unique centre of gravity theorem many things require less ‘proof’ than using geometric methods.” Student 16 is probably expressing a similar thought by claiming that it is “Easier to remember because there isn’t much to it, just moving masses around and finding CGs.” Student 19 found the physical argument easier to remember “Because it was just a known and given fact.”

A few more students found it easier to remember the geometrical proof than the physical one in the case of the Quadrilateral Theorem. For the most part this was because the geometric arguments were more familiar. In one case this was because the geometry was found to be intuitively obvious. Student 19 found the geometric proof easier to remember “Because you don’t have to remember it, you just see the similarities and prove it.”

Question 7: Do the proofs from physics help you understand the nature of proof?

All but one student reported that the proofs from physics helped them to understand the nature of proof, but the responses show that the majority students misread the question. Seven students took the question to be asking whether the proofs from physics helped them to gain insight into these particular theorems. Student 3 said that the physical proofs helped, “Because of the visual aspect,” while Student 5 found that, “They help explain things like the medians.” Student 4 wrote, “The physics proofs also helped me to understand the nature of proof because I noticed some new points when I was using arguments from physics in both theorems and I learned that we can also look into these theorems by this way that we consider masses on each corner.” Student 6 stated that, “Because the physics arguments are easier to explain in everyday language, it’s easier to understand exactly how the things have been proven.” Student 17 said, “The balance and weight that we used help aid my understanding of proof. To figure out problems mathematically and to show it with the balance, proves the basic ideas of this theorem.” Students 18 and 19 pointed out that the physical principles made the theorems obvious. Student 18 said, “Some physics have become common sense knowledge so it made sense,” and Student 19 claimed that physical argument, “Makes it easier to explain certain things.” Although Student 9’s comment was framed in terms of proofs in general, it should also most likely be read as a comment about these particular theorems. It reads, “Because I grasp the physics part I understand the “how’s” and the “why’s” of proofs.” Only Student 12 read the question as we had intended and said that the physical arguments show, “That proof is a step by step process.” The responses to this question indicate that the students in this class were not yet at the stage of making statements about proofs in general.

Question 8: Are These Proofs Explanatory?

Question 8 was phrased as follows: Do the proofs from physics help you understand *why* the theorems are true, not just *that* they are true?

Since explanation is one of the most important functions of proof in the classroom (Dreyfus and Hadas, 1996; de Villiers, 1998; Hanna, 1990), it is instructive to examine the students' answers to this question and understand what constituted an explanation for them.

The vast majority of students found that the physical arguments helped them to understand *why* the theorems were true. Student 3 thought that the physical arguments helped because they "Provide the visual aspect." Student 6 had a similar reaction and wrote that the proofs from physics were helpful because "You can visualize the stuff." A number of students said that they found them helpful because they could be related to experiences of the real world. Student 6 claimed that the physical proofs were helpful "Because with the physics arguments you can visualize the stuff and relate it to your everyday experiences." Student 16 said that these arguments "Help understanding of why they are true… mainly because we know that the whole CG thing is true from real life experience." Student 12 finds them helpful because "They compare math with everyday laws of gravity and other things." The only student who claimed that the physical arguments were not helpful in understanding why the theorems are true gave a very interesting answer. Student 1 wrote that "The physics proofs don't help understand why the theorems are true but rather they show why the theorems are necessary. Doing the math proof seems like a waste of time, the physics arguments bring more application of the proofs."

6. Summary

This study investigated a novel application of statics to geometrical proof in the classroom, seeking to determine if and how it might help students understand and produce a proof of a geometrical theorem. The results show that in constructing their proofs most the students found the use of arguments from statics quite convincing but fewer found them clearer than geometric arguments. The results also show that the broader idea that concepts from statics can be used in proving mathematical theorems was more or less conveyed successfully. Also the use of arguments from physics seemed to have helped some students gain insights into the theorems they were asked to prove and helped them understand *why* the theorems were true. This investigation strengthens the belief that teaching geometrical proofs using concepts and principles from physics can make a significant contribution to creating the sort of rich context in which students can best learn proving. It appears to be a promising approach worthy of further exploration.

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Appendix:

Worksheets 3 to 6 asked the students to supply four proofs. The students broke up into workgroups of three or four students who worked together. The students were first asked to prove that the medians of a triangle are concurrent, once using the principles of statics developed in the class, and again using purely geometric considerations. The students were then asked to supply a geometrical proof that a Varignon quadrilateral is a parallelogram, followed by a statics-based proof of the same property. These worksheets were phrased as follows:

Worksheet 3

A. Triangle Medians Theorem.

Theorem 1: The three medians of a triangle are concurrent (intersect in a single point). The point of intersection divides each median in a ratio 2 : 1.

Statics proof of medians of a triangle theorem

1. Given that D, E and F are the midpoints of their respective sides of triangle ABC, prove the triangle medians theorem using physical reasoning by applying the lever principle (LP), Substitution Principle (SP) and existence and uniqueness of the centre of gravity (UCG).
Hint: locate the centre of mass on each of the three medians.

Worksheet 4

Traditional proof of medians of a triangle theorem

2. Use the theorems developed in class (chapter 5 of your textbook) to prove the medians of triangles theorem. You may use your own ideas to construct a proof or use the following suggestions.

a) Given. ABC with medians BE and CF intersecting at G. (Figure 3)
  i) Prove EF:CB = 1:2
  ii) Prove EG:GB = FG:GC = 1:2
  iii) What does this prove about two medians? Does this prove that all three medians are concurrent (intersect in a single point)? Why or why not?
b) Suppose medians AD and BE, intersecting at G', were drawn instead of CF and BE and you were asked to do questions similar to a) i) and ii) (Figure 4). What would you be able to prove about DG':G'A and EG':G'B?
c) Explain why G and G' must be the same point.
d) What do these questions prove about the medians of a triangle?

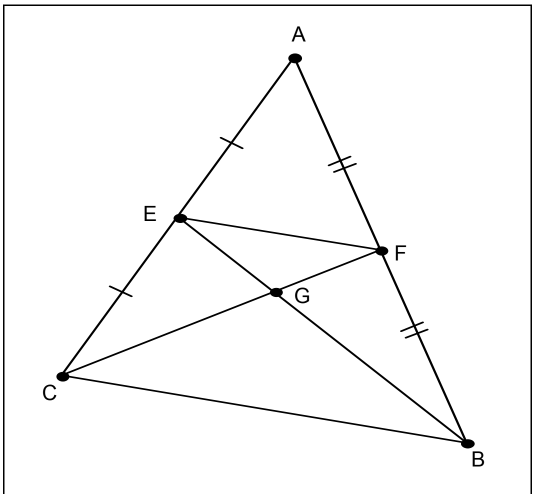

Figure 3: BE and CF intersect at G

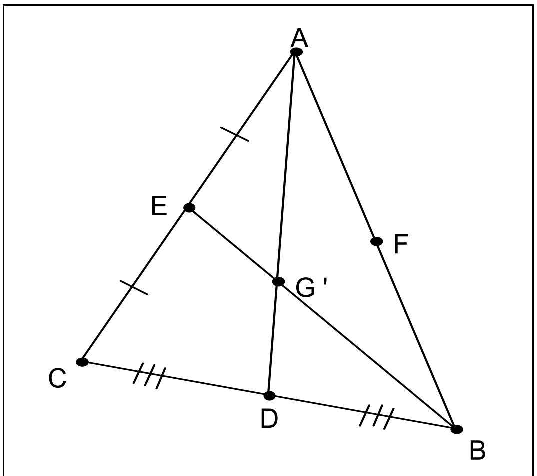

Figure 4: AD and BE intersect at G'

Worksheet 5

B. The Varignon Quadrilateral

Definition. The quadrilateral constructed by joining the midpoints of any given quadrilateral is called a *Varignon quadrilateral*

Theorem **2:** The midpoints of the sides of any quadrilateral are vertices of a parallelogram.

You will be asked to prove this theorem in two ways. Firstly to provide a proof using the traditional geometric theorems and methods, and secondly to prove it by means of the Statics principles you just learned.
Traditional (Euclidean) proof of the Varignon Quadrilateral

1. Prove this theorem by using the methods and theorems you learned at the beginning of the course: Given that E, F, G, H are the midpoints of ABCD, prove that EFGH is a parallelogram by proving that its sides are parallel to diagonals AC and BD (Figure 5).

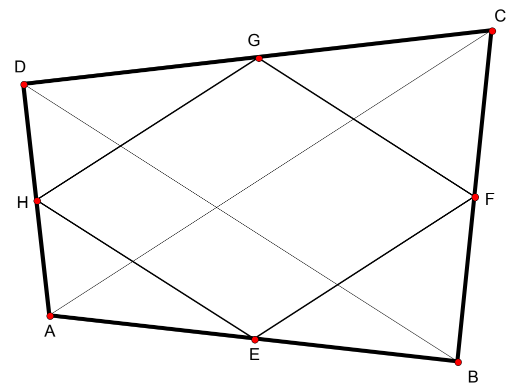

Figure 5: Quadrilateral ABCD with midpoints E, F, G, H

Proof of the Varignon quadrilateral using Statics

2. Prove that the midpoints of the sides of a quadrilateral are the vertices of a parallelogram by means of the principles of statics: *Lever principle, Substitution principle and the existence and uniqueness of the centre of gravity.*

The following are some suggestions to follow in completing the proof, but they can be ignored if you find another way.

Definition: A median of a quadrilateral is a line segment joining the midpoints of a pair of opposite sides of the quadrilateral.

a) Assume equal masses, *m*, located at each of the vertices A, B, C and D of the given quadrilateral (Figure 6).
   i) By applying the substitution principle show that the centre of gravity of ABCD is at the midpoint of each of the two medians of the quadrilateral ABCD.
   ii) How does i) prove that the medians of a quadrilateral intersect each other at their respective midpoints?
   iii) What do you now know about the Varignon quadrilateral EFGH?
   iv) Why do the midpoints of the sides of a quadrilateral form a parallelogram?

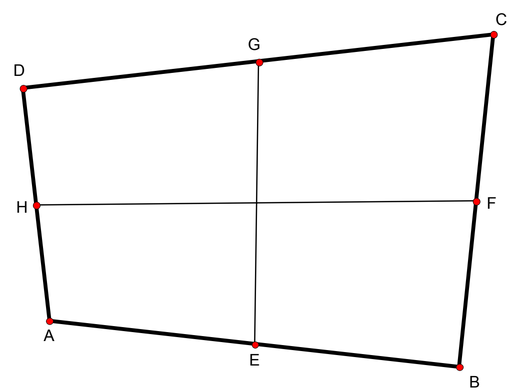

Figure 6: Quadrilateral ABCD with equal masses located at the vertices