# Constructions in ancient Greek *Spherics*: Mathematical spheres and solid globes

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(joint work with Ken Saito)

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# Instruments in Greek mathematics

Aside from some abacus boards and papyri containing calculations, we have no material evidence for practices in Greek mathematics. Nevertheless, there is textual evidence for various instruments. Some examples:

- Plato's and Euratothenes' devices for analog calculation of two mean proportionals.
- Nicomedes' device for producing a conchoid for use in neusis constructions.
- Diocles description of a bone ruler for drafting a parabola.
- Ptolemy's "analemma board" and Heron's "local hemisphere," for analog calculations.

# The geometry of instrumental practice

Certain domains of geometric theory may have been, in some sense, delimited by certain instrumental practices.

- The classic example is Euclid's elementary geometry, which is based on the postulation of operations equivalent to the use of an unmarked ruler and a collapsing compass.
- Pappus's *Collection* VIII gives constructions using an unmarked ruler and a fixed compass.
- The analemma techniques rely on the use of a set square and a non-collapsing compass.
- I will argue in this talk that ancient spherics was a geometry constructed with an unmarked ruler and a non-collapsing compass.

# Instruments in Greek astronomy

We have no surviving observational instruments, but we have many descriptions.

- Mostly from Ptolemy: the meridian quadrant, the equatorial ring, the armillary sphere, etc.
- And others as Archimedes, Geminus and Proclus: the dioptra, the angular size dioptra, the armillary sphere, the gnomon, etc.

We have a fair number of objects related to the general cultural interest in astronomy and astrology.

- Sundials (fixed, portable, geared), the Antikythera mechanism, the parapegmata and other astronomical inscriptions, etc.
- Astrologer's boards and markers, inscribed horoscopes, etc.

# Evidence for Greek globes



We have both material and textual evidence for globes in Greek antiquity. Some examples:

- Ptolemy (*Almagest*) describes the construction of an elaborate star globe, Leontius describes a demonstration globe.
- Geminus refers to "inscribed globes" and Ptolemy (*Plansiphere*) talks about the lines that are found on "struck globes."
- We know of three ornamental globes – the Farnese Atlas and two simple celestial spheres – inscribed with the principle circles and decorated with images of the constellations (all $1^{st}$ or $2^{nd}$ century).

# *Theorems* and *Problems*



The propositions of elementary treatises, like the *Elements* or the *Spherics*, are divided into two types:

**Theorems** Given some set of initial objects, a theorem asserts some property that is true of these objects. (“If... then...”)

**Problems** Given some set of initial objects, a problem shows how to do something (say, to *find*, *draw*, *set out*) and then demonstrates that what has been done is satisfactory. (“To do such-and-such...”)

Some theorems can be intelligibly expressed as problems and the converse. (We will see that theorem *Spher.* I 8 might very well have been a problem.)

# The components of a Greek proposition





Proclus (5th CE) put forward the following six parts of a Greek proposition. (In fact, they are only ever found so complete and clearly divided in Euclid's *Elements*.)

## Introductory components[1]

1. **Enunciation** (πρότασις): A general statement of what is to be shown (done).
2. **Exposition** (ἔκθεσις): A statement setting out the given objects with letter names.
3. **Specification** (διορισμός): A restatement in terms of the specific objects of (1) what is to be shown (theorem), or of (2) what is to be done, including any conditions of solvability (problem).

[1] Introductory division not distinguished by Proclus.

# The components of a Greek proposition





Proclus (5$^{\text{th}}$ CE) put forward the following six parts of a Greek proposition. (In fact, they are only ever found so complete and clearly divided in Euclid's *Elements*.)

## Core components[1]

4. **Construction** (κατασκευή): Statements about the production of *new objects* that will be required in the proof. (Relies on posts. 1-3 and problems.)
5. **Proof** (ἀπόδειξις): A logical argument that the proposition holds (has been done). (Relies on the other assumptions and theorems.)
6. **Conclusion** (συμπέρασμα): A restatement, in general terms, of what has been shown (done).[2]

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[1] Core division not distinguished by Proclus.

[2] Very rare, except in the *Elements*.

# Using constructions to prove theorems





In a geometric theorem, it is sometimes the case that the properties of the objects stated in the enunciation are sufficient to demonstrate the proposition, but more often than not we have to introduce new objects and use their properties in the argument.

These new objects are introduced using constructions. (In fact, the geometer asserts that these objects must have been produced, using the same grammatical constructions, and sometimes even the same verbs as are used when the initially given objects are set out.[3])

[3] In *Spher.* I 8, we will see a case where the exposition is undistinguishable from the construction, except on the basis of the proof structure.

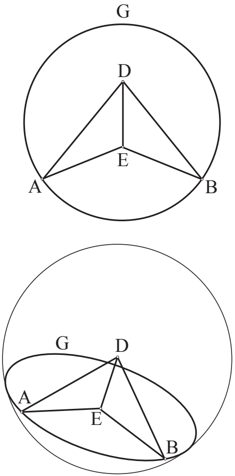
# Using constructions to solve problems



A problem is solved by producing a *specific* geometric object. A problem shows *how* to produce the object using constructions and then uses deductive argumentation to show *why* this object is correct.

This deductive argumentation sometimes requires new constructions in the same way as a theorem. That is, the geometric object that solves the problem together with the initial objects stipulated in the problem are sometimes not sufficient to show that this object *is* a solution. In such cases, we must construct new objects for the proof.

# Example theorem, *Spherics* I 1

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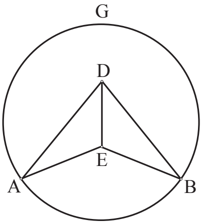
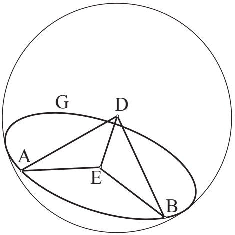
## Introduction

1. “If a plane cuts a spherical surface, the curved line in the spherical surface is a circle.”
2. Let a plane cut a sphere and make curved line *ABG*.
3. “I say,” *ABG* is a circle.

# Example theorem, *Spherics* I 1

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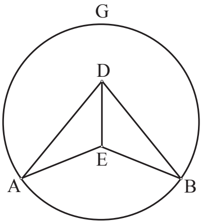
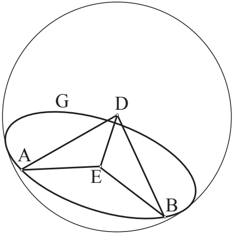
## Core [Case 1]

$5_1$ If the plane goes through the center of the sphere, then every line from the center of the sphere to *ABG* is equal (*definition of a sphere*), so that curved line *ABG* is a circle.

# Example theorem, *Spherics* I 1

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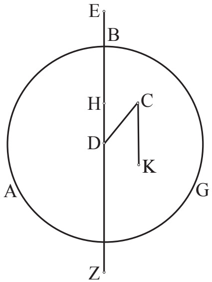
## Core [Case 2]

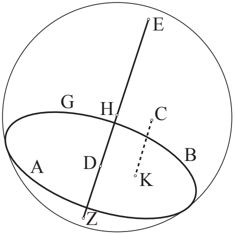
$4_2$ If the plane does *not* go through the center of the sphere, let the center of the sphere be $D$ [?],[a] draw $DE \perp$ to plane $ABG$ [*El*.11.11], join $AE$, $BE$, $AD$ and $BD$ (P.1).

$5_2$ Use *El*.1.47 to show that $EB = EA$. This could be done for any other two points on curved line $ABG$. Hence, $ABG$ is a circle with center $E$.

[a] *How* we locate $D$ is passed over with a simple use of ἔστω. Compare with the next proposition.

# Example problem, *Spherics* I 2

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## Introduction

1. “To find the center of a given sphere.”
2. Let there be a given sphere.[a]
3. “I say,” it is necessary to find its center.

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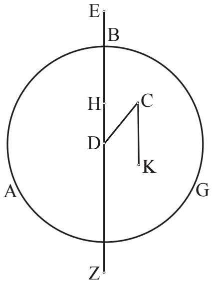
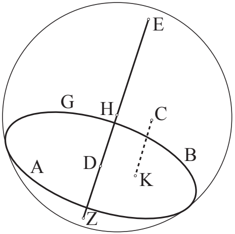
[a] Notice there are no letter names.

# Example problem, *Spherics* I 2



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## Core [solving the problem]

4 Let some plane cut it, making circle *ABG* [?;*Sph*.1.1].[a] Find *D* as the center of *ABG* [*El*.3.1], draw *ZDE* $\perp$ circle *ABG* to meet the sphere at *Z*, *E* [*El*.11.12;P.2]. Bisect *EZ* at *H* [*El*.1.10].

$3_{const}$ "I say," *H* is the center of the sphere.

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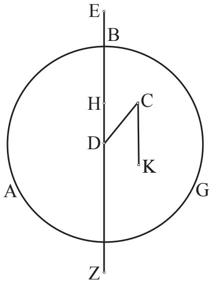
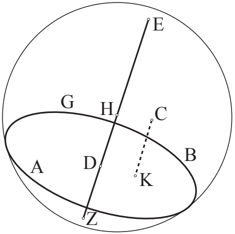
[a] We are not told *how* this is done. Perhaps we can imagine some sort of postulate, "To pass a plane through a solid figure." At any rate, such a construction was regularly performed.

# Example problem, *Spherics* I 2

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## Core [verifying the solution]

5 Assume, if possible, that there is some *other* center, say $C$.

$4_{proof}$ Let a perpendicular be drawn from $C$ to circle $ABG$ at $K$ [*El*.11.11].

Then, $K$ is a center of $ABG$ [*Sph*.1.1], and so is $D$, which is impossible.

# Components of a Geometry Problem

1. $1_p$ **Enunciation:** General statement of what is to be *done*.
2. $2_p$ **Exposition:** Statement of *what is given*, usually using specific, letter names.
3. $3_p$ **First Specification:** Specific statement of *what is to be done*, often with qualifications.
4. $4_p$ **Solution:** Construction of the geometric object which satisfies the requirements of the problem.
5. $5_p$ **Second Specification:** Specific statement of *what is to be shown*.[1]
6. $6_p$ **Construction:** Construction of any *new objects* necessary to the proof.[1]
7. $7_p$ **Proof:** Argument that the *solution* meets the requirements of the proposition; if necessary, using the new objects of the construction.

[1] May be absent.

# The role of constructions in *Spherics* I

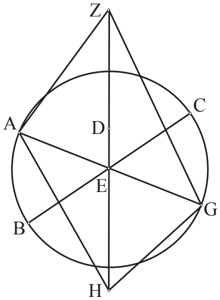
The book opens with a number of propositions that use lines internal to the sphere to demonstrate the properties of great circles and sets of parallel circles and their relations.

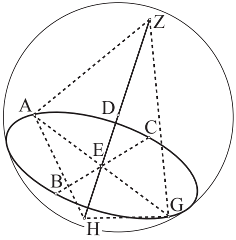
For most of the book, constructions serve the role of supplying objects which are necessary to proofs.

We then have two problems that show us how to draw these internal lines outside of the sphere, into a space where we can *use* them. (From this point on, we can solve problems by actually drawing some object.)

The book ends with two problems that show how to draw a great circle through two points and how to find the pole of a circle working entirely *outside* of the sphere.

# Using internal constructions, *Spherics* I 8

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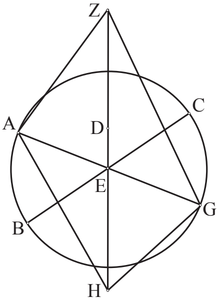
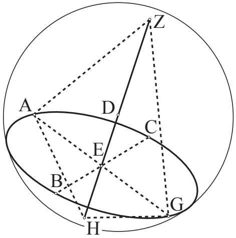
## Introduction

1. "If a circle is in a sphere and a perpendicular is drawn from the center of the sphere and extended in both directions, it will fall on the poles of the circle."
2. Let *ABG* be a sphere, let its center, *D*, be taken [*Sph*.1.2], let *DE* be drawn ⊥ *ABG* [*El*.11.11], and extended to meet the sphere at *Z* and *H* [*El*.P.2].
3. "I say," *Z* and *H* are the poles of *ABG*.

# Using internal constructions, *Spherics* I 8



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## Core

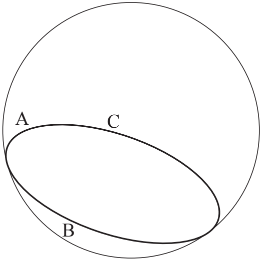
4 Let $AEG$, $BEC$, $AZ$, $GZ$, $AH$ and $HG$ be drawn [*El*.P.2].

5 $AE = EG$, the $\angle$s at $E$ are right and $ED$ is common, so $AZ = GZ$. We could make the same argument for any other point on $ABC$. Likewise with respect to $H$. So $Z$ and $H$ are the poles of circle $ABC$.

6 If a circle is in a sphere, from the center of the sphere, “and so on.”

# Bringing out a diameter, *Spherics* I 18






## Introduction

$1_p$ "To set out (ἐκθέσθαι) the diameter of a given circle in a sphere."

$2_p$ Let the given circle be $ABC$.

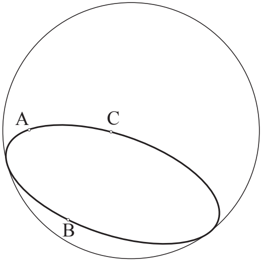
$3_p$ So, it is necessary to set out its diameter.

# Bringing out a diameter, *Spherics* I 18



## Core [solving the problem]

$4_p$ Let three random points, $A$, $B$ and $C$, be taken on the circumference.



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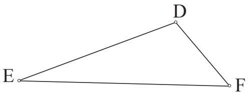
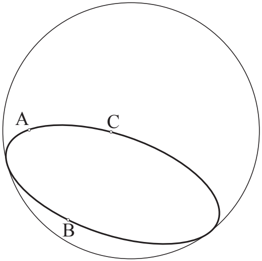


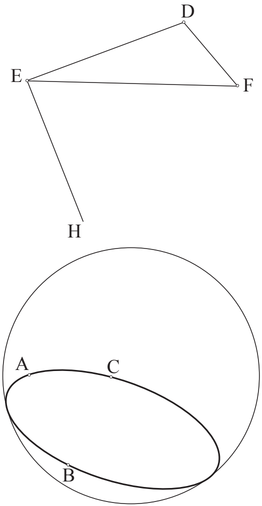

Upper: a plane diagram, outside the sphere

Lower: a solid sphere

## Core [solving the problem]

$4_p$ Let three random points, $A$, $B$ and $C$, be taken on the circumference. Let $\triangle DEZ$ be put together from (συνεστάτω) three lines (such that $DE$ equals that from $A$ to $B$, etc.) [?].

# Bringing out a diameter, *Spherics* I 18

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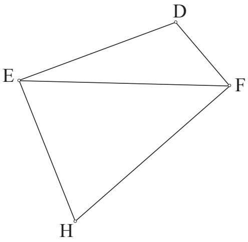


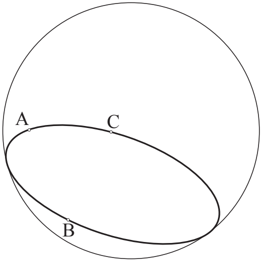


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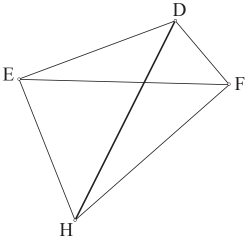


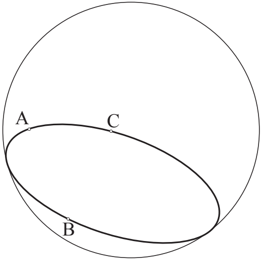


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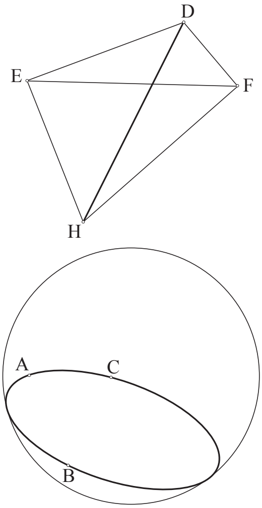
# Bringing out a diameter, *Spherics* I 18

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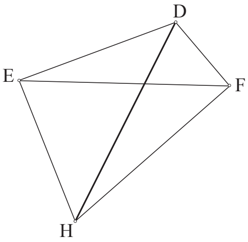
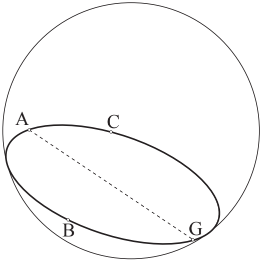
$5_p$ <I say, line $DH$ is equal to the diameter of circle $ABC$.>

# Bringing out a diameter, *Spherics* I 18

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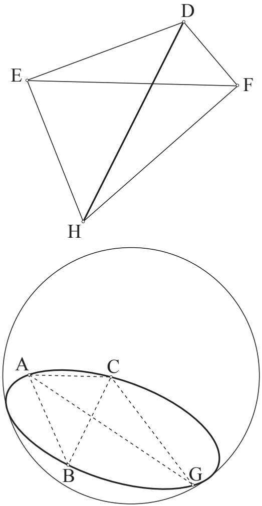
## Core [verifying the solution]

$6_p$ Draw $AG$ the diameter of circle $ABC$ [?; *El*.3.1&P.1],

# Bringing out a diameter, *Spherics* I 18

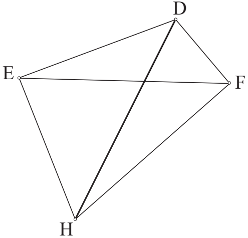


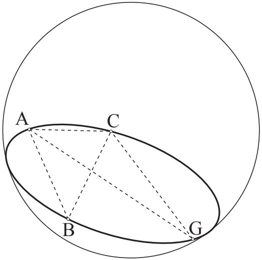



## Core [verifying the solution]

$6_p$ Draw *AG* the diameter of circle *ABC* [?; *El*.3.1&P.1], and join *AB*, *BG*, *GA* and *GC* [*El*.P.1].

# Bringing out a diameter, *Spherics* I 18



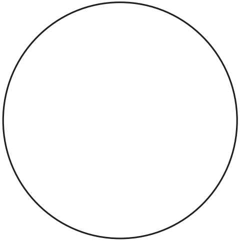

## Core [verifying the solution]

$6_p$ Draw $AG$ the diameter of circle $ABC$ [?; *El*.3.1&P.1], and join $AB$, $BG$, $GA$ and $GC$ [*El*.P.1].

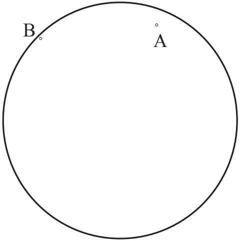
$7_p$ Now, $AB = ED$, $BC = DE$ and $AC = DZ$, $\therefore \angle ABC = \angle DEZ$ [*El*.1.4]. And $\angle ABC = \angle AGC$ while $\angle DEZ = \angle DHZ$ [*El*.3.27], so $\angle AGC = \angle DHZ$. But $\angle ACG = \angle DZH = R$ and $AC = ED$, so $AG = DH$. Therefore, $DH$ is equal to the diameter of the circle.

### Summary of the solution

Let there be a sphere.



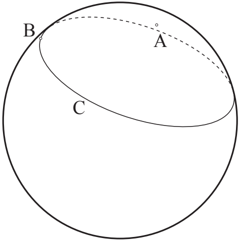
# Bringing out a diameter, *Spherics* I 19




## Summary of the solution

Let there be a sphere. We take two random points, $A$ and $B$.

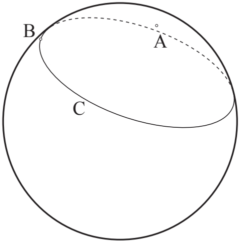
# Bringing out a diameter, *Spherics* I 19




## Summary of the solution

Let there be a sphere. We take two random points, $A$ and $B$. With $A$ as pole and $AB$ as *distance*, we draw circle $ABC$ [?].

# Bringing out a diameter, *Spherics* I 19

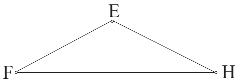

Upper: a plane diagram, outside the sphere

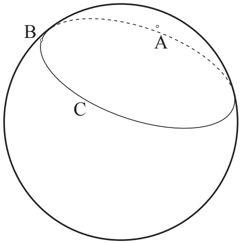
Lower: a solid sphere

## Summary of the solution

Let there be a sphere. We take two random points, $A$ and $B$. With $A$ as pole and $AB$ as *distance*, we draw circle $ABC$ [?]. Then it is possible set out the diameter of this circle as $EF$ [*Sph.*1.18],

# Bringing out a diameter, *Spherics* I 19




Upper: a plane diagram, outside the sphere

Lower: a solid sphere

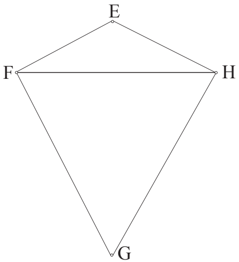
## Summary of the solution

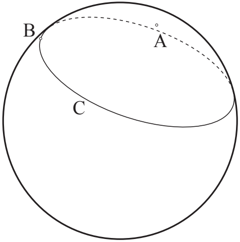
Let there be a sphere. We take two random points, $A$ and $B$. With $A$ as pole and $AB$ as *distance*, we draw circle $ABC$ [?]. Then it is possible set out the diameter of this circle as $EF$ [*Sph.*1.18], so that we can put together $\triangle EFH$ [?;*Ele.*1.22*].

# Bringing out a diameter, *Spherics* I 19







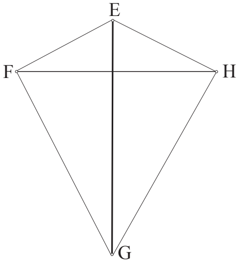

Upper: a plane diagram, outside the sphere

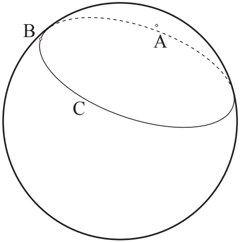
Lower: a solid sphere

## Summary of the solution

Let there be a sphere. We take two random points, $A$ and $B$. With $A$ as pole and $AB$ as *distance*, we draw circle $ABC$ [?]. Then it is possible set out the diameter of this circle as $EF$ [*Sph.*1.18], so that we can put together $\triangle EFH$ [?;*Ele.*1.22*]. We draw $HG \perp EH$ and $FG \perp EF$ [*Ele.*1.11], meeting at point $G$.

# Bringing out a diameter, *Spherics* I 19

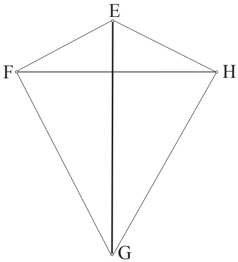


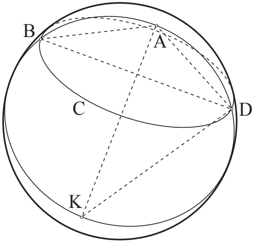




## Summary of the solution

Let there be a sphere. We take two random points, $A$ and $B$. With $A$ as pole and $AB$ as *distance*, we draw circle $ABC$ [?]. Then it is possible set out the diameter of this circle as $EF$ [*Sph*.1.18], so that we can put together $\triangle EFH$ [?;*Ele*.1.22*]. We draw $HG \perp EH$ and $FG \perp EF$ [*Ele*.1.11], meeting at point $G$. We join $EG$ [*Ele*.P.1].

# Bringing out a diameter, *Spherics* I 19



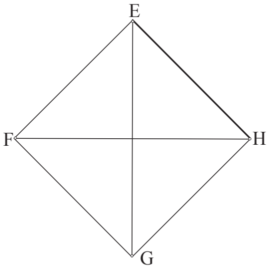



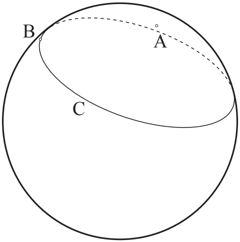

## Summary of the solution

Let there be a sphere. We take two random points, *A* and *B*. With *A* as pole and *AB* as *distance*, we draw circle *ABC* [?]. Then it is possible set out the diameter of this circle as *EF* [*Sph*.1.18], so that we can put together △*EFH* [?;*Ele*.1.22*]. We draw *HG* ⊥ *EH* and *FG* ⊥ *EF* [*Ele*.1.11], meeting at point *G*. We join *EG* [*Ele*.P.1].

We draw the internal lines and show that line *EG* is equal to the diameter of sphere *ABC*.

# An interlude



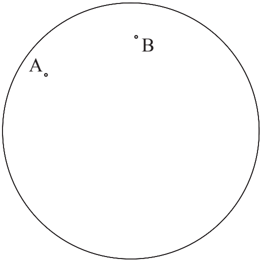




*Spherics* I 19 is never explicitly used in the text.

In a number of propositions, however, it is required to let a circle be drawn with a given point as pole and a pole-distance equal to the side of a square described in a great circle. This construction can be effected using *Spherics* I 19.

We set out the diameter of the sphere, as $EG$. We construct a square on $GE$ as diameter, and use one of its sides, say $EH$, as the pole-distance of the great circle.

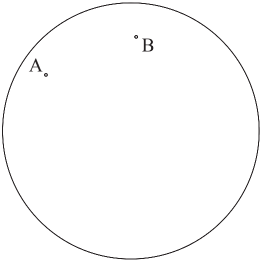
# Drawing a great circle through two points, *Spherics* I 20




## Summary of the solution, 1

Let $A$ and $B$ be two given points.

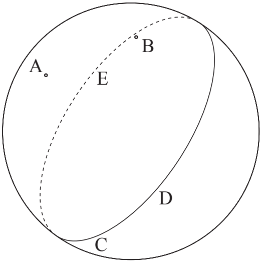
# Drawing a great circle through two points, *Spherics* I 20




## Summary of the solution, 1

Let $A$ and $B$ be two given points. **[Case 1:]** If $A$ and $B$ are the endpoints of a diameter of the sphere, there are an infinite number of great circles through them [?].

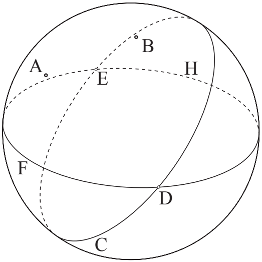
# Drawing a great circle through two points, *Spherics* I 20




## Summary of the solution, 2

Let $A$ and $B$ be two given points.
**[Case 2:]** With $A$ as pole, we draw great circle $CDE$ [*Sph*.1.19 (*inter.)].

# Drawing a great circle through two points, *Spherics* I 20



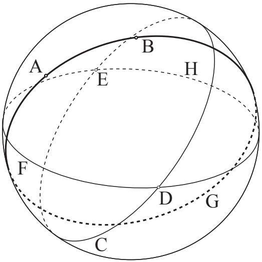

## Summary of the solution, 2

Let $A$ and $B$ be two given points. **[Case 2:]** With $A$ as pole, we draw great circle $CDE$ [*Sph*.1.19 (*inter.)]. With $B$ as pole, we draw great circle $EFH$ [*Sph*.1.19 (*inter.)], intersecting great circle $CDE$ at points $D$ and $E$.

# Drawing a great circle through two points, *Spherics* I 20







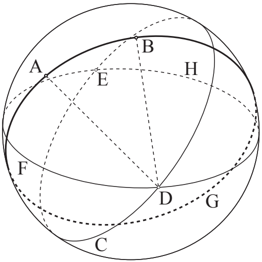

## Summary of the solution, 2

Let $A$ and $B$ be two given points. [**Case 2:**] With $A$ as pole, we draw great circle $CDE$ [*Sph.*1.19 (*inter.)]. With $B$ as pole, we draw great circle $EFH$ [*Sph.*1.19 (*inter.)], intersecting great circle $CDE$ at points $D$ and $E$. With $D$, or $E$, as pole, we draw a circle passing through either $A$ or $B$ [*Ele.*P.3*].

# Drawing a great circle through two points, *Spherics* I 20





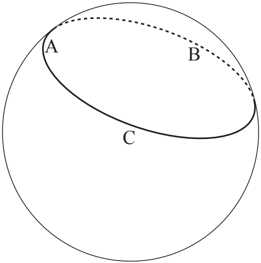



## Summary of the solution, 2

Let $A$ and $B$ be two given points. **[Case 2:]** With $A$ as pole, we draw great circle $CDE$ [*Sph*.1.19 (*inter.)]. With $B$ as pole, we draw great circle $EFH$ [*Sph*.1.19 (*inter.)], intersecting great circle $CDE$ at points $D$ and $E$. With $D$, or $E$, as pole, we draw a circle passing through either $A$ or $B$ [*Ele*.P.3*].

We argue that $AD = BD$ are equal to the side of a square inscribed in a great circle.

# Finding the pole of a given circle, *Spherics* I 21

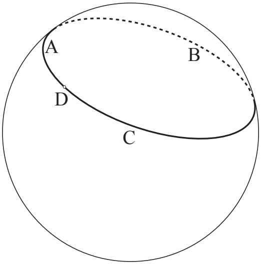



## Summary of the solution, 1

Let $ABC$ be a given circle.

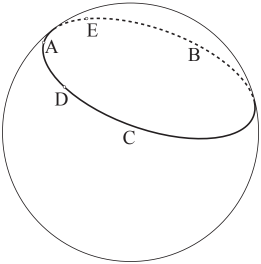
# Finding the pole of a given circle, *Spherics* I 21






## Summary of the solution, 1

Let $ABC$ be a given circle. We take a random point, $D$.

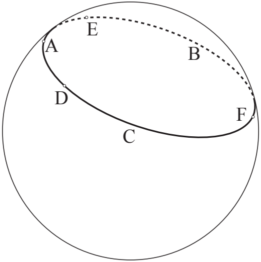



## Summary of the solution, 1

Let $ABC$ be a given circle. We take a random point, $D$. We cut off arcs $AD = AE$ [*Ele*.P.3*].

# Finding the pole of a given circle, *Spherics* I 21






## Summary of the solution, 1

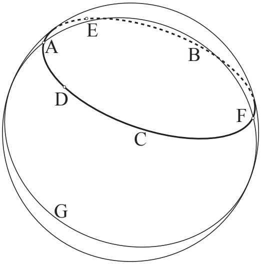
Let $ABC$ be a given circle. We take a random point, $D$. We cut off arcs $AD = AE$ [*Ele*.P.3*]. We bisect arc $DE$ at $F$ [?].[a]

[a] If we *set out* the diameter of $ABC$, we can transfer the arcs $AD = AE$ and bisect the arc using *Ele*.3.30.

# Finding the pole of a given circle, *Spherics* I 21



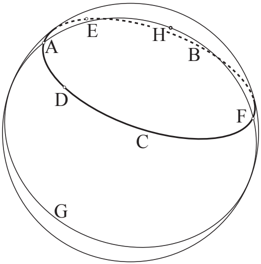





## Summary of the solution, 1

Let $ABC$ be a given circle. We take a random point, $D$. We cut off arcs $AD = AE$ [*Ele*.P.3*]. We bisect arc $DE$ at $F$ [?].

**[Case 1:]** If circle $ABC$ is *not* a great circle, we draw great circle $AFG$ [*Sph*.1.20].




## Summary of the solution, 1

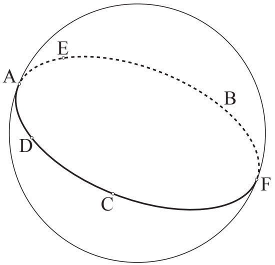
Let *ABC* be a given circle. We take a random point, *D*. We cut off arcs $AD = AE$ [*Ele*.P.3*]. We bisect arc *DE* at *F* [?].

**[Case 1:]** If circle *ABC* is *not* a great circle, we draw great circle *AFG* [*Sph*.1.20]. We bisect arc *AHF* at point *H* [?].[a]

We argue that all the lines drawn from *H* to circle *ABC* are equal.

[a] We *set out* the diameter of the sphere [*Sph*.1.19] and *draw* a great circle. Then we set out the diameter of circle *ABC* [*Sph*.1.18], *fit* it into the great circle [*Ele*.4.1] and bisect the resulting cord [*Ele*.3.30].

# Finding the pole of a given circle, *Spherics* I 21



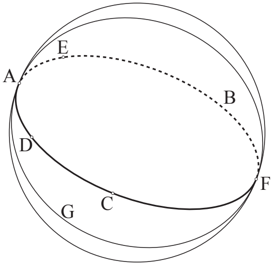

## Summary of the solution, 2

Let $ABC$ be a given circle. We take a random point, $D$. We cut off arcs $AD = AE$ [*Ele*.P.3*]. We bisect arc $DE$ at $F$ [?].

**[Case 2:]** If circle $ABC$ is a great circle, we bisect arc $AF$ at $C$ [?].[a]

[a] We *set out* the diameter of the sphere [*Sph*.1.19], *draw* a great circle, and bisect a semicircle [*Ele*.3.30].

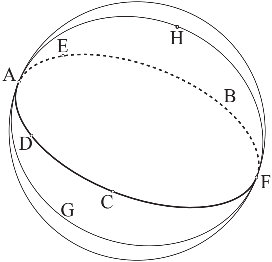
# Finding the pole of a given circle, *Spherics* I 21




## Summary of the solution, 2

Let $ABC$ be a given circle. We take a random point, $D$. We cut off arcs $AD = AE$ [*Ele*.P.3*]. We bisect arc $DE$ at $F$ [?].

**[Case 2:]** If circle $ABC$ is a great circle, we bisect arc $AF$ at $C$ [?]. With $C$ as pole, we draw a great circle through $A$ and $F$ [*Sph*.1.19 (*inter.)].

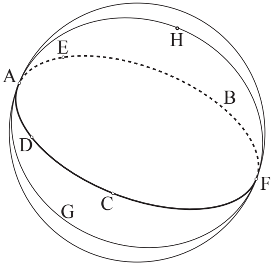



## Summary of the solution, 2

Let *ABC* be a given circle. We take a random point, *D*. We cut off arcs *AD* = *AE* [*Ele*.P.3*]. We bisect arc *DE* at *F* [?].

**[Case 2:]** If circle *ABC* is a great circle, we bisect arc *AF* at *C* [?]. With *C* as pole, we draw a great circle through *A* and *F* [*Sph*.1.19 (*inter.)]. We bisect arc *AF* at *H* [?].[a]

[a] Again, we reproduce the arc outside the sphere, *bisect* it and then transfer the distance back to the sphere.

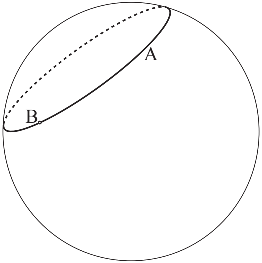



## Summary of the solution, 2

Let $ABC$ be a given circle. We take a random point, $D$. We cut off arcs $AD = AE$ [*Ele*.P.3*]. We bisect arc $DE$ at $F$ [?].
**[Case 2:]** If circle $ABC$ is a great circle, we bisect arc $AF$ at $C$ [?]. With $C$ as pole, we draw a great circle through $A$ and $F$ [*Sph*.1.19 (*inter.)]. We bisect arc $AF$ at $H$ [?].

We argue that all the lines drawn from $H$ to circle $ABC$ are equal.

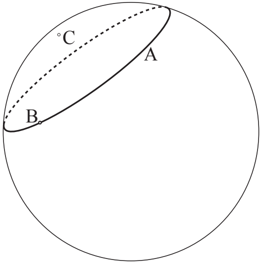
# Drawing a tangent great circle, *Spherics* II 14




## Summary of the solution

Let there be a given lesser circle $AB$ with point $B$ given on it.

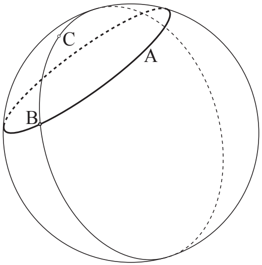
# Drawing a tangent great circle, *Spherics* II 14




## Summary of the solution

Let there be a given lesser circle $AB$ with point $B$ given on it. We find the pole of circle $AB$, as point $C$ [*Sph*.1.21].

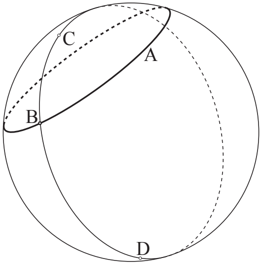
# Drawing a tangent great circle, *Spherics* II 14




## Summary of the solution

Let there be a given lesser circle *AB* with point *B* given on it. We find the pole of circle *AB*, as point *C* [*Sph*.1.21]. We draw a great circle through *C* and *B* [*Sph*.1.21],

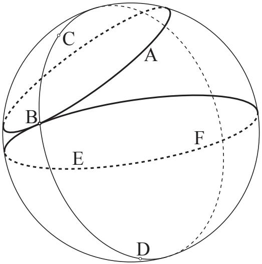
# Drawing a tangent great circle, *Spherics* II 14




## Summary of the solution

Let there be a given lesser circle $AB$ with point $B$ given on it. We find the pole of circle $AB$, as point $C$ [*Sph*.1.21]. We draw a great circle through $C$ and $B$ [*Sph*.1.21], and cut off arc $BD$ equal to a quadrant [*Sph*.1.19 (*inter.)].

# Drawing a tangent great circle, *Spherics* II 14



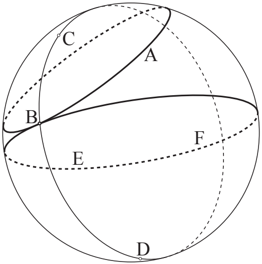

## Summary of the solution

Let there be a given lesser circle $AB$ with point $B$ given on it. We find the pole of circle $AB$, as point $C$ [*Sph*.1.21]. We draw a great circle through C and $B$ [*Sph*.1.21], and cut off arc $BD$ equal to a quadrant [*Sph*.1.19 (*inter.)]. With pole $D$, we draw great circle $BEF$ [*Ele*.P.3*].

# Drawing a tangent great circle, *Spherics* II 14







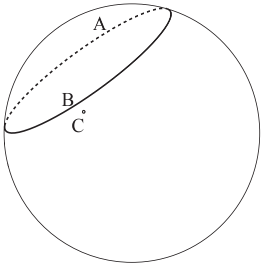

## Summary of the solution

Let there be a given lesser circle $AB$ with point $B$ given on it. We find the pole of circle $AB$, as point $C$ [*Sph*.1.21]. We draw a great circle through $C$ and $B$ [*Sph*.1.21], and cut off arc $BD$ equal to a quadrant [*Sph*.1.19 (*inter.)]. With pole $D$, we draw great circle $BEF$ [*Ele*.P.3*].

The proof follows immediately from the properties of tangency [*Sph*.2.3-5].

# Drawing a tangent great circle, *Spherics* II 15



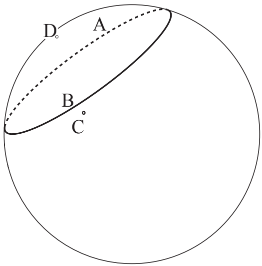



## Summary of the solution, 1

Let $AB$ be a given lesser circle and $C$ a given point, not on it.[a]

[a] Point $C$ must be between $AB$ and the circle equal and parallel to it.

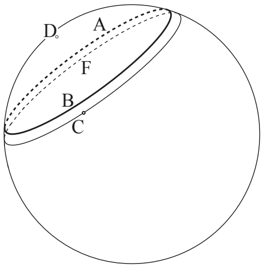
# Drawing a tangent great circle, *Spherics* II 15




## Summary of the solution, 1

Let $AB$ be a given lesser circle and $C$ a given point, not on it. We find the pole of circle $AB$, as point $D$ [*Sph*.1.21].

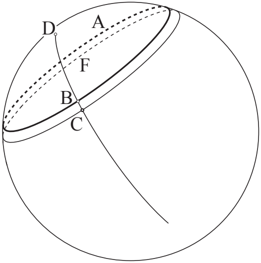
# Drawing a tangent great circle, *Spherics* II 15




## Summary of the solution, 1

Let *AB* be a given lesser circle and *C* a given point, not on it. We find the pole of circle *AB*, as point *D* [*Sph*.1.21]. With *D* as pole, we draw lesser circle *CF* [*Ele*.P.3*].

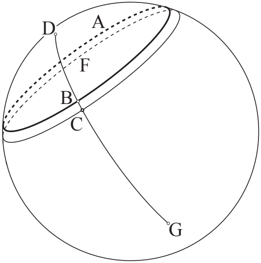
# Drawing a tangent great circle, *Spherics* II 15




## Summary of the solution, 1

Let *AB* be a given lesser circle and *C* a given point, not on it. We find the pole of circle *AB*, as point *D* [*Sph*.1.21]. With *D* as pole, we draw lesser circle *CF* [*Ele*.P.3*]. We draw great circle *DC*, meeting circle *AB* at point *B* [*Sph*.1.20],

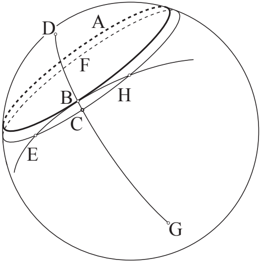
# Drawing a tangent great circle, *Spherics* II 15




## Summary of the solution, 1

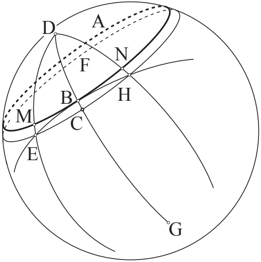
Let *AB* be a given lesser circle and *C* a given point, not on it. We find the pole of circle *AB*, as point *D* [*Sph*.1.21]. With *D* as pole, we draw lesser circle *CF* [*Ele*.P.3*]. We draw great circle *DC*, meeting circle *AB* at point *B* [*Sph*.1.20], and cut off quadrant *BG* [*Sph*.1.19 (*inter.)].

# Drawing a tangent great circle, *Spherics* II 15




## Summary of the solution, 1

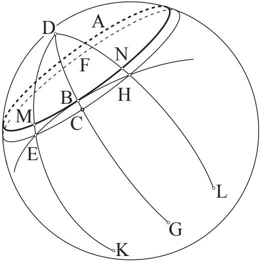
Let *AB* be a given lesser circle and *C* a given point, not on it. We find the pole of circle *AB*, as point *D* [*Sph*.1.21]. With *D* as pole, we draw lesser circle *CF* [*Ele*.P.3*]. We draw great circle *DC*, meeting circle *AB* at point *B* [*Sph*.1.20], and cut off quadrant *BG* [*Sph*.1.19 (*inter.)]. With *G* as pole, we draw great circle *EBH* [*Ele*.P.3*].




## Summary of the solution, 2

We draw two great circles, *DH* and *DE*, meeting circle *AB* at points *N* and *M* [*Sph*.1.20].

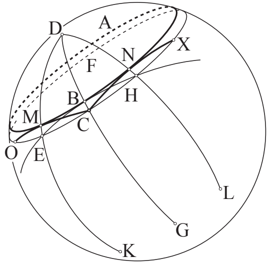
# Drawing a tangent great circle, *Spherics* II 15




## Summary of the solution, 2

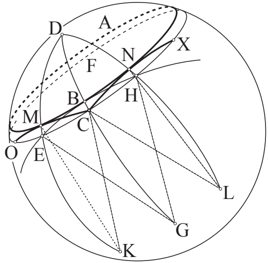
We draw two great circles, *DH* and *DE*, meeting circle *AB* at points *N* and *M* [*Sph*.1.20]. We cut off arcs *HL* and *EK* equal to arc *CG* [*Ele*.P.3*].

# Drawing a tangent great circle, *Spherics* II 15




## Summary of the solution, 2

We draw two great circles, *DH* and *DE*, meeting circle *AB* at points *N* and *M* [*Sph*.1.20]. We cut off arcs *HL* and *EK* equal to arc *CG* [*Ele*.P.3*]. With *L* as pole and distance *LN*, we draw circle *XNC*, and with *K* as pole and *KM* as distance, we draw circle *OMC* [*Ele*.P.3*].

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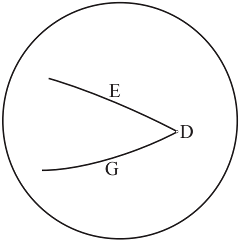
We draw in the internal lines and show that the are all equal to the side of a great square.

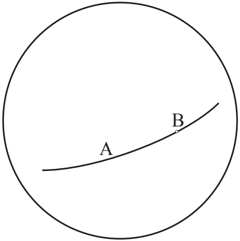
"One sees he considers Theodosius inadequate in his treatise *On Spheres* and thinks that the way he followed is other than satisfactory, since there is difficulty in it, and the setting out of many lines, and he did not adhere, in it, to the properties of the figures that occur on the sphere, namely the conditions of the angles that arise from the intersection of the circles.

Upon my life, Menelaus has easily shown everything that Theodosius proved in that book and he intends that the proof be straightforward (بطريق الإستقامة)[4] without using straight lines."

[4] That is "by the correct way," the "by the straight route." Maybe *direct*, as opposed to *indirect*.

# To construct equal angles, Menelaus *Spher.* I 1



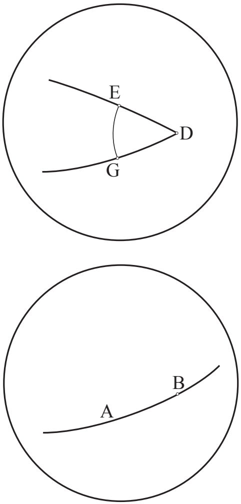




## Introduction

$1_p$ "We want to make an angle at a given point on the circumference of a great circle that is equal to a given angle."

$2_p$ Let $B$ be the given point on great-circle arc $AB$ and let the given angle be angle $GDE$.

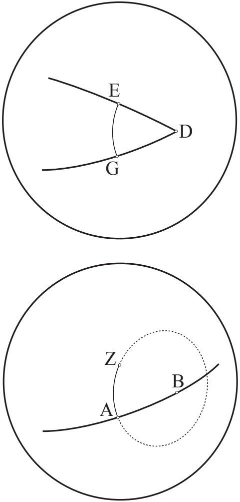
$3_p$ So, it is necessary to make an angle at $B$ equal to angle $GDE$.




## Core [solving the problem]

$4_p$ With $D$ as pole, we draw arc $GE$ with “whatever distance” (بايّ بعد).

# To construct equal angles, Menelaus *Spher.* I 1



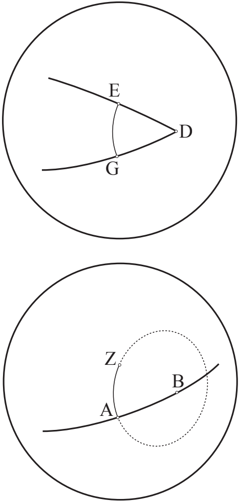

## Core [solving the problem]

$4_p$ With $D$ as pole, we draw arc $GE$ with “whatever distance” (بايّ بعد). With $B$ as pole, we draw arc $AZ$ with *the same* distance,

# To construct equal angles, Menelaus *Spher.* I 1



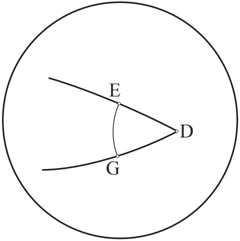


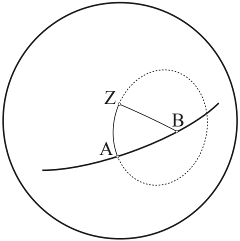



## Core [solving the problem]

$4_p$ With $D$ as pole, we draw arc $GE$ with “whatever distance” (بايّ بعد). With $B$ as pole, we draw arc $AZ$ with *the same* distance, and we cut off arc $AZ =$ arc $GE$.

# To construct equal angles, Menelaus *Spher.* I 1

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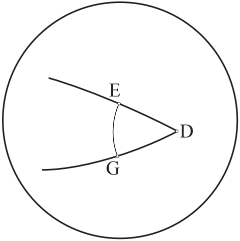
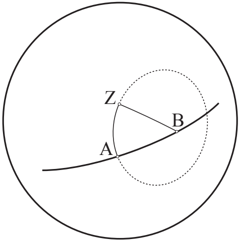
$4_p$ With $D$ as pole, we draw arc $GE$ with "whatever distance" (بايّ بعد). With $B$ as pole, we draw arc $AZ$ with *the same* distance, and we cut off arc $AZ$ = arc $GE$. We join great-circle arc $BZ$.

# To construct equal angles, Menelaus *Spher.* I 1



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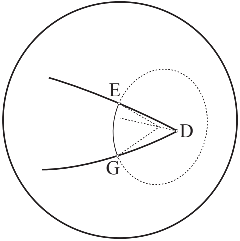
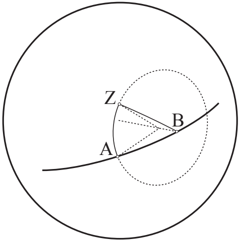
$4_p$ With $D$ as pole, we draw arc $GE$ with "whatever distance" (بايّ بعد). With $B$ as pole, we draw arc $AZ$ with *the same* distance, and we cut off arc $AZ =$ arc $GE$. We join great-circle arc $BZ$.

$5_p$ I say, angle $ABZ$ is equal to angle $GDE$.

# To construct equal angles, Menelaus *Spher.* I 1

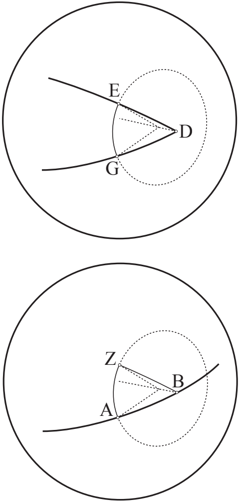





## Core [verifying the solution]

$6_p$ Two great circles, *GD* & *DE*, pass through the pole of circle *GE*, so their planes are ⊥ to circle *GE* [T*Sph*.1.15]. The two intersections of great circles *GE* & *DE* with circle *GE*, pass through the center of circle *GE*, while intersection of circles *GD* & *DE* is ⊥ to circle *GE* at its center, hence the intersections of circles *GE* & *DE* with circle *GE* are each ⊥ the intersection of circles *GD* & *DE*...




## Core [verifying the solution]

$6_p$ ... The same argument can be made for the internal objects of triangle $ABZ$. But circle $AZ$ = circle $GE$ and arc $AZ$ = arc $GE$, so the inclination of plane $BZ$ on plane $AB$ is equal to the inclination of plane $ED$ on plane $GD$. Therefore, $\angle ABZ = \angle GDE$ [def. of equal inclination].

# Final Remarks

- All of the constructions used to solve problems in ancient spherics can be carried out through *physical manipulations* of an unmarked ruler and a non-colapsing compass.
  - In the texts on spherics, an instrument like a compass must be used to *transfer* distances.
- The use of practical constructions appears to have guided not only the solution to problems but also to have influenced the overall, theoretical approach.
  - Theodosius moves from considerations of internal lines to considerations of the objects on the surface of the sphere.
  - Menelaus attempts to restrict his attention to the objects on the surface. (Of course, he still uses internal lines, but he does not *name* them; moreover, he uses Theodosius's theorems that make use of internal lines.)