Constructions in Euclid's *Elements* I–VI

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Paris, June 2015

Constructive aspects of Greek geometry

In the early 20th century, mathematicians, following Hilbert, axiomatized geometry using predicate logic, which provides no natural treatment of geometric constructions. This lead to a number of misconceptions about Greek *problems*—they are existence proofs of geometric objects, they are logically convertible with *theorems*, etc.

In the late 1960s, logicians began to develop the framework to naturally express constructions [Pambuccian, 2008]. This work lead to logical formulations of Euclidean construction procedures [Mäenpää and Von Plato, 1990], a clear distinction between *problems* and *theorems* [Mäenpää, 1997], formal systems for Greek geometry involving constructions and diagrammatic inferences [Avigad, Dean and Mumma, 2009], programable systems for Euclid's constructions [Beeson, 2010], and so on.

Historians of Greek mathematics, however, have largely neglected these developments.

Theorems and *problems*

It is well known that propositions of elementary treatises, like the *Elements* or the *Spherics*, are divided into *theorems* and *problems*. The distinction is discussed by Pappus and Proclus, and we are told that a number of Greek mathematicians held strong views favoring one type over the other.

Theorems: Given some set of initial objects, a *theorem* asserts some property that is true of these objects. (“If... then...”)

Problems: Given some set of initial objects, a *problem* shows how to do something (say, how *to find*, *to draw*, *to set out*) and then demonstrates that what has been done is satisfactory. (“To do such-and-such...”)

Some *theorems* can be intelligibly expressed as *problems* and the converse.

The first six books of the *Elements* presents a blend of *theorems* and *problems*. In the *Elements*, constructions are used in both *theorems* and in *problems*, but they are used in different ways in the two types of propositions. I will argue that Euclid’s *problems* cannot be understood as simply auxiliary to *theorems*, but constitute a mathematical project in their own right.

Using constructions to prove *theorems*

In a geometric ***theorem***, it is sometimes the case that the properties of the objects stated in the enunciation are sufficient to demonstrate the proposition, but more often than not we have to introduce new objects and use their properties in the argument.

These new objects are introduced using constructions. (In fact, the geometer usually asserts that these objects must have been produced, using the same grammatical forms, and often the same verbs as are used when the initially given objects are set out.) These constructions can sometimes be carried out using the postulates, or previously demonstrated *problems*, but Euclid also demonstrates *theorems* with various types of constructive processes that cannot be so explained.

Using constructions to complete *problems*

A ***problem*** is solved by producing a *specific* geometric object that meets certain conditions. A *problem* (a) shows *how* to produce the object using construction postulates and previously established *problems* and then, (b) through deductive argumentation, using first principles (all three) and previously established *theorems*, shows *why* this object is the one we set out to produce.

This deductive argumentation sometimes requires new constructions in the same way as a *theorem*. That is, the geometric objects that complete the *problem*, together with the initial objects stipulated in the *problem* itself, are sometimes not sufficient to show that this constructed object satisfies the requirements of the *problem*. In such cases, we must construct new objects for the proof—as we will see, this can be done using a variety of different means. (These "unnecessary" objects are often simply rolled into the original construction without any explicit mention that this is being done.)

Proclus' division of a Greek proposition

Proclus (5th CE) put forward the following six parts of a Greek proposition. (In fact, they are usually only found so complete and clearly divided in Euclid's *Elements* I.)

Introductory components

1. **Enunciation** (πρότασις): A general statement of what is to be shown (done).
2. **Exposition** (ἔκθεσις): A statement setting out the given objects with letter names.
3. **Specification** (διορισμός): A restatement in terms of the specific objects of (1) what is to be shown (*theorem*), or of (2) what is to be done, including any conditions of solvability (*problem*).

Proclus’ division of a Greek proposition

Proclus (5$^{\text{th}}$ CE) put forward the following six parts of a Greek proposition. (In fact, they are usually only found so complete and clearly divided in Euclid’s *Elements* I.)

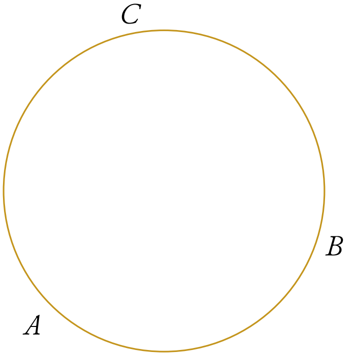
Core components

4 **Construction** (κατασκευή): Statements about the production of *new objects* that will be required in the proof. (Often using Posts. 1–3 and *problems*, but in *theorems* there are also counterfactual constructions in this section.)

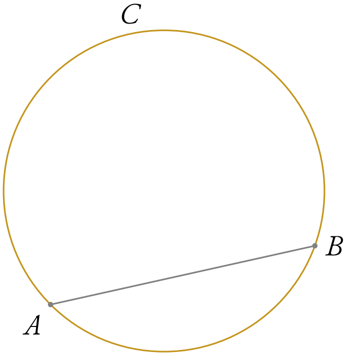
5 **Proof** (ἀπόδειξις): A logical argument that the proposition holds (has been done). (Relies on the other assumptions and *theorems*.)

6 **Conclusion** (συμπέρασμα): A restatement, in general terms, of what has been shown (done).[1]

[1] Rare, except in the *Elements*.

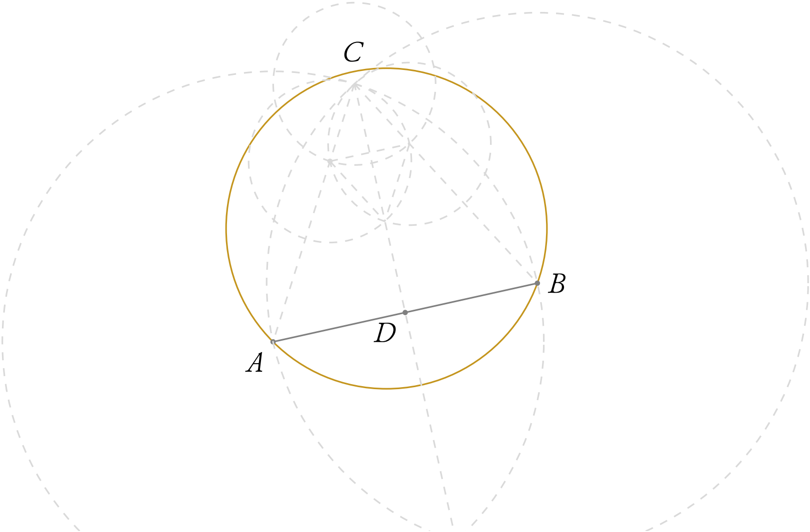

C
B
A

Elements III.1



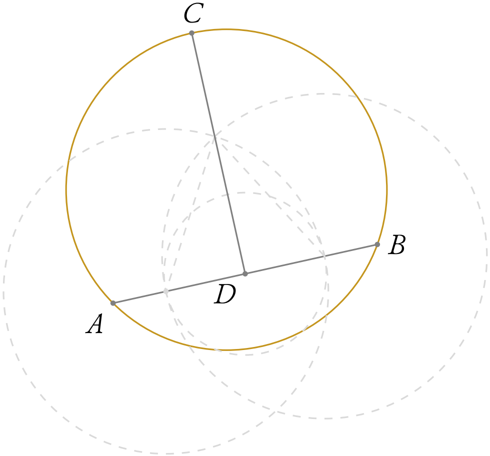
Unpostulated

Elements III.1



*Elem.*I.10

Elements III.1



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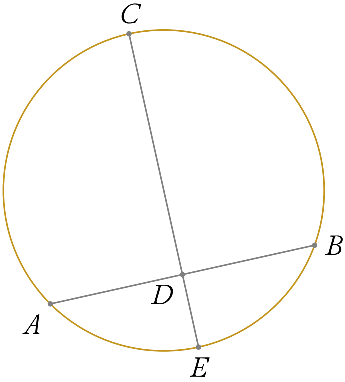
Elements III.1



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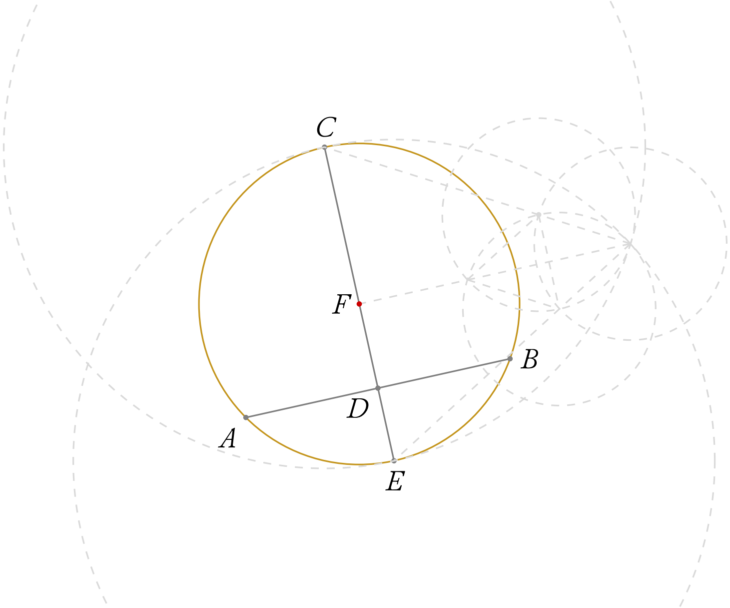
Elements III.1



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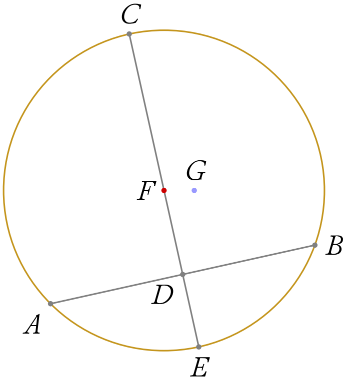

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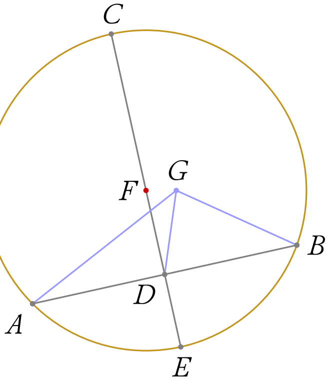


Elem.post.1

Elements III.1, Observations

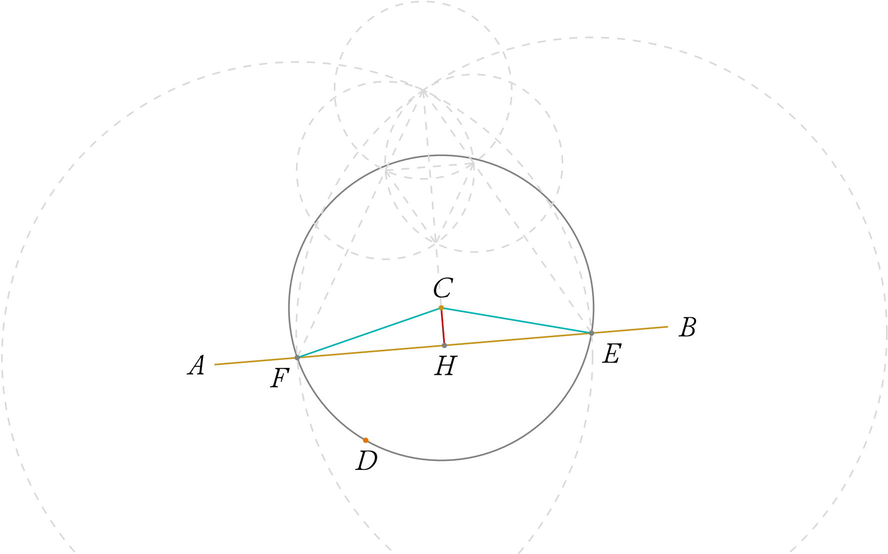
Assumptions: Some of the constructive steps are assumed without any postulate. This is true elsewhere in the *Elements*, and in the *Data* as well. For example, we can take a point or a line at random, including a line that goes through a circle. We also assume that constructed objects intersect in various ways without any axioms.

Constructions: Constructions are used in different ways for the *problem* itself and for the *proof*.

- Problem: These rely on the postulates and other *problems*.
- Proof: These can be pure assumptions or counterfactual—that is, not constructed in any previously established *problem* or even impossible to actually construct.

Structure: *Problems*, in general, have a different structure than Proclus' schema. (The structure of *Elements* III.1 may be required by the presence of counterfactual constructions.)

Elements I.12



Components of a Greek geometric *problem*

1_p **Enunciation:** General statement of what is to be *done*.

2_p **Exposition:** Statement of *what is given*, usually using specific, letter names.

3_p **First Specification:** Specific statement of *what is to be done*, often with qualifications.

4_p **Solution:** Construction of the geometric object which satisfies the requirements of the *problem*.

5_p **Second Specification:** Specific statement of *what is to be shown*.[1]

6_p **Construction:** Construction of any *new objects* necessary to the proof.[1]

7_p **Proof:** Argument that the *solution* meets the requirements of the proposition; if necessary, using the new objects of the construction.

8_p **Conclusion**: A restatement, in general terms, of what has been done.[2]

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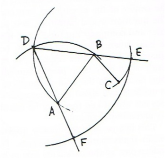
[1] May be absent.

[2] Rare, except in the *Elements*.

Euclid's *problems* are not...

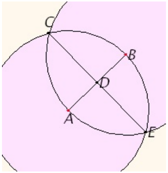
[1] A series of practical instructions for using a straight-edge and compass to draw points, lines, circles, etc.

- Circles, triangles, rectangles, etc., are drawn in full.



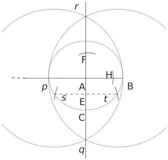
[2] A list of instructions for performing an optimal solution for producing the sought objects.

- Obviously simpler constructions are overlooked in favor of constructions that call on previously demonstrated *problems*. The goal is to show feasibility, not elegance.



[3] A series of manual instructions for producing the minimal graphical requirement for the sought objects.

- The construction of a *problem* is a series of references to previously demonstrated *problems* and it produces something new out of the objects produced in those *problems*.



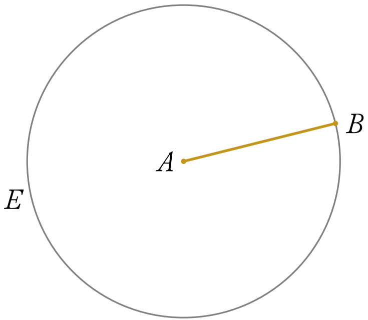
Euclid's *problems* as routines

Each of Euclid's *problems* presents a routine of constructive operations that acts upon *given* objects and produces a new object according to well-defined rules. Arbitrary points, lines (and later planes) can also be introduced at the geometer's discretion, in which case these objects are also *given*.

New objects are produced with *Elements* I.post.1–3, and previously demonstrated *problems*. When a previously demonstrated *problem* is called upon, it acts as a subroutine. That is, only the final, constructed object of the *problem* is involved in the new construction—all of the steps required to produce this object are simply black-boxed.

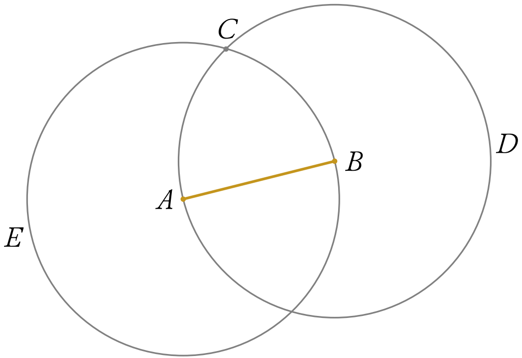
In this sense, a *problem* is similar to a *theorem*. A *theorem* uses the knowledge claims of previously established *theorems*, while essentially ignoring their internal arguments. Likewise, a *problem* introduces the objects constructed by previously demonstrated *problems* without any reference to the actual mechanics of construction.

Elements I.1



*Elem.*I.post.3

Elements I.1



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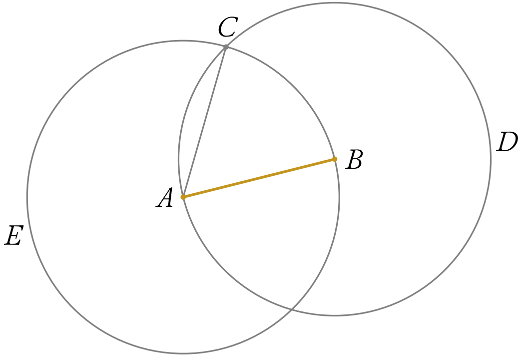
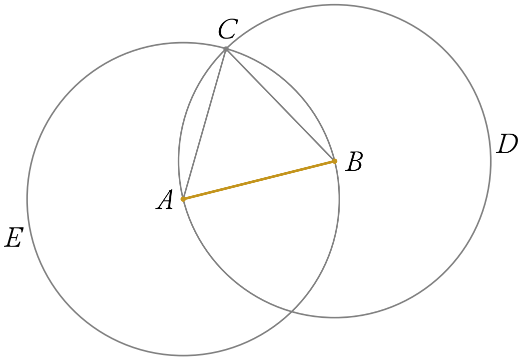
Elements I.1



Elem.I.post.1

Elements I.1



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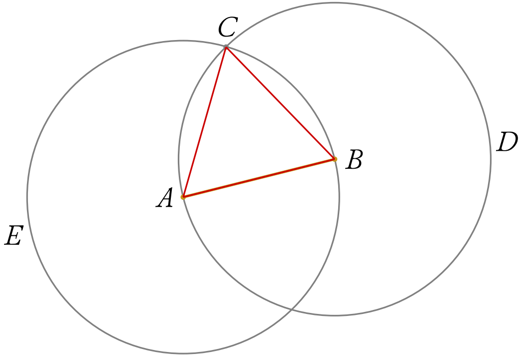
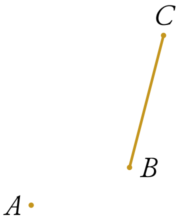
Elements I.1



Elements I.2



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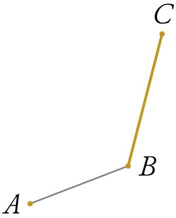
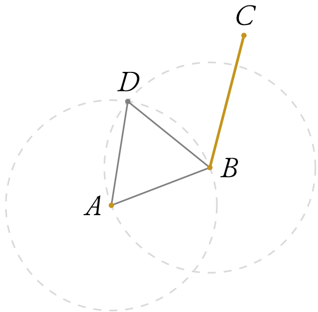
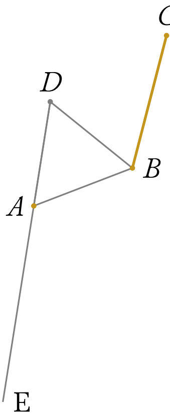


Elem.I.post.1



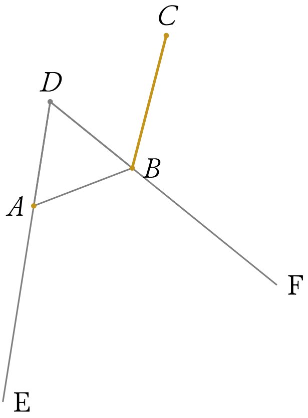
*Elem.*I.1

Elements I.2



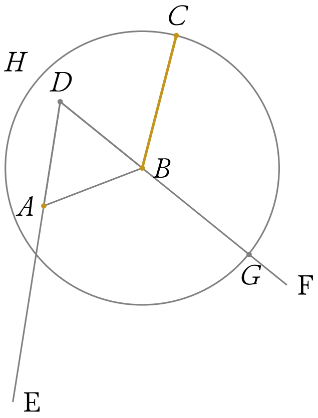
Elem.I.post.2

Elements I.2



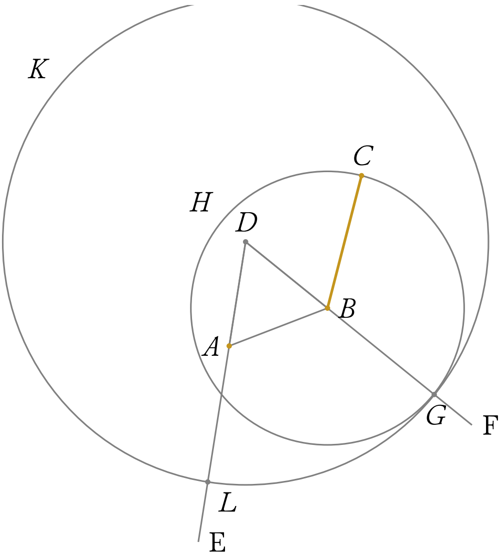
Elem.I.post.2

Elements I.2



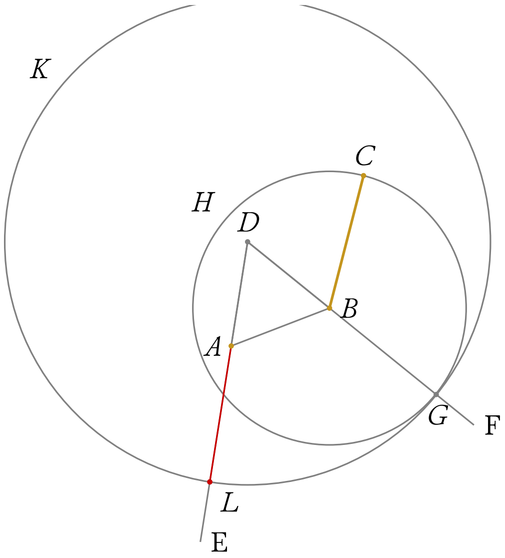
Elem.I.post.3

Elements I.2

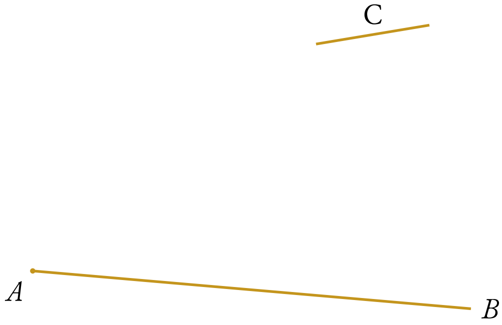


Elem.I.post.3

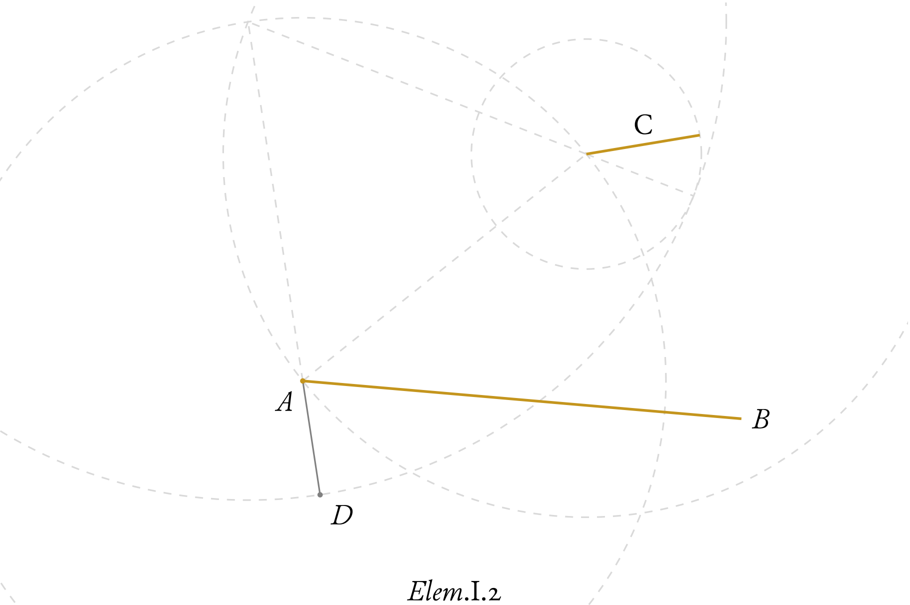
Elements I.2



Elements I.3



Elements I.3



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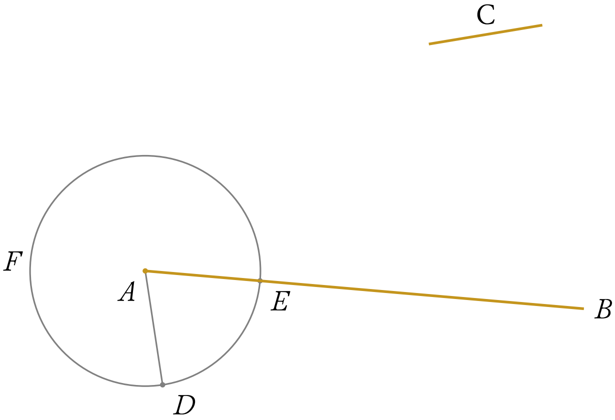
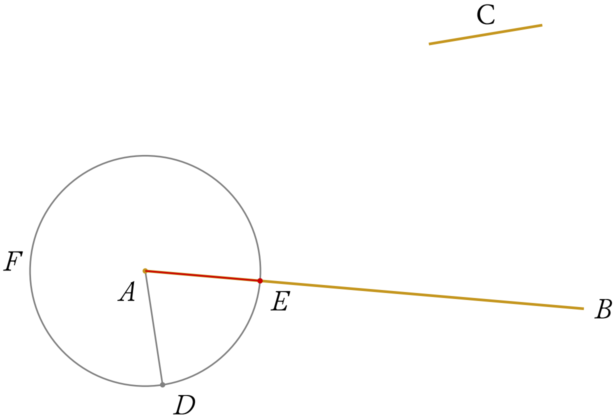
Elements I.3



Elem.I.post.3

Elements I.3



Problems as routines calling subroutines

Routine: A *problem* works with given objects, or objects assumed as given, and then performs a well-ordered series of new constructions—each of which depends on either a postulate, or a previously demonstrated *problem*—that results in the sought object.

Subroutine: When a previously demonstrated *problem* is called in, it is effectively treated as a black-boxed subroutine—no information is given about how the construction is performed, and no sub-constructions are found in the diagram. Euclid is directly invoking geometric objects, which we must assume are constructed using the same series of steps provided in the original *problem*.

These constructions are not carried out by straight-edge and compass—although they could be. They are carried out by directly invoking objects whose construction has already been demonstrated.

Constructive functions

Problems are constructive functions that take certain objects as given, including arbitrary objects, and produce some new object as an output.

I.post.1: A segment is produced from *Point A* to *Point B*.

$$joinSegment(A, B)$$

I.post.2: A segment is produced from *Point A* though *Point B*, extending the distance of an arbitrary segment ℓ.

$$extendSegment(AB, \ell)$$

I.post.3: A circle is produced around *Point A* and passing through *Point B*.

$$drawCircle(A, B)$$

The function of *Elements* I.2 and 3

These *problems* allow us to move a given segment and place it at a given point, and to cut off a given line equal to a given segment. We can express equivalent ideas as follows:

***Elem*.I.2:** A circle is produced around *Point A* with radius equal to *Segment BC*.

$$drawCircle(A, BC)$$

***Elem*.I.3:** A *Segment CD* is cut off from the greater *Segment AB*.

$$cutSegment(AB, CD), \text{ where } AB > CD$$

The fact that these propositions are built up from *Elements* I.1 and the postulates, means that we cannot simply move a circle as we do with a compass. Hence, in making fully explicit figures, we will see a lot of background clutter—especially from Elements I.2.

The function of *Elements* I.2 and 3

These *problems* allow us to move a given segment and place it at a given point, and to cut off a given line equal to a given segment. We can express equivalent ideas as follows:

***Elem.*I.2:** *Point D* is produced such that *Segment AD* = *Segment BC*, and *Segment AD* is joined.

$$producePoint(A, BC)(D, E),\ joinSegment(A, D) \text{ or } joinSegment(A, E)$$

***Elem.*I.3:** A *Segment CD* is cut off from the greater *Segment AB*.

$$cutSegment(AB, CD),\ \text{where } AB > CD$$

The fact that these propositions are built up from *Elements* I.1 and the postulates, means that we cannot simply move a circle as we do with a compass. Hence, in making fully explicit figures, we will see a lot of background clutter—especially from Elements I.2.

Elements II.11

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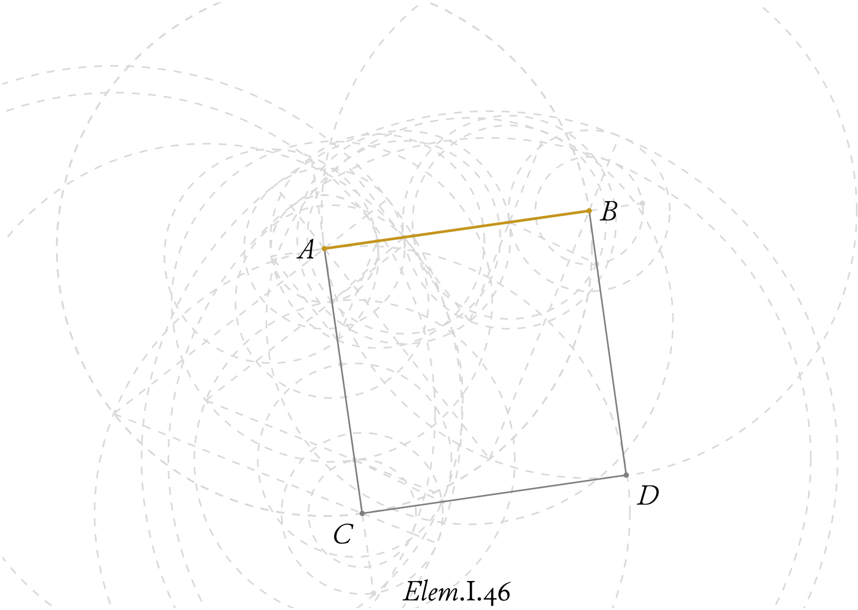
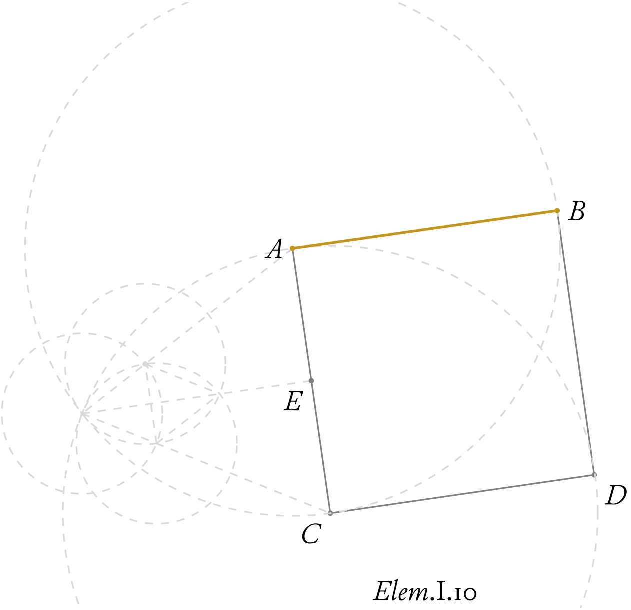
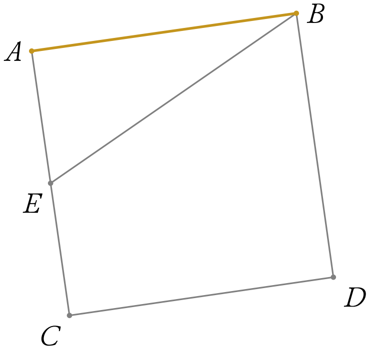


Elem.I.46

Elements II.11



*Elem.*I.10



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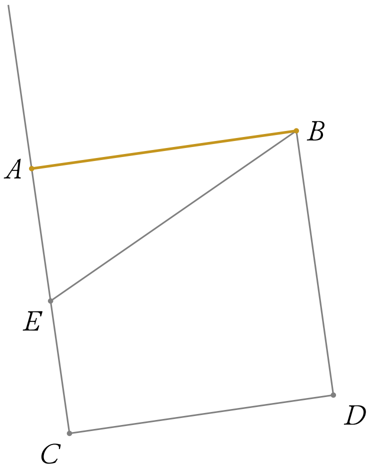
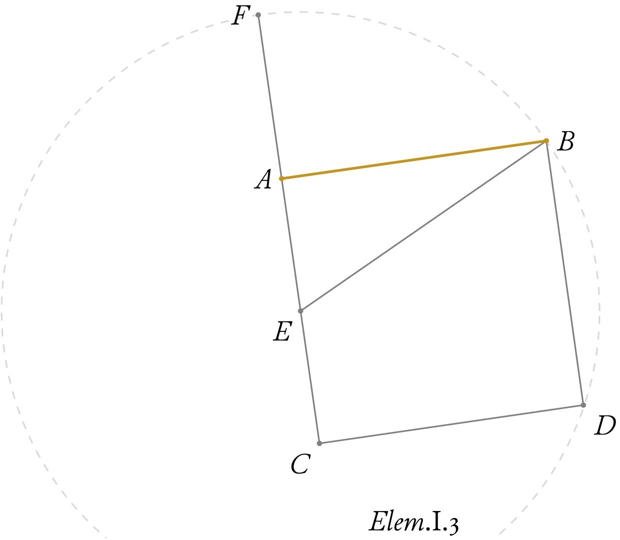


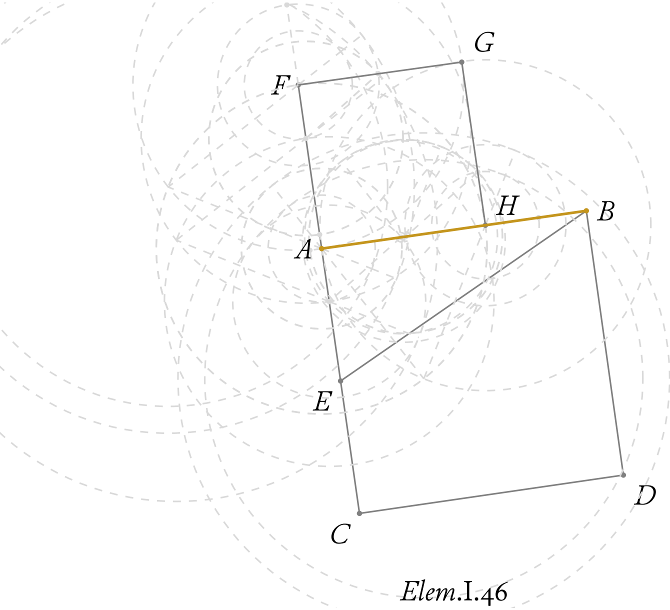
Elem.I.post.2

Elements II.11

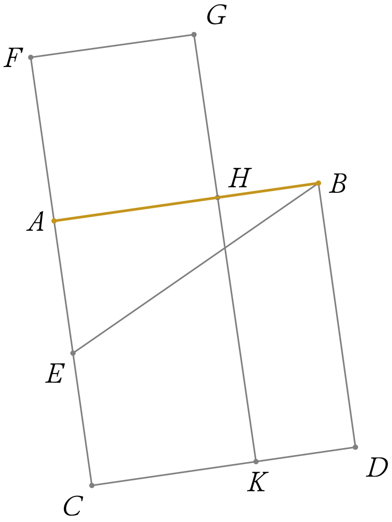


*Elem.*I.3

Elements II.11



*Elem.*I.46



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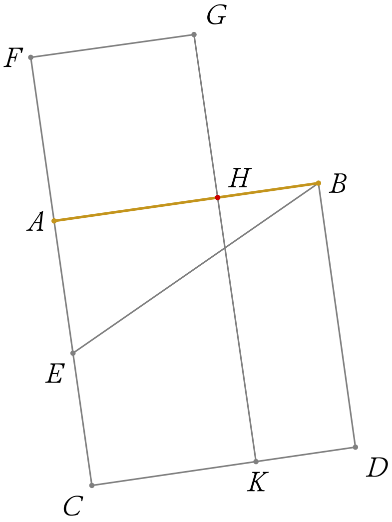
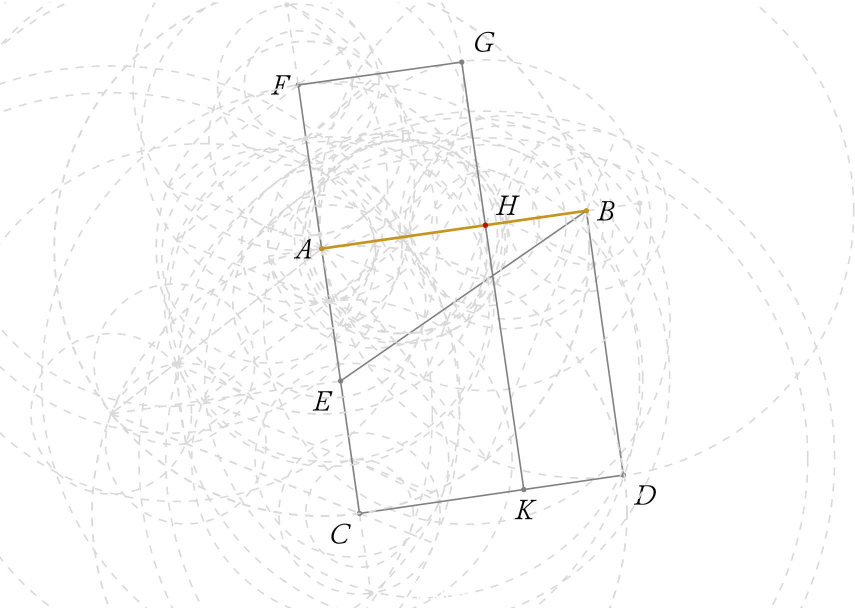
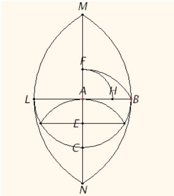


Elements II.11

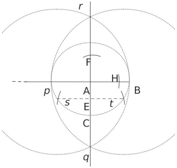


Issues with *Elements* II.11

A number of scholars have argued that Euclid's diagram includes unnecessary objects (points D, G, and K) and does not follow the steps of the construction given in the text. In place of Euclid's construction they propose a simpler set of procedures, using straight-edge and compass that result in a point which is mathematically—but not conceptually—equivalent to H.



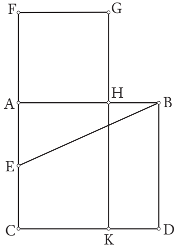
Euclid's construction, however, directs us to previous *problems* and postulates. In order to see how the squares are constructed, for example, we must look back to *Elements* I.46, which leads us back through the various *problems* in a systematic way. The only constructions allowed for in the text is the series of subroutines that I have used in my figures. (There is some flexibility, since we can decide what side to place the equilateral triangle on, and so on.)

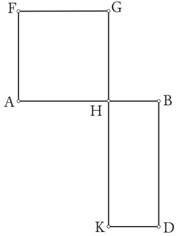


Euclid's approach to *Elements* II.11

Furthermore, these simpler constructions do not produce a point that divides the line *such that the square on one part is equal to the rectangle formed by the whole and the other part*. Of course, the point that they find is the same point, but the proposed construction does not exhibit the geometrical properties that we seek.

In order for a Euclidean *problem* to be effected, it must, at the very least, produce all of the objects explicitly stated in the enunciation—while other objects may also be necessary to either produce these, or for the proof.



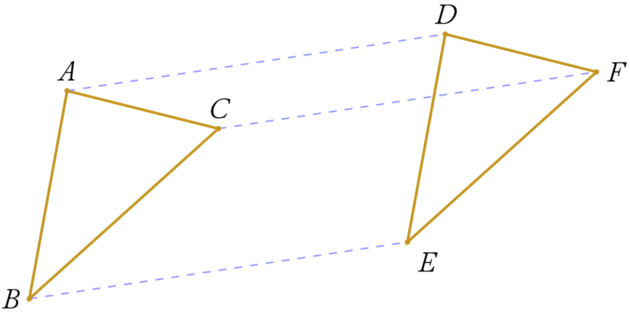


Constructions in *theorems*

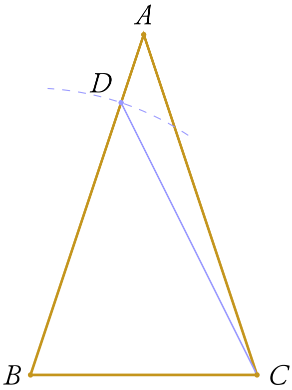
When we turn to the use of constructions in *theorems*, however, we find that they function in very different ways. Although the construction part of most *theorems* employ construction steps that are each supplied by a postulate or a previously established *problem*, such as *Elements* I.5, others do not. In the first few propositions we find, for example,

a. a triangle "fit on" (ἐφαρμόζειν ἐπί) another, points "placed on" (τιθέναι ἐπί) each other (genitive absolute, *Elements* I.4),
b. a counterfactual use of *Elements* I.3 to produce a line that is assumed to be equal to a line that it must actually be less than (perfect imperative, *Elements* I.6),
c. the assumption that a construction has been carried out (συνιστάναι), for which no *problem* has been provided and which is counterfactual (perfect imperative, *Elements* I.7),
d. and, a triangle "fit on" another and points "placed on" each other, such that lines either will "fit on" each other or will be "transposed" (παραλλάσσειν), all of which are either imaginary or counterfactual (genitive absolute, future, *Elements* I.8).

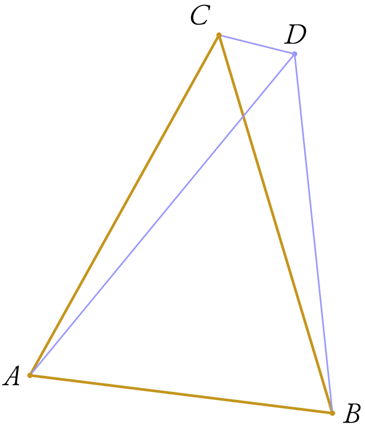
Elements I.4



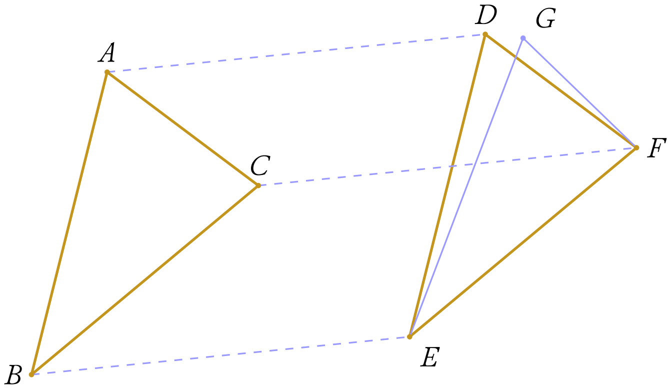
Elements I.6



Elements I.7



Elements I.8



Overview

We must distinguish between constructions used to complete *problems*, and constructions used to demonstrate *theorems* (or for the proof part of *problems*).

Constructions that complete, or solve, *problems* are well-formed routines that use postulates and previously established *problems*, as subroutines, to produce geometric objects. Constructions that are used to demonstrate *theorems* are not bound by these constraints. Where possible Euclid does use *problem*-constructions for this purpose, but he also employs a wider range of constructive procedures, such as conceptual constructions, counterfactual constructions, assumed constructions that have not already been established, and so on.

Most commentators have sought to explain *problems* by appealing to their use in *theorems*. But this will not do. The establishment of *problems* should be understood as goal in its own right. The *Elements* is meant to provide the fundamental tools necessary to *do* geometry—that is, writing new *theorems*, and completing new *problems*. We should seek the motivation for Euclid's *problems* in their use in *geometrical analysis*.

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