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# The function of diorism in ancient Greek analysis

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**Abstract**

This paper is a contribution to our knowledge of Greek *geometric analysis*. In particular, we investigate the aspect of analysis know as *diorism*, which treats the conditions, arrangement, and totality of solutions to a given geometric problem, and we claim that diorism must be understood in a broader sense than historians of mathematics have generally admitted. In particular, we show that diorism was a type of mathematical investigation, not only of the limitation of a geometric solution, but also of the total number of solutions and of their arrangement. Because of the logical assumptions made in the analysis, the diorism was necessarily a separate investigation which could only be carried out after the analysis was complete.


**Résumé**

Cet article vise à contribuer à notre compréhension de l'*analyse géométrique* grecque. En particulier, nous examinons un aspect de l'analyse désigné par le terme *diorisme*, qui traite des conditions, de l'arrangement et de la totalité des solutions d'un problème géométrique donné, et nous affirmons que le diorisme doit être saisi dans un sens plus large que celui précédemment admis par les historiens des mathématiques. En raison des suppositions logiques faites dans l'analyse, le diorisme était nécessairement une recherche indépendante qui ne pouvait être effectuée qu'une fois l'analyse achevée.


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*Keywords:* Ancient Greek analysis; Diorism; Archimedes; Apollonius; Pappus

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## 1. Introduction

This paper investigates an aspect of ancient Greek geometry known as the diorism (διορισμός),[1] which is little discussed in modern scholarship, but which served an important role in the ancient problem-solving art known as *the field of analysis*.[2] In its most basic form, a diorism provides a discussion of the possibilities of solving a particular geometric problem, usually in the form of some limit to its solvability. Hence, diorism is a feature of problematic analysis, which Pappus describes as "the supplying of what is proposed" (τὸ ποριστικὸν τοῦ προταθέντος), in contrast to theoretic analysis, which is "the investigation of the truth" (τὸ ζητητικὸν τἀληθοῦς) [Jones, 1986, 83].

A Greek geometric problem is solved by the construction of some geometric object. A problem proposes a certain construction, shows how to do it, and then demonstrates that it has been done. Hence, there is generally not a clear linguistic distinction between a problem and its solution. Solving a problem in Greek geometry means doing what one set out to do, and this is usually expressed by saying that the problem *itself* has been done. When we read about certain attributes of a problem in a Greek mathematical text, we are sometimes reading about attributes of what we would call the solution, that is, attributes of the geometric construction that solves the problem. If the production of the problem is limited in some way, or if it can be effected by a number of different objects, these issues are handled in a section of the proposition called the diorism. According to Pappus, in his general description of various aspects of the field of analysis,

> Diorism is the preliminary treatment of when, how and in how many ways the problem will be possible. [Jones, 1986, 85]

Most of the diorisms that we find in the ancient texts are more or less straightforward specifications of certain limits to the solvability of a given problem and, hence, diorism itself has often been treated briefly by modern scholars.[3] As we will see below in an example from Apollonius's *Cutting off a Ratio*, however, the diorism could also be quite involved, treat multiple solutions, and contain internal propositions.

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[1] In fact, there are two, at best only vaguely related, uses of διορισμός in Greek mathematics [Mugler, 1958, 141–142]. Diorism as the part of the division of a proposition put forward in Proclus's commentary on the *Elements* I, however, does not concern us in this paper [Friedlein, 1873, 203–210; Morrow, 1970, 159–164; Netz, 1999]. Mahoney [1969, 327–329] reads the two meanings of diorism as closely related, but his reasons for doing so are not clearly stated.

[2] See Jones [1986, 377–379] for a discussion of ὁ ἀναλυόμενος τόπος, and related terminology, as the name of a particular field of intellectual activity. Although ancient authors used the term *analysis*, ἀνάλυσις, in a fairly restricted sense for that part of an argument in which the mathematician assumed what was to be done and made arguments on the basis of this supposition [Mugler, 1958, 57–58], while Pappus, at least, used *the field of analysis*, ὁ ἀναλυόμενος (τόπος), for the more general range of mathematical activity in which such arguments were produced or the works that were written to facilitate this activity, modern authors have had a tendency to conflate the two into the single term *analysis*. While we generally distinguish between the two, we also occasionally use *analysis* in the broader sense, especially when discussing modern scholarship.

[3] For examples, only about one page of the 144-page monograph on analysis by Hintikka and Remes [1974, 58–59] treats diorism, Behboud [1994, 71] in his study of the logical structure of problematic analysis only mentions diorism once, and Berggren and Van Brummelen [2000, 3] eliminate the diorism from their discussion of one of Archimedes' analyses.

That diorism played an important role in the field of analysis is made clear by the fact that one of Apollonius's motivations for the organization of *Conics* IV is that its propositions are "useful both for the syntheses of problems and for the diorisms" [Heiberg, 1891–1893, Vols. 2, 4]. The subject of *Conics* IV is, primarily, the number and arrangement of the meeting points of a conic section with another conic section, opposite sections, or with a circle.[4] Since many of the most interesting and difficult problems in Greek geometry were solved by finding the intersections of conic sections with other conic sections or circles, it is intuitively clear that the subject matter of *Conics* IV would be useful for the diorisms of such problems.[5] Apollonius, however, wanted to make sure that this point was not lost on his readers. A sentence later he returns to the matter, stating,

> ...For even if it is possible to fully exhibit (κατὰ ... ἄποδίδοσθαι) the diorism without these [propositions of Book IV], there are surely still some things that are more readily grasped through them, such as that it might be produced (ἂν γένοιτο) in various ways or in so many ways, or again that it might not be produced.[6] While such prior knowledge (πρόγνωσις), contributes a sufficient starting point for the investigations (τὰς ζητήσεις), the theorems are also useful for the analysis of diorisms. [Heiberg, 1891–1893, Vols. 2, 4]

This passage makes a number of points about the role of diorism, which we will examine further in this paper.[7] The diorism, as well as stating the limit to the solvability of a problem, may state how many solutions there are and how they are arranged.[8] Moreover, the diorism itself was sometimes treated using analysis, and this was done frequently enough so that Apollonius organized a book of his *Conics* in such a way as to be useful for this process. We will examine an example of one of these extended diorisms below, in the section on Apollonius's *Cutting off a Ratio*.

After setting out the current, standard model of analysis, we will investigate a number of the diorisms preserved in the ancient texts. We will argue that the word *diorism* was used by ancient geometers to signify a type of mathematical argument, or investigation, regularly undertaken during the course of doing analysis, but which was distinguished from both the analysis proper and the synthesis. As we will argue, there were both practical and logical reasons for this distinction. An analysis, since it is the investigation of a single,

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[4] Apollonius calls a branch of a hyperbola simply a "hyperbola" (ὑπερβολή) and what we would regard as a complete hyperbola he calls "opposite [sections]" (ἀντικείμεναι). See Heath [1921, Vol. 2, 157–158] and Fried and Unguru [2001, 117–139] for rather different assessments of the contents of *Conics* IV. Fried [2002, xi–xxvi] also gives a nice overview of the whole book. None of these accounts, however, takes special notice of Apollonius's claim that Book IV will be useful for treating diorisms, nor makes any attempt to understand the book in this light.

[5] See Knorr [1986] for discussions of many such problems.

[6] The grammatical subject, which is contained in the verbs, and which we have translated vaguely as *it*, is presumably either the problem or the object that solves the problem, if, indeed, these were not thought of as the same.

[7] Fried reads Apollonius as attributing this passage to Nicoteles as part of his overall criticism of Conon's investigations of the number of points at which conic sections meet [Fried, 2002, xii]. Apollonius, however, believes that Nicoteles was wrong to make the criticism that these studies are useless for diorisms and here, beginning with the phrase καὶ γὰρ εἰ, he gives the reason that he believes Nicoteles was mistaken.

[8] The fact that diorism also concerns the number of solutions has been noted already by Toomer [1990, lxxxv] and Fried and Unguru [2001, 287]; however, these scholars have not developed the full implications of this broader conception of diorism.

assumed solution, is not the place to explore the more general questions of whether or not there is a limit to the possible solutions, more than one solution, and so on. A synthesis, on the other hand, since it is the construction of an actual solution, must be carried out under the restrictions of any limitations, or divided into cases according to the number of possible solutions. As we will argue, the diorism was a type of analytical investigation, in which the geometer used certain properties of the objects constructed in the analysis to determine the overall properties of the solutions. Hence, in the diorism, the mathematician engaged in activities such as *specifying* the limits of a solution, *dividing* the possible cases of solution, and *elaborating* all the possible solutions.

Indeed, this broad range of activities is expressed by the variety of terms used to render διορισμός in the Arabic translation of Apollonius's *Conics*. In the translation of the preface to *Conics* I, the translator has rendered the Greek term διορισμός with two different Arabic words, تحديد (*taḥdīd*, "specification") and تفصيل (*tafṣīl*, "elaboration" or "diversification"), and the closely related Greek term διοριστικός with yet another Arabic term, قسمة (*qismah*, "division" or "distribution") [Rashed, 2008a, 253].[9] In the preface to *Conics* IV, the translator uses a related term, تقسيم (*taqsīm*, "division" or "sectioning"), three times to translate the Greek διορισμός [Rashed, 2009, 119]. Finally, in the preface to *Conics* V, Apollonius tells us that this book will also be of use in diorisms. He says that the propositions relating to minimum lines are important,

> ...Because of our opinion that a student (طالبي) of this science requires them in the comprehension of the division (تقسيم) and elaboration (تفصيل) of problems, and in their synthesis. [Toomer, 1990, 5; Rashed, 2008b, 225]

The translation of this passage is problematic.[10] The Arabic words that we have translated with "division" and "elaboration" are both used elsewhere in the text to render διορισμός [Rashed, 2008a, 253; Rashed, 2009, 119]. As Hogendidjk [1985, 43, n. 4] points out, the translators seem not always to have understood what the Greek term meant. In fact, what the translators may have understood is that the term *diorism* represents a range of mathematical activities, and they may have attempted to describe these different activities with different words.[11] In this passage, Apollonius is probably using *problem* to refer to the geometric objects that solve the problem and he is claiming that the propositions on the minimum lines drawn from certain specified points to conic sections will be relevant to

[9] See Hogendijk [1985, 42, n. 4] for a discussion of these terms.

[10] The phrase تقسيم وتفصيل المسائل has been translated variously with "Divisones et διορισμοὺς Problematum" by Halley [1710, 2, 1], "der Eintheilung und Determination der Aufgaben" by Nix [1889, 20–21], "division (into cases) and the diversification of problems" by Hogendidjk [1985, 43, n. 4], "analysis and determination of problems" by Toomer [1990, 4], and "la détermination et l'analyse des problèmes" by Rashed [2008b, 224]. In the absence of the Greek text for *Conics* V, and before we have a fully articulated understanding of the Arabic translators' views of the mathematical meaning of the term διορισμός, we think it best to translate the Arabic and not try to guess at the lost Greek. Hence, for the time being we have chosen to translate rather literally.

[11] For example, the repeated use of تقسيم to translate διορισμός in the preface to *Conics* IV [Rashed, 2009, 119], where the term is clearly being used to refer to a study of the number and distribution of the intersections of conic sections in the diorisms of problems that are solved by the intersection of conics, may indicate that the translator was attempting to make explicit this particular aspect of diorism. See Section 5 for further discussion of this topic. This issue may have some bearing on Rashed's frequency argument that the use of تقسيم in the preface of *Conics* V must be a translation of διορισμός [Rashed, 2008b, 495].

certain aspects of these objects, one of which is almost certainly a translation of *diorism*, and also to the synthetic construction of these objects. In the course of this paper, we will see why the Arabic translators, rightly, saw diorism as a range of mathematical activities that they believed was not well conveyed by a single Arabic word.

## 2. The standard model of analysis

Most modern discussions of the field of analysis begin with a treatment of Pappus's description at the beginning of *Collection* VII, and, indeed, many of them develop an analysis of Pappus's remarks as a logical interpretation of geometric analysis.[12] We will not follow this course for two reasons. The first is that Pappus was not the author of any of the works that he regarded as belonging to the field, nor one of its most creative practitioners. His study of the field of analysis, *Collection* VII, which is essential to *our* understanding, was based on his reading of the texts of the masters. Indeed, as Acerbi [2007, 442] stresses, Pappus's account was written some five centuries after the last of those works he discusses and his vision of the field of analysis and its purpose may well have been different from that of the authors he treats. Hence, we believe it is possible to learn about the field of analysis by directly reading the works of the Hellenistic geometers.[13] The second reason is that we do not believe that Pappus intended his remarks in the opening passages of *Collection* VII to be read as a comprehensive treatment of the subject. As Berggren and Van Brummelen [2000, 9] point out, Pappus spoke rather vaguely, and he meant these remarks to introduce a broad field expounded in a number of treatises that he expected his reader to be able to read.[14] Pappus, no doubt, expected his readers to learn about the field of analysis primarily from the works he discussed, and only secondarily from his treatment of them.

The ancient field of analysis, as it has been transmitted to us, covered a range of mathematical approaches using a number of different types of argumentation. Only three of the treatises that Pappus considered to be part of the field have survived: Euclid's *Data*, in Greek, Apollonius's *Conics*, partly in Greek and partly in Arabic, and Apollonius's *Cutting off a Ratio*, in Arabic. Of these, the *Data* contains no analyses as such, although it was written for sake of doing analysis, and the *Conics* contains relatively few analyses, although parts of it were written for the sake of doing analysis and the final, apparently analytical book has been lost, while in *Cutting off a Ratio* everything that can be presented using analysis is so presented.

Thus, whereas modern scholars have a tendency to regard ancient analysis as the production of a particular type of proposition having a certain structure, ancient mathematicians apparently regarded the field of analysis as a set of methodological approaches to doing mathematics along with all of the mathematical apparatus developed to facilitate these practices.[15] Nevertheless, the structure of an ancient analysis that has been articulated by

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[12] See, for some examples, Cornford [1932, 43–47], Robinson [1936], Hintikka and Remes [1974, 7–21], and Behboud [1994, 53–57].

[13] Berggren and Van Brummelen [2000] also take this approach.

[14] On the other hand, Acerbi [2007, 442–449] argues that, by the elimination of some passages as spurious, Pappus's account can be read as a fairly accurate description of both problematic and theoretic analysis.

[15] Acerbi [2007, 439–534] gives a broader account of analysis than is generally found in modern scholarship. It should be noted, however, that he does not discuss the special function of diorism in analysis.

modern scholars is useful and we will refer to it frequently. There seems, now, to be a general consensus on the structure of what we will call an *analyzed proposition*, as first put forward by Hankel [1874, 137–150].[16]

In the first place, an analyzed proposition is divided into two parts, an *analysis* and a *synthesis*. It is the presence of the analysis that makes these propositions analyzed propositions; the synthesis is in no way distinguishable from a normal problem.[17] In some cases, the synthesis is omitted. Presumably, in these cases the geometer regarded the production of the synthetic argument as too simple to warrant further attention.[18] In the works of the Hellenistic geometers, an analyzed proposition is usually presented with two different diagrams, one for the analysis and one for the synthesis.[19] This practice, which probably originated from oral presentation, makes it quite clear that the two different parts of the proposition start from different sets of assumed geometric objects and relations.

The analysis is divided into two parts. In the first of these, which modern scholars generally call the *transformation*, the geometer assumes that the objects required by the problem have already been constructed, or in the case of theoretic analysis, that the sought relations obtain.[20] The transformation proceeds by constructing auxiliary objects and using straightforward deductive argumentation, as canonically expressed by Euclid's *Elements*, to show that these new objects imply some configuration of objects, or some construction, that the geometer takes to be sufficiently manageable—such as the intersection of two given conic sections, that a given area is applied to a given line falling short by a square, or that a line has a given ratio to a given line.

In the second part of the analysis, which modern scholars generally call the *resolution*, the geometer uses the mathematical apparatus of Euclid's *Data* to show that if the sufficiently manageable objects that were derived in the transformation are given, then the objects that solve the problem are also given.[21] In the language of the *Data*, this means that if we assume the objects derived in the transformation are given, then it will be possible "to supply" (πορίσασθαι) the objects that complete the problem [Menge, 1896, 2]. The procedures of proof used in the *Data* make it clear that objects are supplied when they are geometrically constructed. This accords well with the use of this idiom in the field of analysis. In the resolution, the geometer shows that if some set of readily obtainable objects are given, then the objects that complete the problem are fully determinate, that is, constructible on this basis.

[16] Berggren and Van Brummelen [2000, 9–13] provide a useful description of the four-part division of an analyzed proposition from which we have drawn guidance.
[17] Here, we use the term *problem* in the technical sense of a proposition of Greek geometry that sets out to perform a specific construction and then demonstrates that this construction is valid.
[18] In the case of theoretic analyzed propositions, in which all of the steps of the analysis were known to be convertible, it is also possible that the analysis was taken as a full proof. See Acerbi [2007, 485–497] for a discussion of a number of theoretic analyzed propositions of this type.
[19] For example, all of the analyzed propositions of Apollonius's *Cutting off a Ratio* and most of those of Archimedes' *Sphere and Cylinder* II are of this type. Moreover, the synthesis is often accompanied by auxiliary figures or objects. For example, the analyzed propositions of Apollonius's *Conics* have a number of different figures for the syntheses and the syntheses in *Cutting off a Ratio* introduce objects that are not necessary in the analyses.
[20] Hintikka and Remes [1974, 24] call this part the "analysis proper" and Fournarakis and Christianidis [2006, 47] call it the "hypothetical part." Nevertheless, it is clear that, although the views of analysis put forward by these authors differs from ours, they identify this part of the argument as a particular component of an analyzed proposition in the same way that we do.
[21] Acerbi [2007, 455] appropriately calls this section the "*catena dei dati*" ("chain of givens").

The synthesis, which also has two parts, is then a sort of reversal of this process. This is not necessarily in a strictly logical sense, but rather in the sense that the set of objects that are taken as a starting point, and the objects that are provided on this basis, are essentially the opposite of those in the analysis. The first part of the synthesis is the *construction*, in which the geometer begins with some basic construction – such as two conic sections intersecting at a point or an area applied to a given line and falling short by a square – and then proceeds to construct the objects that complete the problem on this basis. The construction proceeds in essentially the same direction as the resolution. That is, starting with the objects achieved at the end of the transformation, we construct the objects that solve the problem. Finally, the *proof* demonstrates that the objects so constructed have the properties stipulated in the enunciation of the problem. The proof proceeds in roughly the opposite direction from the transformation. That is, starting with the geometric properties of the given objects, together with those of the auxiliary objects introduced in the construction, we demonstrate that the objects so constructed have the properties that were assumed at the beginning of the transformation.

This schematic is, however, an idealization that is not always found so well structured in the ancient sources and was probably rarely so precisely implemented by the ancient mathematicians. Nevertheless, these divisions of the analyzed proposition are useful interpretive categories, which may also be applied to the somewhat "irregular" propositions that we generally encounter in our sources.

As an example, we take an analyzed proposition from Pappus's *Collection* IV, which treats a neusis problem.[22] In the course of discussing a number of the classical problems of Greek geometry, Pappus gives a solution to the problem of trisecting an angle by means of a particular neusis construction. His approach is divided into two parts. In *Collection* IV Prop. 31, he gives an analyzed proposition that provides the construction of a certain neusis line. Then, in the next proposition, which is a straightforward synthetic problem, he uses this neusis line to trisect a given angle. In this way, he transforms the problem of trisecting a given angle into a problem that can be more readily handled, namely that of constructing a particular neusis line. Here, we will only consider the analyzed problem of constructing the neusis line. In the conclusion, we will return to this problem and examine the broader question of the relationship between the original problem and its transformed solution using the ideas developed in this paper about the role of diorism in ancient analysis.

In *Collection* IV Prop. 31, Pappus solves the following problem. Considering Fig. 1, where rectangle $ABGD$ is given and line $BG$ extended, the problem is to pass a line through point $A$ and intersecting side $GD$, such that the extended segment $EZ$ is equal to some given line [Hultsch, 1876, Vol. 1, 172–174].

The transformation proceeds by assuming that the problem has already been solved; that is, that line $EZ$ is also given in magnitude.[23] Then two auxiliary lines are constructed such that $DH \parallel EZ$ and $ZH \parallel ED$ [*Elements* I 31]. Then, since $EZ$ is given in magnitude, $DH$ is

[22] A neusis was a category of construction in Greek geometry in which a line was passed through a given point in such a way that some segment of it was cut off between two given objects so as to have a given length. For example, see Knorr [1986, 178–194] for a discussion of Archimedes' use of neusis constructions.

[23] According to Definitions 1 and 4 of the *Data*, a line may be said to be given in either *in magnitude* or *in position* [Menge, 1896, 2; Taisbak, 2003, 17]. Where it is given in both magnitude and position, we simply say it is given. In this case, it is given in magnitude, by the analytic assumption, but it is not yet given in position, because, although we know the position of $A$, we do not yet know the position of either $E$ or $Z$.

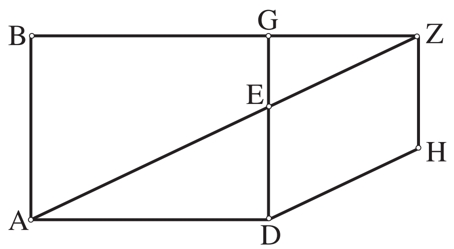


Fig. 1. *Collection* IV Prop. 31, analysis.

given in magnitude. (1) Since point $D$ is given, point $H$ is on the circumference of a circle given in position [*Data* Def. 4]. (2) Moreover, since $(BG \times GD)$ is given and is equal to $(BZ \times ED)$,[24] $(BZ \times ED) = (BZ \times ZH)$ is given. Therefore, point $H$ is on a given hyperbola [*Conics* II 12]. Point $H$, however, is also on the circumference of a given circle. Therefore, it is given.

In this proposition, the transformation uses the auxiliary construction of lines $ZH$ and $DH$ to change the problem into that of the construction of line $DH$. It then breaks into two short parts; (1) shows that $H$ lies on a given circle and (2) shows that it also lies on a given hyperbola. Because of this structure, there are trivial resolution steps mixed into the transformation. That is, there are trivial statements about what objects are given and on what basis. In this example, however, there is no general resolution, probably because it is so obvious. Nevertheless, on the basis of the construction found in the text, we can reconstruct one as follows. Since point $H$ is given, line $HZ$ is given in position [*Data* 28], and hence point $Z$ is given [*Data* 25]. Therefore, line $AZ$ is given [*Data* 26].

The synthesis, as is often the case, begins with a new diagram. The construction is as follows. In Fig. 2, let the given rectangle be $ABGD$ and the given line $M$. Line $AD$ is extended and cut off so that $DK = M$ [*Elements* I 3]. Through point $D$, a hyperbola is drawn with asymptotes $AB$ and $BG$ [*Conics* II 4], and, with center $D$ and distance $DK$, circle $KH$ is drawn [*Elements* I post. 3], cutting the hyperbola at $H$. Through point $H$, line $HZ$ is drawn such that $HZ \parallel DG$ [*Elements* I 31], and it meets $BG$ extended at $Z$. Line $AZ$ is joined.

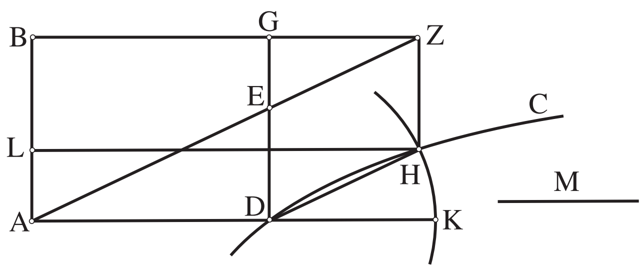


Fig. 2. *Collection* IV Prop. 31, synthesis.

[24] Since triangle $AED$ is similar to triangle $ABZ$, $BA : BZ = ED : AD$, but $BA = GD$ and $AD = BG$; hence $(BG \times GD) = (BZ \times ED)$. In this paper, we use the expression $(AB \times CD)$ for the *rectangle contained by lines AB and CD* and the expression $AB^2$ for the *square on line AB*. In the texts discussed in this paper, these expressions always refer to geometric objects, and no reference to arithmetic operations is intended by our convention.

The proof, then, is to show that $EZ = M$. Line $HD$ is joined and line $HL$ is drawn such that $HL \parallel KA$ [*Elements* I 31]. Then $(BZ \times ZH) = (BG \times GD)$ [*Conics* II 12], so that, since $ZB : BG = GD : DE$ [*Elements* VI 4], and $ZB : BG = GD : ZH$ [*Elements* VI 22], therefore, line $ED$ is equal to line $ZH$, and $DEZH$ is a parallelogram. Therefore, line $EZ$ is equal to line $DH$ [*Elements* I 33], that is $EZ = M$.

This example may be used to illustrate a number of features that are common in the analyses that survive in our sources. (a) The analysis is an analysis of a specific figure and the synthesis is the construction of a different, although closely related, figure. (b) Although the division between the analysis and the synthesis is quite distinct, that between the transformation and the resolution, on the one hand, and the construction and the proof, on the other, can be less absolute and the steps of these are often interwoven. (c) A particular section may be very brief, or missing, presumably if the geometer regarded it as sufficiently obvious. (d) Finally, the direction of the argument in the analysis and synthesis, mentioned by Pappus in his general description of the field of analysis and much discussed by modern scholars,[25] is opposite only in the loose sense that they have opposite premises and goals, not in the strict sense that one is the logical reversal of all of the steps of the other.[26]

In the example given, as in many of the analyzed propositions in our sources, there is no diorism. Indeed in this case, it can be shown that the problem can always be completed by a single line, so that it was not necessary to include any diorism in a formal exposition of the problem.[27] Although the diorism is not always present in ancient analyzed propositions, it is clear that when it is present, it must follow the resolution. Modern scholars have often stated the purpose of the diorism in purely logical terms, for example as establishing the "reversibility of certain one-way implications" or the "necessary ... conditions of solvability" [Mahoney, 1969, 328–329; Hintikka and Remes 1974, 58–59]. This, however, does not fully explain its role within the constructive framework of ancient problem solving. An analysis begins by assuming that some specific object solves the proposed problem and then proceeds by an investigation of the properties and implications of this particular object.[28] Hence, it cannot itself develop a study of the overall possibilities of solvability; it cannot present an assessment of the general limits of solvability or show that there may be multiple solutions that are arranged in various ways. Nevertheless, as we will see below, the geometric properties of the auxiliary objects introduced in the analysis can be utilized to this effect. In this paper, we will examine various examples of how ancient geometers produced diorisms as investigations of the auxiliary objects used in the analysis.

We will argue that the standard division of an analyzed proposition should include a diorism, although in practice this section is often missing:

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[25] See Jones [1986, 83–85] for the text of Pappus's description. There are a number of translations and readings of this passage, as for example, Mahoney [1969, 322], Hintikka and Remes [1974, 8–10], and Jones [1986, 82–84]. The most recent translation, and comprehensive analysis, of Pappus's description is given by Acerbi [2007, 439–446].

[26] Acerbi [2007, 454–463] has also insisted on this point.

[27] We will return to the issues raised by this problem in Section 5.

[28] Although Hintikka and Remes [1974, 32] and Behboud [1994, 58–59] have different logical accounts of ancient analysis, they do agree that the analysis is an investigation of a specific instantiation of geometric objects.

1. Analysis
   (a) Transformation
   (b) Resolution

2. Diorism

3. Synthesis
   (a) Construction
   (b) Proof

Both the diorism and the synthesis may be absent—the synthesis in the case where the geometer believes its production is too obvious to warrant full treatment, and the diorism in the case where the problem always has a single solution. As we saw above, the resolution may be so brief as to appear to be missing, we will see below that the same is true of the transformation.[29] Finally, as we will show in this paper, since the diorism is a study of the solutions, insofar as they are solutions, it can also be longer than the other two parts and contain internal propositions.

In fact, we will argue in this paper that these three parts of a complete analyzed proposition, whether or not they are all found in the final written work, are reflective of three stages of mathematical investigation that the mathematician should have gone through in the course of any complete investigation of a problem. In the analytic stage, the mathematician assumes the existence of a single solution and derives from this various auxiliary constructions that will be of use in affecting the actual solution and which can be used to investigate the overall possibilities for solution. In the diorismic stage, the mathematician uses the auxiliary objects of the analytical stage to show that a single solution is always possible, or if not, how it is limited or how many solutions there may be and how they are arranged. Finally, in the synthetic stage, the mathematician must effect a construction that solves the problem, stating at the outset any restrictions that are necessary or dividing the construction into cases according to the total possibility of solutions. In Section 4, we will argue that one of the principal motivations behind Apollonius's systematic approach in *Cutting off a Ratio* was his desire to produce a paradigmatic case of the analytical treatment of a relatively simple problem in which each of these three stages is exhaustively carried out. We turn now to examples of diorism in ancient authors.

## 3. Analysis and diorism

In this section, we examine diorisms in two of the most canonical mathematical authors of the Hellenistic period, Archimedes and Apollonius. As noted above, the auxiliary constructions and assumptions used in an analysis could sometimes be used to show that a solution was only possible under certain circumstances. This limitation might be due to some inherent geometric property of the given objects, or to the arrangement of the objects. In this section, we will look at two diorisms that characterize limitations to the solvability of the problem inherent in the objects themselves, while in the following section we will discuss more generalized diorisms, which are due to the arrangement of the objects under consideration.

[29] Compare the discussions of *Collection* IV Prop. 31 (above) and Cutting off a Ratio 1.1 (below).

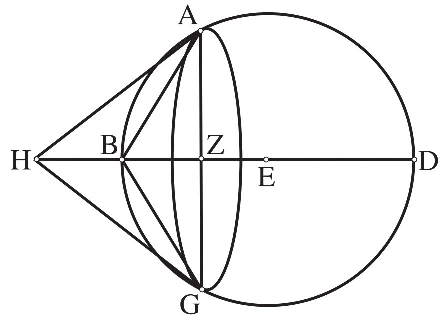



Fig. 3. *Sphere and Cylinder* II 7, analysis.

The presentation of the most common type of diorism, that is, the specification of the limits, or possibilities, of solvability, could be carried out in two ways. In general, it occurs within the context of an analyzed proposition, after the analysis and before the synthesis. We will examine one example of this type from Archimedes' *Sphere and Cylinder*. In some cases, however, the diorism is due to a specific property of the objects under discussion that is significant enough to be presented in a separate proposition. A number of the diorisms given in the *Elements* are of this type.[30] Even in these cases, however, although the text simply refers to this other proposition, the discovery of the property itself was probably often made in the course of the analysis. The deductive structure of the ancient treatises generally prevents us from being certain of the geometer's heuristic process; nevertheless, we will present an example from Apollonius's *Conics*, in which we are fairly sure that this is what happened.

### *3.1. Diorism in Sphere and Cylinder II 7*

In order to understand the role of diorism within a standard analyzed proposition, we may look at the case of Archimedes' *Sphere and Cylinder* II 7. This problem demonstrates how

> To cut a section from a given sphere with a plane, such that the section has a given ratio to the cone that has the same base and height as the section. [Heiberg, 1910–1915, Vol. 1, 206]

In Fig. 3, assuming that the plane through line $AG$ that is perpendicular to line $BD$ cuts sphere $ABG$ such that the ratio of spherical segment $ABG$ to cone $ABG$ is some given ratio, the transformation is as follows. Using a relation between spherical segments and cones that stand on the same base, put forward in *Sphere and Cylinder* II 2, a cone is constructed equal to spherical segment $ABG$ by finding $H$ such that $(ED + DZ) : DZ = HZ : ZB$ [*Elements* VI 12]. This transforms the situation into one in which the ratio of cone $ABG$ to cone $AHG$ is given; that is, $BZ : HZ$ is given [*Elements* XII 14]. The resolution is then simply a matter of showing that, on this basis, lines $DZ$ and $AG$ are also given.

The means of constructing point $H$, however, reveals a limit to the solvability of the problem, which Archimedes states in the diorism.

[30] For example, consider the relationship of *Elements* I 20 to I 22, *Elements* VI 27 to VI 28, or *Elements* XI 20 to XI 22 and 23. As Fried and Unguru [2001, 284–293 and 298–306] have pointed out, it is probably in reference to theorems of this sort, which state inherently interesting properties that will later be used to limit the scope of problems, that Apollonius, in the preface to *Conics* I, states that *Conics* VII contains *diorismic theorems* [Heiberg, 1891–1893, Vol. 1, 4].

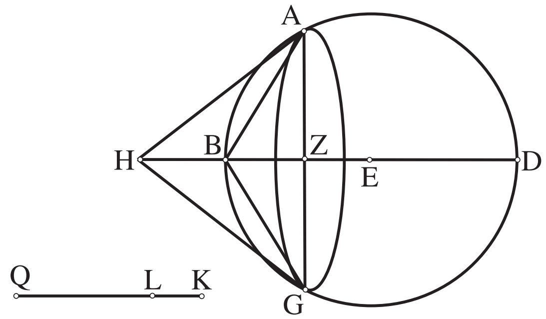



Fig. 4. *Sphere and Cylinder* II 7, synthesis.

> Since the sum of $ED$, $DZ$[31] has to $DZ$ a greater ratio than the sum of $ED$, $DB$ to $DB$,[32] while $BD$ is twice $ED$, therefore the sum of $ED$, $DZ$ has to $DZ$ a greater ratio than that which three has to two.[33] Since the ratio of the sum of $ED$, $DZ$ to $DZ$ is the same as the given, therefore it is necessary that the ratio given for the synthesis be greater than that which three has to two. [Heiberg, 1910–1915, Vol. 1, 208]

The construction of point $H$ as the vertex of the cone equal to spherical segment $ABG$ then provides necessary and sufficient conditions for solving the problem. The necessary condition is demonstrated in the diorism, while the synthesis shows that, given the condition stated in the diorism, the construction set out in the analysis is sufficient to solve the problem.

Again, the synthesis is clearly distinguished from the analysis by the fact that it is based on a new diagram. The fact that a new diagram was used, despite being essentially the same as the first, reveals an essential characteristic of the problem solving art. Ancient geometers started the synthesis by starting over with a new figure. The synthesis begins from a different set of given objects and redrawing the diagram makes this quite clear. Moreover, although the finished texts always refer to the diagrams as having already been drawn, geometric practice, both in teaching and in research, almost certainly involved actually drawing out the diagrams and then reasoning about them. Thus, in working through an analyzed proposition, a Greek geometer probably thought of the constructive process of the synthesis as different enough from that of the analysis to require a new diagram.

In Fig. 4, the construction is as follows. Where sphere $ABGD$, about center $E$, and ratio $QK : KL > 3 : 2$ are given, point $Z$ is determined by setting $QL : LK = ED : DZ$ [*Elements* VI 12]; hence point $Z$ falls between points $D$ and $B$. Line $AG$ is drawn perpendicular to $BD$ at $Z$ [*Elements* I 11], and a plane is passed through line $AG$ perpendicular to line $BD$. The proof then proceeds by constructing cone $AHG$ equal to the spherical segment $ABG$, in the same way as in the transformation and then shows that this cone has the given ratio, $QK : KL > 3 : 2$, to cone $ABG$.

In this case, the diorism expresses in a straightforward way a limit made clear by the auxiliary construction used in the transformation. Indeed, this way of motivating the diorism

31 Here and following, the expression we have translated as "the sum of $ED$, $DZ$" (συναμφότερος ἡ ΕΔΖ) would more literally be rendered as "both of $EDZ$," that is, *$ED$ and $DZ$ taken together*.

32 Eutocius explains this as follows [Heiberg, 1910–1915, Vol. 3, 190]. Since $DB > DZ$, $ED : DZ > ED : DB$ [*Elements* V 8], and by composition, $(ED + DZ) : DZ > (ED + DB) : DB$ [extension of *Elements* V 18]. The extension of *Elements* V 18 to ratio inequalities was provided, much later, by Pappus, in the beginning of *Collection* VII, along with similar extensions of the other fundamental operations on proportions [Jones, 1986, 128–130].

33 Since $ED$ is the radius, and $BD$ the diameter, of circle $ABGD$.

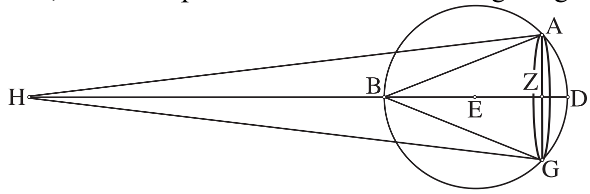
may be one of the reasons that this proposition was presented as an analyzed proposition. We cannot know whether or not Archimedes thought along the following lines; nevertheless, it is clear that he had some way of thinking about these things that lead to the same conclusion. Intuitive considerations of the figure will lead to the nonrigorous conclusion that there is no upper limit to the given ratio. As line $AG$ approaches point $D$, the spherical segment approaches the sphere itself, while the cone under the same hight becomes indefinitely small.[34] Hence, since one term always approaches an upper limit, which happens to be the volume of the sphere, while the other term approaches a lower limit of zero, there is no upper limit to the ratio. In the other direction, however, as line $AG$ approaches point $B$, the fact that the lower limit to both objects is zero, that they are always approaching this limit, and that the segment is always greater than the cone will lead to the nonrigorous conclusion that there is some limiting ratio greater than, or equal to, the ratio of equal to equal.

The transformation of the problem into one of comparing cones, however, by setting $(ED + DZ) : DZ = HZ : ZB$, reveals this upper limit by a simple consideration of the geometry of the circle. Since the upper limit to the given ratio, however, is based on the trivial fact that the diameter of a circle is twice its radius, it does not warrant separate treatment. Hence, it was included in the diorism of the analyzed proposition. In the next section, we look at a pair of propositions in which the investigation of the diorism almost certainly lead to a significant geometric result, which was then written up in a separate theorem, distinct from the final solution to the problem.

### 3.2. Diorism in Conics II 51, 52, and 53

The second book of Apollonius's *Conics* ends with a series of problems, most of which are carried out using analysis-style arguments. From the prospective of the standard model, the structure of a number of these propositions is somewhat loose;[35] nevertheless, the last five propositions, *Conics* II 49–53, form a group that treats the construction of tangents to conic sections. *Conics* II 49 shows how to draw a tangent to a conic section from a given point not within the section; *Conics* II 50 shows how to construct a tangent that makes a given angle with the axis of the section; and the final three propositions, *Conics* II 51–53, show how to construct a tangent to a given conic section that makes a given angle with the diameter that passes through the point of tangency. In fact, the enunciation for the overall project of these last three propositions is stated at the beginning of *Conics* II 51 as follows:

[34] In other words, although the manuscript figures depict point $Z$ between $E$ and $B$, it may fall anywhere on the segment $BD$; for example consider the following diagram:




[35] For example, *Conics* II 45 is a sort of sketch that is a corollary to II 44, while *Conics* II 48 is actually a proof of the completeness of the solution given in II 47. As we will see in the following section, this proof of the completeness of a solution, which we call the *enumeration*, is a part of the complete analysis of a problem as set out in Apollonius's *Cutting off a Ratio*.

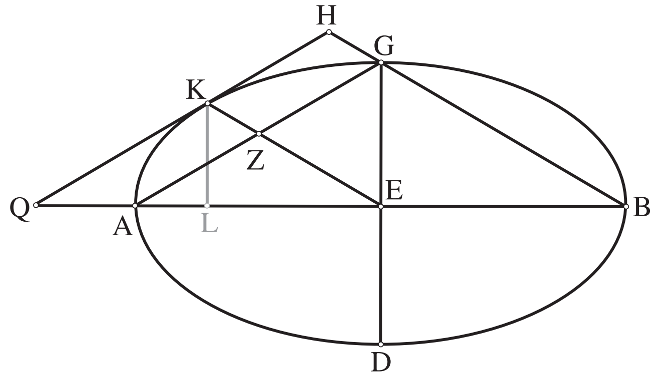



Fig. 5. *Conics* II 52, diagram 1.

> To draw a tangent to a given conic section that contains, with the diameter drawn through the point of tangency, an angle equal to a given acute [angle]. [Heiberg, 1891–1893, Vol. 1, 289]

Since the tangent at an axis is perpendicular to that axis, this upper limit to the given ratio is stated in the enunciation of the problem by the assertion that the given angle must be acute. For the parabola and the hyperbola, there is no lower limit to acute angles contained by a tangent and the diameter at the point of tangency, and hence there is no diorism for these two figures. The problem is solved for the parabola and the hyperbola using analyzed propositions in *Conics* II 51. This is followed, however, by a theorem, *Conics* II 52, establishing, in the case of an ellipse, the lower limit for the angle between a tangent and the diameter through the point of tangency. This theorem is then used in *Conics* II 53, a purely synthetic problem, which gives the solution for an ellipse given an acute angle with a stated lower limit.

The existence of a limit to the solvability of the problem in the case of the ellipse is intuitively obvious. Consider Fig. 5.[36] It is clear that as the point of tangency, $K$, approaches point $A$, the angle $EKH$ is acute but approaches a right angle, while as $K$ approaches $G$, it is again acute and approaches a right angle. Since it cannot be arbitrarily small, there must be some least value between the two right-angle upper limits. *Conics* II 52 demonstrates that this least angle is not less than the supplement to the angle contained by the lines joining the endpoints of the major axis with those of the minor axis. That is, in Fig. 5, angle $EKH \geqq$ angle $AGH$. The proof of *Conics* II 52, however, involves an auxiliary circle, the motivation for which is not entirely clear in the context of *Conics* II 52 alone. This theorem, however, follows the analyzed proposition *Conics* II 51, which introduces a similar circle in the analysis of the problem for the hyperbola. Although *Conics* II 53, which solves the problem for the ellipse, provides no analysis, if we reconstruct an analysis for this problem, along the lines of that given for the hyperbola in *Conics* II 51, we will understand the motivation for introducing the auxiliary circle and see how the limit to the solvability of the problem may have been derived.

In Fig. 6, we begin, as usual, by assuming that the problem has been solved and that a specific angle $HKE$ has been constructed equal to some given angle, so that its complement, angle $QKE$, is also given. The goal of the analysis is, then, to show that the geometry of the ellipse together with the assumption that angle $QKE$ is given implies that triangle $QKE$ is

[36] The labeling of this figure has been altered to agree with that in *Conics* II 53 in order to facilitate comparison with the discussion below. The line $KL$, shown in gray, is not found in the manuscript figures, but has been added for the sake of our discussion.

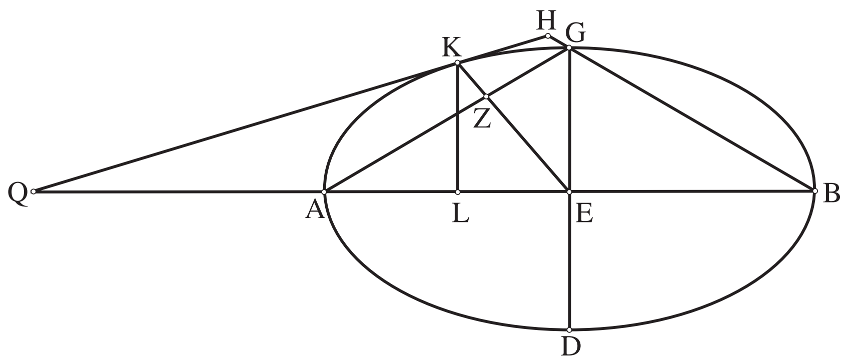



Fig. 6. *Conics* II 53, reconstructed analysis, ellipse.

given in form. According to Euclid's *Data*, rectilinear figures are said to be given in form when each of the angles are given and the sides have given ratios to one another. Angles, on the other hand, are said to be given when it is possible "to supply" equals [Menge, 1896, 2]. In the context of the constructive geometry of the *Data*, this means that it must be shown how it is possible to construct angles equal to these. Hence, we must show that it is possible to construct a triangle similar to triangle $QKE$.

The triangle $QKE$, however, as well as containing a given angle at $K$, can be cut by the perpendicular $KL$ so that, by the property of the tangent to an ellipse asserted in *Conics* I 37,

$$\text{transverse side} : \text{upright side} = (QL \times LE) : KL^2.$$

In this way, the problem is transformed into the construction of a triangle satisfying the two conditions that both this proportion and angle $QKE$ are given.

Since only angle $QKE$ of the triangle is given, we begin by constructing the segment of a circle containing this angle. In Fig. 7, we set out some given line $MP$ and construct the circle $MNPR$ such that angle $MNP =$ angle $QKE$ [*Elements* III 33]. We then assume that the transformed problem has been solved; that is, that triangle $MCP$ has been set out similar to triangle $QKE$ such that

$$\text{transverse side} : \text{upright side} = (MS \times SP) : CS^2.$$

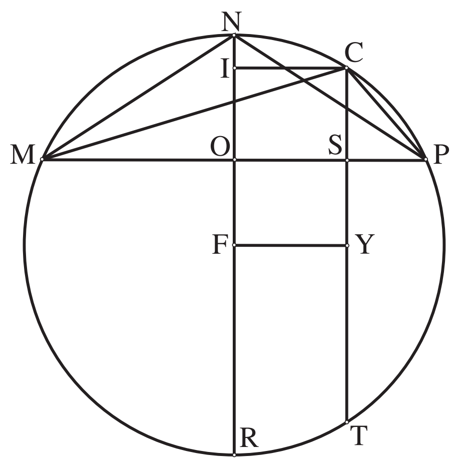



Fig. 7. *Conics* II 53, reconstructed analysis, circle.

Then if line $CS$ is extended to meet the circle at $T$, by *Elements* III 35, $(MS \times SP) = (CS \times ST)$, so that

$$\begin{aligned}\text{transverse side} : \text{upright side} &= (CS \times ST) : CS^2\\ &= ST : CS.\end{aligned}$$

Since the ellipse is given, the ratio of the transverse side to the upright side is given, hence $ST : CS$ is given. Then it remains to be shown that this can be determined in the circle.

The resolution can be reconstructed as follows. Since $ST : CS$ is given, by composition, $TC : CS$ is given [*Data* 6], and thus, since $YC$ is half of $TC$, $YC : CS$ is given [*Data* 8], and by separation, $YS : CS$ is given.[37] Where $F$ is the center of the circle, $FO = YS$ is given, and hence $SC$ is given [*Data* 2]. Then, the point $C$ will be determined by a parallel line when we set out point $I$ such that $FO : OI = YS : SC$, which is given.[38]

With point $C$ given, the triangle $MCP$ is given. Hence, the original problem is solved by constructing triangle $QKE$ similar to triangle $MCP$, that is, by setting out angle $QEK$ similar to angle $MPC$ [*Elements* I 23].

This analysis, however, also serves to reveal a limit to the solvability of the problem. By transforming the problem in this way, it is clear that no solution will exist if line $OI$ is greater than line $ON$. Hence, the lower limit to the given angle is obtained when triangle $QKE$, in Fig. 6, is similar to triangle $MNP$, in Fig. 7. By considering, once again, the properties of the ellipse, and comparing Figs. 5–7, we will see how Apollonius almost certainly discovered the least angle that a tangent can make with a diameter for a given ellipse.

In the limiting case, since $MO = OP$ (Fig. 7),[39] and hence $QL = LE$ (Figs. 5 and 6), from the property of the tangent to an ellipse asserted in *Conics* I 37,

$$\text{transverse side} : \text{upright side} = (QL \times LE) : KL^2 = QL^2 : KL^2.$$

On the other hand, a basic property of the ellipse, asserted in *Conics* I 21, states that the ratio of the upright side to the transverse side is that of the square of an ordinate to the rectangle contained by the two parts of the diameter cut by the ordinate. That is, as shown in Fig. 5,

$$\text{transverse side} : \text{upright side} = (AE \times EB) : EG^2 = AE^2 : EG^2;$$

hence,

$$\begin{aligned}QL^2 : KL^2 &= AE^2 : EG^2,\\ QL : KL &= AE : EG.\end{aligned}$$

Thus, triangle $AEG$ is similar to triangle $QLK$ and tangent $QK$ is parallel to line $AG$.

[37] Although no theorem for this step is demonstrated in the *Data*, a proof along the lines of those we find for the other ratio operations would be fairly simple and the mathematical fact was well known in practice. In this case, we have the analytic complement to *Elements* V 17.

[38] In the synthesis of the problem as set out in *Conics* II 53, however, it is not sufficient to simply state that $YS : SC$ is given, it must also be constructed [Heiberg, 1891–1893, Vol. 1, 310–316]. This is realized by constructing an auxiliary figure, line $WJ$, cut at point $A'$, such that $WA' : A'J =$ transverse side : upright side. Then, if line $WJ$ is bisected at $V$, from the argument given above it is clear that $VA' : A'J = YS : SC$. When Apollonius introduces these objects, he actually uses inequalities, because, having already established the limit to the solvability of the problem in the previous proposition, he now gives a construction for all the possible solutions Heiberg [1891–1893, Vol. 1, 314].

[39] That is, $S$ and $O$ will coincide.

Although we cannot be certain that this is the derivation Apollonius made, we are confident that he must have worked along these general lines. This reconstruction follows that given for the hyperbola in *Conics* II 51 and it furnishes a motivation for all of the mathematical objects we find in *Conics* II 52 and 53. It appears, however, that Apollonius considered that the limit to the solvability of the problem in the case of the ellipse was based on a property of the ellipse that is itself interesting enough to demand special treatment. Hence, instead of presenting the full analysis, diorism, and synthesis of an analyzed proposition, such as we saw in *Sphere and Cylinder* II 7, Apollonius wrote up the limit of solvability as a theorem, in *Conics* II 52, and then simply solved the problem with a synthesis, in *Conics* II 53. *Conics* II 52, which states the least angle that a tangent can make with the diameter through the point of tangency, is almost certainly an example of the type of proposition that Apollonius calls a *diorismic theorem* [Heiberg, 1891–1893, Vol. 1, 4; Fried and Unguru, 2001, 284–293].

The material that we have examined thus far was written in the context of generally synthetic treatises and hence, although some analyzed propositions are present, they are part of the overall deductive strategy of the work and are not meant to provide a full treatment of the analytic path taken by the geometer. In the next section, we will turn to a treatise that we believe was written as a programatic introduction to analysis, and hence gives a much more systematic survey of the analytical approach.

## 4. Diorism in Apollonius's *Cutting off a Ratio*

There are no known Greek sources for Apollonius's *Cutting off a Ratio*, but an Arabic translation is extant in two manucripts, Aya Sofia 4830 and Bodliean Arch. Seld. 32.[40] A Latin translation of Bodliean Arch. Seld. 32, which was begun by Edward Bernard, was completed and published by Halley [1706], and this has served as the basis for what little modern study there has been of this text.[41]

Because it exhaustively solves a simple, and seemingly pointless, problem using a style of mathematics far removed from our own, most modern readers will probably agree with Hogendijk [1986, 224] that the text of *Cutting off a Ratio* is "long and dreary." Nevertheless, since the treatise is the only complete work, beyond Euclid's *Data*, devoted to the field of ancient analysis and was originally written by one of the great practitioners of that discipline, it is deserving of more attention from historians than it has hitherto received. Indeed, as we will show in this section, *Cutting off a Ratio* reveals a number of features of ancient analysis that are obscure or absent in other sources.

The overall project of *Cutting off a Ratio* is stated at the outset as follows.

> When two unlimited lines are positioned (موضوعان) in a plane, either parallel or intersecting, and a point is marked (تعلّم) on each of them, and a ratio is determined (مفروضة), and a point is marked (معلومة) that is not on either of them, how, from the marked point, we pro-

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[40] For purposes of citation, we will refer to these as *CR* A and *CR* B. The library shelf marks are given in the bibliography.

[41] Macierowski and Schmidt [1987] also made an English version of this text on the basis of both Arabic manuscripts. As the subtitle states, however, this is "an attempt to recover the original argumentation through a critical translation," and, hence, should not be read as a faithful translation of the Arabic. We are told that an edition of the Arabic, with French translation, has been been made by R. Rashed and H. Bellosta and will appear in print soon [Rashed and Bellosta, forthcoming].

> duce a line that crosses the two positioned lines and sections them such that the ratio of one of the two sections to the other of them, with respect to what is adjacent to the two marked points on the two lines, is equal (متساوية) to the determined ratio.[42] (*CR* A, 2v; *CR* B, 1v)

That is, given two lines $\ell_1$ and $\ell_2$, point $E$ on $\ell_1$, point $Z$ on $\ell_2$, point $T$ on neither $\ell_1$ nor $\ell_2$, and given ratio $r$, to draw a line, $TKL$, such that

$$EK : ZL = r.$$

The treatise then proceeds to systematically investigate all arrangements of the given objects that differ in some geometrically significant way. The first book solves the problem when $\ell_1$ and $\ell_2$ are parallel and when $E$ or $Z$ is the intersection of $\ell_1$ and $\ell_2$, and the second book reduces the construction of all other arrangements to these.

*Cutting off a Ratio* was probably written as a sort of training exercise in the techniques of ancient analysis, and its repetitive structure makes rather explicit certain processes of the problem-solving art that appear to have generally been left out of published works—such as the systematic arrangement of all possible cases and the subsequent reduction of more complicated cases to simpler ones, the systematic investigations of the limits of solution, and the complete enumeration of all possible solutions.[43]

Probably the most striking difference between *Cutting off a Ratio* and other Greek works of pure geometry is the structural arrangement. In general, the works of Greek geometry—such as Euclid's *Elements*, Archimedes' *Sphere and Cylinder*, or Apollonius's *Conics*—are divided into relatively small units of text which each establish a particular theorem or solve a certain problem. Since *Cutting off a Ratio*, however, solves a single problem, the text is divided first according to the geometrically significant ways in which the given objects can be arranged and then according to the ways in which the sought line can fall on the given lines so that the problem is solved. The first division, which we translate as *disposition*,[44] is determined by the position of the original objects, and the

[42] Although Greek mathematicians generally asserted a proportion by saying that one ratio was the *same* as another, Arabic mathematician retained no such distinction, employing instead a number of different terms with a range of meanings such as *similar*, *as*, and *equal*.

[43] Acerbi [2007, 442] doubts that *Cutting off a Ratio* was meant as a preparatory text for the practice of analysis in the same sense as the *Data*. Indeed, if the *Data* was even meant to be studied thoroughly, certainly its primary purpose was to act as a repository for theorems justifying the kind of steps that were required in the resolution of an analyzed proposition. Although one, in some sense, needs the *Data* to *do* analysis, one cannot learn much about *how* to do analysis by reading it. We believe that Apollonius probably composed exhaustive solutions to simple problems, such as *Cutting off a Ratio*, to address the question of how to structure analytic investigations. Nevertheless, we do not mean to suggest that Apollonius had students who were interested in studying analysis, but rather that the exhaustion and repetition of the text seems to indicate that Apollonius planned the work around what he considered to be the proper approach when investigating problems. The problem solved is neither very interesting nor difficult; hence he is probably demonstrating a general strategy with a simple case. On the other hand, whether or not he knew any of them personally, it is clear from the opening of *Conics* V, quoted in the Introduction, that Apollonius thought there would be people interested in learning how to do analysis, and he, naturally, called them students.

[44] The Arabic term is وضع and Pappus uses τόπος, which can both mean "place" and are often translated with *locus* (see Section 4.2, below). Since this usage is different from what we generally mean when we say "locus," however, we have followed Jones [1986, 87] and translated with *disposition*.

second division, which we translate as *occurrence*,[45] is determined by the possible arrangements of the solution.

Each occurrence, in turn has four parts—analysis, diorism, synthesis, and enumeration. The analyses and syntheses usually begin with short contrivances or constructions that one may think of in terms of the transformation and construction of a standard analyzed proposition, but since these are usually so short, and since contrivances and constructions are also used in the other parts of the text, it is not particularly useful to impose these parts of the standard structure on *Cutting off a Ratio*. Finally, each disposition concludes with a summary of all possible solutions that draws on the diorisms and enumerations given in each occurrence. Hence, we may outline the overall structure of the text as follows:

1. Problem [The project of the entire text]
   - (a) Disposition 1 [A geometrically significant arrangement of the given objects]
     - i. Occurrence 1 [An investigation of one of the possible configurations for a solution of Disposition 1]
       - A. Analysis [Assuming that a particular solution has been found, an investigation of what is known on this basis]
       - B. Diorism [A statement about, or study of, the solutions themselves]
       - C. Synthesis [The construction of the line(s) that solve the problem and a proof that the construction solves the problem]
       - D. Enumeration [A proof that only the line(s) constructed in the synthesis solve(s) the problem]
     - ii. Occurrence 2 [An investigation of a different possible configuration for a solution of Disposition 1]
     - iii. ...
     - iv. Summary [A discussion of the total number of possible solutions found in all of the occurrences of Disposition 1]
   - (b) Disposition 2 [A different, geometrically significant arrangement of the given objects]
   - (c) ...

As this list shows, in one sense the basic unit of text in *Cutting off a Ratio* can be taken as the occurrence, which is, in some ways, equivalent to an individual analyzed proposition. The most variable section of an occurrence is the diorism. For an occurrence in which the problem can always be solved, the diorism, or lack thereof, is stated in a single line,

[45] The Arabic term is وقوع, which literally means "the place where something falls" or "position," and Pappus uses πτῶσις, which in mathematical texts usually denotes a case (see Section 4.2, below). Since the division in question here, however, is according to the possible ways in which solutions can be arranged, it has a different meaning than what we generally mean by "case." Hence, we have translated the Arabic term somewhat abstractly as *occurrence*, in the sense of a particular setting out of geometric objects. In the Arabic text, another word, وجه, is sometimes used synonymously with وقوع, although it also has more general uses. Although these words may in fact be fully synonymous, for the time being, we differentiate and translate وجه with *case*.

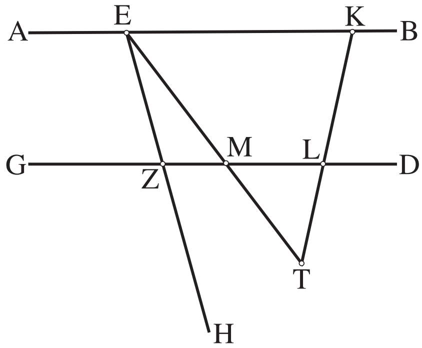


Fig. 8. *Cutting off a Ratio* 1.1, analysis.

or omitted altogether. In other cases, however, the diorism can be rather involved, using internal analyzed propositions and becoming the longest section of the occurrence. In another sense, however, the basic unit of the text is the disposition. In fact, the disposition includes a final summary that gives a treatment of all possible solutions and the limits of solvability for the particular arrangement of objects set out in the disposition. The study of the solutions that was developed in the diorisms reappears in this final summary. In the following two sections, we will examine one example of a standard diorism and one example of an extended, analytical diorism.

### 4.1. A standard diorism in Cutting off a Ratio

For a standard diorism, we will look at *Cutting off a Ratio* Disposition 1, Occurrence 1. In this example we will sketch all four parts of the analyzed proposition in order to make clear the structure of the text. In *Cutting off a Ratio* 1, Apollonius assumes that the given lines, $\ell_1$ and $\ell_2$, are parallel and that the given point, $T$, is somewhere outside the pair of parallel lines.

In Fig. 8, let $\ell_1$ be $AB$, $\ell_2$ be $GD$, point $E$ given on $AB$, point $Z$ given on $GD$, and point $T$ given outside $AB$ and $GD$. *Cutting off a Ratio* 1.1, then, solves the problem of constructing a line through $T$ falling on $EB$ and $ZD$, as say $TLK$, such that

$$EK : ZL = r,$$

where $r$ is a given ratio.

The analysis proceeds as follows. We begin with the assumption that the problem has been solved; that is, that $EK : ZL$ is given.[46] Then we join line $ET$. Since points $E$ and $T$ are given, line $ET$ is given [*Data* 26], and since line $GD$ is given in position, point $M$ is given [*Data* 25]; therefore ratio $EM : MT$ is given [*Data* 1 and 26]. So, by composition, ratio $ET : TM$ is given [*Data* 6].

[46] Our interpretation of this proposition is different from that recently put forward by Panza [2007, 107–109]. Part of the problem arises from the ambiguity in the term *given*, which can mean either given by the conditions of the problem or assumed at the geometer's discretion. For example, Panza [2007, 108] states that $EK : ZL$ (his $AM : BN$) is "given as the condition of the problem," whereas this is the analytical assumption. Moreover, because, as we will argue below, this proposition intermingles the transformation and the resolution, Panza's attempt to restructure the order of the argument is unnecessary.

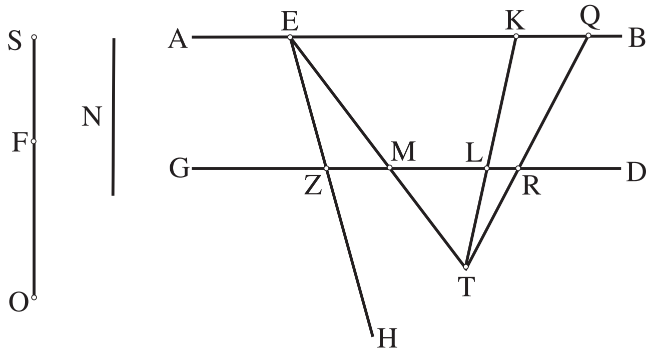


Fig. 9. *Cutting off a Ratio* 1.1, synthesis.

By the geometry of the figure, however, $ET : TM = EK : ML$ [*Elements* VI 4], so ratio $EK : ML$ is given. By hypothesis, however, ratio $EK : ZL$ is given, so ratio $ZL : LM$ is given [*Data* 8]. Hence, by separation, ratio $ZM : ML$ is given.[47] Line $ZM$, however, is given [*Data* 26], so line $ML$ is given [*Data* 2]. Point $M$, however, is given, so point $L$ is given [*Data* 27].[48] Point $T$, again, is given, so line $TK$ is given [*Data* 26].

In this proposition, because the transformation is geometrically so simple, the steps of the transformation and resolution have been combined into a single argument. The transformation component of the argument is the construction of line $ET$ and a demonstration that $ET : TM = EK : ML$. The resolution component of the argument is the proof that, given the initial configuration and $ET : TM = EK : ML$, point $L$ is given, so that line $TK$ is given. Moreover, $ET$ will be essential to the synthesis and, hence, it is the auxiliary construction that solves the problem.[49] Because $ET$ will be used in the synthesis to solve the problem, its geometric properties lead directly to an investigation of the limits of solvability. The diorism, which we quote in full, reads as follows:

> Because line $LM$ is less than line $LZ$, ratio $EK$ to $ML$ is greater than ratio $EK$ to $ZL$ [*Elements* V 8]. Ratio $EK$ to $ML$ is as ratio $ET$ to $TM$ [*Elements* VI 4], so ratio $ET$ to $TM$ is greater than ratio $EK$ to $ZL$, and ratio $EK$ to $ZL$ is the determined ratio. Therefore, it is necessary that the determined ratio is less than ratio $ET$ to $TM$. [*CR* A, 5r; *CR* B, 2r]

Here we see that the diorism simply uses the properties of the constructed line to show a necessary, but not sufficient, condition for solvability. In order to show that $TK$ also provides a sufficient condition for solvability, Apollonius, following standard practice, introduces a synthesis.

As usual, the synthesis for *Cutting off a Ratio* 1.1 begins with a new diagram, Fig. 9. The argument is fairly brief and can be sketched as follows. Let lines $AB$, $GD$ and points $E$, $Z$, $T$

[47] See note 37, above.

[48] Panza [2007, 108] claims that concluding by showing that point $L$ (his $N$) is given seems to contradict Pappus's "structural description." The statement that point $L$ is given, however, is the conclusion of the resolution, so whatever Pappus may say about structure, it must be the conclusion of a deductive chain of givens that begins with the analytic assumption, the geometry of the given objects and the properties of the auxiliary constructions. Hence, the argument Panza [2007, 108] reconstructs is the inverse of that Apollonius gives.

[49] See Hintikka and Remes [1974, 41–48] for a discussion of the role of auxiliary constructions in analyses.

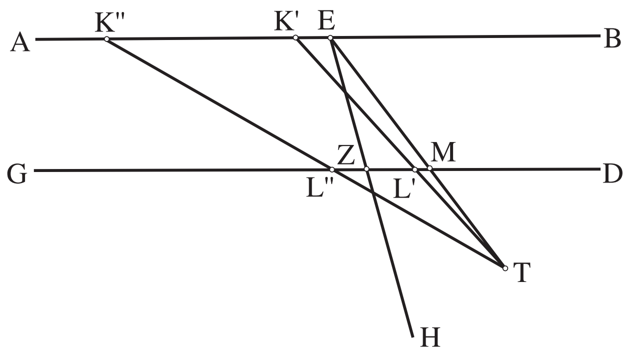


Fig. 10. *Cutting off a Ratio* 1.2 and 1.3.

be given as before. Let $N : SO = r$ be the given ratio, but satisfying the condition, determined in the diorism, that $N : SO < ET : TM$. Then line $ET$ is drawn and points $F$ and $L$ are taken such that $ET : TM = N : SF$, and $OF : FS = ZM : ML$ [*Elements* VI 12]. It is then a straightforward matter of ratio manipulation to show that $EK : ZL = N : SO = r$, and hence, that line $TK$ solves the problem.

In the synthesis, the given ratio is made explicit with the two lines $N$ and $SO$, not found in the original configuration. Hence, it becomes necessary for Apollonius to argue that $OF : FS = ZM : ML$ implies $EK : ZL = N : SO$, following a series of steps that does not correspond to anything in the analysis. In this way, the synthesis is burdened with the full task of explicitly setting out the given ratio and showing its relation to the arrangement of the given lines. The motivation for this division of labor can, again, be understood in terms of the diorism. In the analysis, we assume the existence of some actual solution; hence it will necessarily fall within the limits of the possible solutions. Moreover, the geometric properties of this solution can be used to understand the limits of the possibility of solution as presented in the diorism. Setting out the given ratio, independent from a specific configuration, however, involves the geometer in the use of the results of the diorism. Hence, it was natural to put this task off until after the diorism had been completed.

Following the synthesis, Apollonius gives an *enumeration* of the solutions by showing that, for this occurrence, $TK$ is the only line that solves the problem. He begins, in Fig. 9, by assuming that some other line, say $TQ$, also solves the problem. Then, since $LM < LZ$, $RL : LM > RL : LZ$ [*Elements* V 8], and ratio manipulation is used to show that $QE : RZ > EK : ZL = r$. Hence, line $TQ$ does not solve the problem and the same argument could be used to show that any line closer to $Z$ than $TK$ cuts off a lesser ratio than $EK : ZL = r$.

In the context of *Cutting off a Ratio* 1.1, the diorism establishes a necessary condition for the given ratio, which is then simply assumed in the synthesis. The results of the diorism, however, reappear in the final section of *Cutting off a Ratio* 1, in which this particular arrangement of given objects is considered in full generality.

For *Cutting off a Ratio* 1, there are three occurrences. In Fig. 10, *Cutting off a Ratio* 1.2 solves the problem when $TK'$ falls on lines $EA$ and $ZD$, cutting off $EK' : ZL' = r$, and *Cutting off a Ratio* 1.3 solves the problem when $TK''$ falls on lines $EA$ and $ZG$, cutting off $EK'' : ZL'' = r$. For *Cutting off a Ratio* 1.2, the problem can always be solved, while for *Cutting off a Ratio* 1.3, Apollonius shows that the given ratio, $r$, must be greater than $ET : TM$.

Following the three occurrences, in the *summary*, Apollonius considers some initial configuration of the given objects satisfying the conditions of *Cutting off a Ratio* 1, draws the line $TE$ intersecting line $GD$ at point $M$, and points out that

$$r \gtreqless ET : TM.$$

Thus,

where $r < ET : TM$, the problem is solvable by Occurrences 1 and 2;
where $r = ET : TM$, the problem is only solvable by Occurrence 2; and
where $r > ET : TM$, the problem is solvable by Occurrences 2 and 3.

In this way, the diorisms, in conjunction with the analyses, establish the necessary and sufficient conditions for solvability. A fully general statement of the solutions for *Cutting off a Ratio* 1 can only be given on the basis of the solutions given in the syntheses structured according the claims of the diorisms.

### 4.2. An analytical diorism in Cutting off a Ratio

In the first book of *Cutting off a Ratio*, 18 of the occurrences have the basic structure we have seen above, containing a short diorism situated between the analysis and the synthesis. For five occurrences, however, Apollonius produced a much more extended diorism, containing internal problems and theorems treated with analyzed propositions, and resulting in a full treatment of the possibilities of solvability that is much longer than the combined passages of the general synthesis and analysis.[50] In fact, for Pappus, these extended diorisms were the only ones that he regarded *as* diorisms. In his summary of the work, he tells us that

> ...The first book of *Cutting off a Ratio* has seven dispositions (τόπους), twenty-four occurrences (πτώσεις), and five diorisms (διορισμοὺς), of which three are maxima and two minima. [Jones, 1986, 87]

As we will see, this is because these diorisms are treated using analyzed propositions, first to demonstrate whether the limit of the solvability of the problem is a maximum or minimum and then to show how the possible solutions are arranged around the limiting solution.

In order to examine the more developed role of diorism in these occurrences, we take, as an example, *Cutting off a Ratio* Disposition 6, Occurrence 4. Because the structure of *Cutting off a Ratio* 6.4 is rather convoluted, it may be useful to give a summary of the argument before going into the details.

In Fig. 11, let the given lines be $AB$ and $GD$, meeting at point $E$, and let the given point not on the lines be $H$. Let the given points on the lines be the intersection $E$ and some other point on $GE$, say $Z$. *Cutting off a Ratio* 6.4 solves the problem of constructing a line through $H$, say $HL$, falling on $EA$ and $ZD$ such that

$$EK : ZL = r,$$

where $r$ is a given ratio.

[50] These five occurrences are *Cutting off a Ratio* 5.3, 6.2, 6.4, 7.2, and 7.4. The dispositions of the second book are reduced in one way or another to those in the first book, so that the argument is generally much shorter and the extended diorisms that are the focus of our attention in this section, in particular, are absent. This is explained by Pappus in his summary of the work in *Collection* VII [Jones, 1986, 87].

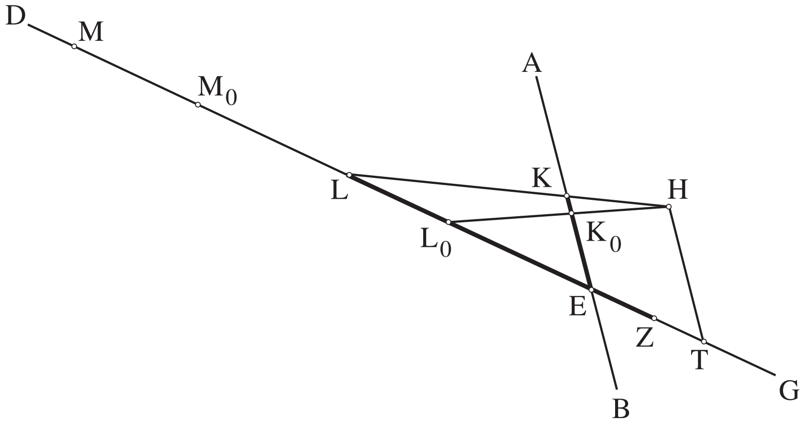


Fig. 11. *Cutting off a Ratio* 6.4, overview.

We may summarize the full occurrence as follows. (I) A short, general analysis shows that if $HT \parallel AB$, and $TH : ZM = EK : ZL = r$, then the point $L$ is given; that is, line $HL$ is given. (II) Because, in the analysis, $(TL \times LM) = (ZM \times TE)$ is applied to line $TM$ and deficient by $LM^2$, it is not always possible to solve the problem and there will be some limit when $TL = LM$. (IIa) Hence, there is one principal case where the problem is solved, where $TL_0 = L_0M_0$, $TH : ZM_0 = EK_0 : ZL_0 = r_0$ and $(TL_0 \times L_0M_0) = (ZM_0 \times TE)$.[51] It is shown through an analyzed proposition that, in the principal case, points $L_0$ and $M_0$ are given when $TE : L_0E = L_0E : EZ$. (IIb) Since $(TL_0 \times L_0M_0)$ is applied to line $TM_0$ and deficient by $L_0M_0^2$, the principal case must be a limiting case and it is shown through an analyzed proposition that, in the principal case, line $HL_0$ cuts off the greatest ratio of all lines drawn from point $H$ and cutting lines $EA$ and $ZD$. (IIc) It is, then, shown through another analyzed proposition that lines closer to line $HL_0$ always cut off greater ratios on lines $EA$ and $ZD$ than those more distant from $HL_0$. (III) Using the principal case, the synthesis of the problem is developed in three cases. (IIIa) Where $r = EK_0 : EL_0$, line $HL_0$ alone solves the problem. (IIIb) Where $r > EK_0 : EL_0$, the problem cannot be solved. (IIIc) Where $r < EK_0 : EL_0$, the auxiliary construction provided in the analysis is used to construct two points, one on either side of $L_0$, that solve the problem. (IV) The limiting ratio is shown to be given as $r_0 = EK_0 : EL_0 = TH : \left(TE + EZ + \sqrt{4(TE \times EZ)}\right)$.

With this overview as a guide, we now examine the details of *Cutting off a Ratio* 6.4. In the 18 standard occurrences of *Cutting off a Ratio* I, there are two diagrams, one for the analysis and one for the synthesis. In each of the five occurrences with an extended diorism, however, we find a third diagram specifically for the diorism. In *Cutting off a Ratio* 6.4, the first diagram serves the general analysis, (I), and the principal case of the diorism, (IIa), and the second diagram serves the rest of the diorism, (IIb) and (IIc), while the third diagram serves the general synthesis, (III), and the metrical determination of the upper limit of the given ratio, (IV). Because the full text of *Cutting off a Ratio* 6.4 is long, in the following, we give a summary of the argument, providing the full justification only for key steps.

[51] In this summary, we distinguish between points $L$ and $M$, introduced in the analysis, and points $L_0$ and $M_0$, which are special cases of these points introduced in the diorism. In the more detailed discussion that follows, just as in the text, the difference between these $L$s and $M$s must be distinguished by context.

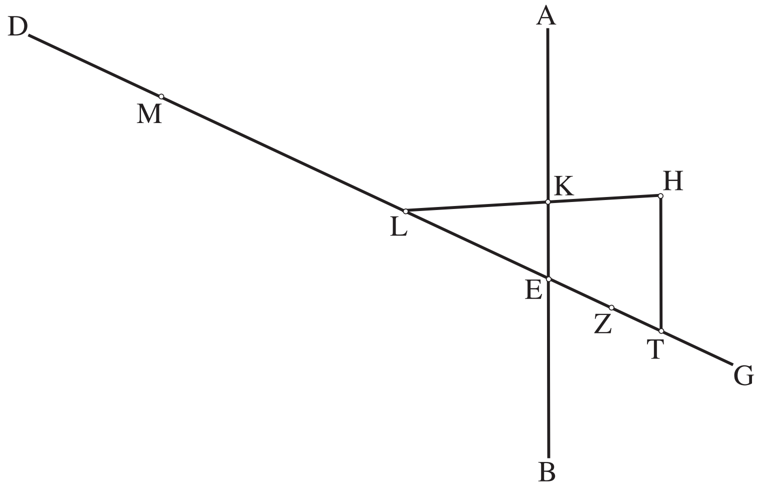


Fig. 12. *Cutting off a Ratio* 6.4, analysis.

In Fig. 12, the general analysis, Section (I), proceeds as follows. Where lines $AB$, $GD$ and points $H$, $E$, and $Z$ are given, assume $HL$ is drawn cutting off a given ratio $EK : ZL$. Line $HT$ is drawn parallel to line $AB$ and point $M$ is taken such that $TH : ZM = EK : ZL$. By alternation, $TH : EK$ [$=TL : LE$] $= ZM : ZL$ [*Elements* V 16], and by conversion, $TL : TE = MZ : ML$ [*Elements* V conversion*];[52] therefore [*Elements* VI 16],

$$(MZ \times TE) = (TL \times LM). \tag{1}$$

Then, since both $TE$ and $MZ$ are given, a given rectangle, $(TL \times LM)$, has been applied to a given line $TM$, deficient by a square; therefore point $L$ is given [*Data* 58], and hence line $HL$ is given.

The diorism, Section (II), begins by pointing out that the auxiliary construction, used to complete the analysis, also reveals that the problem cannot always be solved. Since, $TE$ and $MZ$ are given by the geometry of the initial configuration,[53] it may not always be possible to locate a point $L$ between $E$ and $M$, such that $(MZ \times TE) = (TL \times LM)$. The text reads, "We had taken point $L$, and this is not always possible in every case. Therefore, the synthesis of the problem is not possible in every case (*CR* A, 13r; *CR* B, 25r)."

Since, for a solution to be possible, $L$ must be between $E$ and $M$, there will be at least one case in which it will certainly be possible to find the appropriate point $L$; that is, when $L$ is the midpoint of $TM$. The first problem of the diorism, Section (IIa), begins with the statement, "The relation (القياس) does work, however, according to one principal case (على وجه واحد اول) (*CR* A, 13r; *CR* B, 25r)."

This leads to the following problem: how to find points $L$ and $M$ such that $EK : ZL = TH : ZM$, $TL = LM$ and $(MZ \times TE) = (TL \times LM)$. Hence, for this problem, we abandon the original assumption of a given ratio. Indeed, the construction of $L$ and $M$ satisfying the three stated conditions will produce a limiting case of $EK : ZL$, as will be shown in the next sec-

[52] An argument for the legitimacy of this operation is included in the *Elements* as the corollary to *Elements* V 19, but as Heath [1926, 175] has argued, this corollary does not follow directly from the proposition and most scholars agree that it is a later addition to the text. Vitrac [1990–2001, Vol. 2, 113–114], on the other hand, shows that an argument can be restored, although it is somewhat involved. As far back as Clavius, however, it has been noted that conversion is simply successive application of composition, inversion, and separation. This is the simplest explanation for the fact that no justification of this operation was provided in *Elements* V.

[53] Since $TE : ME = r$ and $TE$ is given, $MZ$ is given.

tion of diorism. The problem is solved using a standard analyzed proposition. The transformation shows that if the three conditions are assumed, $EL$ will be a mean proportional between lines $TE$ and $EZ$, such that $TE : EL = EL : EZ$, and the resolution shows that this implies that both points $L$ and $M$ are given. The synthesis, then, proceeds to construct point $L$, straightforwardly, by setting $TE : EL = EL : EZ$ [*Elements* VI 13], and $M$ by setting $TL = LM$ [*Elements* I 3], and using ratio manipulation to show that $(MZ \times TE) = (TL \times LM)$ and $EK : ZL = TH : ZM$. The text then asserts that, "Before the synthesis, it is necessary that we take a line between $TE$ and $EZ$ with respect to ratio, and join $HL$ (*CR* A, 14r; *CR* B, 25v)."[54] That is, the diorism has provided a construction necessary to the synthesis and not derived in the analysis.

Since $TH : ZM = EK : ZL$, then $L$ will fall between $T$ and $M$. Therefore, since $(TL \times LM)$ is greatest when $TL = LM$,[55] while $TE$ is given by the initial configuration of the figure, by Eq. (1), $MZ$ is also greatest when $TL = LM$. Since $M$ was taken by setting $TH : ZM = EK : ZL$, however, while $TH$ is determined by the initial configuration, the principal case will be some limiting case of ratio $EK : ZL$. Apollonius simply assumed that this was obvious and asks, "Does line $HL$ cut off ratio $EK$ to $ZL$ less or greater than all the [other] lines that are produced from point $H$ and cut $EA$ and $ZD$?" [*CR* A, 14r; *CR* B, 25v]

This is followed by two theoretic analyses that show that, when point $L$ is taken according to the conditions of the principal case, line $HL$ cuts off the greatest ratio of all lines drawn from point $H$ falling on lines $EA$ and $ZD$, Section (IIb), and that lines closer to $HL$ always cut off greater ratios then those further from it, Section (IIc). Because the extended diorisms in *Cutting off a Ratio* contain the earliest example, of which we are aware, of a style of argumentation that we call *comparative analysis*, it will be valuable to look at the first of these theorems in some detail.[56]

In Fig. 13, we let line $HL$ satisfy the requirements of the principal case and another line, such as $HN$, is drawn. The text then reads,

> So, it is necessary to link $EK$ to $ZL$ and $ES$ to $ZN$.[57] Ratio $EK$ to $ZL$ is equal to $TH$ to $ZM$, so $TH$ to $ZM$ and $ES$ to $ZN$ will be linked, and by alternation, $TH$ to $SE$ and $ZM$

[54] The expression "between $AB$ and $GD$ with respect to ratio" (بين اب جد على نسبة) is the idiom used in the text to denote a mean proportional.

[55] Although the fact that a regular figure has the greatest area of any other polygon of the same number of sides was demonstrated in a treatment of isoperimetric figures, generally attributed to Zenodorus, for the case of four sided figures, however, this follows directly from *Elements* II 5 (see Note 59, below). We have three treatments of isoperimetric figures in Greek texts, probably all from the same source but with significant discrepancies among them: Theon of Alexandria, *Commentary on the Almagest* I 3; Pappus; *Collection* V; and the unpublished anonymous introduction to the *Almagest* [Rome, 1931; Hultsch, 1876]. (The anonymous introduction is now being edited by Fabio Acerbi and Bernard Vitrac.)

[56] Comparative analysis is found, as well, in Pappus's *Collection* VI, Props. 16–20 [Hultsch, 1876, 494–512]. A linguistically related passage is *Collection* VII, Prop. 26 [Jones, 1986, 147]. We will discuss all of these related passages in a separate paper, which sets out the structure of comparative analysis and distinguishes it from standard theoretic analysis.

[57] The expression "it is necessary to link" (ينبغي أن يقرن) means that we must determine the mathematical relation between one object (such as a rectangle), or relation (such as a ratio), and another. Another expression that more literally means "it is necessary to compare" (ينبغي أن يقيس) is also used in this text to express the same idea. Although the two expressions appear to be synonymous, for the time being we translate rather literally. Although we do not know what the original Greek formulation was, stylistically related passages in Pappus's *Collection* VI and VII suggest that it may have relied on prepositional phrases, without the use of a specific verb [Hultsch, 1876, 498–500; Jones, 1986, 147].

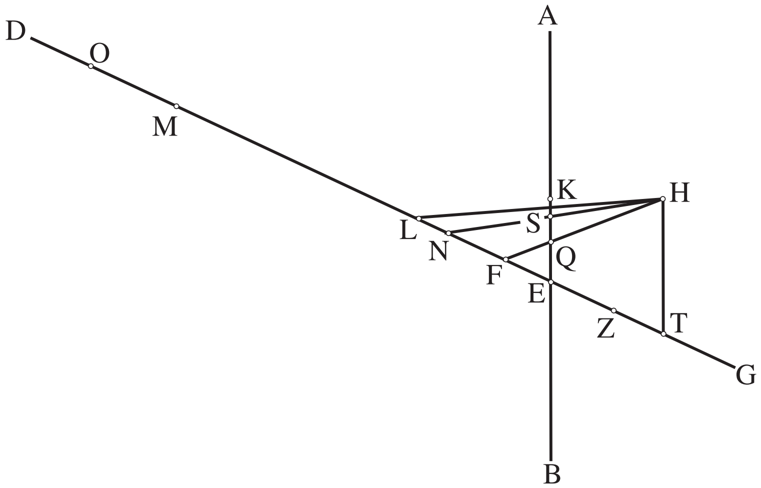


Fig. 13. *Cutting off a Ratio* 6.4, diorism.

> to $ZN$ will be linked [*Elements* V 16*].[58] Ratio $TH$ to $SE$ is equal to $TN$ to $EN$ [*Elements* VI 4], so $TN$ to $EN$ and $ZM$ to $ZN$ will be linked. Then we convert, so $TN$ to $TE$ and $ZM$ to $NM$ will be linked [*Elements* V conversion*], and rectangle $ZM$ by $TE$ and $TN$ by $NM$ will be linked [*Elements* VI 16*]. Rectangle $ZM$ by $TE$ is equal to rectangle $TL$ by $LM$ [Eq. (1)], so it is necessary to link rectangle $TL$ by $LM$ and rectangle $TN$ by $NM$. Its relation (قياسه) is that $TL$ by $LM$ is greater than $TN$ by $NM$, because point $L$ is at half of $TM$.[59] [*CR* A, 13v; *CR* B, 26v–27r]

The analysis begins by assuming that some relation holds between the ratios that need to be compared and then uses standard techniques of ratio manipulation to reduce this to a relation that is obvious on the basis of the geometry of the figure. In this process, ratio manipulations are applied to as yet undetermined relations in the same way that they are generally applied to proportions and ratio inequalities. This furnishes a means for the geometer to reduce the relation that must be determined to one that can be readily asserted on the basis of the geometry of the figure or previously determined results.

The synthesis is a reversal of the steps of the analysis. By starting with the observation that $(TL \times LM) > (TN \times NM)$ [*Elements* II 5], it is a simple matter to show that $EK : ZL > ES : ZN$. Since $(TL \times LM) > (TN \times NM)$ holds as long as point $N$ is any point other than $L$ on line $TM$, ratio $EK : ZL$ is the greatest ratio cut off by any line falling from point $H$ onto lines $EA$ and $ZD$.

Again in Fig. 13, in order to show that the ratios continuously increase as the line approaches $HL$, Apollonius draws another line, $HF$, cutting $AB$ and $GD$ at $Q$ and $F$, respectively, and shows that $ES : ZN > EQ : ZF$. For this purpose, he sets out point $O$, as analogous to point $M$, such that $ES : ZN = TH : ZO$, and uses the same type of comparative analysis to show that

$$(TE \times ZO) < (TL \times LO). \qquad (2)$$

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[58] Note that these ratio manipulations are carried out on proportions or ratio inequalities that, for the moment, we know nothing about, $\gtreqless$. Pappus, much later, in *Collection* VII demonstrated the applicability of the standard ratio operations to inequalities (see note 32.)

[59] This inequality is a direct consequence of *Elements* II 5, which shows that $(TL \times LM) = (TN \times NM) + LN^2$.

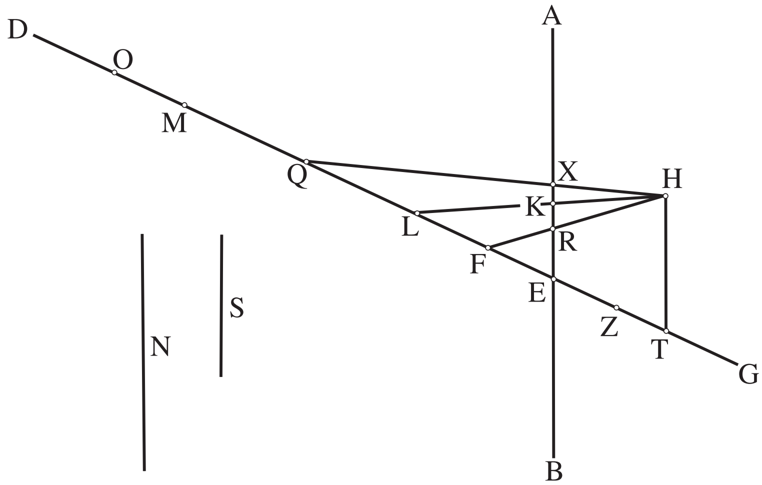


Fig. 14. *Cutting off a Ratio* 6.4, synthesis.

The synthesis of this theorem, again, begins by asserting that $(TE \times ZO) < (TL \times LO)$ and then working backwards to show that $ES : ZN > EQ : ZF$. The relation, in this case, follows from the fact that point $F$ was taken between points $N$ and $T$, such that $(TF \times FO) < (TN \times NO)$. Hence, again, it does not matter on what side of point $L$ the two other lines are taken, it is only the ordering of the lines that determines the relation between the ratios they cut off.

These two theorems can be taken as a full proof that, where $L$ bisects $TM$, line $HL$ cuts off the greatest ratio drawn from point $H$ to lines $EA$ and $ZD$ and that all other lines cut off ratios that always approach the maximum ratio cut off by $HL$ as the lines themselves approach $HL$. Again, we see that the diorism is an investigation of the general mathematical properties of the construction that solves the problem. Whereas the analysis assumes the existence of one particular solution and then explores its properties, the diorism is a study of all possible solutions carried out by a series of analyzed propositions that demonstrate the limiting solution and the behavior of the solutions around the limit.

In this way, following standard practice, the geometrical properties of the auxiliary construction are then used to establish the conditions for the solvability of the problem as set out in the synthesis.

These theorems can be compared with propositions that demonstrate the properties of minimum and maximum lines drawn from a given point to a given curve, such as *Elements* III 7 and 8 and most of the theorems in *Conics* V.[60] The treatment in *Cutting off a Ratio* is much longer, however, for two reasons. In this text, the comparison is between ratios, as opposed to line segments, and the demonstrations are carried out using analyzed propositions, as opposed to pure syntheses. The fact that these theorems are carried out with analyzed propositions can be taken as further evidence that *Cutting off a Ratio* was written for the sake of instructing readers in the techniques of the analytical approach.

In Fig. 14, the construction of the synthesis, section (III), proceeds as follows. Let ratio $N : S$, lines $AB$ and $GD$ intersecting at $E$, and points $H$ and $Z$ be given and take point $L$ such that $TE : LE = LE : ZE$ [*Elements* VI 13]. Then $N : S \gtreqless EK : EL$.

[60] See Fried and Unguru [2001, 177–185] for a discussion of Apollonius's concept of minimal and maximal lines in *Conics* V and the analogy between this book and *Elements* III.

Thus,

(1) where $N : S = EK : EL$, the problem is solved by $HL$ alone;
(2) where $N : S > EK : EL$, there is no solution; and
(3) where $N : S < EK : EL$, the problem is solved as follows.

We set $TL = LM$, so that, by the principal case, $(TL \times LM) = (TE \times ZM)$ and $EK : ZL = TH : ZM$. Then, since $N : S < EK : ZL = TH : ZM$, we take point $O$ such that $N : S = TH : ZO$ [*Elements* VI 12]. Then, since

$$\begin{aligned}(TL \times LM) &= (TL \times LO) - (TL \times MO)\\ &= (TE \times ZM) = (TE \times ZO) - (TE \times MO),\end{aligned}$$

while $(TL \times MO) > (TE \times MO)$, therefore $(TL \times LO) > (TE \times ZO)$. Hence, it is possible to apply a rectangle equal to $(TE \times ZO)$ to line $TO$ being deficient by a square at two points equally distant from the midpoint of $TO$, and thus on either side of point $L$.[61] Let these points be constructed as $F$ and $Q$.

The proof of the synthesis of the third case uses ratio manipulation to show that if $(TE \times ZO) = (TF \times FO) = (TQ \times QO)$, then $N : S = TH : ZO = ER : ZF = EX : ZQ$, so that both lines $HF$ and $HQ$ solve the problem.

In this case, the synthesis itself is structured around the possibilities for the solution determined in the diorism. The diorism not only sets out the limit of solvability but also shows that the problem should be approached in cases based on the circumstances of the given lines and the given ratio.

The synthesis is then followed by a passage that points out that the "boundary" of the ratio can be understood in terms of lines that are given by the geometry of the figure.[62] It is a relatively straightforward matter of substitutions to show that where $M$ is taken such that $EK : EL = TH : ML$, then $EK : EL = TH : \left(TE + EZ + \sqrt{4(TE \times EZ)}\right)$. This gives a metrical determination of the ratio $EK : EL$ in terms of the givens of the problem. It is interesting to note that Apollonius waits until the problem has been completely solved before showing that the bounding ratio, developed in the diorism and utilized in the synthesis, is itself given. Indeed, because he proceeds by analysis and synthesis, there is no reason to show that this ratio is also given before the problem has been solved.

In problems such as *Cutting off a Ratio* 6.4, the diorism takes on an expanded role and is used to structure the cases of the synthesis. As usual, the analysis begins by the assumption of a single solution to the problem. An investigation of this specific solution, however, leads to an auxiliary construction that leads both to a limit to the solvability and to the possibility that there may be more than one solution. In this case, the auxiliary construction is the application of a given area to a given line deficient by a square [*Elements* VI 28], the limitation is that the greatest area that can be so applied is the square on half the given line [*Elements* II 5], and the possibility of more than one solution is due to the fact that the deficient square my be situated at either end of the line. The diorism then uses the properties of

[61] This, again, follows as a direct consequence of *Elements* II 5 and would have been intuitively obvious to mathematicians working within the Euclidean theory of the application of areas.
[62] The Arabic term we translate as *boundary* is حدّ, whose plural often translates the plural of ὅρος and means mathematical *definitions*. Both terms, however, also have the more literal meaning of *boundary*.

this auxiliary construction to investigate the limit and to show how possible solutions are symmetrically arranged around the limiting case. This discussion of the limits and conditions of solvability, then, becomes the key to structuring the synthesis of the problem. Moreover, the arguments developed in the synthesis rely as much, on the diorism as on the analysis, if not more.

## 5. Conclusion

In this paper, we have seen that although the majority of the diorisms that survive in ancient texts simply state a limit to the solvability of the problem, the ancient mathematicians themselves regarded the investigation of the diorism as an important aspect of analytical problem solving. Indeed, it is only on the basis of this expanded view of diorism that we can understand Pappus's general description of diorism or Apollonius's claim to have put together *Conics* IV so as to be useful in the analysis of diorisms (see Section 1). This distinction between diorism as a study of the limits of solvability and the total number and arrangement of solutions, on the one hand, and diorism as part of an analyzed proposition, on the other hand, should make clear an obvious, but sometimes overlooked, characteristic of the Greek mathematical texts. The texts that we possess are the finished product of an attempt to articulate the most interesting results of a probably messy, but now largely inaccessible, process of mathematical research.[63] Hence, diorism would have meant something different to a working mathematician than to someone who was simply interested in reading the ancient mathematical texts and appreciating their results.

In order to develop a concrete sense for this distinction, we take the liberty of an imaginative reconstruction. Let us suppose that we have been asked to carry out a dioristic investigation at about the same level of detail and exhaustion as we find used in *Cutting off a Ratio*. For this purpose, we return again to the example analyzed proposition that we presented in the introduction, *Collection* IV Prop. 31. In our sources, this neusis problem has no diorism, because it is always soluble with only one solution.

Nevertheless, we suppose that a full treatment is required, which will provide an argued account explaining why it is unnecessary to include a diorism in the final version of this problem. In Fig. 15, given rectangle ($AB \times BG$), the problem is to draw a line through point $A$ that intersects segment $GD$ and $BGZ$ extended such that line $EZ$ is equal to some given length, $l$. This is solved by the construction of a parallel line, which transforms the problem into that of finding the intersection of the hyperbola with asymptotes $BA$ and $BG$ that passes through point $D$ and the circle drawn about center $D$ with radius $l$.

We propose the following as an exhaustive account of the diorism for these problems. In fact, the transformed problem has more solutions than the original problem. Indeed, there will always be at least two solutions, since a circle centered on a hyperbola will intersect it in at least two points, $H$ and $H'$. The circle, however, will not intersect the same branch of hyperbola in more than two points, again because its center is on the hyperbola so that it is clear that there will not be two points of intersection in the same direction from the center. Moreover, even if we consider the other branch of the hyperbola, which the ancients called the opposite section, there will be no more than four solutions, since a circle will not intersect opposite sections at more than four points [*Conics* IV 38].[64] Finally, it is also pos-

[63] Furthermore, the majority of our texts can be shown to have been edited and variously altered some unknown number of times in the course of their transmission.
[64] See note 4 above.

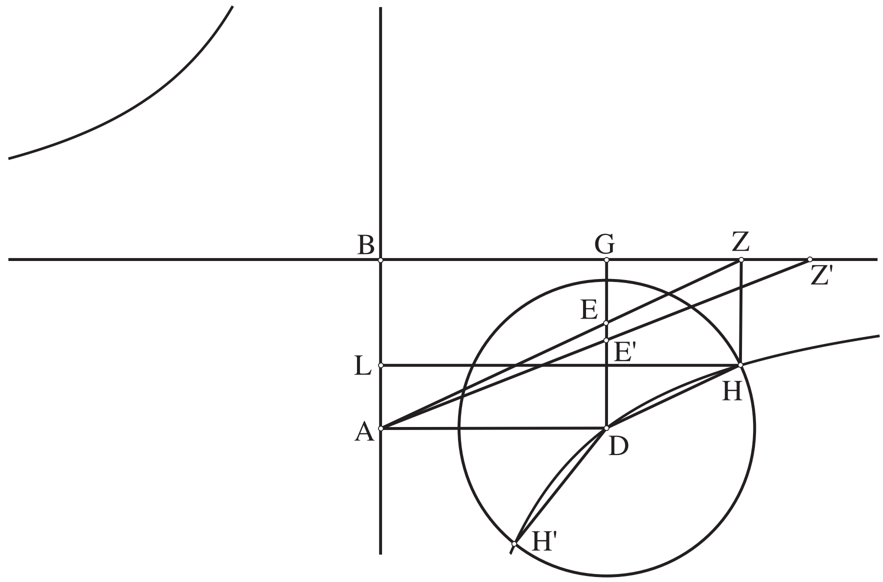


Fig. 15. *Collection* IV Prop. 31, hyperbola and circle.

sible that the circle will be of such a size as to touch the opposite branch of the hyperbola in one point, and intersect the other in two; however, given the tools of ancient conic theory, the determination of this magnitude would be nontrivial. Thus, it is a simple matter to use the theorems of *Conics* IV to show that the transformed problem has between two and four solutions, depending on whether we consider the branch of the hyperbola independent or not, and to see where they are arranged with respect to the original rectangle, ($AB \times GB$).

We must then consider whether or not any of these other possible solutions to the intersection problem can be transformed into another solution for the neusis problem. We can show by indirect argument, however, that there can only be one solution to the neusis problem. If we assume that $EZ$ solves the problem and is equal to $l$, then if one of the other solutions to the secondary problem, for example $DH'$, is transformed in some way to provide another solution, let it fall as $AE'Z'$. It can then be shown that $E'Z'$ does not equal $EZ = l$, and hence cannot, in fact, be another solution of the original problem.[65]

Moreover, since $BZ$ is an asymptote of the hyperbola, the two figures may be continued indefinitely, so that $ZH$ can be taken as less than any given length [*Conics* II 14], and line $DH$ can be taken as greater than any given length. Hence, there is no limit to the solvability of the problem. Nevertheless, in order to be certain that it is not necessary to provide a diorism in the text, we had to carry out a mathematical argument that is neither an analysis or a synthesis, but which, rather, concerns the general limits of solubility and the total number of possible solutions.

Although the specific neusis problem that Pappus solves has no interesting diorism, if we consider the original problem of trisecting an angle, which the neusis problem was designed

[65] This method of showing that there is only one solution to the problem is similar to what we find in Apollonius's *Cutting off a Ratio* and when we consider that there would have often been a concern about whether or not solutions of a transformed problem were applicable to the original problem (for example, see the discussion of Archimedes' *Sphere and Cylinder* II 4, below), it becomes understandable why Apollonius seems so insistent on providing deductive proofs for the number of possible solutions in *Cutting off a Ratio*.

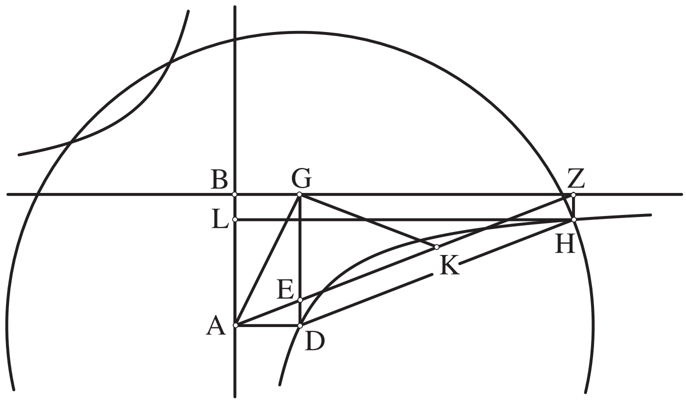


Fig. 16. *Collection* IV Prop. 32, reconstructed diorism, case 1.

to treat, we will see that the diorism discussed above is indeed related in some interesting ways. Considering Fig. 16, in *Collection* IV 32, Pappus shows that given some acute angle, $GAD$, if we complete the rectangle $GBAD$ and construct the neusis line $ZE$ as twice $AG$, then line $ZEA$ will trisect angle $GAD$. The argument is as follows. If we bisect line $EZ$ at $K$ and join $GK$, then since $G$, $Z$ and $E$ can be regarded as three points on the circumference of a circle about diameter $EZ$, $GK = KE = KZ$ [*Elements* III 31]. Hence, $GK = AG$, so that the angles $GAK$ and $GKA$ are equal [*Elements* I 5]. But, if we consider a circle through $G$, $Z$ and $E$, angle $GKA$ is twice angle $GZA$ [*Elements* III 20], so that angle $GAK$ is twice angle $EAD$ [*Elements* I 29].

The key to the solution of this problem lies in passing a line through point $A$ such that the segment of it that is cut off between the extensions of the two lines $BG$ and $GD$ is twice line $AG$. If we consider these analogous objects for the other points of intersection of the circle and the opposite sections, we will see that they effect a number of related constructions.

In Fig. 17a, we consider the other intersection of the circle with the same branch of the hyperbola. We set out $ZE = 2AG$ cut off between lines $BG$ and $GE$. We take the midpoint of $ZE$ at $K$ and join $GK$. Hence, when we consider the circle through points $E$, $Z$ and $G$, angle $ZKG$ is twice angle $ZEG$ [*Elements* III 20]. Since $GE$ is parallel to $BA$, angle $ZAG$ is twice angle $BAZ$ [*Elements* I 29]. Therefore, line $EA$ trisects angle $BAG$, which is the complement of angle $GAD$.

In Fig. 17b, we show the trivial case in which $DH$ passes through point $B$ and point $A$ is the midpoint of $ZE$. In this case, the line equal to twice $AG$ that passes through $A$ is also bisected by $A$ and it is a simple matter to show that the triangles $BZH$, $ZBA$, $ADE$, $ADG$ and $GBA$ are all equal. Hence, there is no related solution because line $ZE$ coincides with the reflection of line $AG$ about line $AB$.

In fact, however, there is another position in which a line equal to twice $GA$ will pass through point $A$ and be cut off between lines $GBZ$ and $GDE$ such that the line drawn from $G$ to its midpoint is equal to line $GA$. In Fig. 18, we consider the final intersection of the circle and the hyperbola. We set out $ZAE = 2GA$ and join the midpoint, $K$, with $G$, such that $GK = KZ = KE$ [*Elements* III 31]. In this construction, we can show that line $ZAE$ cuts off an angle, $DAE$, that is one-third of the supplement of the original angle $GAD$. If we consider the circle through points $G$, $Z$ and $E$, it is clear that angle $GKA$ is twice angle $BZA$ [*Elements* III 20]. But, if we extend $GA$ to $X$, angles $GKA$, $GAK$ and $EAX$ are equal

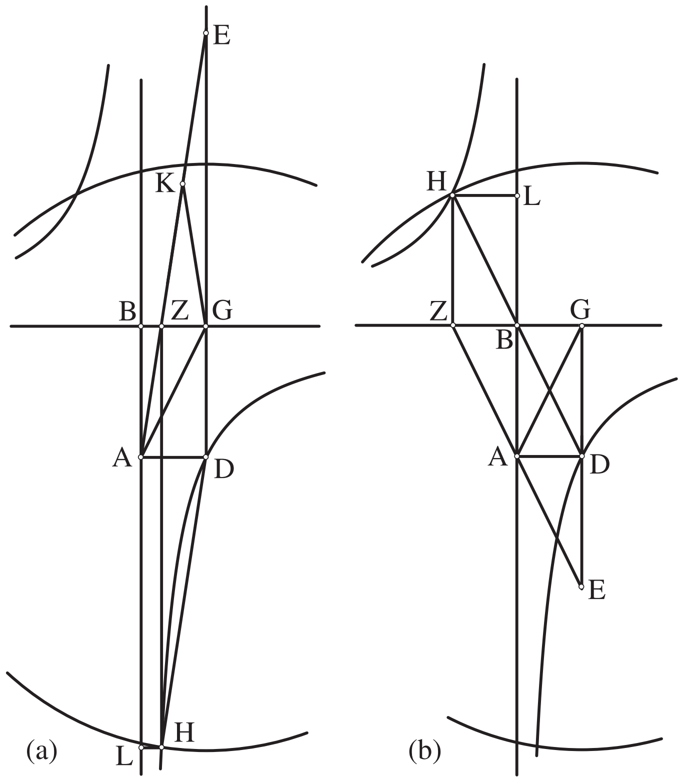


Fig. 17. *Collection* IV Prop. 32, reconstructed diorism cases 2 and 3.

[*Elements* I 5, 15], while angles $DAE$ and $GZE$ are equal [*Elements* I 28]. Hence, angle $DAE$ is one-third of angle $DAX$, which is the supplement of angle $GAD$.

Although these constructions do not solve the stated problem, they show how diorism may have been used as a way to investigate the relationship between a problem and its transformed solution. The point of this fanciful exercise was to give an example of how diorism, although not well represented in our sources, may have been used by ancient mathematicians to analyze the more general conditions of geometric problems. Moreover,

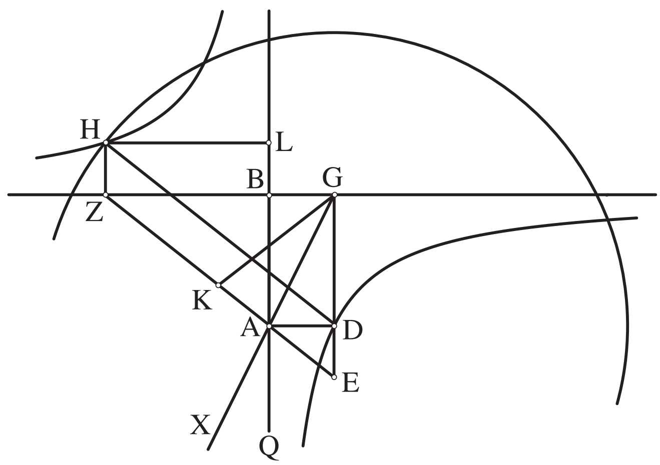


Fig. 18. *Collection* IV Prop. 32, reconstructed diorism case 4.

we have shown how theorems of *Conics* IV could have been useful in these investigations, just as Apollonius claims they were.

As we have seen in this example, analysis often transforms a given problem into one that is more abstract and general. Hence, this more general, transformed problem may have a diorism that is not relevant to the original problem. In this way, the expanded form of diorism that we saw in *Cutting off a Ratio* may have often been used in the investigation of general problems, such as the one solved in that text, without being necessary for the final version of the specific problems we generally encounter in our sources. Although there are a number of other examples in the ancient literature of problems for which the transformed problem may have had an extended diorism that was left out of the final version, we have at least one example in which this was certainly so. This is the well-known case of Archimedes' *Sphere and Cylinder* II 4, in which the transformed problem, which was provided by Eutocius, has a diorism, whereas the original problem does not. Archimedes, in reference to the transformed problem says, "This, stated generally, in this way, has a diorism, but with the suppositions of the forgoing problem included … it has no diorism" [Heiberg, 1910–1915, Vol. 1, 190]. Indeed, the transformed problem, as found in Eutocius's revision of a solution to this problem that he found in "an old book," is solved by the intersection of a parabola and a hyperbola [Heiberg, 1910–1915, Vol. 3, 130–131]. Hence, not only is there a limit to the solvability of the problem but there may be multiple solutions and the number and arrangement of the solutions can once again be addressed by the theorems of *Conics* IV. Hence, we see that here Archimedes also intended diorism to have the sense of a general treatment of the possibility, arrangement and totality of solutions.

A diorism then, is an investigation of the geometric objects used to solve a problem, which provides more general insight into the circumstances of the solutions themselves, such as their limitations, number and arrangement. In this paper, we have seen a number of different types of problems which involve diorism. Although it is possible that there were other types as well, since most of the diorisms in the Greek mathematical corpus fall into one of the categories discussed here, it will be useful to briefly enumerate them. (1) A straightforward case of the diorism is that in which some inherent property of the geometric objects involved in the problem presents a limit to possibility of solving the problem [*Sphere and Cylinder* II 7, *Conics* II 53]. Most of the diorisms in the surviving corpus are of this type. (2) Another fairly common form of the diorism arrises when the solution of the problem is found by the intersection of two conics [*Sphere and Cylinder* II 4, *Collection* IV 31]. When Apollonius says that *Conics* IV is useful in the investigation of diorisms, he probably had this sort of diorism in mind. (3) When a given area is applied to a given line and deficient by a given figure, an upper limit to the given area must be stated and there will be two positions for the solutions [*Elements* VI 28, *Cutting off a Ratio* 6.4]. Although there are relatively few of these diorisms in our sources, since the application of areas was a widely used technique in Greek geometry, it is possible that this type was more often encountered than the cases in our surviving sources suggest.

As this list shows, the diorisms discussed thus far constituted a geometric study of the auxiliary objects introduced in the transformation. Although it would only have been necessary to actually write up relatively few diorismic investigations, these studies would have been essential for the mathematicians in understanding the relationship between the more general transformed problem and the original specific problem that it was devised to treat. Indeed, it is only on the basis of this broader understanding of diorism that we can make full sense of a number of the remarks about diorism that we have read in Archimedes, Apollonius, and Pappus.

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