Diagrams and arguments in ancient Greek mathematics: lessons drawn from comparisons of the manuscript diagrams with those in modern critical editions

KEN SAITO AND NATHAN SIDOLI

Introduction

In some ways, the works of ancient Greek geometry can be regarded as arguments about diagrams. Anyone who has ever looked at a medieval manuscript containing a copy of an ancient geometrical text knows that the most conspicuous characteristic of these works is the constant presence of diagrams.\(^1\) Anyone who has ever read a Greek mathematical text, in any language, knows that the most prevalent feature of Greek mathematical prose is the constant use of letter names, which refer the reader’s attention to the accompanying diagrams.

In recent years, particularly due to a chapter in Netz’s The Shaping of Deduction in Greek Mathematics entitled “The lettered diagram”, historians of Greek mathematics have had a renewed interest in the relationship between the argument in the text and the figure that accompanies it.\(^2\) Research projects that were motivated by this interest, however, quickly had to come to grips with the fact that the edited texts of canonical works of Greek geometry, although they contained a wealth of information about the manuscript evidence for the text itself, often said nothing at all about the diagrams. For years, the classical works of Apollonius, Archimedes and, most importantly, the Elements of Euclid have been read in edited Greek texts and modern translations that contain diagrams having little or no relation to the diagrams in the manuscript sources. Because they are essentially mathematical reconstructions, the diagrams in modern editions are often mathematically more intelligible than those in the manuscripts, but they are often historically misleading and occasionally even mathematically misleading.\(^3\)

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1. In some cases, the diagrams were never actually drawn, but even their absence is immediately evident from the rectangular boxes that were left for them.
3. In this chapter, we will see a number of examples of modern diagrams that are more mathematically consistent with our understanding of the argument and a few that may have
In fact, a few scholars of the ancient mathematical sciences have for
many years made critical studies of the manuscript figures, and Neugebauer
often called for the critical and conceptual study of ancient and medieval
diagrams. These scholars, however, were mostly working on the exact
sciences, particularly astronomy and, perhaps due to the tendency of his-
torians of science to divide their research along contemporary disciplinary
lines that would have made little sense to ancient mathematicians, these
works have generally formed a minority interest for historians of ancient
mathematics. Indeed, in his later editions, Heiberg paid more attention
to the manuscript figures than he did in his earlier work, but by this time
his editions of the canonical works were already complete. In fact, for his
edition of Euclid’s Elements, it appears that the diagrams were adopted from
the tradition of printed texts without consulting the manuscript sources.

In this chapter, after briefly sketching the rise of scholarly interest in
producing critical diagrams, we investigate the characteristics of manuscript
diagrams in contrast to modern reconstructions. To the extent that the
evidence will allow, we distinguish between those features of the manu-
script diagrams that can be attributed to ancient practice and those that are
probably the result of the medieval manuscript tradition, through which
we have received the ancient texts. We close with some speculations about
what this implies for the conceptual relationship between the figure and the
text in ancient Greek mathematical works.

Heiberg’s edition of Euclid’s Elements

Heiberg (1883–8), on the basis of a study of manuscripts held in European
libraries, prepared his edition of the Elements from seven manuscripts and
the critical apparatus accompanying his text makes constant reference to
these sources. Nevertheless, there is usually no apparatus for the diagrams
and hence no mention of their source. An examination of the previous

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Figure 2.1. Diagrams for Euclid’s Elements, Book xi, Proposition 12.

printed editions of the text, however, makes it clear that the diagrams
accompanying Heiberg’s edition were drawn entirely, or for the most part,
by copying those in the edition of August (1826–9). The August edition
would have been particularly convenient for copying the diagrams, since,
as was typical for a German technical publication of its time, the diagrams
were printed together in fold-out pages at the end of the volumes.

Although nearly all the diagrams appear to have been so copied, a single
example may be used to demonstrate this point. For Elem. xi.12, concerning
the construction of a perpendicular to a given plane, the diagrams in all the
manuscripts consist simply of two equal lines, ΔA and ∆B, placed side by
side and labelled such that points A and B mark the top of the two lines. In
Figure 2.1, we compare the diagram for Elem. xi.12 in Vatican 190, as rep-
resentative of all the manuscripts, with that in both the August and Heiberg
editions. While Vatican 190 is typical of the manuscript diagrams, that in
Heiberg’s text is clearly copied from the August diagram. Although the given
plane is not shown in the manuscript figures, it appears in both the printed
ditions and it is used with the techniques of linear perspective to make the
two lines appear to be in different planes from the plane of the drawing.
Most significantly, however, there is a labelling error in the line ∆B. Point
Γ is supposed to be in the given plane, and hence must be at the bottom of
line ∆B, as in Vatican 190. This error was transmitted when the diagram was

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1. The diagrams to the arithmetical books are a clear exception. The August diagrams are
dotted lines, whereas Heiberg’s edition returns to the lines we find in the manuscripts. There
also other, individual cases where the diagrams were redrawn, presumably because those in
the August edition were considered to be mathematically unsatisfactory. For example, the
diagram to Elem. xi.38 has been redrawn for Heiberg’s edition, whereas all the surrounding
diagrams are clearly copied. See also the diagram for Elem. xii.17. Compare Heiberg and
Stamatis 1969–77: iv 75 and 128 with August 1826–9: Tab. iix and Tab. x.

2. In this chapter, we refer to manuscripts by an abbreviated name in italic. Full library shelf
marks are given in the references. For the Euclidean manuscripts see also Vitruv’s chapter in
this volume.
commentaries to the Elements, the diagrams were taken directly from Leiden 399, and hence often quite different from those printed in his edition of the Greek. By the time he edited Theodosius’ Spheres, he must have become convinced of the importance of giving the diagrams critical attention, because the finished work includes diagrams based on the manuscripts, generally accompanied with a critical note beginning ‘In fig.’

Editions of manuscript diagrams

Because the manuscript diagrams for spherical geometry are so strikingly different from what we have grown to expect since the advent of the consistent application of techniques of linear perspective in the early modern period, the editions of ancient Greek works in spherical astronomy were some of the first in which the editors began to apply critical techniques to the figures. For example in the eighth, and last, volume of the complete works of Euclid, for his edition of the Phenomena, Menge (1916) provided diagrams based on the manuscript sources and in some cases included critical notes.

One of the most influential editions with regard to the critical treatment of diagrams was that made by Rome (1931–43) of the commentaries by Pappus and Theon to Ptolemy’s Almagest. The diagrams in this long work were taken from the manuscript sources and their variants are discussed in critical notes placed directly below the figures themselves.11 Rome’s practices influenced other scholars working in French and the editions by Mogenet (1950), of Autolycus’ works in spherical astronomy, and Lejeune (1956), of the Latin translation of Ptolemy’s Optics, both contain manuscript figures with critical notes.

More recently, the majority tendency has been to provide manuscript diagrams with critical assessment. For example, the editions by Jones (1986) and Czinczenheim (2000) of Book vii of Pappus’ Collection and

9 See Saito 2006: 110 for further images of the manuscript figures.

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10 Besthorn et al. 1897–1932.

11 Heiberg 1927. In fact, these critical notes are difficult to notice, since they are found among the notes for the Greek text. The notes for the Greek text, however, are prefaced by numbers referring to the lines of the text, whereas the diagrams are always located in the Latin translation, which has no line numbers. Neugebauer 1975: 751–5 seems to have missed them, since he makes no mention of them in his criticism of the failure of classical scholars to pay sufficient attention to the manuscript diagrams of the works of spherical astronomy.

12 In connection with the early interest that Rome and Neugebauer showed in manuscript figures, we should mention the papers they wrote on Heron’s Dioptris, the interpretation of which depends in vital ways on understanding the diagram. Rome 1923; Neugebauer 1938–9; Sidoli 2005.
Theodosius' *Spherics*, respectively, both contain critical diagrams, and a recent translation of Archimedes' *Sphere and Cylinder* also includes a critical assessment of the manuscript figures.\(^\text{13}\)

Nevertheless, although there are critically edited diagrams for many works, especially those of the exact sciences, the most canonical works – the works of Archimedes and Apollonius, the *Elements* of Euclid and the *Almagest* of Ptolemy – because they were edited by Heiberg early in his career, are accompanied by modern, redrawn diagrams. Hence, because a study of Greek mathematics almost always begins with the *Elements*, and because the manuscript diagrams of this work contain many distinctive and unexpected features, it is essential that we reassess the manuscript evidence.

**Characteristics of manuscript diagrams**

In this section, focusing largely on the *Elements*, we examine some of the characteristic features of the manuscript diagrams as material objects that distinguish them from their modern counterparts. Manuscript diagrams are historically contingent objects which were read and copied and redrawn many times over the centuries. In some cases, they may tell us about ancient practice, in other cases, about medieval interpretations of ancient practice, and in some few cases, they simply tell us about the idiosyncratic reading of a single scribe. In the following sections, we begin with broad general tendencies that can almost certainly be ascribed to the whole history of the transmission, and then move into more individual cases where the tradition shows modification and interpretation. In this chapter, we present summary overviews, not systematic studies.

**Overspecification**

One of the most pervasive features of the manuscript figures is the tendency to represent more regularity among the geometric objects than is demanded by the argument. For example, we find rectangles representing parallelograms, isosceles triangles representing arbitrary triangles.

\(^\text{13}\) Netz 2004. In fact, however, the figures printed by Grünzweig contain some peculiar features. Although she claims to have based her diagrams on those of Vatican 204, they often contain curved lines of a sort almost never seen in Greek mathematical manuscripts and certainly not in Vatican 204. Thus, although her critical notes are useful, the visual representation of the figures is often misleading.

![Diagrams and arguments in Greek mathematics](image)

**Figure 2.3** Diagrams for Euclid's *Elements*, Book 1, Proposition 7.

squares representing rectangles, and symmetry in the figure where none is required by the text.\(^\text{14}\) This tendency towards greater regularity, which we call ‘overspecification’, is so prevalent in the Greek, Arabic and Latin transmissions of the *Elements* that it almost certainly reflects ancient practice.

We begin with an example of a manuscript diagram portraying more symmetry than is required by the text. *Elem.* 1.7 demonstrates that two given straight lines constructed from the extremities of a given line, on the same side of it, will meet in one and only one point. In Figure 2.3, where the given lines are $AI$ and $BI$, the proof proceeds indirectly by assuming some lines equal to these, say $A\Delta$ and $B\Delta$, meet at some other point, $\Delta$, and then showing this to be impossible. As long as they are on the same side of line $AB$, points $\Gamma$ and $\Delta$ may be assumed to be anywhere and the proof is still valid. Heiberg, following the modern tradition, depicts this as shown in Figure 2.3. All of the manuscripts used by Heiberg agree, however, in placing points $\Gamma$ and $\Delta$ on a line parallel to line $AB$ and arranged such that triangle $AB\Delta$ and triangle $AB\Gamma$ appear to be equal.\(^\text{15}\) In this way, the figure becomes perfectly symmetrical and, to our modern taste, fails to convey the arbitrariness that the text allows in the relative positions of points $\Gamma$ and $\Delta$.

We turn now to a case of the tendency of arbitrary angles to be represented as orthogonal. *Elem.* 1.35 shows that parallelograms that stand on the same base between the same parallels are equal to each other. In Figure 2.4, the proof that parallelogram $AB\Gamma\Delta$ equals parallelogram $EB\Gamma Z$ follows from the addition and subtraction of areas represented in the figure and would make no sense without an appeal to the figure in order to understand these operations. In the modern figures that culminate in Heiberg's edition, the parallelograms are both depicted with oblique angles, whereas

\(^\text{14}\) In this chapter, we give only a few select examples. Many more examples, however, can be seen by consulting the manuscript diagrams themselves. For Book 1 of the *Elements*, see Saito 2006. For Books 11–13 of the *Elements*, as well as Euclid's *Phenomena* and *Optics*, see the report of a three-year research project on manuscript diagrams, carried out by Saito, available online at www.hs.osakako-u.ac.jp/~ken/saito/.

\(^\text{15}\) See Saito 2006: 103 for further images of the manuscript figures.
any shape. In the modern figure, because the pentagons are irregular, we somehow imagine that they could represent any pair of polygons, although, in fact a certain specific pair of irregular pentagons are depicted.

The presence of overspecification is so prevalent in the diagrams of the medieval transmission of geometric texts that we believe it must be representative of ancient practice. Moreover, there is no mathematical reason why the use of overspecified diagrams should not have been part of the ancient tradition. For us, the lack of regularity in the modern figures is suggestive of greater generality. The ancient and medieval scholars, however, apparently did not have this association between irregularity and greater generality, and, except perhaps from a statistical standpoint, there is no reason why these concepts should be so linked. The drawing printed by Heiberg is not a drawing of 'any' pair of polygons, it is a drawing of two particular irregular pentagons. Since the text states that the two polygons are similar, they could be represented by any two similar polygons, as say those in Bodleian 301 which also happen to be equal and regular. Of course statistically, an arbitrarily chosen pair of similar polygons is more likely to be irregular and unequal, but statistical considerations, aside from being anachronistic, are hardly relevant. The diagram is simply a representation of the objects under discussion. For us, an irregular triangle is somehow a more satisfying representation of 'any' triangle, whereas for the ancient and medieval mathematical scholars an arbitrary triangle might be just as well, if not better, depicted by a regular triangle.

Indifference to visual accuracy

Another widespread tendency that we find in the manuscripts is the use of diagrams that are not graphically accurate depictions of the mathematical objects discussed in the text. For example, unequal lines may be depicted as equal, equal angles may be depicted as unequal, the bisection of a line may look more like a quadrature, an arc of a parabola may be represented with the arc of a circle, or straight lines may be depicted as curved. These tendencies show a certain indifference to graphical accuracy and can be divided into two types, which we call 'indifference to metrical accuracy' and 'indifference to geometric shape'.

We begin with an example that exhibits both overspecification and indifference to metrical accuracy. Elem. 1.24 is a problem that shows how to

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16 See Saito 2006: 131 for further images of the manuscript figures.
17 In fact, the proof given in the proposition is also about a more specific polygon in that it has five sides and is divided into three similar triangles, but it achieves generality by being generally applicable for any given pair of rectilinear figures. This proof is an example of the type of proof that Freudenthal 1953 called quasi-general.
construct, on a given line, a parallelogram that contains a given angle and is equal to a given triangle. As exemplified by Vatican 190 in Figure 2.6, in all the manuscripts, the parallelogram is represented by a rectangle, and in the majority of the manuscripts that Heiberg used for his edition there is no correlation between the magnitudes of the given angle and triangle and those of the constructed angle and parallelogram. In the modern figure, printed by Heiberg and seen in Figure 2.6, however, not only is the constructed figure depicted as an oblique parallelogram, but the magnitudes of the given and constructed objects have been set out as equal.

We turn now to an occurrence of metrical indifference that is, in a sense, the opposite of overspecification. In Elem. i.7, Euclid demonstrates a proposition asserting the metrical relationship obtaining between squares and rectangles constructed on a given line cut at random. The overall geometric object is stated to be a square and it contains two internal squares. Nevertheless, as seen in the examples of Vatican 190 and Bodleian 301 in Figure 2.7, the majority of Heiberg’s manuscripts show these squares as rectangles. We should note also the extreme overspecification of Bodleian 301, in which all of the internal rectangles appear to be equal. In general, there seems to be a basic indifference as to whether or not the diagram should visually represent the most essential metrical properties of the geometric objects it depicts.

In this chapter, when we speak of the majority of the manuscripts, we mean the majority of the manuscripts selected by the text editor as independent witnesses for the establishment of the text. We should be wary of assuming, however, that the majority reading is the best, or most pristine. See Saito 2006:1-16, for further images of the manuscript figures. In Vienna 31, as is often the case with this manuscript, we find the magnitudes have been drawn so as to accurately represent the stipulations of the text (see the discussion of this manuscript in ‘Correcting the diagram’, below).

See Saito 2006 for further images of the manuscript diagrams. In Vienna 31 and Bologna 18-18, the squares, indeed, look like squares.

As well as metrical indifference, the manuscript diagrams often seem to reveal an indifference toward the geometric shape of the objects as specified by the text. The most prevalent example of this is the use of circular arcs to portray all curved lines. As an example, we may take the diagram for Apollonius Con. i.16. As seen in Figure 2.8, the diagram in Vatican 206 shows the two branches of an hyperbola as two semicircles. Indeed, all the diagrams in this manuscript portray conic sections with circular arcs. Heiberg’s diagram, on the other hand, depicts the hyperbolas with hyperbolas.

This diagram, however, is also interesting because it includes a case of overspecification, despite the fact that Eutocius, already in the sixth century, noticed this overspecification and suggested that it be avoided. In Figure 2.8, the line AB appears to be drawn as the axis of the hyperbola, such that HK and ΘA are shown as orthogonal ordinates, whereas the theorem treats the properties of any diameter, such that HK and ΘA could also be oblique ordinates. Eutocius suggested that they be so drawn in order to make it clear that the proposition is about diameters, not the axis. Nevertheless, despite Eutocius’ remarks, the overspecification of this figure was preserved into the medieval period, and indeed was maintained by Heiberg in his edition of the text. This episode indicates that overspecification was indeed in effect in the ancient period and that although Eutocius objected to this particular instance of it, he was not generally opposed, and even here his objection was ignored.

As well as being used to represent the more complicated curves of the conics sections, circular arcs are also used to represent straight lines. As Netz has shown, this practice was consistently applied in the diagrams for

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19 A more general figure, which would no doubt have pleased Eutocius, is given in Taliферro, Desmoules and Donahue 1999: 34.
Archimedes' *Sphere and Cylinder* for a polygon with short sides that might be visually confused with the arcs of the circumscribed circle. 23

In the manuscript diagrams of *Elem.*, vi.16, however, we have good evidence that the curved lines are the result of later intervention by the scribes. *Elem.* vi.16 is a problem that shows how to construct a regular 15-gon in a circle (Figure 2.9). The manuscript evidence for this figure is rather involved and, in fact, none of the manuscripts that Heiberg used contain the same diagram in the place of the primary diagram, although there is some obvious cross-contamination in the secondary, marginal diagrams. 24 Nevertheless, it is most likely that the archetype was a metrically inexact representation of the sides of the auxiliary equilateral triangle and regular pentagon depicted with straight lines, as found in *Bologna 18–19* and in the erased part of *Florence 28*. 25 In *Bodleian 301* and *Paris 2466* we see examples in which the scribe has made an effort to draw lines AB and AB' so as to portray more accurately the sides of a regular pentagon and an equilateral triangle, respectively. In *Bodleian 301*, the external sides of the figures are clearly curved, while in *Paris 2466* this curvature is slight. In *Vienna 31*, the original four lines were straight and metrically accurate, as is usual for this manuscript, and a later hand added further curved lines. In *Vatican 190*, it appears that all the sides of the auxiliary triangle and pentagon were drawn in at some point and then later erased, presumably so as to bring the figure into conformity with the evidence of some other source.

Not only were circles used for straight lines, but we also have at least one example of straight lines being used to represent a curved line. This rather interesting example of indifference to visual accuracy comes from one of the most fascinating manuscripts of Greek mathematics, the so-called *Archimedes Palimpsest*, a tenth-century manuscript containing various Hellenistic treatises including technical works by Archimedes that was...
palimpsested as a prayer book some centuries later. In the section of the treatise that Heiberg called Method 14, Archimedes discusses the metrical relationships that obtain between a prism, a cylinder and a parabolic solid that are constructed within the same square base. In Figure 2.10, the base of the prism is rectangle \(EATH\), that of the cylinder is semicircle \(EZH\), while that of the parabolic solid is triangle \(EZH\). Thus, in this diagram, a parabola is represented by an isosceles triangle. Since the parabola is defined in the text by the relationship between the ordinates and abscissa, and since the triangle intersects and meets the same lines as the parabola, this was apparently seen as a perfectly acceptable representation. In this way, the triangle functions as a purely schematic representation of the parabola. Indeed, without the text we would have no way to know that the diagram represents a parabola.

Diagrams in solid geometry

The schematic nature of ancient and medieval diagrams becomes most obvious when we consider the figures of solid geometry. Although there are some diagrams in the manuscripts of solid geometry that attempt to give a pictorial representation of the geometric objects, for the most part, they forego linear perspective in favour of schematic representation. This means that they do not serve to convey a sense of the overall spacial relationships obtaining among the objects, but rather to convey specific mathematical relationships that are essential to the argument.

Some conspicuous exceptions to this general tendency should be mentioned. For example, the diagrams for the rectilinear solids treated in \(E\)lem.\(\text{x}\) and \(\text{xi}\) and the early derivations of the conic sections in the cone, in \(\text{Con.}\ 1\), appear to use techniques of linear perspective to convey a sense of the three-dimensionality of the objects. In Figure 2.11, we reproduce the diagram for \(E\)lem.\(\text{xi}\).\(3\) from \textit{Vatican} \(190\) and that for \textit{Con.}\ 1.\(1\) from \textit{Vatican} \(206\).

In all of these cases, however, it is possible to represent the three-dimensionality of the objects simply and without introducing any object not explicitly named in the proof merely for the sake of the diagram. For example, in Figure 2.1 above, the plane upon which the perpendicular is to be constructed does not appear in the manuscript figure. Hence, even in these three-dimensional diagrams, techniques of linear perspective are used only to the extent that they do not conflict with the schematic nature of the diagram. Auxiliary, purely graphical elements are not used, nor is there any attempt to convey the visual impression of the mathematical objects through graphical techniques. An example of this is the case of circles seen at an angle. Although it is not clear that there was a consistent theory of linear perspective in antiquity, ancient artists regularly drew circles as ovals and Ptolemy, in his \textit{Geography}, describes the depiction of circles seen from an angle as represented by ovals,26 nevertheless, in the medieval manuscripts such oblique circles are always drawn with two

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26 The circuitous story of this manuscript is told by Netz and Noel 2007.

27 This section of the \textit{Method} is discussed by Netz, Saito and Tchernetska 2003–2. The diagram found in the palimpsest is difficult to see in the original. Here, we include two images developed by researchers in the Archimedes Palimpsest Project. The diagram is in the left-hand column of the text spanning pages 159v–158r. These images, licensed under the Creative Commons Attribution 3.0 Unported Access Rights, are available online at www.archimedespalimpsest.org.

circular arcs that meet at cusps, as seen in Figure 2.11. This confirms that the diagrams were not meant to be a visual depiction of the objects, but rather a representation of certain essential mathematical properties.

Likewise, in the figures of spherical geometry, if the sphere itself is not named or required by the proof, we will often see the objects themselves simply drawn free-floating in the plane, to all appearances as though they were actually located in the plane of the figure. Theodorus' Sph. II.6 shows that if, in a sphere, a great circle is tangent to a lesser circle, then it is also tangent to another lesser circle that is equal and parallel to the first. In Figure 2.12, we find the great circle in the sphere, ABI, and the two equal and parallel lesser circles that are tangent to it, IΔ and BH, all lying flat in the same plane, with no attempt to portray their spacial relationships to each other or the sphere in which they are located.

The diagram for Sph. II.6 thus highlights the schematic nature of diagrams in the works of spherical geometry. The theorem is about the type of tangency that obtains between a great circle and two equal lesser circles and this tangency is essentially the only thing conveyed by the figure. The actual spacial arrangement of the circles on the sphere must either be imagined by the reader or drawn out on some real globe.

Figure 2.12 Diagrams for Theodorus' Spheres, Book II, Proposition 6.

Figure 2.13 Diagrams for Theodorus' Spheres, Book II, Proposition 15.

The schematic role of diagrams in spherical geometry becomes unmistakable when we compare the diagram of one of the more involved propositions as found in the manuscripts with one intended to portray the same objects using principles of linear perspective. Sph. II.15 is a problem that demonstrates the construction of a great circle passing through a given point and tangent to a given lesser circle. As can be seen in Figure 2.13, merely by looking at the manuscript diagram, without any discussion of the objects and their arrangement, it is rather difficult to get an overall sense of what the diagram is meant to represent. Nevertheless, certain essential features are conveyed, such as the conspolarity of parallel circles, the tangency and intersection of key circles, and so on. It is clear that the manuscript diagram is meant to be read in conjunction with the text as referring to some other object, either an imagined sphere or more likely a real sphere on which the lines and circles were actually drawn. It tells the reader how to understand the labelling and arrangement of the objects under discussion, so that the text can then be read as referring to these objects. The modern figure, on the other hand, by selecting a particular vantage point as most opportune and then allowing the reader to see the objects from this point, does a better job of conveying the overall spacial relationships that obtain among the objects.

With respect to linear perspective, there is still a debate as to whether or not the concept of the vanishing point was consistently applied in antiquity. See Andersen 1987 and Knorr 1991. As Jones 2000, 55–6 has pointed out, Pappus' commentary to Euclid, Optics 35 includes a vanishing point, but it is not located in accordance with the modern principles of linear perspective.

We argue elsewhere that Theodorus was, indeed, concerned with the practical aspects of drawing figures on solid globes, but that this practice was not explicitly discussed in the Spheres; Sidoli and Saito 2009.

We should point out, however, that the modern diagram in Figure 2.13, as well as being in linear perspective, employs a number of graphical techniques that we do not find in the manuscript sources, such as the use of non-circular curves, dotted lines, highlighted points, and so on.
One diagram for multiple cases

In the foregoing three sections, we have discussed characteristics of the medieval diagrams that are so prevalent that they almost certainly reflect ancient practice. We turn now to characteristics that are more individual but which, nevertheless, form an essential part of the material transmission through which we must understand the ancient texts.

For a few propositions that are divided into multiple cases, we find, nevertheless, the use of a single diagram to represent the cases. There is some question about the originality of most of these, and in fact it appears that, in general, Euclid did not include multiple cases and that those propositions that do have cases were altered in late antiquity. Nevertheless, even if the cases are all due to late ancient authors, they are historically interesting and the manuscript tradition shows considerable variety in the diagrams. This indicates that single diagrams for multiple cases were probably in the text at least by late antiquity and that the medieval scribes had difficulty understanding them and hence introduced the variety that we now find.

As an example, we consider Elem. iii.36. The proposition shows that if, from a point outside a circle, a line is drawn cutting the circle, it will be cut by the circle such that the rectangle contained by its parts will be equal to the square drawn on the tangent from the point to the circle. That is, in Figure 2.14, the rectangle contained by $\Delta A$ and $\Delta \Gamma$ is equal to the square on $\Delta B$. In the text, as we now have it, this is proved in two cases, first where line $\Delta A$ passes through the centre of the circle and second where it

32 See Saito 2006: 85–90 for the case of a single figure containing two cases in Elem. iii.25, in which the division into cases was almost certainly not due to Euclid. The Arabic transmission of the Elements gives further evidence for the elaboration of a single figure into multiple figures. In the eastern Arabic tradition, we find a single figure for both Elem. iii.31 and iv.5 (see for example, Epitome 20: 42v and 39v), while in the Andalusian Arabic tradition, which was also transmitted into Latin, we find multiple figures for these propositions (compare Rabat 53: 126–8 and 145–6 with Busard 1984: 83–5 and 102–5).

does not. In Heiberg's edition, and Vienna 31 (which often has corrected diagrams), there is an individual figure for each case. In the majority of Heiberg's manuscripts, however, there is only a single figure and it contains two different points that represent the centre, one for each case. In Figure 2.14, we reproduce the two diagrams from Heiberg's edition, which are mathematically the same as those in Vienna 31, and an example of the single figure taken from Bodleian 301. In the single diagram, as found in Bodleian 301, there are two centres, points $E$ and $Z$, and neither of them lies at the centre of the circle. Nevertheless, if we suppose that they are indeed centres, the proof can be read and understood on the basis of this figure.

Despite these peculiarities, there are a number of reasons for thinking that this figure is close to the original on which the others were based. It appears in the majority of Heiberg's manuscripts, and the other diagrams contain minor problems, such as missing or misplaced lines, or are obviously corrected. Moreover, the single figure appears to have caused widespread confusion in the manuscript tradition. In most of the manuscripts, there are also marginal figures which either correct the primary figure or provide a figure that is clearly meant for a single case.

Hence, although we cannot, at present, be certain of the history of this theorem and its figure, the characteristics and variety of the figures should be used in any analysis of the text that seeks to establish its authenticity or authorship. This holds true for nearly all of the propositions that were clearly subject to modification in the tradition.

Correcting the diagrams

Medieval scribes also made what they, no doubt, considered to be corrections to the diagrams both by redrawing the figures according to their own interpretation of the mathematics involved and by checking the diagrams against those in other versions of the same treatise and, if they were different, correcting on this basis. We will call the first practice 'redrawing' and the latter 'cross-contamination'. We have already seen the example of Elem. iv.16, on the construction of the regular 15-gon (see Figure 2.9), in which the scribes corrected for metrical indifference and drew the lines of the polygon as curved lines to distinguish them better from the arcs of the circumscribing circle.

33 See Saito 2008: 78–9 for a discussion of variants of this diagram in the manuscripts of the Elements.
In a number of cases, the tendencies toward overspecification and graphical indifference resulted in a figure that was difficult to interpret as a graphical object. For example, we may refer again to Figure 2.14 in which two different centres of the circle are depicted, neither of which appears to lie at the centre of the circle. In such cases, the scribes often tried to correct the figure so that it could be more readily interpreted without ambiguity.

As an example of a redrawn diagram, we take *Elem.* i.21, which proves that, in a circle, angles that subtend the same arc are equal to one another. As seen in Figure 2.15, *Vatican* 190 portrays the situation by showing the two angles $BA\Delta$ and $BE\Delta$ as clearly separated from the angle at the centre, angle $BZA$, which is twice both of them. In the majority of Heiberg's manuscripts, however, as seen in *Bodleian* 301 and *Vienna* 31, through overspecification the lines $BA$ and $EA$ have been drawn parallel to each other and at right angles to $BA$, so that the lines $AA$ and $BE$ appear to intersect at the centre of the circle. In the course of the proposition, however, centre $Z$ is found and lines $BZ$ and $Z\Delta$ are joined. In order to depict centre $Z$ as distinct from the intersection of lines $AA$ and $BE$, centre $Z$ has been placed off centre, often by later hands, as seen in the examples of *Bodleian* 301 and *Vienna* 31.\(^{34}\) Because of the variety of the manuscript figures, it does not seem possible to be certain of the archetype, but it probably had point $Z$ as the intersection of $AA$ and $BE$, as in the example of *Vienna* 31, or it had a second centre called $Z$ but not located at the centre of the circle, as in the example of *Bodleian* 301.\(^{35}\) Later readers, however, found this situation confusing and corrected the diagrams accordingly. In this case, the redrawing was done directly on top of the original figure.

\(^{34}\) See Saito 2008: 67 for further discussion of this diagram.

\(^{35}\) In *Bodleian* 301, a later hand appears to have crossed out this original second centre, $Z$, and moved it closer to the centre of the circle.

Figure 2.15 Diagrams for Euclid's *Elements*, Book III, Proposition 21.

The redrawing, however, might also be done at the time when the text was copied and the figures drafted. In this case, the source diagram is lost in this part of the tradition. Of the manuscripts used by Heiberg, the diagrams in *Vienna* 31 are often redrawn for metrical accuracy, but less often for overspecification.\(^{36}\) For the diagram accompanying *Elem.* i.44, the figure in *Vienna* 31 (see Figure 2.16) should be compared with that in *Vatican* 190 (see Figure 2.6). As can be seen, the given area $\Gamma$ is indeed the size of the parallelogram constructed on line $AB$, but the parallelogram is depicted as a rectangle and this is reflected in the fact that the given angle, $\Delta$, is depicted as right. In this case, the diagram is metrically accurate but it still represents any parallelogram with a rectangle.

For an example in which the diagram in *Vienna* 31 has been corrected both for metrical accuracy and overspecification, we consider *Elem.* i.22, which demonstrates the construction of a triangle with three given sides. As seen in Figure 2.17, the older tradition, here exemplified by *Vatican* 190, represents the constructed triangle with the isosceles triangle $ZKH$, and the given lines with the equal lines $A$, $B$ and $\Gamma$. In some of the manuscripts, however, the constructed triangle $ZKH$ is drawn as an irregular acute triangle.\(^{37}\) In Figure 2.17 we see the example of *Vienna* 31, in which the

\(^{36}\) As we saw in the foregoing example, in the case of *Elem.* i.21, however, the original scribe of *Vienna* 31 did not correct the diagram, but a correction was added by a later hand.

\(^{37}\) See Saito 2006: 118 for a larger selection of the manuscript figures. The fact that *Vatican* 190 belongs to the older tradition is confirmed by the Arabic transmission.
constructed triangle is depicted as an irregular, acute triangle and all of its sides are depicted as the same length as the sides that have been given for the construction. Indeed, here we have a figure that is fully in accord with modern tastes.

For *Elem. 1.22*, of the manuscripts used by Heiberg in his edition, *Bodleian 301* also depicts the constructed triangle as an irregular, acute triangle, similar to that in *Vienna 31*. The fact that *Vienna 31* and *Bodleian 301* have a similar irregular, acute triangle could either indicate that scribes in both traditions independently had the idea to draw an irregular, acute triangle and randomly drew one of the same shape or, more likely, a scribe in one tradition saw the figure in the other and copied it. There is considerable evidence that this kind of cross-contamination took place. As another example that we have already seen, we may mention *Elem. III.21* in which both *Vienna 31* and *Bodleian 301* show a second centre drawn in freehand at some time after the original drawing was complete. Moreover, in the case of *Elem. III.21*, in *Florence 28*, which has the same primary diagram as *Bodleian 301*, we find a marginal diagram like that in *Vatican 190*, while in *Bologna 18-19*, which has the same primary diagram as *Vatican 190*, we find a marginal diagram like that in *Florence 28*.

Hence, as well as being used as a cross-reference for the primary diagram, the figures of a second or third manuscript were often drawn into the margin as a secondary diagram. Although we are now only at the beginning stages of such studies, this process of cross-contamination suggests the possibility of analysing the transmission dependencies of the diagrams themselves without necessarily relying on those of the text. Indeed, there is now increasing evidence that the figures, like the scholia, were sometimes transmitted independently of the text. The process of cross-contamination has left important clues in the manuscript sources that should be exploited to help us understand how the manuscript diagrams were used and read.

Ancient and medieval manuscript diagrams

Since the ancient and medieval diagrams are material objects that were transmitted along with the text, we should consider the ways they were copied, read and understood with respect to the transmission of the text.

Although, for the most part, the text and diagrams appear to have been copied as faithfully as possible, at various times in the Greek transmission, and perhaps more often in the Arabic tradition, mathematically minded individuals re-edited the texts and redrew the diagrams.

For the most part, in Greek manuscripts the diagrams are drawn into boxes that were left blank when the text was copied, whereas in the Arabic and Latin manuscripts the diagrams were often drawn by the same scribe as copied the text, as is evident from the fact that the text wraps around the diagram. Nevertheless, except during periods of cultural transmission and appropriation, the diagrams appear to have been generally transmitted by scribes who based their drawings on those in their source manuscripts, despite the fact that the diagrams can largely be redrawn on the basis of a knowledge of the mathematics contained in the text. Hence, the diagrams in the medieval manuscripts give evidence for two, in some sense conflicting, tendencies: (1) the scribal transmission of ancient treatises based on a concept of the sanctity of the text and (2) the use of the ancient works in the mathematical sciences for teaching and developing those sciences and the consequent criticism of the received text from the perspective of a mathematical reading.

For these reasons, when we use the medieval diagrams as evidence for ancient practices, when we base our understanding of the intended uses of the diagrams on these sources, we should look for general tendencies and not become overly distracted by the evidence of idiosyncratic sources.

Diagrams and generality

The two most prevalent characteristics of the manuscript diagrams are what we have called overspecification and indifference to visual accuracy. The consistent use of overspecification implies that the diagram was not meant to convey an idea of the level of generality discussed in the text. The diagram simply depicts some representative example of the objects under discussion and the fact that this example is more regular than is required was apparently not considered to be a problem. In the case of research, discussion or presentation, a speaker could of course refer to the level of generality addressed by the text, or, in fact, could simply redraw the diagram. The indifference to visual accuracy implies that the diagram was not meant to be a visual depiction of the objects under discussion but rather to use visual cues to communicate the important mathematical relationships. In this sense, the diagrams are schematic representations. They help the reader navigate
In fact, we will probably never know much with certainty about the parabolas that were drawn by mathematicians investigating conic theory or the circles that were drawn on globes by teachers discussing spherical geometry. Nevertheless, insofar as mathematical teaching and research are human activities, we should not doubt that the real learning and research was done by drawing diagrams and reasoning about them, not simply by reading books or copying them out. Hence, the diagrams in the manuscripts were meant to serve as signposts indicating how to draw these figures and mediating the reader's understanding of the propositions about them.

We may think of the manuscript diagrams as schematic guides for drawing figures and for navigating their geometric properties. In some cases, and for individuals with a highly developed geometric imagination, these secondary diagrams might simply be imagined, but for the most part they would actually have been drawn out. The diagrams achieve their generality in a similar way as the text, by presenting a particular instantiation of the geometric objects, which shows the readers how they are laid out and labelled so that the readers can themselves draw other figures in such a way that the proposition still holds. Hence, just as the words of the text refer to any geometric objects which have the same conditions, so the diagrams of the text refer to any diagrams that have the same configurations.

We may think of the way we use the diagram of a difficult proposition, such as that of the manuscript diagram for Spher. 11.15 in Figure 2.13, in the same way that we think of the way we use the subway map of the Tokyo Metro. We may look at the manuscript diagram in Figure 2.13 before we have worked through the proposition to get a sense of how things are laid out, just as we may look at the Tokyo subway map before we set out for a new place, to see where we will transfer and so forth. Although this may help orientate our thinking, in neither case does it fully prepare us for the actual experience. The schematic representation of the sphere in Figure 2.13 tells us nothing of its orientation in space, an intuition of which we will need to develop in order to actually understand the proposition. The Tokyo subway map tells us nothing about trains, platforms and tickets, all of which we will need to negotiate to actually go anywhere in Tokyo. In both cases, the image is a schematic that conveys only information essential to an activity that the reader is assumed to be undertaking.

There is, however, also an important distinction. The Tokyo subway map points towards a very specific object — or rather a system of objects that are

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30 See Sidoli 2007 for some examples of mathematically minded readers who drew the figures in the treatises they were transmitting.

31 Sidoli and Saito 2009.


34 The Tokyo subway map, in a number of different languages, can be downloaded from www.tokyo-metro.jp/o/.
always in flux, and probably not nearly as determinate as we would like to believe—nevertheless, a system of objects with a very specific locality and temporality. A Tokyo subway map is useless for Paris. If it was drawn this year, it will contain stations and lines that did not exist ten years ago and ten years from now it will again be out of date. The manuscript diagram in Figure 2.13, however, has no such specificity. It can refer to any sphere and does. Anyone who wants to draw a great circle on a sphere tangent to a given line and through a given point can use this diagram in conjunction with its proposition to do so. In the centuries since this proposition was written, a great many readers must have drawn figures of this construction—on the plane, on the sphere, in their mind’s eye—and this diagram, strange and awkward as it is, somehow referred to all of them. It is in such a way that the overspecified, graphically inaccurate diagrams that we find in the manuscript tradition achieve the generality for which they were intended.

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