2 Diagrams and arguments in ancient Greek mathematics: lessons drawn from comparisons of the manuscript diagrams with those in modern critical editions

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Introduction

In some ways, the works of ancient Greek geometry can be regarded as arguments about diagrams. Anyone who has ever looked at a medieval manuscript containing a copy of an ancient geometrical text knows that the most conspicuous characteristic of these works is the constant presence of diagrams.[1] Anyone who has ever read a Greek mathematical text, in any language, knows that the most prevalent feature of Greek mathematical prose is the constant use of letter names, which refer the reader's attention to the accompanying diagrams.

In recent years, particularly due to a chapter in Netz's *The Shaping of Deduction in Greek Mathematics* entitled 'The lettered diagram', historians of Greek mathematics have had a renewed interest in the relationship between the argument in the text and the figure that accompanies it.[2] Research projects that were motivated by this interest, however, quickly had to come to grips with the fact that the edited texts of canonical works of Greek geometry, although they contained a wealth of information about the manuscript evidence for the text itself, often said nothing at all about the diagrams. For years, the classical works of Apollonius, Archimedes and, most importantly, the *Elements* of Euclid have been read in edited Greek texts and modern translations that contain diagrams having little or no relation to the diagrams in the manuscript sources. Because they are essentially mathematical reconstructions, the diagrams in modern editions are often mathematically more intelligible than those in the manuscripts, but they are often historically misleading and occasionally even mathematically misleading.[3]

[1] In some cases, the diagrams were never actually drawn, but even their absence is immediately evident from the rectangular boxes that were left for them.

[2] N1999: 12–67.

[3] In this chapter, we will see a number of examples of modern diagrams that are more mathematically consistent with our understanding of the argument and a few that may have

In fact, a few scholars of the ancient mathematical sciences have for many years made critical studies of the manuscript figures, and Neugebauer often called for the critical and conceptual study of ancient and medieval diagrams.[4] These scholars, however, were mostly working on the exact sciences, particularly astronomy and, perhaps due to the tendency of historians of science to divide their research along contemporary disciplinary lines that would have made little sense to ancient mathematicians, these works have generally formed a minority interest for historians of ancient mathematics. Indeed, in his later editions, Heiberg paid more attention to the manuscript figures than he did in his earlier work, but by this time his editions of the canonical works were already complete. In fact, for his edition of Euclid's *Elements*, it appears that the diagrams were adopted from the tradition of printed texts without consulting the manuscript sources.

In this chapter, after briefly sketching the rise of scholarly interest in producing critical diagrams, we investigate the characteristics of manuscript diagrams in contrast to modern reconstructions. To the extent that the evidence will allow, we distinguish between those features of the manuscript diagrams that can be attributed to ancient practice and those that are probably the result of the medieval manuscript tradition, through which we have received the ancient texts. We close with some speculations about what this implies for the conceptual relationship between the figure and the text in ancient Greek mathematical works.

Heiberg's edition of Euclid's *Elements*

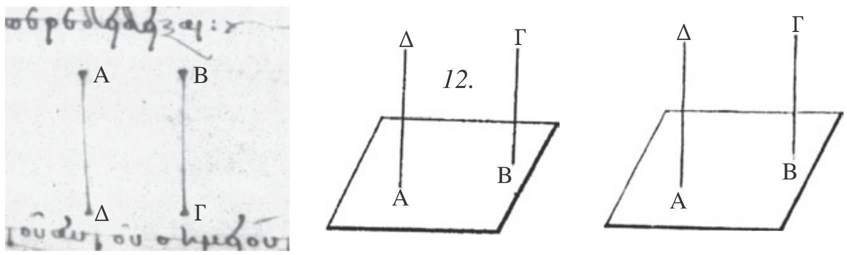
Heiberg (1883–8), on the basis of a study of manuscripts held in European libraries, prepared his edition of the *Elements* from seven manuscripts and the critical apparatus accompanying his text makes constant reference to these sources.[5] Nevertheless, there is usually no apparatus for the diagrams and hence no mention of their source.[6] An examination of the previous

led to historical misunderstandings for this reason. Mathematically misleading modern diagrams, on the other hand, are relatively rare; Neugebauer discusses one example from the edition of Theodosius' *On Days and Nights* prepared by Fecht. Neugebauer 1975: 752; Fecht 1927.

4 For example see the section IV D, 2, 'Figures in Texts' in his *A History of Ancient Mathematical Astronomy*. Neugebauer 1975: 751–5.

5 Heiberg 1903 later published a more detailed account of the manuscript sources and the reasons for his editorial choices. For a more extended discussion of Heiberg's work on the *Elements* and a discussion of the overall history of the text see Vitrac's contribution in this volume.

6 While this is largely the case there are some exceptions. For example, the diagrams for *Elem.* XI.39 and XIII.15 are accompanied with apparatus. Heiberg and Stamatis 1969–77: IV, 73 and 166.



Vatican 190 August Heiberg

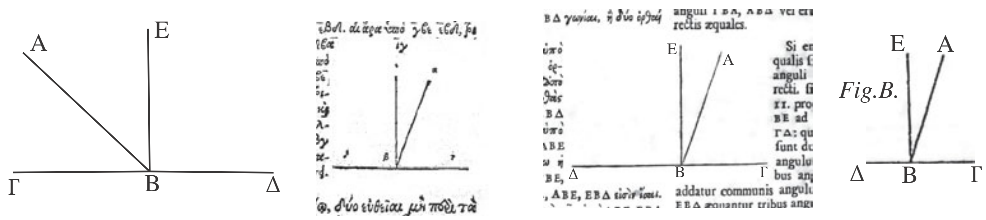
Figure 2.1 Diagrams for Euclid's *Elements*, Book xi, Proposition 12.

printed editions of the text, however, makes it clear that the diagrams accompanying Heiberg's edition were drawn entirely, or for the most part, by copying those in the edition of August (1826–9).[7] The August edition would have been particularly convenient for copying the diagrams, since, as was typical for a German technical publication of its time, the diagrams were printed together in fold-out pages at the end of the volumes.

Although nearly all the diagrams appear to have been so copied, a single example may be used to demonstrate this point. For *Elem*. xi.12, concerning the construction of a perpendicular to a given plane, the diagrams in all the manuscripts consist simply of two equal lines, ΔA and BΓ, placed side by side and labelled such that points Δ and B mark the top of the two lines. In Figure 2.1, we compare the diagram for *Elem*. xi.12 in *Vatican 190*, as representative of all the manuscripts, with that in both the August and Heiberg editions.[8] While *Vatican 190* is typical of the manuscript diagrams, that in Heiberg's text is clearly copied from the August diagram. Although the given plane is not shown in the manuscript figures, it appears in both the printed editions and it is used with the techniques of linear perspective to make the two lines appear to be in different planes from the plane of the drawing. Most significantly, however, there is a labelling error in the line BΓ. Point Γ is supposed to be in the given plane, and hence must be at the bottom of line BΓ, as in *Vatican 190*. This error was transmitted when the diagram was

[7] The diagrams to the arithmetical books are a clear exception. The August diagrams are dotted lines, whereas Heiberg's edition returns to the lines we find in the manuscripts. There also other, individual cases where the diagrams were redrawn, presumably because those in the August edition were considered to be mathematically unsatisfactory. For example, the diagram to *Elem*. xi.38 has been redrawn for Heiberg's edition, whereas all the surrounding diagrams are clearly copied. See also the diagram for *Elem*. xii.17. Compare Heiberg and Stamatis 1969–77: iv 75 and 128 with August 1826–9: Tab. ix and Tab. x.

[8] In this chapter, we refer to manuscripts by an abbreviated name in *italics*. Full library shelf marks are given in the references. For the Euclidian manuscripts see also Vitrac's chapter in this volume.



Vatican 190 Grynée Gregory August

Figure 2.2 Diagrams for Euclid's *Elements*, Book I, Proposition 13.

copied, despite the fact that it could have been easily corrected from considerations of the orientation required by the text.

Indeed, whereas through the course of the modern period, following the general trends of classical scholarship, the editors of successive publications of the *Elements* tended to consult a wider and wider range of manuscripts and give their readers more and more information about these manuscripts, the diagrams that accompanied these editions were generally made on the basis of the diagrams in the previous editions.

As an example of this practice, we may take *Elem*. I.13, which concerns the sum of the angles on either side of a straight line that falls on another straight line. The manuscripts all agree in depicting angle ΑΒΓ as opening to the left, as shown in Figure 2.2 by the example of *Vatican 190*.[9] Nevertheless, all printed editions, following the *editio princeps* of Grynée (1533), print angle ΑΒΓ opening to the right.

In some sense, this may have been a result of the division of labour of the publishers themselves. Whereas the editions were prepared by classical scholars and typeset by printers who were knowledgeable in the classical languages and generally had some sensitivity to the historical issues involved in producing a printed text from manuscript sources, the diagrams were almost certainly drafted by professional illustrators, who would have been skilled in the techniques of visual reproduction but perhaps uninterested in the historical issues at hand. Nevertheless, the fact that the scholars who prepared these editions and the editors who printed them were content to use the diagrams of the previous editions as their primary sources says a great deal about their views of the relative importance of the historical sanctity of the text and of the diagrams in Greek mathematical works.

Already, during the course of Heiberg's career, the attitudes of scholars towards the importance of the manuscript diagrams began to change. In the late 1890s, in the edition he prepared with Besthorn of al-Nayrīzī's

[9] See Saito 2006: 110 for further images of the manuscript figures.

commentaries to the *Elements*, the diagrams were taken directly from *Leiden 399*, and hence often quite different from those printed in his edition of the Greek.[10] By the time he edited Theodosius' *Spherics*, he must have become convinced of the importance of giving the diagrams critical attention, because the finished work includes diagrams based on the manuscripts, generally accompanied with a critical note beginning 'In fig.'[11]

Editions of manuscript diagrams

Because the manuscript diagrams for spherical geometry are so strikingly different from what we have grown to expect since the advent of the consistent application of techniques of linear perspective in the early modern period, the editions of ancient Greek works in spherical astronomy were some of the first in which the editors began to apply critical techniques to the figures. For example in the eighth, and last, volume of the complete works of Euclid, for his edition of the *Phenomena*, Menge (1916) provided diagrams based on the manuscript sources and in some cases included critical notes.

One of the most influential editions with regard to the critical treatment of diagrams was that made by Rome (1931–43) of the commentaries by Pappus and Theon to Ptolemy's *Almagest*. The diagrams in this long work were taken from the manuscript sources and their variants are discussed in critical notes placed directly below the figures themselves.[12] Rome's practices influenced other scholars working in French and the editions by Mogenet (1950), of Autolycus' works in spherical astronomy, and Lejeune (1956), of the Latin translation of Ptolemy's *Optics*, both contain manuscript figures with critical notes.

More recently, the majority tendency has been to provide manuscript diagrams with critical assessment. For example, the editions by Jones (1986) and Czinczenheim (2000) of Book vii of Pappus' *Collection* and

[10] Besthorn *et al*. 1897–1932.

[11] Heiberg 1927. In fact, these critical notes are difficult to notice, since they are found among the notes for the Greek text. The notes for the Greek text, however, are prefaced by numbers referring to the lines of the text, whereas the diagrams are always located in the Latin translation, which has no line numbers. Neugebauer 1975: 751–5 seems to have missed them, since he makes no mention of them in his criticism of the failure of classical scholars to pay sufficient attention to the manuscript diagrams of the works of spherical astronomy.

[12] In connection with the early interest that Rome and Neugebauer showed in manuscript figures, we should mention the papers they wrote on Heron's *Dioptra*, the interpretation of which depends in vital ways on understanding the diagram. Rome 1923; Neugebauer 1938–9; Sidoli 2005.

Theodosius' *Spherics*, respectively, both contain critical diagrams, and a recent translation of Archimedes' *Sphere and Cylinder* also includes a critical assessment of the manuscript figures.[13]

Nevertheless, although there are critically edited diagrams for many works, especially those of the exact sciences, the most canonical works – the works of Archimedes and Apollonius, the *Elements* of Euclid and the *Almagest* of Ptolemy – because they were edited by Heiberg early in his career, are accompanied by modern, redrawn diagrams. Hence, because a study of Greek mathematics almost always begins with the *Elements*, and because the manuscript diagrams of this work contain many distinctive and unexpected features, it is essential that we reassess the manuscript evidence.

Characteristics of manuscript diagrams

In this section, focusing largely on the *Elements*, we examine some of the characteristic features of the manuscript diagrams as material objects that distinguish them from their modern counterparts. Manuscript diagrams are historically contingent objects which were read and copied and redrawn many times over the centuries. In some cases, they may tell us about ancient practice, in other cases, about medieval interpretations of ancient practice, and in some few cases, they simply tell us about the idiosyncratic reading of a single scribe. In the following sections, we begin with broad general tendencies that can almost certainly be ascribed to the whole history of the transmission, and then move into more individual cases where the tradition shows modification and interpretation. In this chapter, we present summary overviews, not systematic studies.

Overspecification

One of the most pervasive features of the manuscript figures is the tendency to represent more regularity among the geometric objects than is demanded by the argument. For example, we find rectangles representing parallelograms, isosceles triangles representing arbitrary triangles,

[13] Netz 2004. In fact, however, the figures printed by Czinczenheim contain some peculiar features. Although she claims to have based her diagrams on those of *Vatican 204*, they often contain curved lines of a sort almost never seen in Greek mathematical manuscripts and certainly not in *Vatican 204*. Thus, although her critical notes are useful, the visual representation of the figures is often misleading.

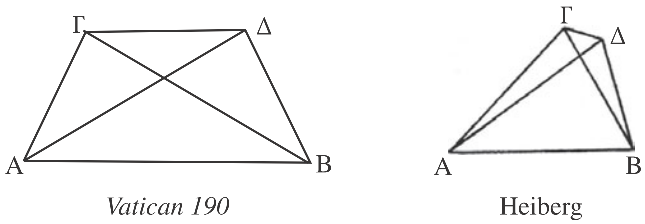


Figure 2.3 Diagrams for Euclid's *Elements*, Book I, Proposition 7.

squares representing rectangles, and symmetry in the figure where none is required by the text.[14] This tendency towards greater regularity, which we call 'overspecification', is so prevalent in the Greek, Arabic and Latin transmissions of the *Elements* that it almost certainly reflects ancient practice.

We begin with an example of a manuscript diagram portraying more symmetry than is required by the text. *Elem*. I.7 demonstrates that two given straight lines constructed from the extremities of a given line, on the same side of it, will meet in one and only one point. In Figure 2.3, where the given lines are AΓ and BΓ, the proof proceeds indirectly by assuming some lines equal to these, say AΔ and BΔ, meet at some other point, Δ, and then showing this to be impossible. As long as they are on the same side of line AB, points Γ and Δ may be assumed to be anywhere and the proof is still valid. Heiberg, following the modern tradition, depicts this as shown in Figure 2.3. All of the manuscripts used by Heiberg agree, however, in placing points Γ and Δ on a line parallel to line AB and arranged such that triangle ABΔ and triangle ABΓ appear to be equal.[15] In this way, the figure becomes perfectly symmetrical and, to our modern taste, fails to convey the arbitrariness that the text allows in the relative positions of points Γ and Δ.

We turn now to a case of the tendency of arbitrary angles to be represented as orthogonal. *Elem*. I.35 shows that parallelograms that stand on the same base between the same parallels are equal to each other. In Figure 2.4, the proof that parallelogram ABΓΔ equals parallelogram EBΓZ follows from the addition and subtraction of areas represented in the figure and would make no sense without an appeal to the figure in order to understand these operations. In the modern figures that culminate in Heiberg's edition, the parallelograms are both depicted with oblique angles, whereas

[14] In this chapter, we give only a few select examples. Many more examples, however, can be seen by consulting the manuscript diagrams themselves. For Book I of the *Elements*, see Saito 2006. For Books II–VI of the *Elements*, as well as Euclid's *Phenomena* and *Optics*, see the report of a three-year research project on manuscript diagrams, carried out by Saito, available online at www.hs.osakafu-u.ac.jp/~ken.saito/.

[15] See Saito 2006: 103 for further images of the manuscript figures.

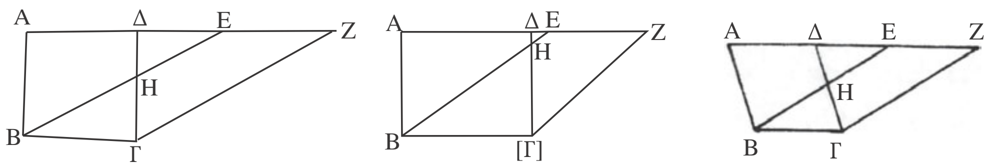


Figure 2.4 Diagrams for Euclid's *Elements*, Book I, Proposition 35.

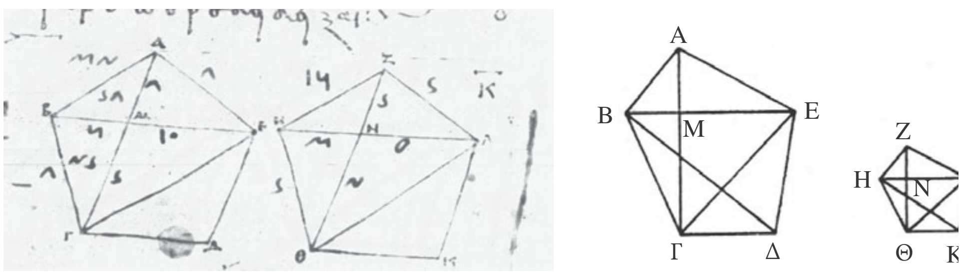


Figure 2.5 Diagrams for Euclid's *Elements*, Book VI, Proposition 20.

in the manuscripts the base parallelogram ΑΒΓΔ is always depicted as a rectangle, as seen in *Bodleian 301*, and often even as a square, as seen in *Vatican 190*.[16] Once again, to our modern sensibility, the diagrams appear to convey more regularity than is required by the proof. That is, the angles need not be right and the sides need not be the same size, and yet they are so depicted in the manuscripts.

We close with one rather extreme example of overspecification. *Elem.* VI.20 shows that similar polygons are divided into an equal number of triangles, of which corresponding triangles in each polygon are similar, and that the ratio of the polygons to one another is equal to the ratio of corresponding triangles to one another, and that the ratio of the polygons to one another is the duplicate of the ratio of a pair of corresponding sides. Although the enunciation is given in such general terms, following the usual practice of Greek geometers, the enunciation and proof is made for a particular instantiation of these objects; in this case, a pair of pentagons. In Figure 2.5, the modern diagram printed by Heiberg depicts two similar, but unequal, irregular pentagons. In *Bodleian 301*, on the other hand, we find two pentagons that are both regular and equal. This diagram strikes the modern eye as inappropriate for this situation because the proposition is not about equal, regular pentagons, but rather similar polygons of

[16] See Saito 2006: 131 for further images of the manuscript figures.

any shape.[17] In the modern figure, because the pentagons are irregular, we somehow imagine that they could represent any pair of polygons, although, in fact a certain specific pair of irregular pentagons are depicted.

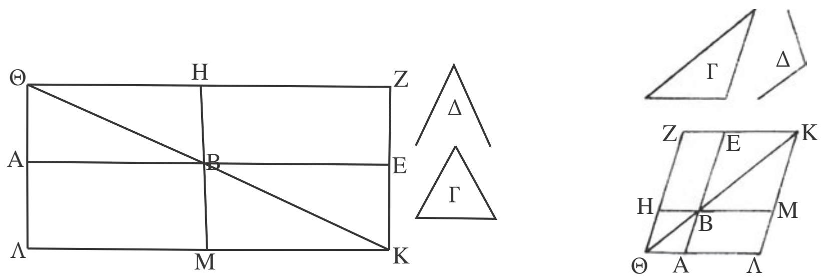
The presence of overspecification is so prevalent in the diagrams of the medieval transmission of geometric texts that we believe it must be representative of ancient practice. Moreover, there is no mathematical reason why the use of overspecified diagrams should not have been part of the ancient tradition. For us, the lack of regularity in the modern figures is suggestive of greater generality. The ancient and medieval scholars, however, apparently did not have this association between irregularity and greater generality, and, except perhaps from a statistical standpoint, there is no reason why these concepts should be so linked. The drawing printed by Heiberg is not a drawing of 'any' pair of polygons, it is a drawing of two particular irregular pentagons. Since the text states that the two polygons are similar, they could be represented by any two similar polygons, as say those in *Bodleian 301* which also happen to be equal and regular. Of course statistically, an arbitrarily chosen pair of similar polygons is more likely to be irregular and unequal, but statistical considerations, aside from being anachronistic, are hardly relevant. The diagram is simply a representation of the objects under discussion. For us, an irregular triangle is somehow a more satisfying representation of 'any' triangle, whereas for the ancient and medieval mathematical scholars an arbitrary triangle might be just as well, if not better, depicted by a regular triangle.

Indifference to visual accuracy

Another widespread tendency that we find in the manuscripts is the use of diagrams that are not graphically accurate depictions of the mathematical objects discussed in the text. For example, unequal lines may be depicted as equal, equal angles may be depicted as unequal, the bisection of a line may look more like a quadrature, an arc of a parabola may be represented with the arc of a circle, or straight lines may be depicted as curved. These tendencies show a certain indifference to graphical accuracy and can be divided into two types, which we call 'indifference to metrical accuracy' and 'indifference to geometric shape'.

We begin with an example that exhibits both overspecification and indifference to metrical accuracy. *Elem*. I.44 is a problem that shows how to

[17] In fact, the proof given in the proposition is also about a more specific polygon in that it has five sides and is divided into three similar triangles, but it achieves generality by being generally applicable for any given pair of rectilinear figures. This proof is an example of the type of proof that Freudenthal 1953 called *quasi-general*.



Vatican 190 Heiberg

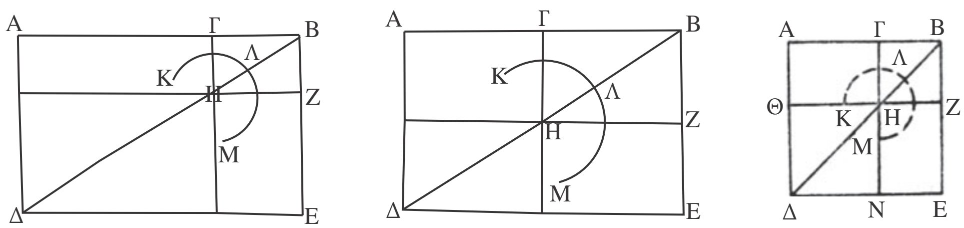
Figure 2.6 Diagrams for Euclid's *Elements*, Book I, Proposition 44.

construct, on a given line, a parallelogram that contains a given angle and is equal to a given triangle. As exemplified by *Vatican 190* in Figure 2.6, in all the manuscripts, the parallelogram is represented by a rectangle, and in the majority of the manuscripts that Heiberg used for his edition there is no correlation between the magnitudes of the given angle and triangle and those of the constructed angle and parallelogram.[18] In the modern figure, printed by Heiberg and seen in Figure 2.6, however, not only is the constructed figure depicted as an oblique parallelogram, but the magnitudes of the given and constructed objects have been set out as equal.

We turn now to an occurrence of metrical indifference that is, in a sense, the opposite of overspecification. In *Elem*. II.7, Euclid demonstrates a proposition asserting the metrical relationship obtaining between squares and rectangles constructed on a given line cut at random. The overall geometric object is stated to be a square and it contains two internal squares. Nevertheless, as seen in the examples of *Vatican 190* and *Bodleian 301* in Figure 2.7, the majority of Heiberg's manuscripts show these squares as rectangles.[19] We should note also the extreme overspecification of *Bodleian 301*, in which all of the internal rectangles appear to be equal. In general, there seems to be a basic indifference as to whether or not the diagram should visually represent the most essential metrical properties of the geometric objects it depicts.

[18] In this chapter, when we speak of the majority of the manuscripts, we mean the majority of the manuscripts selected by the text editor as independent witnesses for the establishment of the text. We should be wary of assuming, however, that the majority reading is the best, or most pristine. See Saito 2006: 140, for further images of the manuscript figures. In *Vienna 31*, as is often the case with this manuscript, we find the magnitudes have been drawn so as to accurately represent the stipulations of the text (see the discussion of this manuscript in 'Correcting the diagram', below).

[19] See Saito 2008 for further images of the manuscript diagrams. In *Vienna 31* and *Bologna 18–19*, the squares, indeed, look like squares.



Vatican 190 *Bodleian 301* Heiberg

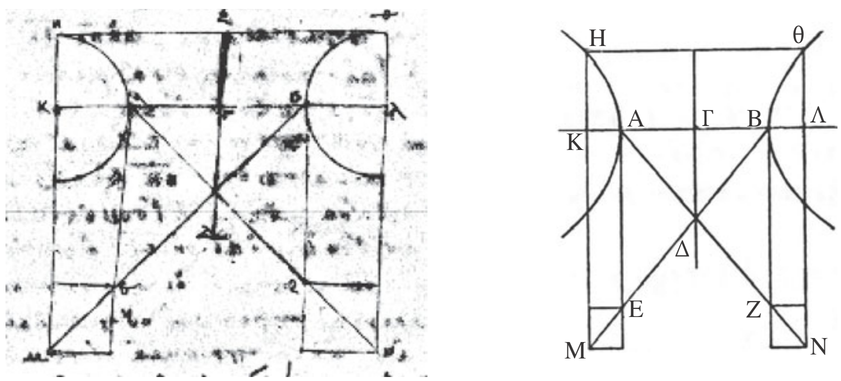
Figure 2.7 Diagrams for Euclid's *Elements*, Book ɪɪ, Proposition 7.

As well as metrical indifference, the manuscript diagrams often seem to reveal an indifference toward the geometric shape of the objects as specified by the text. The most prevalent example of this is the use of circular arcs to portray all curved lines. As an example, we may take the diagram for Apollonius *Con*. ɪ.16. As seen in Figure 2.8, the diagram in *Vatican 206* shows the two branches of an hyperbola as two semicircles. Indeed, all the diagrams in this manuscript portray conic sections with circular arcs. Heiberg's diagram, on the other hand, depicts the hyperbolas with hyperbolas.

This diagram, however, is also interesting because it includes a case of overspecification, despite the fact that Eutocius, already in the sixth century, noticed this overspecification and suggested that it be avoided.[20] In Figure 2.8, the line AB appears to be drawn as the axis of the hyperbola, such that HK and ΘΛ are shown as orthogonal ordinates, whereas the theorem treats the properties of any diameter, such that HK and ΘΛ could also be oblique ordinates. Eutocius suggested that they be so drawn in order to make it clear that the proposition is about diameters, not the axis. Nevertheless, despite Eutocius' remarks, the overspecification of this figure was preserved into the medieval period, and indeed was maintained by Heiberg in his edition of the text.[21] This episode indicates that overspecification was indeed in effect in the ancient period and that although Eutocius objected to this particular instance of it, he was not generally opposed, and even here his objection was ignored.

As well as being used to represent the more complicated curves of the conics sections, circular arcs are also used to represent straight lines. As Netz has shown,[22] this practice was consistently applied in the diagrams for

[20] Heiberg 1891–3: 224; Decorps-Foulquier 1999: 74–5.
[21] A more general figure, which would no doubt have pleased Eutocius, is given in Taliaferro, Densmore and Donahue 1998: 34.
[22] Netz 2004.



Vatican 206 Heiberg

Figure 2.8 Diagrams for Apollonius' *Conica*, Book i, Proposition 16.

Archimedes' *Sphere and Cylinder* for a polygon with short sides that might be visually confused with the arcs of the circumscribed circle.[23]

In the manuscript diagrams of *Elem*. iv.16, however, we have good evidence that the curved lines are the result of later intervention by the scribes. *Elem*. iv.16 is a problem that shows how to construct a regular 15-gon in a circle (Figure 2.9). The manuscript evidence for this figure is rather involved and, in fact, none of the manuscripts that Heiberg used contain the same diagram in the place of the primary diagram, although there is some obvious cross-contamination in the secondary, marginal diagrams.[24] Nevertheless, it is most likely that the archetype was a metrically inexact representation of the sides of the auxiliary equilateral triangle and regular pentagon depicted with straight lines, as found in *Bologna 18–19*

[23] In the present state of the evidence, it is difficult to determine with certainty whether or not the curved lines in the Archimedes tradition go back to antiquity, but there is no good reason to assert that they do not. All of our extant Greek manuscripts for the complete treatise of *Sphere and Cylinder* are based on a single Byzantine manuscript, which is now lost. This is supported by the fragmentary evidence of the oldest manuscript, the so-called Archimedes Palimpsest, whose figures also contain curved lines. The diagrams in an autograph of William of Moerbeke's Latin translation, *Vatican Ottob. 1850*, however, made on the basis of a different Greek codex, also now lost, have straight lines, but this does not prove anything. The source manuscript may have had straight lines or Moerbeke may have changed them. Whatever the case, we now have three witnesses, two of which agree on curved lines and one of which contains straight lines.

[24] See Saito 2008: 171–3 for a full discussion. This previous report, however, was written before the manuscripts could be consulted in person. Since Saito has now examined most of the relevant manuscripts, it is clear from the colour of the lines, the pattern of erasures and so on, that the curved lines are part of the later tradition. See www.hs.osakafu-u.ac.jp/~ken.saito/diagram/ for further updates.

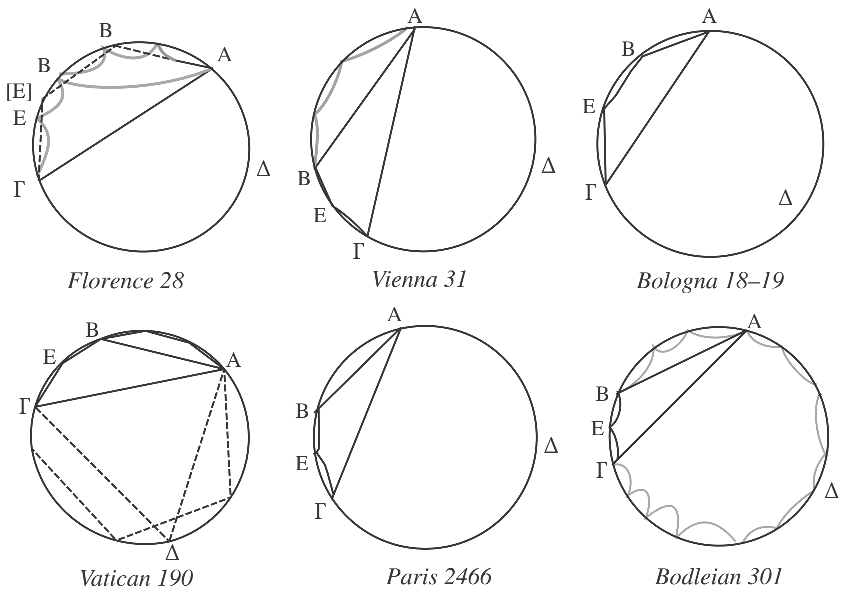
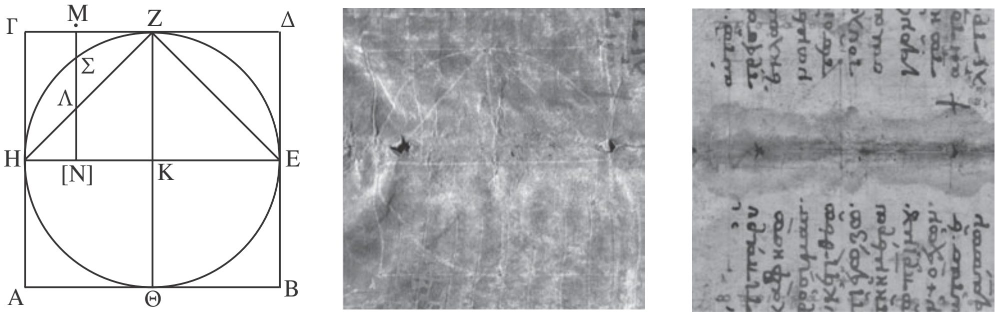


Figure 2.9 Diagrams for Euclid's *Elements*, Book IV, Proposition 16. Dashed lines were drawn in and later erased. Grey lines were drawn in a different ink or with a different instrument.

and in the erased part of *Florence 28*.[25] In *Bodleian 301* and *Paris 2466* we see examples in which the scribe has made an effort to draw lines AB and AΓ so as to portray more accurately the sides of a regular pentagon and an equilateral triangle, respectively. In *Bodleian 301*, the external sides of the figures are clearly curved, while in *Paris 2466* this curvature is slight. In *Vienna 31*, the original four lines were straight and metrically accurate, as is usual for this manuscript, and a later hand added further curved lines. In *Vatican 190*, it appears that all the sides of the auxiliary triangle and pentagon were drawn in at some point and then later erased, presumably so as to bring the figure into conformity with the evidence of some other source.

Not only were circles used for straight lines, but we also have at least one example of straight lines being used to represent a curved line. This rather interesting example of indifference to visual accuracy comes from one of the most fascinating manuscripts of Greek mathematics, the so-called Archimedes Palimpsest, a tenth-century manuscript containing various Hellenistic treatises including technical works by Archimedes that was

[25] In *Florence 28*, the metrically inaccurate figure with straight lines was erased and drawn over with a metrically accurate figure with curved lines. The colour of the ink makes it clear that the rectilinear lines that remain from the original are AΓ and the short part of AB that coincides with the new curved line AB.



Palimpsest 159v–158r

Reversed X-ray fluorescence calcium image

Optical multispectral image

Figure 2.10 Diagrams for Archimedes' *Method*, Proposition 12.

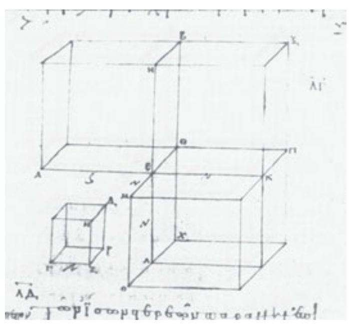
palimpsested as a prayer book some centuries later.[26] In the section of the treatise that Heiberg called *Method* 14, Archimedes discusses the metrical relationships that obtain between a prism, a cylinder and a parabolic solid that are constructed within the same square base.[27] In Figure 2.10, the base of the prism is rectangle ΕΔΓΗ, that of the cylinder is semicircle ΕΖΗ, while that of the parabolic solid is triangle ΕΖΗ. Thus, in this diagram, a parabola is represented by an isosceles triangle. Since the parabola is defined in the text by the relationship between the ordinates and abscissa, and since the triangle intersects and meets the same lines as the parabola, this was apparently seen as a perfectly acceptable representation. In this way, the triangle functions as a purely schematic representation of the parabola. Indeed, without the text we would have no way to know that the diagram represents a parabola.

Diagrams in solid geometry

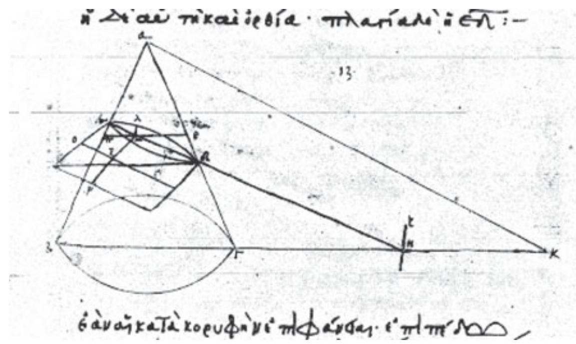
The schematic nature of ancient and medieval diagrams becomes most obvious when we consider the figures of solid geometry. Although there are some diagrams in the manuscripts of solid geometry that attempt to give a pictorial representation of the geometric objects, for the most part, they forego linear perspective in favour of schematic representation. This means that they do not serve to convey a sense of the overall spacial relationships

[26] The circuitous story of this manuscript is told by Netz and Noel 2007.

[27] This section of the *Method* is discussed by Netz, Saito and Tchernetska 2001–2. The diagram found in the palimpsest is difficult to see in the original. Here, we include two images developed by researchers in the Archimedes Palimpsest Project. The diagram is in the left-hand column of the text spanning pages 159v–158r. These images, licensed under the Creative Commons Attribution 3.0 Unported Access Rights, are available online at www.archimedespalimpsest.org.



Vatican 190



Vatican 206

Figure 2.11 Diagrams for Euclid's *Elements*, Book XI, Proposition 33 and Apollonius' *Conica*, Book I, Proposition 13.

obtaining among the objects, but rather to convey specific mathematical relationships that are essential to the argument.

Some conspicuous exceptions to this general tendency should be mentioned. For example, the diagrams for the rectilinear solids treated in *Elem.* XI and XII and the early derivations of the conic sections in the cone, in *Con.* I, appear to use techniques of linear perspective to convey a sense of the three-dimensionality of the objects. In Figure 2.11, we reproduce the diagram for *Elem.* XI.33 from *Vatican 190* and that for *Con.* I.13 from *Vatican 206*.

In all of these cases, however, it is possible to represent the three-dimensionality of the objects simply and without introducing any object not explicitly named in the proof merely for the sake of the diagram. For example, in Figure 2.1 above, the plane upon which the perpendicular is to be constructed does not appear in the manuscript figure. Hence, even in these three-dimensional diagrams, techniques of linear perspective are used only to the extent that they do not conflict with the schematic nature of the diagram. Auxiliary, purely graphical elements are not used, nor is there any attempt to convey the visual impression of the mathematical objects through graphical techniques. An example of this is the case of circles seen at an angle. Although it is not clear that there was a consistent theory of linear perspective in antiquity, ancient artists regularly drew circles as ovals and Ptolemy, in his *Geography*, describes the depiction of circles seen from an angle as represented by ovals,[28] nevertheless, in the medieval manuscripts such oblique circles are always drawn with two

[28] Knorr 1992: 280–91; Berggren and Jones 2000: 116.

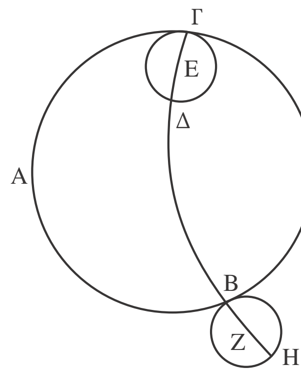


Figure 2.12 Diagrams for Theodosius' *Spherics*, Book II, Proposition 6.

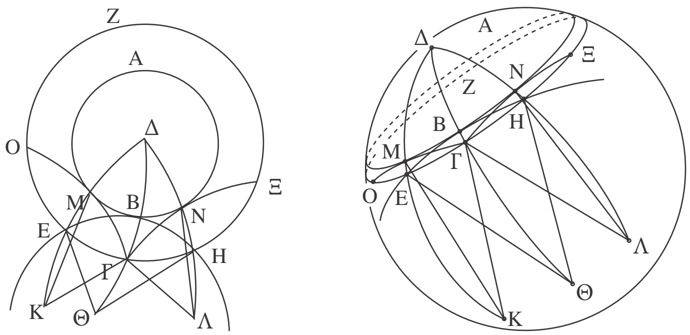
circular arcs that meet at cusps, as seen in Figure 2.11.[29] This confirms that the diagrams were not meant to be a visual depiction of the objects, but rather a representation of certain essential mathematical properties.

Likewise, in the figures of spherical geometry, if the sphere itself is not named or required by the proof, we will often see the objects themselves simply drawn free-floating in the plane, to all appearances as though they were actually located in the plane of the figure. Theodosius' *Spher*. II.6 shows that if, in a sphere, a great circle is tangent to a lesser circle, then it is also tangent to another lesser circle that is equal and parallel to the first. In Figure 2.12, we find the great circle in the sphere, ΑΒΓ, and the two equal and parallel lesser circles that are tangent to it, ΓΔ and ΒΗ, all lying flat in the same plane, with no attempt to portray their spacial relationships to each other or the sphere in which they are located.

The diagram for *Spher*. II.6 thus highlights the schematic nature of diagrams in the works of spherical geometry. The theorem is about the type of tangency that obtains between a great circle and two equal lesser circles and this tangency is essentially the only thing conveyed by the figure. The actual spacial arrangement of the circles on the sphere must either be imagined by the reader or drawn out on some real globe.[30]

[29] With respect to linear perspective, there is still a debate as to whether or not the concept of the vanishing point was consistently applied in antiquity. See Andersen 1987 and Knorr 1991. As Jones 2000: 55–6 has pointed out, Pappus' commentary to Euclid's *Optics* 35 includes a vanishing point, but it is not located in accordance with the modern principles of linear perspective.

[30] We argue elsewhere that Theodosius was, indeed, concerned with the practical aspects of drawing figures on solid globes, but that this practice was not explicitly discussed in the *Spherics*; Sidoli and Saito 2009.

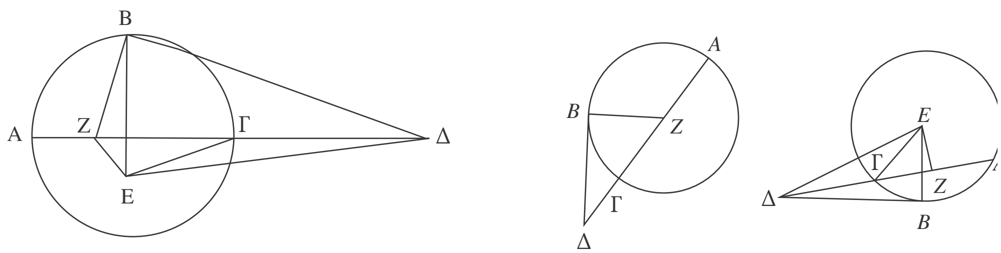


Vatican 204 Perspective diagram

Figure 2.13 Diagrams for Theodosius' *Spherics*, Book II, Proposition 15.

The schematic role of diagrams in spherical geometry becomes unmistakable when we compare the diagram of one of the more involved propositions as found in the manuscripts with one intended to portray the same objects using principles of linear perspective. *Spher*. II.15 is a problem that demonstrates the construction of a great circle passing through a given point and tangent to a given lesser circle. As can be seen in Figure 2.13, merely by looking at the manuscript diagram, without any discussion of the objects and their arrangement, it is rather difficult to get an overall sense of what the diagram is meant to represent. Nevertheless, certain essential features are conveyed, such as the conpolarity of parallel circles, the tangency and intersection of key circles, and so on. It is clear that the manuscript diagram is meant to be read in conjunction with the text as referring to some other object, either an imagined sphere or more likely a real sphere on which the lines and circles were actually drawn. It tells the reader how to understand the labelling and arrangement of the objects under discussion, so that the text can then be read as referring to these objects. The modern figure, on the other hand, by selecting a particular vantage point as most opportune and then allowing the reader to see the objects from this point, does a better job of conveying the overall spacial relationships that obtain among the objects.[31]

[31] We should point out, however, that the modern diagram in Figure 2.13, as well as being in linear perspective, employs a number of graphical techniques that we do not find in the manuscript sources, such as the use of non-circular curves, dotted lines, highlighted points, and so on.



Bodleian 301 Heiberg

Figure 2.14 Diagrams for Euclid's *Elements*, Book III, Proposition 36.

One diagram for multiple cases

In the foregoing three sections, we have discussed characteristics of the medieval diagrams that are so prevalent that they almost certainly reflect ancient practice. We turn now to characteristics that are more individual but which, nevertheless, form an essential part of the material transmission through which we must understand the ancient texts.

For a few propositions that are divided into multiple cases, we find, nevertheless, the use of a single diagram to represent the cases. There is some question about the originality of most of these, and in fact it appears that, in general, Euclid did not include multiple cases and that those propositions that do have cases were altered in late antiquity.[32] Nevertheless, even if the cases are all due to late ancient authors, they are historically interesting and the manuscript tradition shows considerable variety in the diagrams. This indicates that single diagrams for multiple cases were probably in the text at least by late antiquity and that the medieval scribes had difficulty understanding them and hence introduced the variety that we now find.

As an example, we consider *Elem*. III.36. The proposition shows that if, from a point outside a circle, a line is drawn cutting the circle, it will be cut by the circle such that the rectangle contained by its parts will be equal to the square drawn on the tangent from the point to the circle. That is, in Figure 2.14, the rectangle contained by AΔ and ΔΓ is equal to the square on ΔB. In the text, as we now have it, this is proved in two cases, first where line AΔ passes through the centre of the circle and second where it

[32] See Saito 2006: 85–90 for the case of a single figure containing two cases in *Elem*. III.25, in which the division into cases was almost certainly not due to Euclid. The Arabic transmission of the *Elements* gives further evidence for the elaboration of a single figure into multiple figures. In the eastern Arabic tradition, we find a single figure for both *Elem*. III.31 and IV.5 (see for example, *Uppsala 20*: 42v and 38v), while in the Andalusian Arabic tradition, which was also transmitted into Latin, we find multiple figures for these propositions (compare *Rabāṭ* 53: 126–8 and 145–6 with Busard 1984: 83–5 and 102–5).

does not. In Heiberg's edition, and *Vienna 31* (which often has corrected diagrams), there is an individual figure for each case. In the majority of Heiberg's manuscripts, however, there is only a single figure and it contains two different points that represent the centre, one for each case. In Figure 2.14, we reproduce the two diagrams from Heiberg's edition, which are mathematically the same as those in *Vienna 31*, and an example of the single figure taken from *Bodleian 301*. In the single diagram, as found in *Bodleian 301*, there are two centres, points E and Z, and neither of them lies at the centre of the circle. Nevertheless, if we suppose that they are indeed centres, the proof can be read and understood on the basis of this figure.

Despite these peculiarities, there are a number of reasons for thinking that this figure is close to the original on which the others were based. It appears in the majority of Heiberg's manuscripts, and the other diagrams contain minor problems, such as missing or misplaced lines, or are obviously corrected.[33] Moreover, the single figure appears to have caused widespread confusion in the manuscript tradition. In most of the manuscripts, there are also marginal figures which either correct the primary figure or provide a figure that is clearly meant for a single case.

Hence, although we cannot, at present, be certain of the history of this theorem and its figure, the characteristics and variety of the figures should be used in any analysis of the text that seeks to establish its authenticity or authorship. This holds true for nearly all of the propositions that were clearly subject to modification in the tradition.

Correcting the diagrams

Medieval scribes also made what they, no doubt, considered to be corrections to the diagrams both by redrawing the figures according to their own interpretation of the mathematics involved and by checking the diagrams against those in other versions of the same treatise and, if they were different, correcting on this basis. We will call the first practice 'redrawing' and the latter 'cross-contamination'. We have already seen the example of *Elem*. IV.16, on the construction of the regular 15-gon (see Figure 2.9), in which the scribes corrected for metrical indifference and drew the lines of the polygon as curved lines to distinguish them better from the arcs of the circumscribing circle.

[33] See Saito 2008: 78–9 for a discussion of variants of this diagram in the manuscripts of the *Elements*.

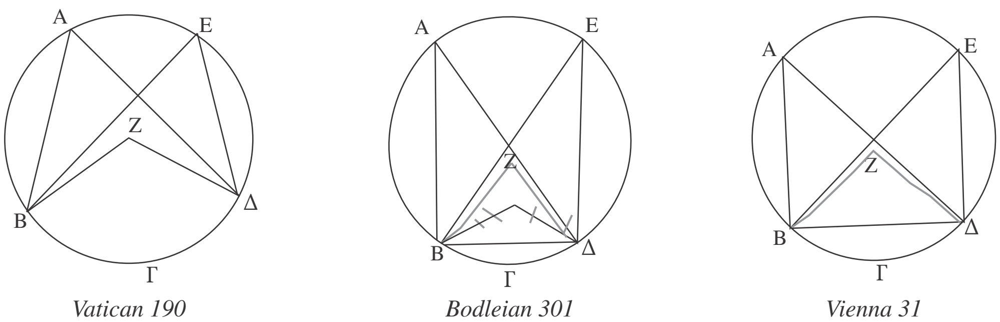


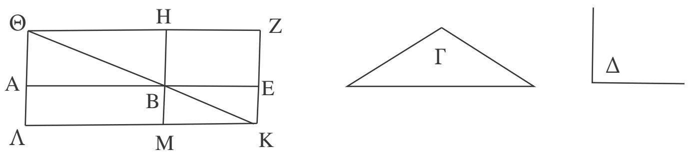
Figure 2.15 Diagrams for Euclid's *Elements*, Book III, Proposition 21.

In a number of cases, the tendencies toward overspecification and graphical indifference resulted in a figure that was difficult to interpret as a graphical object. For example, we may refer again to Figure 2.14 in which two different centres of the circle are depicted, neither of which appears to lie at the centre of the circle. In such cases, the scribes often tried to correct the figure so that it could be more readily interpreted without ambiguity.

As an example of a redrawn diagram, we take *Elem*. III.21, which proves that, in a circle, angles that subtend the same arc are equal to one another. As seen in Figure 2.15, *Vatican 190* portrays the situation by showing the two angles BAΔ and BEΔ as clearly separated from the angle at the centre, angle BZΔ, which is twice both of them. In the majority of Heiberg's manuscripts, however, as seen in *Bodleian 301* and *Vienna 31*, through overspecification the lines BA and EΔ have been drawn parallel to each other and at right angles to BΔ, so that the lines AΔ and BE appear to intersect at the centre of the circle. In the course of the proposition, however, centre Z is found and lines BZ and ZΔ are joined. In order to depict centre Z as distinct from the intersection of lines AΔ and BE, centre Z has been placed off centre, often by later hands, as seen in the examples of *Bodleian 301* and *Vienna 31*.[34] Because of the variety of the manuscript figures, it does not seem possible to be certain of the archetype, but it probably either had point Z as the intersection of AΔ and BE, as in the example of *Vienna 31*, or it had a second centre called Z but not located at the centre of the circle, as in the example of *Bodleian 301*.[35] Later readers, then, found this situation confusing and corrected the diagrams accordingly. In this case, the redrawing was done directly on top of the original figure.

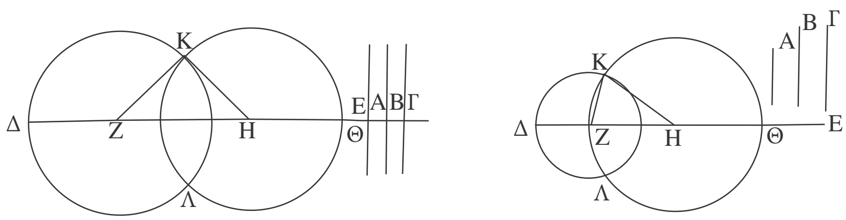
[34] See Saito 2008: 67 for further discussion of this diagram.

[35] In *Bodleian 301*, a later hand appears to have crossed out this original second centre, Z, and moved it closer to the centre of the circle.



Vienna 31

Figure 2.16 Diagrams for Euclid's *Elements*, Book I, Proposition 44.



Vatican 190 *Vienna 31*

Figure 2.17 Diagrams for Euclid's *Elements*, Book I, Proposition 22.

The redrawing, however, might also be done at the time when the text was copied and the figures drafted. In this case, the source diagram is lost in this part of the tradition. Of the manuscripts used by Heiberg, the diagrams in *Vienna 31* are often redrawn for metrical accuracy, but less often for overspecification.[36] For the diagram accompanying *Elem*. I.44, the figure in *Vienna 31* (see Figure 2.16) should be compared with that in *Vatican 190* (see Figure 2.6). As can be seen, the given area Γ is indeed the size of the parallelogram constructed on line AB, but the parallelogram is depicted as a rectangle and this is reflected in the fact that the given angle, Δ, is depicted as right. In this case, the diagram is metrically accurate but it still represents any parallelogram with a rectangle.

For an example in which the diagram in *Vienna 31* has been corrected both for metrical accuracy and overspecification, we consider *Elem*. I.22, which demonstrates the construction of a triangle with three given sides. As seen in Figure 2.17, the older tradition, here exemplified by *Vatican 190*, represents the constructed triangle with the isosceles triangle ZKH, and the given lines with the equal lines A, B and Γ. In some of the manuscripts, however, the constructed triangle ZKH is drawn as an irregular acute triangle.[37] In Figure 2.17 we see the example of *Vienna 31*, in which the

[36] As we saw in the foregoing example, in the case of *Elem*. III.21, however, the original scribe of *Vienna 31* did not correct the diagram, but a correction was added by a later hand.

[37] See Saito 2006: 118 for a larger selection of the manuscript figures. The fact that *Vatican 190* belongs to the older tradition is confirmed by the Arabic transmission.

constructed triangle is depicted as an irregular, acute triangle and all of its sides are depicted as the same length as the sides that have been given for the construction. Indeed, here we have a figure that is fully in accord with modern tastes.

For *Elem*. I.22, of the manuscripts used by Heiberg in his edition, *Bodleian 301* also depicts the constructed triangle as an irregular, acute triangle, similar to that in *Vienna 31*. The fact that *Vienna 31* and *Bodleian 301* have a similar irregular, acute triangle could either indicate that scribes in both traditions independently had the idea to draw an irregular, acute triangle and randomly drew one of the same shape or, more likely, a scribe in one tradition saw the figure in the other and copied it. There is considerable evidence that this kind of cross-contamination took place. As another example that we have already seen, we may mention *Elem*. III.21 in which both *Vienna 31* and *Bodleian 301* show a second centre drawn in freehand at some time after the original drawing was complete. Moreover, in the case of *Elem*. III.21, in *Florence 28*, which has the same primary diagram as *Bodleian 301*, we find a marginal diagram like that in *Vatican 190*, while in *Bologna 18–19*, which has the same primary diagram as *Vatican 190*, we find a marginal diagram like that in *Florence 28*.

Hence, as well as being used as a cross-reference for the primary diagram, the figures of a second or third manuscript were often drawn into the margin as a secondary diagram. Although we are now only at the beginning stages of such studies, this process of cross-contamination suggests the possibility of analysing the transmission dependencies of the diagrams themselves without necessarily relying on those of the text. Indeed, there is now increasing evidence that the figures, like the scholia, were sometimes transmitted independently of the text.[38] The process of cross-contamination has left important clues in the manuscript sources that should be exploited to help us understand how the manuscript diagrams were used and read.

Ancient and medieval manuscript diagrams

Since the ancient and medieval diagrams are material objects that were transmitted along with the text, we should consider the ways they were copied, read and understood with respect to the transmission of the text.

[38] For examples of the independent transmission of the scholia of Aristarchus' *On the Sizes and Distances of the Sun and Moon* and Theodosius' *Spherics* see Noack 1992 and Czinczenheim 2000. The independent transmission of the manuscript figures for Calcidius' Latin translation of Plato's *Timaeus* has been shown by Tak 1972.

Although, for the most part, the text and diagrams appear to have been copied as faithfully as possible, at various times in the Greek transmission, and perhaps more often in the Arabic tradition, mathematically minded individuals re-edited the texts and redrew the diagrams.

For the most part, in Greek manuscripts the diagrams are drawn into boxes that were left blank when the text was copied, whereas in the Arabic and Latin manuscripts the diagrams were often drawn by the same scribe as copied the text, as is evident from the fact that the text wraps around the diagram. Nevertheless, except during periods of cultural transmission and appropriation, the diagrams appear to have been generally transmitted by scribes who based their drawings on those in their source manuscripts, despite the fact that the diagrams can largely be redrawn on the basis of a knowledge of the mathematics contained in the text. Hence, the diagrams in the medieval manuscripts give evidence for two, in some sense conflicting, tendencies: (1) the scribal transmission of ancient treatises based on a concept of the sanctity of the text and (2) the use of the ancient works in the mathematical sciences for teaching and developing those sciences and the consequent criticism of the received text from the perspective of a mathematical reading.

For these reasons, when we use the medieval diagrams as evidence for ancient practices, when we base our understanding of the intended uses of the diagrams on these sources, we should look for general tendencies and not become overly distracted by the evidence of idiosyncratic sources.

Diagrams and generality

The two most prevalent characteristics of the manuscript diagrams are what we have called overspecification and indifference to visual accuracy. The consistent use of overspecification implies that the diagram was not meant to convey an idea of the level of generality discussed in the text. The diagram simply depicts some representative example of the objects under discussion and the fact that this example is more regular than is required was apparently not considered to be a problem. In the case of research, discussion or presentation, a speaker could of course refer to the level of generality addressed by the text, or, in fact, could simply redraw the diagram. The indifference to visual accuracy implies that the diagram was not meant to be a visual depiction of the objects under discussion but rather to use visual cues to communicate the important mathematical relationships. In this sense, the diagrams are schematic representations. They help the reader navigate the thicket of letter names in the text, they relate the letter names to specific objects and they convey the most relevant mathematical characteristics of those objects. Again, in the course of research, discussion or presentation, a speaker could draw attention to other aspects of the objects that are not depicted, or again could simply redraw the diagrams.

We have referred to the fact that the diagrams could have been redrawn in the regular course of mathematical work, and, in fact, the evidence of the medieval transmission of scientific works shows that mathematically minded readers had a tendency to redraw the diagrams in the manuscripts they were transmitting.[39] This brings us to another essential fact of the manuscript diagrams. They were conceived, and hence designed, to be objects of transmission, that is, as a component of the literary transmission of the text. Nevertheless, the extent to which mathematics was a literary activity was changing throughout the ancient and medieval periods and indeed the extent to which individual practitioners would have used books in the course of their study or research is an open question. This much, however, is virtually certain: the total number of people studying the mathematical sciences at any time was much greater than the number of them who owned copies of the canonical texts. Hence, in the process of learning about and discussing mathematics the most usual practice would have been to draw some temporary figure and then to reason about it.

In fact, there is evidence that, contrary to the impression of the diagrams in the manuscript tradition, ancient mathematicians were indeed interested in making drawings that were accurate graphic images of the objects under discussion. We argue elsewhere that the diagrams in spherical geometry, as represented by Theodosius' *Spherics*, were meant to be drawn on real globes and that the problems in the *Spherics* were structured so as to facilitate this process.[40] As is clear from Eutocius' commentary to Archimedes' *Sphere and Cylinder*, Greek mathematicians sometimes designed mechanical devices in order to solve geometric problems and to draw diagrams accurately.[41] In contrast to the triangular parabola we saw in *Method* 14, Diocles, in *On Burning Mirrors*, discusses the use of a horn ruler to draw a graphically accurate parabola through a set of points.[42] Hence, we must distinguish between the diagram as an object of transmission and the diagram as an instrument of mathematical learning and investigation.

[39] See Sidoli 2007 for some examples of mathematically minded readers who redrew the figures in the treatises they were transmitting.
[40] Sidoli and Saito 2009.
[41] Netz 2004: 275–6 and 294–306.
[42] Toomer 1976: 63–7.

In fact, we will probably never know much with certainty about the parabolas that were drawn by mathematicians investigating conic theory or the circles that were drawn on globes by teachers discussing spherical geometry. Nevertheless, insofar as mathematical teaching and research are human activities, we should not doubt that the real learning and research was done by drawing diagrams and reasoning about them, not simply by reading books or copying them out. Hence, the diagrams in the manuscripts were meant to serve as signposts indicating how to draw these figures and mediating the reader's understanding of the propositions about them.

We may think of the manuscript diagrams as schematic guides for drawing figures and for navigating their geometric properties. In some cases, and for individuals with a highly developed geometric imagination, these secondary diagrams might simply be imagined, but for the most part they would actually have been drawn out. The diagrams achieve their generality in a similar way as the text, by presenting a particular instantiation of the geometric objects, which shows the readers how they are laid out and labelled so that the readers can themselves draw other figures in such a way that the proposition still holds. Hence, just as the words of the text refer to any geometric objects which have the same conditions, so the diagrams of the text refer to any diagrams that have the same configurations.

We may think of the way we use the diagram of a difficult proposition, such as that of the manuscript diagram for *Spher*. II.15 in Figure 2.13, in the same way that we think of the way we use the subway map of the Tokyo Metro.[43] We may look at the manuscript diagram in Figure 2.13 before we have worked through the proposition to get a sense of how things are laid out, just as we may look at the Tokyo subway map before we set out for a new place, to see where we will transfer and so forth. Although this may help orientate our thinking, in neither case does it fully prepare us for the actual experience. The schematic representation of the sphere in Figure 2.13 tells us nothing of its orientation in space, an intuition of which we will need to develop in order to actually understand the proposition. The Tokyo subway map tells us nothing about trains, platforms and tickets, all of which we will need to negotiate to actually go anywhere in Tokyo. In both cases, the image is a schematic that conveys only information essential to an activity that the reader is assumed to be undertaking.

There is, however, also an important distinction. The Tokyo subway map points towards a very specific object – or rather a system of objects that are

[43] The Tokyo subway map, in a number of different languages, can be downloaded from www.tokyometro.jp/e/.

always in flux, and probably not nearly as determinate as we would like to believe – nevertheless, a system of objects with a very specific locality and temporality. A Tokyo subway map is useless for Paris. If it was drawn this year, it will contain stations and lines that did not exist ten years ago and ten years from now it will again be out of date. The manuscript diagram in Figure 2.13, however, has no such specificity. It can refer to any sphere and does. Anyone who wants to draw a great circle on a sphere tangent to a given line and through a given point can use this diagram in conjunction with its proposition to do so. In the centuries since this proposition was written, a great many readers must have drawn figures of this construction – on the plane, on the sphere, in their mind's eye – and this diagram, strange and awkward as it is, somehow referred to all of them. It is in such a way that the overspecified, graphically inaccurate diagrams that we find in the manuscript tradition achieve the generality for which they were intended.

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