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HIPPARCHUS AND THE ANCIENT METRICAL METHODS ON THE SPHERE

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Ptolemy solves problems in spherical astronomy involving the arcs of great circles by means of mathematical techniques that modern scholars refer to as spherical trigonometry. In fact, spherical *tetrapleurometry* would be a more accurate appellation. These methods are based on a fundamental theorem, attributed to Menelaus, which relates the arcs of great circles forming a concave quadrilateral.[1] Consider Figure 1. The two most important versions of the basic theorem state that if the arcs of the quadrilateral are not greater than 180°, and $Crd(x)$ is the chord subtending arc x, then

$$\frac{Crd(2\widehat{GA})}{Crd(2\widehat{EA})} = \frac{Crd(2\widehat{GD})}{Crd(2\widehat{DZ})} \times \frac{Crd(2\widehat{ZB})}{Crd(2\widehat{BE})} \tag{1}$$

and

$$\frac{Crd\,(2\widehat{GE})}{Crd\,(2\widehat{EA})} = \frac{Crd\,(2\widehat{GZ})}{Crd\,(2\widehat{DZ})} \times \frac{Crd\,(2\widehat{DB})}{Crd\,(2\widehat{BA})}\,. \tag{2}$$

In general, it is necessary to know five terms in order to solve for the sixth term. However, Ptolemy shows that there are certain cases where it is sufficient to know four terms and the sum or difference of the other two.[2] For the purposes of this paper I shall call the techniques built up around this figure and these compound ratios *ancient spherical trigonometry*.

The standard current view holds that Hipparchus did not possess the methods of ancient spherical trigonometry and that these were later developed by Menelaus. This view follows Neugebauer who simply says that it is evident from "everything we

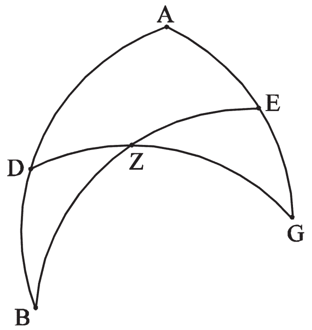

FIG. 1. Figure for the so-called Menelaus Theorem, the fundamental theorem of ancient spherical trigonometry.

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know about Greek 'Spherics' and Menelaus' role in it".[3] Both Björnbo and Rome, however, through their investigations of Greek spherics, had come to the opinion that the ancient trigonometrical methods on the sphere went back at least as far as Hipparchus, and that Hipparchus had used them in his investigation of rising times.[4] The basis of the claim that Menelaus was the founder of ancient spherical trigonometry is the fact that the fundamental theorem of these methods is found in the surviving texts of Menelaus's *Spherics* as theorem *Spherics* III 1.[5] The relationship between what survives of Menelaus's text in Arabic and the Greek tradition is not, however, nearly as straightforward as has previously been assumed.[6]

In fact, there are a number of reasons for being suspicious of the standard view of Menelaus's role in the history of spherics. (1) There is nothing in the mathematics or presentation of the fundamental theorem that compels us to believe it could not have been written before Menelaus's time. (2) It makes use only of rudimentary lemmas that might well have been assumed by Euclid in his more advanced works and relies on no other theorem in the *Spherics*. (3) The lemmas as they appear in the Arabic text may well have found their way in via Ptolemy and not the other way around. (4) The theorem makes no use, or mention, of spherical triangles, the hallmark of Menelaus's approach to spherical geometry. (5) The theorem appears as the first proposition in the third book and introduces a number of metrical theorems that all concern spherical triangles, which are the proper subject of the text. The first proposition in a book is usually a construction or a simple auxiliary theorem, one that may be well known and introduces the more advanced material of the book. It may be fundamental, but it is almost never exciting. *Spherics* III 1 serves precisely this subsidiary function in the context of Book III as a whole.

Quite apart from these circumstantial considerations, however, a close reading of Hipparchus's *Commentary on the* Phaenomena *of Aratus and Eudoxus* provides evidence for two calculations that I will argue can be reconstructed using ancient spherical trigonometry. A reconstruction, however, is not to be taken as evidence. Hence, I will argue that in this case the analemma methods as they are handed down in the ancient and medieval sources cannot be used to make the necessary calculations. Moreover, it will be found that the trigonometrical reconstructions make use of a lemma that is found in Ptolemy's *Almagest* but never used. This unmotivated lemma gives us reason to believe that Ptolemy took his material from a source that still required it and that when he revised this previous work for the purpose of writing the *Almagest* he neglected to excise the now unnecessary lemma. All these contingencies conspire to argue that the source from which Ptolemy drew the fundamental mathematics for his treatment of spherical astronomy went back to Hipparchus's work on simultaneous risings.[7]

Hipparchus's Commentary *II 2, 25–28*

There are numerous places in Hipparchus's *Commentary*, especially in the second, systematic part, where we can suppose that Hipparchus proceeded by metrical analysis but where he may also have used nomographic techniques or manipulations of a star globe. There is one place, however, where we can safely assume that metrical analysis was used because he tells us as much.

The phrase "by means of lines", *dia tōn grammōn*, was first recognized as technical terminology by Luckey.[8] It is used by Ptolemy extensively in the *Almagest* where it generally means "through the techniques of metrical analysis".[9] It is again used by Ptolemy in the *Analemma* where it denotes a problem that is soluble through metrical analysis, in this case using plane trigonometry.[10] Solutions that are described as *dia tōn grammōn*, or with the related adjective *grammikos*, are often contrasted, by both Ptolemy and Theon, to solutions that proceed by means of tables. Tables are generally constructed *dia tōn grammōn*, but once constructed they save the user the hassle of solving a certain set of problems *dia tōn grammōn* by providing a ready numerical algorithm.[11]

Our first evidence of the phrase, however, is a use by Hipparchus in the *Commentary*.[12] The context is a discussion of the points of the ecliptic that rise and culminate simultaneously with the setting of the "southernmost star in the left foot of the Bear Keeper" (υ Boo).[13] Hipparchus gives the values of the points of the ecliptic at the principal points of the local coordinate system and says that he demonstrated these values *dia tōn grammōn* in other works devoted to such things. Since the passage is not long, it will be useful to translate it in full. I have numbered the sentences for later reference. Hipparchus's *Commentary* II 2, 25–28 reads as follows:[14]

> [1] Now, let the southernmost star in the left foot of the Bear Keeper [υ Boo] be imagined as situated on the horizon toward the west. [2] This star is, however, $27\frac{1}{3}°$ north of the equator, where the circle through the poles is 360°. [3] For this reason, the circle drawn through the said star parallel to the equator will clearly have an arc above the Earth of approximately $15 - \frac{1}{20}$ parts, where a whole circle is 24 parts. [4] Half of the said arc, from the meridian to the setting, is therefore approximately $7\frac{1}{2}$ of these parts. [5] But the said star is situated at about 1° ♎ along the parallel circle. [6] When the same star is setting, $23\frac{1}{3}°$ ♑[15] along the parallel circle must culminate. [7] But when this point culminates, 22° ♑ on the ecliptic must culminate. [8] But when 22° ♑ on the ecliptic culminates, around 6° ♉ rises on the underlying horizon. [9] For each of these statements was demonstrated by means of lines (*dia tōn grammōn*) in the treatises we composed generally concerning these things. [10] Therefore the left foot of Bootes sets opposite 6° ♉.

This passage derives three sets of values from the position of a star, given in equatorial coordinates. The first set of values (in [3], [4] and [6]) are the results of the *preliminary problem*, the second set (in [7]) the results of the *primary problem*, and the final set (in [8]) the results of the *secondary problem*.

The essential claim of this passage is that if the position of a star is given in equatorial coordinates then Hipparchus has mathematical methods for determining the points of the zodiac at the principal positions of the local coordinate system. Here, Hipparchus simply gives his results and refers the curious reader to other works where his methods are presented in more detail. It is possible that the special right ascension of υ Boo ($\alpha = 180°$) allows a shortcut to the values given in [7] and [8]. However, I take [9] to indicate that Hipparchus elsewhere set out general methods that would solve these kinds of problems, not that he there solved these particular problems. It will be useful to follow through Hipparchus's argument in detail.

After setting the star on the western horizon in [1], Hipparchus gives the star's declination in the equatorial coordinate system in [2]: $\delta = 27\frac{1}{3}°$ N.[16] From this, Hipparchus calculates the arc of the equatorial parallel, the δ-circle, through the star which is above the horizon and gives its value in [3] as $15 - \frac{1}{20}$ twenty-fourths of a circle, or $224\frac{1}{4}°$. In this passage, Hipparchus uses two different units to measure arc length: divisions of the circle into 360°, and 24 parts, or 15° arcs. Dropping the $-\frac{1}{20}$ twenty-fourths, or rounding up $\frac{3}{4}°$, in [4], Hipparchus approximates half of the arc of the δ-circle above the horizon as $7\frac{1}{2}$ twenty-fourths of a circle, or $112\frac{1}{2}°$. Neugebauer takes the calculation of the value stated in [3] to be what Hipparchus refers to as having determined *dia tōn grammōn* and has demonstrated that it can be done using analemma methods.[17] The analemma methods are the best, and probably the only, ancient means of performing this calculation. In fact, however, this is merely the preliminary calculation. In [5], Hipparchus introduces the right ascension of υ Boo, pointing out that the star lies on its δ-circle at around 1°♎, in other words, $\alpha = 180°$. Hipparchus designates arc length on any circle parallel to either the ecliptic or the equator by a system of signs and degrees, where the following signs denote 30° multiples. In this way, a great circle drawn through the poles of one of the principal great circles will cut all of the parallel circles at the same sign and degree as it cuts the principal great circle. Moreover, he starts each sign with 1°, so we convert from his degree positions to ours by subtracting 1° and adding the multiple of 30° indicated by the sign.[18] Hence, as [6] states, when υ Boo sets, $23\frac{1}{2}°$♑ of the equatorial parallel, and of the equator, culminates. In other words, $180° + 112\frac{1}{2}° = 292\frac{1}{2}°$, or $23\frac{1}{2}°$♑. In [7] and [8] Hipparchus tells us that when this degree of the parallel circle culminates, 22°♑ of the ecliptic culminates and 6°♉ of the ecliptic rises. He follows these claims, in [9], with the remark, "For each of these statements was demonstrated *dia tōn grammōn* in the treatises we composed generally concerning such things".[19] The primary and secondary problems are the determination of the degrees of the ecliptic rising and culminating, given the culminating point of the equator.

Hipparchus may include the preliminary calculation, that of the arc of the δ-circle that is above the horizon, as one of the things that he showed *dia tōn grammōn*, but more particularly he refers to the determination of the culminating and rising points of the ecliptic. Both of these problems can be solved with a table of rising times. For the primary calculation, we use the column for *sphaera recta* and allow the horizon to serve as the meridian.[20] For the secondary calculation, we follow a procedure like that given by Ptolemy in *Almagest* II 9.[21] His use of the phrase *dia tōn grammōn*, however, makes it clear that Hipparchus did not proceed in this manner.

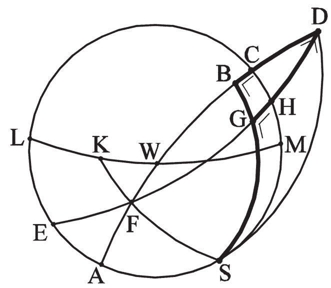

Fig. 2. Figure for the reconstruction of the primary problem, based on the figure in *Almagest* II 11. Circle *ASML* is the meridian, *AWCD* is the equator, *EFHD* is the ecliptic, *W* is the west point, *F* the autumnal equinox, *G* the winter solstice, and *C* is point of the equator which is culminating, $23\frac{1}{2}°$ ♑ or $\alpha = 293;30°$. The problem is to determine the degree of the ecliptic that is rising at point *H*.

The analemma methods are the best ancient methods for making determinations on parallel circles, but when it comes to calculating arc lengths on oblique great-circles they are not generally useful.

The next three sections lay out the reconstruction of the solution to the primary problem using ancient spherical trigonometry and show why the analemma is unable to solve problems of this sort. A penultimate section discusses the solution to the secondary problem, the details of which are too similar to the primary problem to warrant full treatment.

Reconstruction of the Approach to the Primary Problem

The primary problem is the determination of the culminating point of the ecliptic given that $23\frac{1}{2}°$ ♑ of the equator, our $\alpha = 292;30°$, is culminating. This section will show that the ancient trigonometrical methods are able to make this calculation. Consider Figure 2. Let *ALCM* be the meridian, *LM* the horizon, *ACD* the equator, and *EHD* the ecliptic. The arcs $\widehat{CD}$ and $\widehat{HD}$ are the continuation of the equator and the ecliptic on the other side of the meridian. This method of drawing the figure follows what we find in the *Almagest*.[22]

Since the star is setting, we let *K* represent υ Boo so that *W* is the west point and *S* the south pole. Hence, the movement of the sphere is from *B* toward *W*, so that the order of the following signs is in the opposite direction. Since, by the preliminary calculation, 23;30° ♑ on the equator is culminating, 1° ♑ on the equator will be 22;30° in advance of *C*. Let *B* represent this point so that $\widehat{BC} = 22;30°$. Hence, if we join the arc of the solstitial colure $S\widehat{G}B$, point *G* will be the winter solstice, 1° ♑, and angle SBD = angle $SGD = 90°$. Then *D* is the vernal and *F* the autumnal equinox. We join $\widehat{SD}$ so that angle $SDB = 90°$. Hence, $\widehat{SB} = \widehat{BD} = \widehat{GD} = 90°$, and $\widehat{BG} = \varepsilon = 23;51,20°$.[23] The fundamental theorem of ancient spherical trigonometry, given in Equation 2, states[24] that

$$\frac{Crd(2\overset{\frown}{CD})}{Crd(2\overset{\frown}{CB})} = \frac{Crd(2\overset{\frown}{HD})}{Crd(2\overset{\frown}{HG})} \times \frac{Crd(2\overset{\frown}{SG})}{Crd(2\overset{\frown}{SB})} .$$

Since we are solving for $\overset{\frown}{HG}$ we rewrite as

$$\frac{Crd(2\overset{\frown}{HG})}{Crd(2\overset{\frown}{HD})} = \frac{Crd(45°)}{Crd(135°)} \times \frac{Crd(132;17,20°)}{Crd(180°)} = 0;22,43,44,19,04. \quad (3)$$

Since we also know that $\overset{\frown}{HG} + \overset{\frown}{HD} = 90°$, we can solve for $\overset{\frown}{HG}$. We will take up the completion of this calculation after we have seen how this situation would be modelled in the analemma.

Modelling the Primary Problem in the Analemma

It is difficult to give a complete description of the ancient analemma techniques because we have so few examples of the analemma in mathematical practice. The best we can do is form a picture of the practice based on the ancient and medieval evidence. Ptolemy and Heron are the only ancient authors who preserve texts treating the analemma as a tool for mathematical investigation.[25] Most discussions of the analemma as a calculating device are modern reconstructions as opposed to explications of ancient texts.[26]

The central device of the analemma is the translation of arcs of the sphere into arcs on the plane of the figure. This translation takes place either through rotation or superposition. Both of these are mathematically equivalent to orthogonal projection. Orthogonal projection allows a point on the sphere to be specified in the local frame of reference. In practice, this means that in order to find a relation between two frames of reference, the coordinate circles in these two systems must be perpendicular. Hence, if the equatorial or ecliptic frames of reference can be reduced to orthogonal projection, a point on the sphere can be specified and basic problems of spherical astronomy can be solved. The analemma is useful for (1) relating the equatorial system to the system of the local horizon, (2) relating the ecliptic system to either of these two systems when an equinox is rising, or (3) making determinations on parallel circles. In cases where the ecliptic is oblique to the meridian, however, the analemma is not generally useful.[27] Examples of problems solved using (1) are Heron's determination of the distance between two cities based on simultaneous lunar eclipse observations, and medieval Arabic determinations of the *qibla*.[28] By considering individual points of the ecliptic, Ptolemy's *Analemma* uses (2) to determine the local coordinates of a given point of the ecliptic at a given time. It is not generally possible to use the analemma to transform between ecliptic and equatorial coordinates, but because (1) allows us to transform between the equatorial and local coordinate systems, (2) allows us to specify the equatorial coordinates for points of the ecliptic. It should be noted that this solves the inverse of the primary problem: to determine the culminating point of the equator given the culminating point of the ecliptic. Although there are no ancient examples of (3), this would seem to be an obvious use for the analemma since it allows for the solution of problems that cannot be handled by ancient spherical trigonometry. Neugebauer and Wilson have given reconstructions that solve problems through this use of the analemma.[29]

It can be shown that the analemma is of no use for calculating the culminating point of the ecliptic given the degree of the equator culminating. In fact, in this case, setting up the analemma figure would involve knowing what we are trying to find. Consider Figure 3. Let *ADBC* be the meridian with *AB* the gnomon and *DC* the diameter of the horizon. We let the point *S* be υ Boo setting so that *HG* is the diameter of the δ-circle of υ Boo. Since we are given the equatorial coordinates of *S*, we can calculate $\widehat{SH} = 112;30°$ using the analemma as above.[30] In order to make use of the analemma, we need to draw the ecliptic on the figure. Let *LK* be the diameter of the ecliptic and assume, for the sake of the argument, that we can determine angle *LTF* by analemma techniques.[31] Let point *M* be the autumnal equinox, the intersection of the ecliptic and the equator. For the analemma to be of any use, we need to rotate the ecliptic into the plane of the figure so that *M* maps to some point *P*. We will then solve for $\widehat{KP}$ which, since *P* is the autumnal equinox, will be sufficient to solve the problem. Unfortunately, we cannot set up the analemma figure without assuming a value essentially equivalent to this arc.

Under normal circumstances, in every analemma preserved from the ancient or the medieval periods, the two circles that we wish to compare are perpendicular to one another and the figure allows us to exploit this feature to determine the position of a given point on the two circles. Take for example υ Boo at point *S*. Consider the semicircle *GSH* rotated into its proper position so that it is perpendicular to the plane of the figure. Then *S* is on a point above *I* at distance *SI*. Then the position of

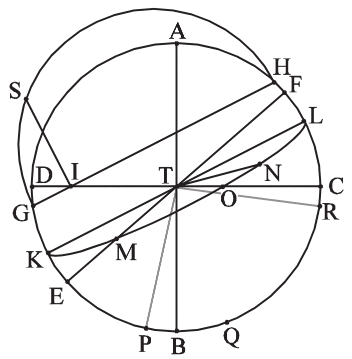

FIG. 3. Figure for the application of the analemma to Hipparchus's calculation, based on the figures of Ptolemy's *Analemma*. Circle *ADBC* is the meridian, *DC* is the diameter of the horizon, *AB* the gnomon, *FE* is the diameter of the equator, *HG* the diameter of the δ-circle of υ Boo, *LK* is the diameter of the ecliptic, and semicircle *LNMK* is the arc of the ecliptic west of the meridian. Point *S* is υ Boo, *M* is the autumnal equinox, *N* the winter solstice, *O* the intersection of the ecliptic and the horizon, and *L* the point of the ecliptic that is culminating.

the star can easily be determined on either the horizon or its δ-circle by dropping perpendiculars. In our case, however, the semicircle of the ecliptic, *LNMK*, is not perpendicular to the meridian and the only thing we know about its position is the arc from the autumnal equinox to the horizon.

Let *M* be the autumnal equinox, *O* the intersection of the ecliptic and the horizon, and *N* the winter solstice. Then, by the preliminary calculation on the analemma, we know that $\widehat{MT}$ on the equator is 22;30°, as above. Let the ecliptic be rotated into the plane of the figure so that *M* maps to *P*, *O* to *Q* and *N* to *R*. Then $\widehat{RL}$ will be the arc distance between the culminating point of the ecliptic and the winter solstice. Unfortunately, the analemma figure gives us no help in determining any of the points *P*, *Q* or *R* on the figure. In fact, if we knew the position of any one of these points in relation to the cardinal points of the meridian, the problem would already be solved. The reason that the analemma is of no use in this situation is simple: the ecliptic is not perpendicular to the meridian.

This discussion has shown that although the present problem can be modelled on the analemma figure, the values that we need to make the model exact are the same values that we are using the model to find. Hence, the analemma is not a serviceable tool for problems of this sort.

Reconstruction of Hipparchus's Calculation

In *Almagest* I 13, in the course of laying the foundation of ancient spherical trigonometry, Ptolemy gives two short lemmas of metrical analysis that he himself never needs. These lemmas increase the scope of the fundamental compound ratios of the trigonometrical methods. Since there are six terms in the compound ratio, five terms will generally be required to solve for any one term. In fact, in the *Almagest* five terms are always used to solve for an unknown term.[32] It is clear, however, that it will not always be possible to obtain five terms as given. The lemmas given in *Almagest* I 13 provide for certain cases where four terms and either the sum or the difference of the two unknown terms are given.

Only the first of these lemmas concerns us here. Consider Figure 4. The first lemma states that if $\widehat{AG}$ and the ratio $Crd(2\ \widehat{AB})/Crd(2\ \widehat{BG})$ are given, then each of

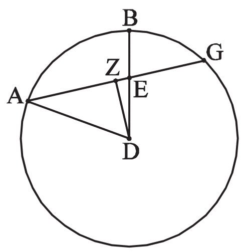

FIG. 4. Figure for the first lemma of numerical analysis in *Almagest* I 13. This is the lemma required to carry out Hipparchus's calculation.

$\widehat{AB}$ and $\widehat{BG}$ will be given. Join *AG* and *BD* and drop *DZ* perpendicular to *AG*. Now, if $\widehat{AG}$ is given, then angle *ADZ* is given, because it is half of $\widehat{AG}$ [*Data* 2]. Hence, triangle *ADZ* will be given [*Data* 40]. Now, since chord *AG* is given [by the Chord Table, *Almagest* I 11], and $AE/EG = Crd(2\,\widehat{AB})/Crd(2\,\widehat{BG})$ is given [this equality is established in the previous theorem in *Almagest* I 13], then *AE* will be given [*Data* 7], and so will *ZE* by subtraction [*Data* 2 and 4]. Hence, since *DZ* is given in the right triangle *EDZ* [*Elements* I 47], angle *EDZ* will be given [*Data* 1 and 41],[33] and hence the whole of angle *ADB* will be given [*Data* 40]. Therefore, $\widehat{AB}$ will be given, and $\widehat{BG}$ will be given by subtraction [*Data* 4].[34]

In our case, we have $\widehat{AG} = 90°$ and, by Equation 3,

$$\frac{Crd(2\widehat{AB})}{Crd(2\widehat{BG})} = 0;22,43,44^{o}.$$

It is required to find $\widehat{AB}$ and $\widehat{BG}$. Following Ptolemy's convention, we will set $\widehat{AD}$ = 60p. *AG* is given by the Chord Table as 84;51,09p, so that *AZ* = 42;25,34p. *Data* 6 and 7 show us how to find *AE*. We set

$$\frac{x}{y} = \frac{AE}{AG} = \frac{Chd(2\widehat{AB})}{Chd(2\widehat{BG})},$$

where *x* is given, say *x* = 1. Thus *y* is given and, since *x*/*y* = *AE*/*EG*, and *AE*/*AG* = 1/(1+*y*) [*Elements* V 18], therefore

$$AE = \frac{AG}{1+y} = 61;32,25\text{p}.$$

Then, $$EZ = AE - AZ = 19;06,50\text{p},$$

but $$DZ = \sqrt{AD^2 + AZ^2} = 42;25,35\text{p},$$

so that $$DE = \sqrt{DZ^2 + EZ^2} = 46;31,59\text{p}.$$

Although *Data* 41 tells us that the triangle *ZDE* is known, it gives us no help in assigning values to the triangle's angles. To find the value of angle *ZDE* we set *DE* = 120p and we use the Chord Table to find $\widehat{ZE}$ in the circle around triangle *ZDE*; this will be twice angle *ZDE*, hence *ZDE* = 24;15°. For angle *ADZ*, we set *AD* = 120p and again use the Chord Table. We find that angle *ADZ* = 45°. Hence,

$$\widehat{AB} = \text{angle } ZDE + \text{angle } ADZ = 69;15° \tag{4}$$

and, by subtraction, $$\widehat{BG} = \widehat{AG} - \widehat{AB} = 20;45°. \tag{5}$$

If we return with the values given by Equations 4 and 5 to the original problem we see that $\widehat{HD} = 69;15°$ and $\widehat{HG} = 20;45°$, see Figure 2. Hence, the culminating point of the ecliptic is 20;45° in advance of the winter solstice. Therefore, since the winter solstice is 1° ♑, the culminating point of the ecliptic by our calculation is 21;45° ♑. This is very close to Hipparchus's value of 22° ♑.

The quarter-degree discrepancy need not worry us since there are a number of possible sources for this disparity. I carried out these calculations according to Ptolemy's chord table and the system of sexagesimal fractions, while we do not know the details of Hipparchus's chord table, and he seems to have preferred to work with fractions.[35] The calculations, as we have seen, are involved, and repeated roundings may well account for this disagreement: I carried out these calculations on a computer to five sexagesimal places, while Hipparchus carried out his calculations by hand and very likely rounded the fractions as he went. Most importantly, Hipparchus does not seem to have been interested in a high degree of precision in this passage. We saw above, in the preliminary calculation of the arc of the δ-circle above the horizon, that he rounded up by 0;45°. If this is taken as an indication of the quantity of rounding Hipparchus saw as permissible in calculations of this sort, then an agreement of 0;15° is quite good.

This reconstruction has shown that Hipparchus uses the phrase *dia tōn grammōn* in a technical sense to signify a calculation that is carried out by means of metrical analysis. This is the same technical meaning that it has in Ptolemy. Since the usage we have in Hipparchus is the earliest in the Greek corpus, the technical meaning likely originated around his time and was appropriated by later writers in the same tradition.

Discussion of the Secondary Problem

The secondary problem can be solved along lines similar to the primary problem by a double application of the fundamental theorem of ancient spherical trigonometry. Since we are now to determine the degree of the ecliptic rising, we will turn our attention to the other side of the sphere. Consider Figure 5. Let *AMNL* be the meridian, *LM* the horizon, *AEDC* the equator, and *FQDH* the ecliptic. The arcs $\widehat{CB}$ and $\widehat{GH}$

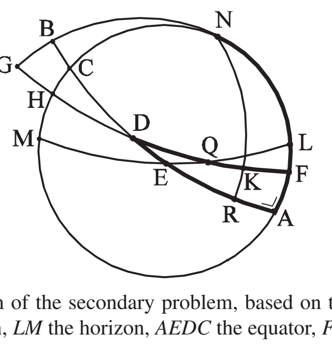

FIG. 5. Figure for the reconstruction of the secondary problem, based on the figure in *Almagest* II 11. Circle *AMNL* is the meridian, *LM* the horizon, *AEDC* the equator, *FQDH* the ecliptic, *E* the east point, *D* the vernal equinox, *G* the winter solstice, *K* the summer solstice, and *C* the point of the equator that is culminating, $23\frac{1}{2}°$ ♑ or $\alpha = 293;30°$. The problem is to determine the degree of the ecliptic that is rising at point *Q*.

are the continuation of the equator and the ecliptic on the other side of the sphere.

Let E be the east point, N the north pole, G the winter solstice, D the vernal equinox, and K the summer solstice. The great circle $GBNKR$ will be the solstitial colure. The sphere will rotate about the celestial poles from A toward C and the order of the signs will be from H toward F. The arcs $\widehat{RA}$ and $\widehat{KF}$ are symmetrical, and hence equal, to the arcs $\widehat{BC}$ and $\widehat{GH}$. Since Q is the point of the ecliptic that is rising the problem will be solved if we can find $\widehat{DQ}$.

With respect to the convex quadrilateral $NADK$, a variant of the fundamental theorem[36] of ancient spherical trigonometry asserts that

$$\frac{Crd\,(2\widehat{NF})}{Crd\,(2\widehat{FA})} = \frac{Crd\,(2\widehat{NK})}{Crd\,(2\widehat{KR})} \times \frac{Crd\,(2\widehat{RD})}{Crd\,(2\widehat{AD})}$$

where $\widehat{KR} = \varepsilon = 23;51,20°$, $\widehat{NK} = 90° - \varepsilon$, $\widehat{RD} = 90°$ and $\widehat{AD} = 90° + \widehat{RA} = 90° + 22;30°$. Moreover, $\widehat{NF} + \widehat{FA} = 90°$. Hence, since the ratio $Crd(2\widehat{NF})/Crd(2\widehat{FA})$ and the sum of the two arcs are given, the two arcs $\widehat{NF}$ and $\widehat{FA}$ are given individually by the lemma demonstrated in the previous section. If we assume the elevation of the north pole to be the ancient value for that at Rhodes, $\phi = 36°$, $\widehat{FL}$ will be given and we can complete the problem.

With respect to the convex quadrilateral $LADQ$, the fundamental theorem asserts that

$$\frac{Crd\,(2\widehat{DQ})}{Crd\,(2\widehat{QF})} = \frac{Crd\,(2\widehat{ED})}{Crd\,(2\widehat{AE})} \times \frac{Crd\,(2\widehat{LA})}{Crd\,(2\widehat{FA})}$$

where $\widehat{ED} = 90° - \widehat{CD} = 22;30°$, and $\widehat{AE} = 90°$ and $\widehat{LA}$, $\widehat{FA}$ are given by the previous application of the fundamental theorem. Moreover, $\widehat{DQ} + \widehat{QF} = 90° + 21°$. Hence, since the ratio $Crd(2\,\widehat{DQ})/Crd(2\,\widehat{QF})$ and the sum of the two arcs are given, the two arcs $\widehat{DQ}$ and $\widehat{QF}$ are given individually.

There is no reason to reproduce the tedium of these calculations. It suffices to say that Hipparchus has again rounded his figure, 6°♉, to the nearest whole degree.

Even without drawing a new figure, it is clear that this problem likewise cannot be solved on the analemma. The same difficulties that we encountered with the primary problem are again operative. Because the ecliptic is oblique to the meridian, we have no ready way of locating the principal points of the ecliptic with respect to the local coordinates. Since Hipparchus tells us that he solved these problems by metrical analysis, we should assume that he used the techniques of spherical trigonometry.

Final Remarks

In the one place in the *Commentary* where we can be sure that Hipparchus proceeded by calculation, we have good reason to believe that he used a combined approach employing both the analemma and the so-called Menelaus Theorem. Given the equatorial coordinates of a star on the western horizon, Hipparchus determines the degree of the equator at the meridian. This calculation is made using the analemma methods. Given the culminating point of the equator, Hipparchus uses the methods of ancient spherical trigonometry to determine both the culminating and rising points of the ecliptic. These calculations, however, can be carried out only if we make use of an unmotivated lemma in Ptolemy's *Almagest*. Ancient spherical trigonometry allows us to carry out the calculation with as much precision as we should expect based on approximations demonstrated elsewhere in the same passage.

We should not underestimate the significance of the fact that the lemma needed for calculating Hipparchus's result is found in Ptolemy's *Almagest* but never used. The sections of the *Almagest* that cover spherical astronomy, *Almagest* I 13–II 13 and VII 5–6, are lean from a logical and mathematical perspective. Very little material is introduced that is not used in later mathematical arguments. It is a tight systematic development of Ptolemy's theory of spherical astronomy, not a collection of tools and theorems that the mathematical astronomer may find useful. The few other instances of material in these sections that is not later used can be attributed to gestures toward historical topics or historical strata in the text. The fact that Ptolemy includes these two lemmas is a good sign that he carried them over from one of his sources. The fact that one of these lemmas is necessary to a probable reconstruction of Hipparchus's calculation makes it likely that Ptolemy based his work on material that originated with, or was derived from, Hipparchus's work on spherical astronomy.

If this reconstruction may be taken as a representative sample of Hipparchus's spherical astronomy, we can notice a fundamental difference between his approach and Ptolemy's. Whereas Hipparchus seems to have combined the analemma with the trigonometrical methods, Ptolemy took pains to base his spherical astronomy on the trigonometrical methods alone. Even in *Almagest* VII, where Ptolemy demonstrates how to find the degrees of the ecliptic and equator that rise, culminate and set with a given star, he uses only the trigonometrical theorem.[37] In fact, when Ptolemy introduces the mathematical theorems of ancient spherical trigonometry he states that they will allow him "to carry out most demonstrations involving spherical theorems in the simplest and most methodological way possible".[38] By calling his approach simple and methodological, he is likely referring to the fact that it makes use of only one of the two ancient metrical methods on the sphere.

Acknowledgements

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REFERENCES

1. See O. Neugebauer, *A history of ancient mathematical astronomy* (New York, 1975; hereafter *HAMA*), 26–29, and O. Pedersen, *A survey of the Almagest* (Odense, 1974), 72–78.
2. See G. Toomer, *Ptolemy's* Almagest (London, 1984), 65–67.
3. See *HAMA* (ref. 1), 301. Neugebauer supports this statement with a footnote that refers to his own discussion of the Menelaus Theorem and a passage in Pappus's *Collection* that mentions Menelaus. Neugebauer's discussion of the Menelaus Theorem is purely mathematical and does not argue that Menelaus was the author of the theorem. See *HAMA* (ref. 1), 26–29. The passage in Pappus

merely states that Menelaus wrote a book on rising times that mentioned, in particular, that the logic of proofs concerning setting times fell within the specifications (διορισμοί) of those for rising times, see F. Hultsch, *Pappi Alexandrini Collectionis quae supersunt* (Berlin, 1876–78), 600–1. Neither of these passages has any bearing on whether or not Menelaus was the originator of the eponymous theorem.

4. A. Björnbo, *Studien über Menelaos' Sphärik: Beiträge zur Geschichte der Sphärik und Trigonometrie der Griechen* (Abhandlungen zur Geschichte der mathematischen Wissenschaften, Heft 14; Leipzig, 1902), 72 ff, and A. Rome, "Les explications de Théon d'Alexandrie sur le théorème de Ménélas", *Annales de la Société Scientifique de Bruxelles*, Série A, liii (1933), 39–50.
5. See Menelaus's *Spherics* in Krause's German translation from Arabic, M. Krause, *Die Sphärik von Menelaos aus Alexandrien in der Verbesserung von Abu Naṣr Manṣur b. ʻAli b. ʻIrāq mit Untersuchungen zur Geschichte des Texts bei den islamischen Mathematikern* (Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen, Philosophisch-Historische Klasse, Dritte Folge, Nr. 17; Berlin, 1936), 192–5.
6. The Greek text of Menelaus's *Spherics* is lost. The text survives in Latin, Hebrew and Arabic. The most thorough discussion of these texts is still Krause, *op. cit.* (ref. 5), 20–86.
7. Hipparchus twice explicitly refers to a work or works on simultaneous rising times. See C. Manitius, *Hipparchi in Arati et Eudoxi Phaenomena Commentariorum libri tres* (Leipzig, 1894), 128 and 148. The passage on p. 184, however, probably also refers to the same general material.
8. διὰ τῶν γραμμῶν. P. Luckey, "Das Analemma von Ptolemäus", *Astronomische Nachrichten*, ccxxx (1927), cols 17–46.
9. I am aware of 16 uses of this terminology in the *Almagest*. See J. L. Heiberg, *Claudii Ptolemae Syntaxis mathematica* (Leipzig, 1916), i, 32, 42, 251, 335, 381, 383, 416, 439, and ii, 193, 198, 201, 210, 321, 426, 427, 429. The phrase διὰ τῶν γραμμῶν is also used in vol. ii, p. 181 in what appears to be a different sense. Taisbak has briefly discussed Ptolemy's use of διὰ τῶν γραμμῶν in the *Almagest*. See C. M. Taisbak, *Euclid's* Data*: The importance of being given* (Copenhagen, 2003), 13 and 29.
10. See J. L. Heiberg, *Claudii Ptolemae Opera astronomica minora* (Leipzig, 1907), 202–3, and D. R. Edwards, "Ptolemy's Περὶ ἀναλήμματος", Ph.D. Thesis, Brown University, 1984, 107–8.
11. See for example *Syntaxis* (ref. 9), 32 and 142, and A. Rome, *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almagest* (Rome, 1931–46), ii, 449. Toomer (*op. cit.* (ref. 2), 48 and 99) generally translates these phrases with "geometrically".
12. See Manitius, *op. cit.* (ref. 7), 150.
13. ... ὁ νοτιώτατος ἀστὴρ τῶν ἐν τῷ ἀριστερῷ ποδὶ τοῦ ᾿Αρκτοφύλακος. See Manitius, *op. cit.* (ref. 7), 148.
14. See Manitius, *op. cit.* (ref. 7), 148–50.
15. The vulgate ms reads τὴν κα δ′ μοῖραν τοῦ Αἰγόκερω, which makes no mathematical sense. Manitius corrects this to [μέσην] τὴν κδ′ μοῖραν τοῦ Αἰγόκερω which he translates as "$23\frac{1}{3}^\circ$ ♑" following the use of the term μέση in the rest of the text; see his *op. cit.* (ref. 7), 128, 152–4 ff. This correction makes sense of the numbers since $7\frac{1}{2}$ twenty-fourths of a circle is $112\frac{1}{2}^\circ$ so that 1° ♎ $+ 112\frac{1}{2}^\circ = 181^\circ + 112\frac{1}{2}^\circ = 293\frac{1}{2}^\circ = 23\frac{1}{2}^\circ$ ♑.
16. For the case that Hipparchus used an equatorial coordinate system, see D. Duke, "Hipparchus' coordinate system", *Archive for history of exact science*, lvi (2002), 427–33.
17. See *HAMA* (ref. 1), 301–2.
18. See *HAMA* (ref. 1), 278–9. Hence the equatorial coordinates he gives for υ Boo are (180°, 26;20°). These are a little off from the coordinates given by Ptolemy in *Almagest* VII 5. Ptolemy gives the ecliptic coordinates as (171;20°, 25°), see Toomer, *op. cit.* (ref. 2), 347. When we transform these to equatorial coordinates we get (177;04°, 26;13°).
19. ἕκαστον γὰρ τῶν εἰρημένων ἀποδείκνυται διὰ τῶν γραμμῶν ἐν ταῖς καθόλου περὶ τῶν τοιούτων ἡμῖν συντεταγμέναις πραγματείαις, Manitius, *op. cit.* (ref. 7), 150.
20. I would like to thank James Evans for pointing this out to me.

21. See Toomer, *op. cit.* (ref. 2), 104.
22. See Toomer, *op. cit.* (ref. 2), 69–130 & 407–417.
23. The value that Hipparchus used for the obliquity of the ecliptic seems to have been either a precise figure close to 23;51,20°, or 24° used when round figures are suitable. See A. Jones, “Eratosthenes, Hipparchus, and the obliquity of the ecliptic”, *Journal for the history of astronomy*, xxxiii (2002), 15–19. In this case it makes little difference which value we choose.
24. See Toomer, *op. cit.* (ref. 2), 68–69, or Krause, *op. cit.* (ref. 5), 164–169.
25. See Heiberg and Edwards, *op. cit.* (ref. 10) for Ptolemy’s *Analemma*. Heron uses the analemma in his determination of the distance between two cities, see H. Schöne, *Heron von Alexandria: Vermessungslehre und Dioptra* (Leipzig, 1903), 302–6. Vitruvius also discusses the analemma but does not use it for any mathematical purpose, see J. Soubiran, *Vitruve: De l’Architecture, Livre IX* (Paris, 1969), 26–30.
26. For examples of reconstructed determinations using the analemma, see *HAMA* (ref. 1), 301–4 and 850–2. Another analemma reconstruction is given by C. Wilson, “Hipparchus and spherical trigonometry”, *DIO*, vii/1 (1997), 14–15.
27. When the ecliptic is oblique to the meridian only one point of the ecliptic can be treated by considering its equatorial coordinates. This is the problem of Ptolemy’s *Analemma*.
28. A. Rome, “Le problème de la distance entre deux villes dans la *Dioptra* de Héron”, *Annales de la Société Scientifique de Bruxelles*, xlii (1923), 234–58. O. Neugebauer, “Über eine Methode zur Distanzbestimung Alexandria-Rom bei Heron 1 & 2”, *Det Kongelige Danske Videnskabernes Selskab*, xxvi, nos 2 (1938), 3–26, and 7 (1939), 3–11. E. S. Kennedy, “A letter of al-Bīrūnī: Ḥabash al-Ḥāsib’s Analemma for the Qibla”, *Historia mathematica*, i (1974), 3–11.
29. See *HAMA* (ref. 1), 301–4, and Wilson, *op. cit.* (ref. 26).
30. See *HAMA* (ref. 1), 301–2.
31. The angle that the diameter of the ecliptic makes with the diameter of the equator in the plane of the meridian goes from 0 to ε and back again in each sidereal day. It is possible that one could establish a diurnal table like *Almagest* I 15 for the variation in this angle using only analemma methods. This would be a modification of the determination of solar declination shown by Neugebauer, see *HAMA* (ref. 1), 302–4. There is, of course, no ancient evidence of such a procedure.
32. The fundamental theorem is used sixteen times in the *Almagest*. In every case, Ptolemy uses five terms to solve for the sixth. These sixteen instances are in the following sections: *Almagest* I 14; I 16; II 2; II 3, three instances; II 7, twice; II 10; II 11; II 12, twice; VII 5, twice; and VII 6, twice. In some cases, for example *Almagest* I 16, the same arrangement is used to make two calculations. I have counted these as a single instance. Rome also counts sixteen instances of the theorem, see Rome, “Les explications” (ref. 4), 47.
33. This step seems to require a theorem to the following effect: *if in a right triangle the two sides about the right angle are given, then the whole triangle will be given in form*. There is no such theorem in the *Data*. A more general justification of this step can be fashioned out of *Data* 1 and 41.
34. See Toomer, *op. cit.* (ref. 2), 66–67.
35. Although Toomer gave a reconstruction of Hipparchus’s chord table in G. Toomer, “The Chord Table of Hipparchus and the early history of Greek trigonometry”, *Centaurus*, xviii (1973), 6–18, he later recanted the details of this reconstruction. See Toomer, *op. cit.* (ref. 2), 215, n. 75.
36. Although this variant is not stated by either Menelaus or Ptolemy, it is used once by Ptolemy in the *Almagest*, see Toomer, *op. cit.* (ref. 2), 95. The rectilinear lemma necessary to show this variant is given by Theon in his *Commentay*, see Rome, *op. cit.* (ref. 11), 543.
37. The fact that Ptolemy designates the stars by their ecliptic coordinates means that he will have to use spherical trigonometry in this section, since the analemma is not serviceable for the transformation of coordinates.
38. See Toomer, *op. cit.* (ref. 2), 64.