Heron's *Dioptra* 35 and Analemma Methods: An Astronomical Determination of the Distance between Two Cities

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Abstract. Heron's *Dioptra* 35 is the unique witness of an ancient mathematical procedure for finding the great arc distance between two cities using methods of ancient spherical astronomy and simultaneous observations of a lunar eclipse. This paper provides a new study of the text, with mathematical and historical commentary. I argue that Heron's account is a summary of some longer work of mathematical astronomy or geography, which made extensive use of the analemma, an ancient model of the celestial sphere. Heron's text can be used to show the utility of the analemma model, both as a theoretical device and as a computational tool.

Heron's *Dioptra* 35 is both fascinating and corrupt.[1] On the one hand, it is the unique witness of an involved mathematical procedure for finding the great arc distance between two cities using techniques of ancient spherical astronomy based on simultaneous observations of a lunar eclipse. On the other hand, the text itself would be almost unreadable were it not for numerous corrections made by two of its editors, Vincent and Schöne (Vincent 1858; Schöne 1903, pp. 302–306). Even with these, Schöne did not completely understand the matter; he was unable to provide sufficient figures based on those in the manuscripts, and he acknowledged that his translation required clemency (Schöne 1903, p. 303, n. 1).

The foundation for our current understanding was laid in three papers published between the great wars. Rome (1923) and Neugebauer (1938) independently gave interpretations of the text that agreed in general but differed in a few key details. After he was made aware of Rome's work and the existence of figures in the oldest manuscript, Neugebauer (1939) published a short addendum, bringing his views into line with Rome's and providing a facsimile reproduction of the figures. Schöne's text should be read in conjunction with the figures provided by Neugebauer in his second paper (Neugebauer 1939, pp. 6–7). A French and partial German translation can be found in these papers. Recently, an English translation was made by Irby-Massie and Keyser (2002), although they appear to have followed Rome on most of the difficult issues. Moreover, they only give a modern figure for the primary construction and it contains some extraneous points transferred from the analemma figure.[2]

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Heron's presentation, in *Dioptra* 35, is a summary of a mathematical procedure which he must have found in a work of mathematical astronomy or geography. Although Rome and Neugebauer have given a good general interpretation of Heron's meaning, no one has yet attempted to explicate the underlying mathematics that the text implies. Moreover, in order to form a complete picture of the text and its diagrams, one must consult a number of different papers, some in obscure journals. This paper is primarily a new textual study of *Dioptra* 35, giving full attention to issues of the mathematical details as they would have been understood in an ancient context.

It has generally been assumed that Heron is describing a nomographic technique which must be carried out on a physical apparatus (Rome 1923, pp. 241–242; Neugebauer 1975, pp. 8–9).[3] Neugebauer goes so far as to call the procedure 'totally unmathematical' (Neugebauer 1975, p. 846). Heron's treatment, however, is clearly a summary of some other work, which presumably contained the missing mathematical details. In my reading of the text, I will show that these details can be supplied in a way which is consistent with our knowledge of the ancient traditions of spherical geometry. Moreover, I will argue that the procedure sketched in *Dioptra* 35 could have been used to calculate a great arc distance using techniques of metrical geometry familiar to us from other ancient and medieval sources. I make no claim that anyone ever used these techniques to calculate a great arc distance between two places based on actual records of simultaneous observations. Nevertheless, I believe *Dioptra* 35 should be read as a vestige of the wealth of the ancient traditions of spherical astronomy.

Dioptra 35 combines two different Greek traditions of spherical geometry. The first involves drawing great circles, and circles parallel to these, directly on the sphere. These methods are represented in the theoretical tradition by the *Spherics* of Theodosius and Menelaus (Heiberg 1927; Krause 1936; Berggren 1991). The second uses a model of the sphere known as the analemma which involves the representation of circles in a receiving plane through orthogonal projection or rotation. Unfortunately, we have very few ancient texts on analemma methods. They are unique to ancient and medieval mathematics and warrant some words of introduction.

1. Preliminaries on the Analemma Model

There are three ancient texts pertinent to a study of the analemma. These are Vitruvius' *Architecture* IX 7, Heron's *Dioptra* 35, and Ptolemy's *Analemma* (Soubiran 1969; Rowland and Howe 1999; Heiberg 1907; Edwards 1984).[4] There are also a number of analemma constructions preserved in the medieval Arabic tradition.[5] In this section, I will give a sketch of the analemma model with little attempt to engage the texts themselves.

The analemma appears to have had its origin in gnomonics, the theory of sundials. At some point, however, mathematicians realized its value for studying general problems of spherical astronomy. In particular, it was useful for comparing the coordinate systems of two orthogonal great circles and for determining arc lengths on circles parallel to these. On a practical level, it allowed mathematicians to calculate arc lengths of circles on a sphere using the techniques of plane trigonometry built around the use of chord tables.

A sphere is imagined resting on the plane of the horizon, AB (see Figure 1). A normal, OΠ, is drawn through the tangent point of the sphere and the horizon, Π. This line is called the *gnomon*.[6] A great circle, ΞONΠ, is taken in the plane of the local meridian. This circle represents the meridian. In the plane of the meridian, a line, NΞ, is drawn through the center, E, parallel to the horizon; it represents the local horizon. Hence, the sphere of the analemma can be used to model the celestial sphere; everything above NΞ represents the parts of the heavens above the earth and everything below NΞ, the parts below. The line ΨΩ is drawn, making an angle with NΞ which equals the angular height of the celestial pole above the local horizon, φ. Hence, ΨΩ represents the celestial axis; assuming we are in the northern hemisphere, Ω is the north pole and Ψ the south pole. The line DG is drawn through E, perpendicular to the axis, and represents a diameter of the celestial equator. The line HI is drawn so as to make an angle with DG which equals the obliquity of the ecliptic, ε. Hence, HI represents a diameter of the ecliptic. Line HM is drawn through H, parallel to the diameter of the equator. Hence, HM represents a diameter of a circle of equal declination, parallel to the plane of the equator. We will call such circles, *δ-circles*. In the ancient literature, they are called *day circles*, *month circles*, or simply *parallels*. In fact, HM is the δ-circle of the summer solstice. A semicircle is drawn with HM as diameter. This circle should be imagined as actually positioned perpendicular to the plane of the figure. It represents the diurnal motion of the sun at the summer solstice. The line QR is drawn perpendicular from the intersections of the horizon line and the line of the solstitial δ-circle. This line is called the *dioron*, the *separator*.[7] It represents the intersection of a given solar δ-circle with the horizon.

From this it is clear that the analemma can be used to model the daily motion of the sun given the time, h; solar declination, $\delta_{\odot}$; and geographic latitude, φ. The latitude of the model is adjusting by setting $\overset{\frown}{\Xi\Omega} = \varphi$ to the local value. The declination is given by situating HM $\parallel$ DG such that $\overset{\frown}{\mathrm{DH}} = \delta_{\odot}$. Taking the summer solstice as an example, the time is then marked off on the semicircle MRH. In practice, the analemma was used with seasonal hours.[8] Hence, the seasonal hour, h_{s}, was marked off on $\overset{\frown}{\mathrm{RH}}$, for the daytime, or on $\overset{\frown}{\mathrm{RM}}$, for the nighttime. Hence, on the day of the summer solstice, the sun moves twice along semicircle MRH; along $\overset{\frown}{\mathrm{RH}}$ from sunrise to noon, $\overset{\frown}{\mathrm{HR}}$ from noon to sunset, $\overset{\frown}{\mathrm{RM}}$ from sunset to midnight, and $\overset{\frown}{\mathrm{MR}}$ from midnight to sunrise.

The analemma does not provide a way to precisely model the proper, longitudinal motion of the sun. If, however, the solar longitude is given, the analemma can be used to find the local position of the sun for any geographic place and time. This is done as follows. A perpendicular, KM, is dropped onto the line of the equator. A circle is then drawn with

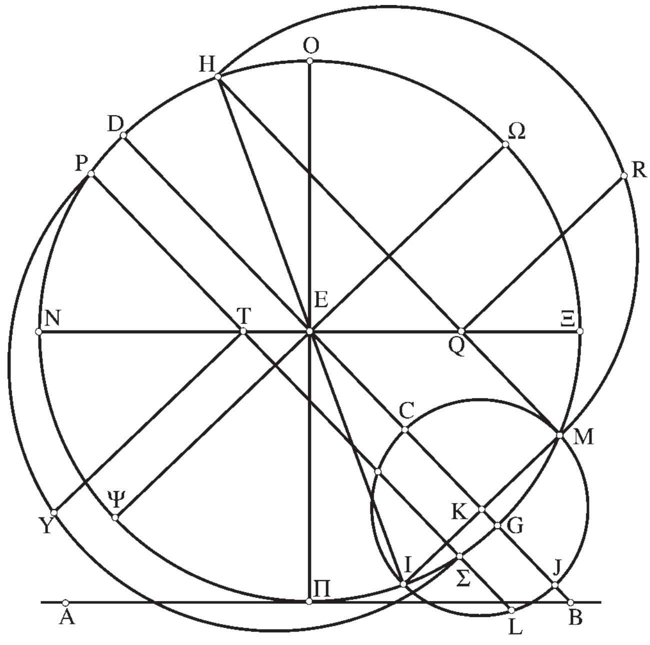

Fig. 1. The analemma figure.

K as center and KM as radius. This circle is called the *menaeus*, or monthly circle. The menaeus circle models the zodiacal circle, however, not in the same way as the rest of the analemma; it is not a rotation of the great circle around HI into the plane of the figure. Nevertheless, in circle JMCI, J corresponds to the vernal equinox; M, the summer solstice; C, the autumnal equinox; and I, the winter solstice. A given solar longitude, $\overset{\frown}{JCL}$, is then laid off on the menaeus circle. A line is drawn through L, parallel to the line of the equator and cutting the meridian at Σ and P. Then, PΣ is the diameter of the δ-circle of the sun when the sun is at the given longitude.[9] Hence, on the day in question, the sun travels twice through $\overset{\frown}{P\Sigma}$. The name of the menaeus circle indicates that in practice the solar longitude was likely modeled under the rough assumption that the sun was simply in a given 'month', that is in a given sign of the zodiac. More precisely, one may assume that the longitudinal displacement of the sun is about a degree a day. These simplifications would make the analemma a complete system for modeling the motion of the sun with respect to the local coordinates.

An examination of the diagram shows that all of the arcs in any given analemma configuration are represented in the same plane in such a way that they can be determined by the methods of plane trigonometry through a chord table. One of the great advantages of the analemma model is that it makes a number of problems of spherical astronomy readily accessible to calculation by means of plane trigonometry.

2. *Heron's* Dioptra *35*

2.1 Translation

It will be useful to begin with a translation of the text. The sentences are numbered for later reference. *Dioptra* 35 reads as follows (see Figure 2) (Schöne 1903, pp. 302–306).[10]

> [1] Now then, the distance between as many places as are accessible by foot is found either by means of the constructed dioptra or the hodometer discussed.[11] [2] But since it is also useful, in reality, to know the size of the path between two regions (κλιμάτων)—even if islands, seas, or, say, inaccessible regions fall upon it—it is necessary to have some method besides this, so that our published treatise will be complete.
>
> [3] Let it be necessary to measure, say, the path between Alexandria and Rome along a line—or rather along a great circle arc on the earth—if it has been agreed that the circumference of the earth is 252,000 stades—as Eratosthenes, having worked rather more accurately than others, showed in his book entitled *On the Measurement of the Earth.*
>
> [4] Now, let the same lunar eclipse have been observed at Alexandria and Rome. [5] If one is found in the records, we will use that, or, if not, it will be possible for us to state (εἰπεῖν) our own observations because lunar eclipses occur at 5- and 6-month intervals.
>
> [6] Now, let the same eclipse be found in the stated regions; in Alexandria at the fifth hour of the night, but at the third hour of the same night in Rome—obviously the same night. [7] And let the night—that is the day circle with respect to which the sun moves on the said night—be 10 days from the vernal equinox in the direction of the winter solstice.
>
> [8] Let a hemisphere through the tropics (ἡμισφαίριον τὸ διὰ τῶν τροπικῶν)[12] have been inscribed, if we are in Alexandria, with respect to the latitude (κλίμα) of Alexandria, if in Rome, with respect to that in Rome.

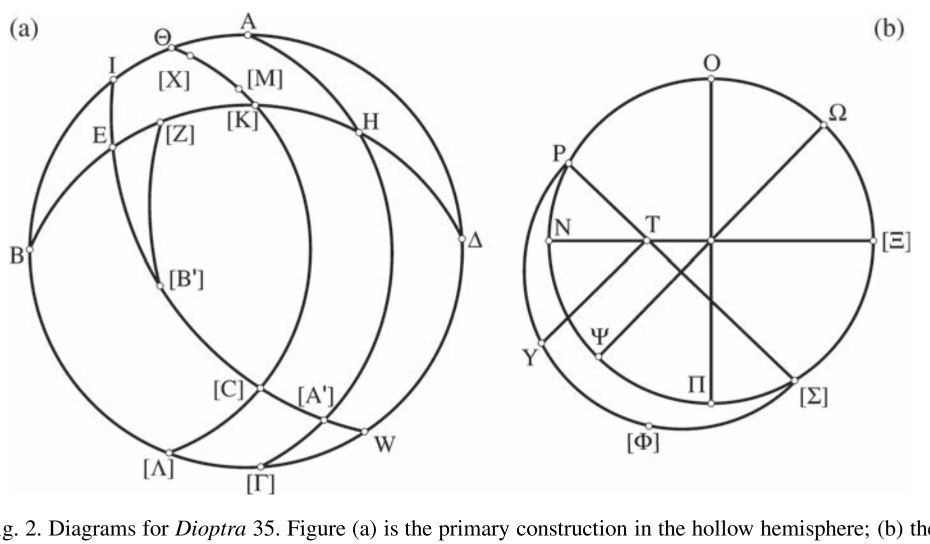

Fig. 2. Diagrams for *Dioptra* 35. Figure (a) is the primary construction in the hollow hemisphere; (b) the auxiliary construction of the analemma for Rome. A facsimile of the diagrams in the oldest manuscript was printed by Neugebauer (1939, p. 4). The figure for (a) is badly misdrawn in the manuscript. In (b), points P and N are mislabled. The labels in brackets are missing in the manuscript.

[9] And let some hollow hemisphere be set up, inscribed through the tropics with respect to the latitude of Alexandria. [10] And let ΑΒΓΔ be the circle around its lip, and let ΒΕΖΗΔ be the meridian in it; ΑΗΓ, the equator; Ε, the pole of the parallels (to the equator); and Ζ, the pole of the circle around the lip of the hemisphere. [11] And in it, let a correspondent circle have been arranged (ἐντετάχθω ὁματαγὴς) for the circle with respect to which the sun moves at the fifth hour of the said night when it is 10 days away from the vernal equinox in the direction of the winter solstice; and let it be ΘΚΛ. [12] And let arc ΘΚΛ be divided into 12 parts, and let ΘΜ be the fifth of them, since the eclipse was observed at Alexandria at the fifth hour. [13] Therefore, Μ will be the correspondent point for the place where the sun was during the occurrence of the eclipse.

[14] And let the analemma through Rome (τὸ διὰ Ῥώμης ἀνάλημμα)[13] have been drawn, in which, let the day circle—the correspondent to ΘΚΛ—have had been drawn. [15] ⟨And let ΨΩ be the axis in the analemma⟩,[14] and let ΝΞ be the diameter of the horizon; ΟΠ, the gnomon; ΡΣ, the diameter of the day circle; and ΤΥ, the separator (δίορον). [16] And arc ΥΦΣ will be 6 of the daily hours,[15] of which ΥΦ is 3, because the observation occurred in Rome at the third hour. [17] And let ΜΧ be set out similar to arc ΥΦ; therefore, point Χ is on the horizon through Rome. [18] And let ΧΚC have been set out similar to arc ΥΦΣ; clearly, C will be on the meridian through Rome.

[19] But Ε is the pole of the parallels [to the equator]. [20] Let the great circle ΕC⟨Α′⟩[16] have been drawn through Ε and C. [21] Clearly, this will be the stated meridian through Rome. [22] And let Α′Β′ have been set out similar to ΞΩ, but let Α′Β′ have been set out from the quadrant ΕΑ′.[17] [23] Therefore, point Β′ will be the pole of the horizon through Rome, while Ζ is that of the one through Alexandria.

[24] Then, let Β′Ζ—the arc of the great circle through Β′ and Ζ—have been drawn. [25] And let it have been investigated (ἐξητάσθω) how many degrees it is with respect to the circle ΑΒΓΔ. [26] And let it have been found to be, say, 20°.

[27] Then, the arc of the earth cut off between Rome and Alexandria will be 20°, where a great circle is 360°. [28] But 1° of the earth has 700 stades, if the whole circumference is 252,000. Therefore, 20° comes to 14,000. Clearly, we have exhibited the number of stades and the length of the stated path.

[29][18] But if point Α′ projects over (ὑπερπίπτῃ) the ... [30] The overprojecting arc ... which we will set ... the [arc] ⟨?⟩G [19] ... [31] And $\overline{\mathrm{B}}$ will be diametrically opposite to the overprojecting point ... [32] Setting out ⟨Ε⟩Β′ of the quadrant we will again have point Β′.[20]

3. *Mathematical Commentary*

Heron's approach is to find the length of the great circle between the nadirs of Alexandria and Rome on a model of the heavens under the horizon at Alexandria. He does this by effecting a transformation between the local coordinates of Alexandria and Rome. The key to this transformation is a pair of simultaneous observations of the same lunar eclipse. Much of Heron's language is vague and hypothetical. He is giving a loose overview of an exact procedure; he assumes round observations as his starting point and skims over a number of steps. This commentary will flesh out the steps of the mathematical argument, simply assuming the observations as given.

As is usually the case in Greek geometry, the diagrams preserved in the medieval manuscripts are purely schematic; they make no attempt to produce a visual representation of the objects they portray. Although, it is possible to make sense of the text with only these diagrams, most modern readers will find a diagram in linear perspective helpful.

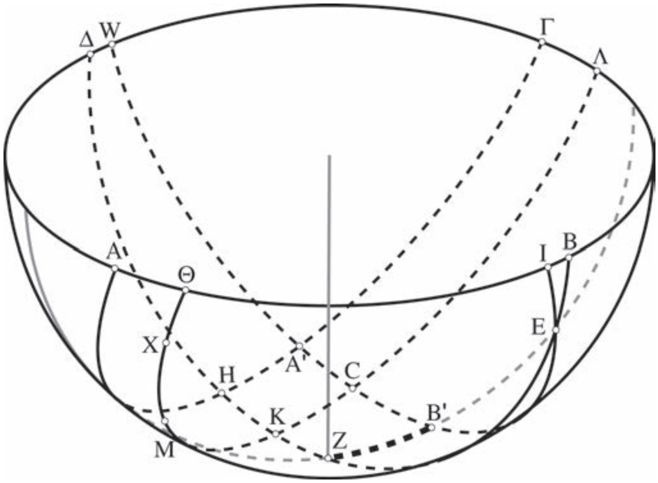

Fig. 3. Diagram, in linear perspective, of *Dioptra* 35; the primary construction in the hollow hemisphere. Lines drawn in gray do not appear in the medieval figure.

Each method of representation has its advantages (compare Figures 2a and 3). Figure 3, in linear perspective, allows us to read information directly off the diagram; for example, that Z is the local nadir or that circle ΘΚΛ is parallel to great circle ΑΗΓ. On the other hand, the modern figure is cluttered, and it is only because the lines are varied in tone and quality that the eye can parse the objects. The diagram in the manuscript, Figure 2a, avoids this difficulty and gives the eye direct access to the topological relationships essential to the proof.

Heron begins by assuming that a lunar eclipse has been observed simultaneously at the fifth seasonal hour of the night in Alexandria and at the third seasonal hour in Rome, 10 days before the vernal equinox. He then models the position of the sun at the time of this eclipse with respect to each of the local coordinate systems. This model functions on two levels. (1) It can be understood as a purely geometric construction using the methods of the analemma and the *Spherics* of Theodosius and Menelaus. (2) The geometric model can be used, in conjunction with the techniques of ancient trigonometry, to derive numerical parameters. Although Heron only explicitly mentions the analemma for Rome, a number of steps in the construction of the hemisphere for Alexandria are best explained by reference to an analemma for Alexandria. This will become clear in what follows.

A hemisphere is constructed and configured like the heavens below the horizon at Alexandria. Heron refers to this hemisphere, in [8] and [9], as having been inscribed through the tropics, but in fact the tropic circles do not appear in it. This expression probably refers to the fact that the equator is aligned in the hemisphere as it is in the heavens below Alexandria. In a hemisphere of given radius, the four cardinal points are found by constructing a square in the great circle of the lip [*Elem.* IV 6].[21] Let A be the east; B, the south; Γ, the west; and Δ, the north. The meridian is constructed by drawing a great circle through B and Δ about A or Γ as pole [Theo. *Spher.* I 20].

For a few locations the angular height of the pole above the horizon, φ, would have been known; in most cases, however, the latitude is specified as either (1) the ratio of the length of a gnomon to its shadow at the time of a solstice or equinox, $s : g$, or (2) the ratio of the length of the longest day to the shortest day $M : m$. The analemma would have been used to convert these conventional measures to an arc length. Let an analemma be established for the horizon at Alexandria with the same radius as the hemisphere. Consider Figure 4. Then, $s : g = \mathrm{E\Pi} : \mathrm{\Pi B}$ and $M : m = \widehat{\mathrm{HR}} : \widehat{\mathrm{RM}}$; since, by reasons of symmetry, $\widehat{\mathrm{RM}}$, the path of the sun on the shortest night, is equal to the path of the sun of the shortest day. It is clear that $\varphi_{\mathrm{A}} = \angle \Omega\mathrm{E}\Xi$ can be found by purely geometric means; it can also be found as an angular value, using ancient trigonometry.[22] (1) Given the radius of the analemma and $\mathrm{E\Pi} : \mathrm{\Pi B}$, chord table methods immediately yield $\angle \mathrm{BE\Pi} = \varphi_{\mathrm{A}}$ [*Data* 43]. (2) Given the radius of the analemma, ε and $\widehat{\mathrm{HR}} : \widehat{\mathrm{RM}}$, $\angle \Omega\mathrm{E}\Xi$ is determined as follows. Given ε and HE, EF and HF are given by the chord table [*Data* 87]. Again, given $\widehat{\mathrm{HR}} : \widehat{\mathrm{RM}}$, HR and RM are given by the chord table [*Data* 87]. Now since $\triangle\mathrm{HRM} \sim \triangle\mathrm{HRQ} \sim \triangle\mathrm{QRM}$ [*Elem.* VI 8], while HR, RM, and HM are given, therefore HQ and QM are given [*Data* 1 and 2]. Hence, FQ is given [*Data* 3 and 4]. Therefore, EQ is given [*Elem.* I 47]. Then, by the chord table, $\angle \Omega\mathrm{E}\Xi = \varphi_{\mathrm{A}}$ is given [*Data* 88]. Hence, the south pole, E, can be situated in the hemisphere (see Figures 2a and 3).[23]

In the hemisphere, the equator, semicircle $A\Gamma$, is found by drawing a great circle through A and Γ with E as pole [Theo. *Spher.* I 20] (see Figures 2a and 3). The nadir, Z, is found by laying off a quadrant along the meridian, BEΔ. This completes the constructions in sentence [10].

The construction of the δ-circle $\Theta K \Lambda$, in [11], can also be effected by means of the analemma (see Figure 4). The menaeus circle, MCIJ, is drawn according to the value for ε. Hence, its radius is given [*Data* 87]. $\widehat{\mathrm{JL}}$ is taken according to the value of the solar longitude predicted by the solar model or, without much loss of accuracy, simply set $\widehat{\mathrm{JL}} = 10^\circ$. Line LP is drawn parallel to the line of the equator, GD; then $\widehat{\mathrm{DP}}$ will be the declination of the δ-circle of the sun on the day in question. Again, it is clear that this arc is given using purely geometric techniques. It remains to be shown that a value for its length is also given through ancient trigonometric methods. Since KL and $\angle$JKL are given, SL is also given by means of the chord table [*Data* 87]. Hence, in $\triangle$EUP, the chord table gives $\angle \mathrm{EPU} = \angle \mathrm{PED}$ [$\mathrm{SL} = \mathrm{EU}$, *Elem.* I 47 and *Data* 88]. But $\delta = \angle \mathrm{PED}$. This arc is then laid off, in the hemisphere, along BEHΔ from H in the direction of the south pole such that $\widehat{\mathrm{HK}}$, in the hemisphere, is equal to $\widehat{\mathrm{PD}}$, on the analemma (compare Figures 2a and 3 with Figure 4). A circle is then drawn through K with E as pole and distance EK.[24] This is the δ-circle $\Theta K \Lambda$.

The next step of the construction, in [12] and [13], is to find the position of the sun, on its δ-circle, at the time of the eclipse (see Figures 2a and 3). Since the observations were

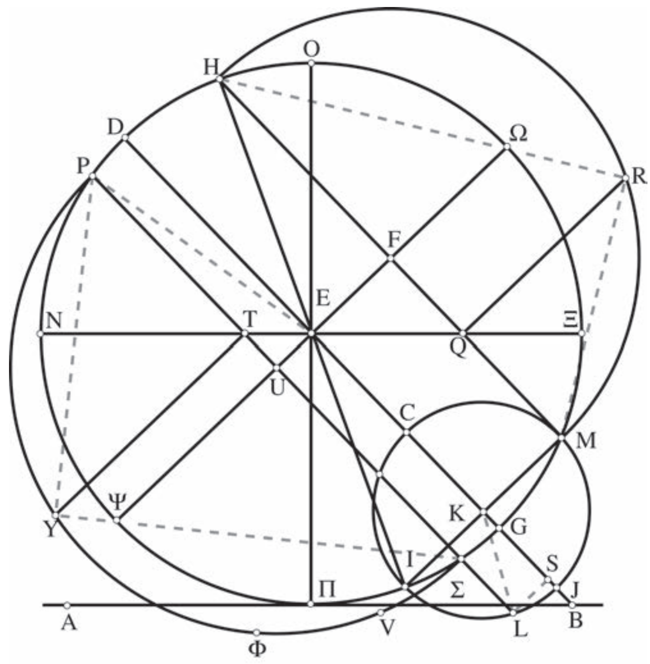

Fig. 4. The analemma figure with auxiliary lines for trigonometric calculation.

made in seasonal hours, this will involve dividing $\widehat{\Theta\mathrm{K}\Lambda}$ into 12 parts. In the analemma, this arc is twice $\widehat{\mathrm{Y}\Sigma}$ (see Figure 4). Hence, $\widehat{\mathrm{Y}\Sigma}$ must be divided into six parts. Using purely geometric means, this division is tricky because it involves trisecting a given angle and there are no Euclidean means of effecting this construction. Greek geometers got around this problem by devising solutions using conics and other, more involved, curves.[25] We know that such divisions were part of the toolbox of analemma techniques because, according to his own report, Pappus gave a trisection procedure in his lost commentary to Diodorus' lost *Analemma* (Hultsch 1876–1878, vol. 1, pp. 244–246). On the other hand, using metrical methods, it will be sufficient to find an angular value for $\widehat{\mathrm{Y}\Sigma}$ and divide this by 6. The procedure can be described as follows. Since $\angle$EPU is given, $\angle$PEU is given and hence chord $\mathrm{P}\Sigma$ is given [*Data* 87]. Again, since EU and $\angle \mathrm{TEU} = \varphi$ are given, TU will also be given by the chord table [Tais. *Data* 87*].[26] Hence, PT and $\mathrm{T}\Sigma$ will be given [*Data* 3, 4, and 7]. Then, since $\triangle \mathrm{PY}\Sigma \sim \triangle \mathrm{YT}\Sigma \sim \triangle \mathrm{PYT}$ [*Elem.* VI 8], TY is given [$\mathrm{TY}^2 = (\mathrm{PT} \times \mathrm{T}\Sigma)$]. Therefore, PY, $\mathrm{Y}\Sigma$, and hence $\widehat{\mathrm{PY}}$ and $\widehat{\mathrm{Y}\Sigma}$ are given [*Elem.* I 47 and *Data* 87]. The position of the sun at the time of the eclipse is found by dividing $\widehat{\mathrm{Y}\Sigma}$ into six parts and laying out $\widehat{\mathrm{YV}}$ equal to five of these parts. In the hemisphere, $\widehat{\Theta\mathrm{M}}$ is laid out similar to $\widehat{\mathrm{YV}} = 5/6\,\widehat{\mathrm{Y}\Sigma}$, on the analemma (compare Figures 2a and 3 with Figure 4). Hence, M represents the position of the sun at the eclipse, as stated in [13].

In [14]–[18], the local coordinates of Rome are found and situated in the hemisphere, using M as a reference point, so that they are given with respect to the local horizon at Alexandria (see Figures 2a and 3). We have seen that the analemma configured for Alexandria has been implicitly necessary to the construction thus far. At this point, Heron explicitly invokes the analemma configured for Rome. This analemma will, in general, be the same as that established for Alexandria, with the exception that $\angle \Omega E \Xi$ will be set for the latitude of Rome, φ_R (see Figure 4). The δ-circle of the sun, $P\Upsilon\Sigma$, is found in the same way as before. $\overset{\frown}{\Upsilon\Sigma}$ is again divided into six parts, and $\overset{\frown}{\Upsilon\Phi}$ is laid out equal to $3/6\ \overset{\frown}{\Upsilon\Sigma}$, corresponding to the time of the eclipse observation at Rome. In the hemisphere, $\overset{\frown}{MX}$ is laid out similar to $\overset{\frown}{\Upsilon\Phi}$ on the analemma (compare Figures 2a and 3 with Figure 4). Hence, X lies on the local horizon at Rome. $\overset{\frown}{XC}$ is then laid out, in the hemisphere, similar to $\overset{\frown}{\Upsilon\Sigma}$, on the analemma, such that C lies on the local meridian at Rome. It is clear from the discussion of the analemma for Alexandria that the angular positions of X and C can be determined either through descriptive geometry or trigonometry.

A great circle is joined through E and C to A′ [Theo. *Spher.* I 20]; hence, it represents the meridian through Rome. In the hemisphere, $\overset{\frown}{A'B'}$ is laid off from the quadrant EA′ such that it is equal to $\overset{\frown}{\Omega\Xi}$, on the analemma (compare Figures 2a and 3 with Figure 4). Since the arc between the equator and the nadir on the analemma, $\overset{\frown}{\Pi G}$, is equal to $\varphi_R = \overset{\frown}{\Omega\Xi}$, B′ will represent the nadir of Rome, in the hemisphere. Z and B′ are then joined by the arc of a great circle [Theo. *Spher.* I 20], and the arc length is measured (see Figures 2a and 3).

The text is vague about how this measurement it carried out. Grammatically, Heron dictates that the measurement has been performed using the passive imperative perfect, the usual locution for a mathematical operation. Since a numeric value results, it must have been obtained either through a nomographic procedure or a calculation. A nomographic measurement would be quite simple. It would involve setting the compass points on Z and B′ and then transferring this span to a measured circle, equal to a great circle of the hemisphere. The verb used to express the procedure, however, argues against such a simple operation. *Exetazō*, in [25], means *to study* or *examine closely*. It can be used in the technical literature to refer to a process of mathematical reasoning which involves calculation.[27] Heron himself uses it this way in his *Definitions* (Heiberg 1912, p. 116). I am aware of no instance where it refers to a nomographic measurement. I will argue below that this arc length can be calculated using analemma methods. Heron, however, simply assumes some measurement procedure and takes the arc as 20°. With this arc length, and a predetermined value for the size of the earth, it is a simple matter to state the physical length of the distance between Rome and Alexandria.

The final passages, [29]–[32], are corrupt and cannot be read as they stand.[28] Nevertheless, enough of the text remains to see that it asserts that the desired point can still be found when certain objects fall above the hemisphere. The overall intent of the passage must be

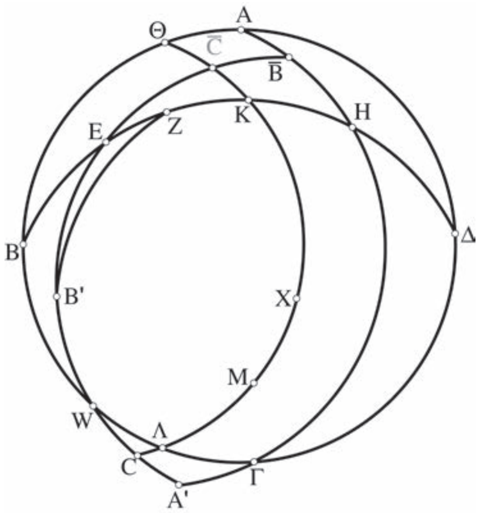

Fig. 5. Diagram for the final passages of *Dioptra* 35 following the medieval model. Points in gray are not mentioned in the text.

to show that the nadir of a distant city can be found by this method, even if it is so far away that the intersection of its meridian with the equator, A′, falls outside the hemisphere of the local horizon.

It is to be shown that, under such circumstances, the construction can still be carried out entirely within the hemisphere. Consider Figures 5 and 6. Since A′ lies on a great circle which intersects the horizon, the point diametrically opposite it, $\overline{\mathrm{B}}$, will fall inside the hemisphere [Theo. *Spher.* I 11]. The construction has two cases: (1) A′ falls above the horizon while C falls below it or (2) both A′ and C fall above the horizon. Point X, on the distant horizon, will be located as before. In (1), point C on the distant meridian is found as before. The great circle EC is joined. It will intersect the equator at $\overline{\mathrm{B}}$, inside the hemisphere [Theo. *Spher.* I 11]. In (2), point $\langle\overline{\mathrm{C}}\rangle$, diametrically opposite C, will lie inside the hemisphere [Theo. *Spher.* I 11]. $\langle\overline{\mathrm{C}}\rangle$ is found by setting out a quadrant of circle XKΘ from point X. The great circle $\mathrm{E}\langle\overline{\mathrm{C}}\rangle\overline{\mathrm{B}}$ is joined and extended to W [Theo. *Spher.* I 20]. Once $\overline{\mathrm{B}}$ is found, the two constructions proceed in the same way. An analemma is established for the distant city. On the analemma, the complement to the latitude, $90^\circ - \varphi$, is determined. If $\widehat{\langle \mathrm{E}\rangle \mathrm{B}'}$ is set out similar to this complement, it will determine the nadir of the distant city as before. Something to this effect is asserted in [32], the final clause.

The final passages can only be of interest if the problem is to be solved by nomographic means. For the purposes of calculation it makes no difference whether or not A′ falls inside the hemisphere.

This commentary has made clear just how cursory is Heron's treatment of the proposed solution. An analemma must be constructed for both cities, despite the fact that Heron only

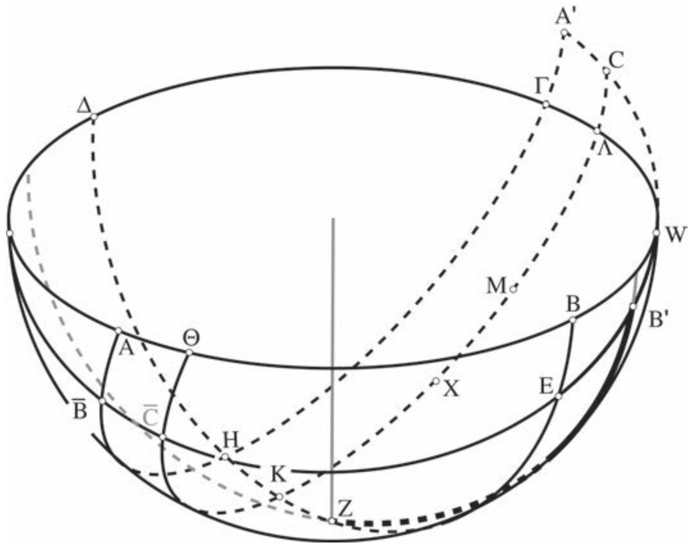

Fig. 6. Perspective diagram for the final passages of *Dioptra* 35. Points and lines in gray are not mentioned in the text.

uses one for Rome. It is clear that, by using analemma for both cities, the problem can be solved in a purely geometric manner. I have argued that all but one step can also be carried through using plane trigonometry to produce a numeric solution. The missing step is the most important one, that is, obtaining a value for the great circle arc between the nadirs of the two cities. I will argue below that this step can also be computed using the analemma and plane trigonometry.

4. *Calculation of the Great Arc Distance*

The problem of finding the great arc distance between two cities is closely related to that of finding the qibla, the direction toward Mecca in which Muslims must pray.[29] Indeed, some of the medieval Arabic solutions to the qibla problem use, as an intermediary step, the computation of the great arc distance to Mecca. A number of analemma solutions to the qibla problem have been found in the medieval Arabic sources.[30] I have adopted the following analemma solution to the great arc distance problem from a solution to the qibla problem put forward by al-Bīrunī (Berggren 1980, pp. 70–74).

An analemma is constructed and configured to the local horizon, in this case, Alexandria. Consider Figure 7. Let ADBC be the plane of the meridian; AB, the line of the horizon; and CD, the gnomon. Then, set $\angle BEH = \varphi_A$, so that HI is the celestial axis; H, the north pole; I, the south pole; and GF, the line of the equator. Then, let $\overset{\frown}{GJ} = \overset{\frown}{KF} = \varphi_R$. Let KJ be joined and semicircle KMJ completed. Now, since the arc of the solstitial colure

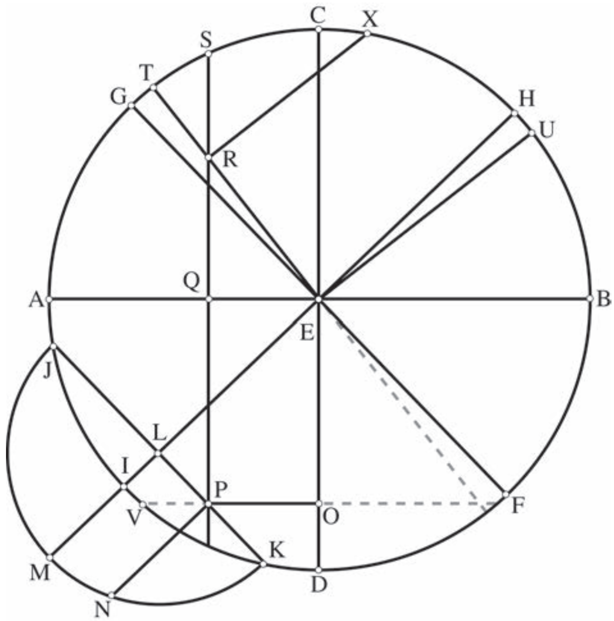

Fig. 7. Analemma construction for finding the great arc distance between two cities.

between the equator and the nadir is equal to that between the horizon and the pole, KJ will be the diameter of the δ-circle which contains the nadir of Rome. Hence, the nadir of Rome will lie on semicircle KMJ. Let $\widehat{JN}$ be set out equal to the difference in longitude between Alexandria and Rome, $L_A - L_R = 15°(h_A - h_R)$.[31] Then, since J lies on the meridian of Alexandria, N, lying on the meridian of Rome, will be the nadir of the latter city. The arc distance between Rome and Alexandria is modeled on the analemma in two steps: the nadir of Rome, N, is first projected onto the horizon of Alexandria and then it is again projected onto the great circle through the two nadirs which is rotated into the plane of the figure.

The semicircle KNJ is imagined turned to its proper position, perpendicular to the plane of the meridian.[32] Then, point N is situated the length OP to the south of the gnomon and the length PN to the east of the meridian.[33] The horizon is rotated onto the plane of the figure such that east maps to C and west to D. In this configuration, line AB is the diameter of the meridian and CD the diameter of the prime vertical. A line is drawn through point P, cutting line AB at Q, such that PQS ∥ CD. Point R is taken such that QR = PN. Hence, point R is south of the gnomon by EQ = OP and east of the meridian by QR = PN. Therefore, R is the orthogonal projection of the nadir of Rome onto the horizon of Alexandria. The diameter of the great circle that joins the nadir of Alexandria with that of Rome goes through E and R and meets the horizon at T.

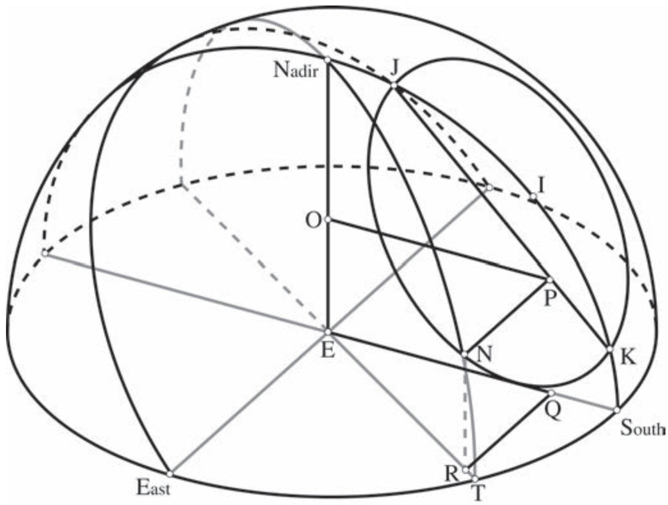

Fig. 8. Perspective diagram of the analemma construction of the great arc distance between two cities.

The great circle through the two nadirs is imagined rotated in the plane of the figure such that the nadirs are found by dropping perpendiculars. The points X and U are determined by XR $\perp$ ET and UE $\perp$ ET. Hence, $\overset{\frown}{\mathrm{XU}}$ is the great arc distance between Alexandria and Rome.

For the purpose of following the mathematical argument, the analemma figure provided above is sufficient. Nevertheless, some readers will find a perspective diagram helpful for imagining the situation on the sphere. Figure 8 shows the hemisphere inverted so that we are looking down on it from the southeast. The nadir, south point, and east point have been labeled as such, because these do not remain constant in the analemma figure. The other relevant points are given the same name as they have in the analemma figure.

A comparison of the two figures shows the advantage of the ancient analemma for solving problems of this kind. The modern figure is cluttered, and both lines and arcs are distorted in accordance with the principles of linear perspective. The analemma figure, on the other hand, is relatively clean, and all lengths and arcs are preserved in their proper proportions. Because of these characteristics, the analemma lends itself to analysis using the techniques of plane geometry.

The solution to this problem shows that the positions of points P and R are completely determined, given $\overset{\frown}{\mathrm{BH}} = \varphi_{\mathrm{A}}$, $\overset{\frown}{\mathrm{GJ}} = \varphi_{\mathrm{R}}$, and $\overset{\frown}{\mathrm{JN}} = (L_{\mathrm{A}} - L_{\mathrm{R}})$. Hence, using the analemma figure, this problem can be solved numerically using the techniques of plane trigonometry. A reconstruction of the ancient trigonometric procedures could be given along the lines of those presented for the hemisphere construction. This would require a number of auxiliary triangles, and a complete computation would be quite lengthy, especially using ancient methods. For our purposes, it is sufficient to simply point out that the trigonometric calculations could be carried through.[34]

5. *Historical Commentary*

5.1 The Provenance of Dioptra *35*

Chapter 35 is peripheral to the overall project of the *Dioptra*. The text as a whole is devoted to surveying practices using the eponymous instrument.[35] Heron tells us that he included the geodesic material in *Dioptra* 35 so that his treatise should be complete. We saw above that Heron's account of the method of mensuration is cursory at best. It is unlikely that Heron devised this complex astronomical method and then proceeded to give such a brief sketch of his own original work. *Dioptra* 35 is almost certainly a summary of a more complete work of mathematical astronomy or geography. Rome, on the basis of passages in Strabo's and Ptolemy's works on geography, believe that the technical origins of Heron's method were in a work by Hipparchus (Rome 1923, p. 249).

According to Strabo, Hipparchus, in a polemical work against Eratosthenes, stated that it is not possible to obtain geographic knowledge without a determination of the heavens and of observations of eclipses (Dicks 1960, p. 6; Jones 1969, vol. 1, pp. 22–24).

> For example, it is not possible to grasp whether Alexandria in Egypt is more northerly or more southerly than Babylon, nor the quantity of the interval, apart from an investigation by means of latitudes (διὰ τῶν κλιμάτων). Similarly, one cannot know, with any accuracy, if the extension (τὰς ... προσκεχωρηκυίας) to the west or east is more or less, except by means of comparisons of solar and lunar eclipses.

In this passage the north-south and east-west intervals between two cities are specified by the same means as in *Dioptra* 35.

Ptolemy, in his *Geography*, again associates Hipparchus with these measures of geographic position. In *Geog.* I 4, he tells us that Hipparchus was the only person who observed the elevation of the pole for certain cities as a measure of latitude. Later in the same paragraph, he goes on to explain, without mentioning Hipparchus, the usefulness of simultaneous eclipse observations for determining longitudinal differences (Berggren and Jones 2001, pp. 62–63).

It is reasonable to agree with Rome that the mathematical methods in *Dioptra* 35 go back at least as far as Hipparchus. We saw above that the great arc distance can be calculated using analemma methods. Neugebauer has shown that Hipparchus made use of such techniques in his work on spherical astronomy (Neugebauer 1975, pp. 301–302). We may take *Dioptra* 35 as another witness to this fruitful tradition of spherical geometry.

5.2 Heron's Date

Neugebauer used the data mentioned in *Dioptra* 35 for the eclipse observations as the basis for dating an actual eclipse and claimed that this real eclipse could be used, in turn, to date Heron (Neugebauer 1938, p. 23). The year which he found, 62 CE, has been agreeable to historians and is now generally accepted without question (see, for examples, Drachman 1950, p. 137; Lewis 2001, p. 54; Irby-Massie and Keyser 2002, p. 117). Neugebauer's claim is that the eclipse data in the text are so 'ill suited' to the problem that Heron must be appealing to the 'recent memory of his readers' (Neugebauer 1975, p. 846).

Neugebauer's argument, and the date which follows from it, is based on three assumptions, none of which is certain. These are as follows: (1) that the problem is solved using purely nomographic techniques, (2) that the data refer to a lunar eclipse which was actually observed, and (3) that the eclipse of 62 March 13 is the only one which satisfies the data.

Assumptions (1) and (2) are interdependent. If the problem is purely nomographic, an eclipse which is only 10 days away from an equinox will be awkward to model on the analemma. On the other hand, if the problem is solved by calculation, then any observed or hypothetical eclipse will do. Since we have shown that it is possible to solve the problem by calculation, we must admit the possibility that Heron is using a hypothetical eclipse. Certainly, the time for one of the two cities must be hypothetical or badly reported, since the two are actually only about an hour and ten minutes apart.

Neugebauer simply asserts that the only lunar eclipse which satisfies the data is that of 62 March 13. An examination of Liu and Fiala (1992), however, shows that although this eclipse possibility is the best fit for the time datum, it is not for that of the date.

Searching Liu and Fiala (1992) for umbral eclipses[36] between −200 and 300, which are both 10 ± 3 days prior to the vernal equinox and within ± 5 hours of the stated times for either Alexandria or Rome,[37] gives three possible eclipses.[38] These margins of error allow for the vagaries of timekeeping in antiquity and possible difficulties in dating the equinox in the ancient calendars. Where the times for the beginning of the eclipses and the equinoxes are local to Alexandria,[39] these are as follows (Table 1).[40]

Table 1. Eclipse possibilities for Heron's data.

Type	Eclipse date			Time		Equinox date		Time	
Um	−133	March	10	4	36	March	21	3	26
Um	−3	March	12	1	41	March	20	16	02
Um	62	March	13	22	50	March	20	10	20

It is clear that, although the time for the eclipse of 62 March 13 is good, the date is too close to the eclipse. For the eclipse of −133 March 10, the date is better, while the time is too late. The errors for both the date and the time of the eclipse of −3 March 12 are more modest than those of the other two eclipses. Most importantly, the differences between the data Heron gives and the eclipse reports in Liu and Fiala (1992) show that the data in *Dioptra* 35 cannot correspond to an accurately recorded real eclipse. As Steele (2003) has recently argued, we must be wary of using dubious or inaccurate observations for dating purposes.

There is nothing in Heron's language to suggest that he actually made the observation he reports. He simply says that it is possible to state eclipse observations due to their relative frequency. He does not say that he himself ever made such an observation. His expression leaves open the possibility that the data he gives are supposed, calculated, or modified from an actual eclipse. In any of these cases, we cannot use the data to date an actual eclipse.

I have no wish to refute Neugebauer's date for Heron. Nor do I wish to offer any other date in its place. Nevertheless, I believe we should admit that the arguments for Neugebauer's date are weak or nonexistent. It is quite possible that Heron was working in the year 62 CE, but if future research should suggest that we prefer a different date, we should not, on the basis of Neugebauer's claim, rule out this possibility.

6. *Final Remarks*

It is clear from Heron's presentation that he is primarily interested in the nomographic aspects of this problem. Indeed, the final, mutilated passage makes little sense outside the context of nomography. On the other hand, it is also clear that Heron is treating the mathematical underpinnings in a summary fashion. His expression for the most important operation in the procedure suggests an involved mathematical process. It is probable that the source he is abridging contained the full mathematical details, including some method of calculation.

Our primary witness to nomographic techniques in antiquity is Ptolemy's *Analemma* (Heiberg 1907, pp. 189–223; Edwards 1984). The nomographic aspects of this text were first elucidated by Luckey (1927), a modern expert in nomography. Ptolemy's primary contribution, in the *Analemma*, is a set of instructions for finding certain arc lengths on a specially prepared plate. Before he presents his nomographic innovations, however, he first demonstrates, using metrical analysis, that the arc lengths in question can also be calculated precisely. The source of *Dioptra* 35 should be seen as an earlier work in the same tradition as the *Analemma*, solving problems in spherical astronomy through both computational and nomographic methods.

In the *Analemma*, the nomographic procedure gives a real practical advantage over calculation. One of the goals of the work is the production of 49 tables containing a total of 2,058 entries. Using ancient trigonometric methods, calculating these individually would be a considerable and tedious labor. In the case of *Dioptra* 35, however, it is unlikely that the nomographic procedures were devised to solve a practical difficulty. Despite the fact that the nomographic measurement would be considerably faster than calculation, the time consumed in constructing an accurately inscribed hemisphere with a graduated scale would have to be weighed against the number of times that the calculation was to be carried out. It is unlikely, however, that this method of calculating terrestrial distances was actually used more than once or twice, if at all. Only three ancient authors mention specific simultaneous eclipse observations: Heron, Pliny, and Ptolemy.[41] Two of these, however, are of the same eclipse and two of the three reported pairs of times contain errors.

We saw above that at least one of the times which Heron reports must be hypothetical. Before he sets out his pair of eclipse times, however, he states that we should first check the recorded observations. The fact that he used at least one arbitrary time suggests that he could not find any actual pairs in the records at his disposal.

Both Ptolemy and Pliny give simultaneous observation times for the famous eclipse that was seen before Alexander defeated Darius at the battle of Arbela. Pliny records the times of this eclipse fairly accurately, but the time he gives for the Arbela observation differs from that in Ptolemy. There are problems with both the observations which Ptolemy reports. They are inconsistent with Pliny, modern computation, and the geographic distances between the places of observation. The observations of this eclipse probably entered the technical literature from the historical accounts of Alexander's conquests. Ptolemy does not explicitly state that this was the only pair of simultaneous eclipse observations of which he was aware, but he does make it clear that he knew of few, if any, others.

Since neither Heron nor Ptolemy knew of enough simultaneous eclipse observations to make any real use of them, it is unlikely that the author of the mathematical methods of *Dioptra* 35 did either. Hence, we must read *Dioptra* 35 as witness to a theoretical tradition of applied mathematics. The exact methods of metrical analysis and trigonometric computation, as well as the nomographic procedures designed to expedite these, were not produced in order to handle an existing set of observations but to solve the problem in principle and to encourage future observational practices. Hence, *Dioptra* 35 demonstrates that nomographic techniques were not simply of interest to Greek mathematicians for the advantages they brought to facilitating calculation but also because they were a rich source of new problems and new methods of solution.

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NOTES

1. Neugebauer (1975, pp. 845–848) gives an overview.
2. Drachman (1950) should also be mentioned in regard to this material. This paper is an attempt to reconcile certain mathematical methods of Heron and Ptolemy. Drachman makes some strange readings of Ptolemy's *Analemma*, and his reconstruction of the mathematics is unlike anything we find in

ancient works on the analemma. For example, he claims that Ptolemy used equinoctial hours in the *Analemma*, and he projects circles orthogonally onto the receiving plane creating ellipses (Drachman 1950, pp. 118 and 123–125).

3. My usage of the term *nomography* follows Neugebauer (1975). It designates a tradition of graphic procedures which are carried out using the theorems of ancient geometry. Through these techniques, line segments or arcs are produced which are then physically measured.
4. For overviews of the ancient analemma, see Evans (1998, pp. 132–141) and Neugebauer (1975, pp. 839–856). In fact, Ptolemy's *Planisphaerium* 18 also shows evidence of an analemma construction (Anagnostakis 1984; Heiberg 1907).
5. Berggren (1980) summarizes four of these (also, see Carandell 1984 and Berggren 1992).
6. The term *gnomon* is adopted from the name of the upright, shadow-casting pointer in a horizontal sundial. The whole line ΟΠ is called the *gnomon* by Heron and Ptolemy (Schöne 1903, p. 304; Heiberg 1907, p. 190 ff.). Vitruvius, on the other hand, uses *gnomon* to designate the line ΕΠ (Soubiran 1969, pp. 26 ff.). Vitruvius is thinking of an actual sundial where *gnomon* has its original meaning. For Heron and Ptolemy, *gnomon* is clearly an abstraction and probably has the same meaning as our term *normal*.
7. This term is used by Heron (Schöne 1903, p. 304). A possible synonym is *loxotomus*, a correction, not elsewhere attested, for the variations on *locothomus* which appear in the Vitruvius MSS (Soubiran 1969, p. 29). Neugebauer (1975, p. 844, n. 7) holds that *loxotomus* denotes the same line as *dioron*. Soubiran (1969, pp. 228–230), on the other hand, following Degering, states that there is a lacuna in the text and a different line is meant.
8. In principal, seasonal hours divided each of the daytime and nighttime periods into 12 equal parts; so that, for all latitudes but the equator, these hours varied throughout the year.
9. Neugebauer (1975, p. 845) gives a proof of this statement.
10. For other translations, see Irby-Massie and Keyser (2002, pp. 136–138) and Rome (1923, pp. 235–241). I have suppressed most of Schöne's editorial marks.
11. Heron treats the two methods of distance measurement mentioned in this sentence previously in the *Dioptra*. That of the dioptra involves the sighting instrument which is the subject of the whole work. That of the hodometer involves a complicated mechanical device designed to solve the problem of measuring linear distance along roads. See Lewis (2001, pp. 97–98, 268–269, and 279–280; 283–285 and 134–139) for a translation and discussion of this material.
12. This is a strange expression. Perhaps, it is a technical idiom for a certain type of sundial construction. On the other hand, it may convey the more general idea that the hemisphere is arranged among the circles of the tropics.
13. This expression corresponds to 'the hemisphere through the tropics', mentioned in note 12. It seems to signify both a general sense of the solid geometry associated with a particular configuration of spherical astronomy and the technical geometric construction this implies.
14. This passage is actually found below, at the beginning of sentence [18]. Rome observed that it was out of place and stated that it should be at the beginning of the analemma construction (Rome 1923, pp. 239, n. (**)). Without argument, however, he stated that if it were so placed, there would be no explanation for its displacement. Hence, he placed it further on, just before the axis is required, at the beginning of sentence [22]. Irby-Massie and Keyser (2002, p. 138) follow Rome without giving the reader any indication that the text has been rearranged. Neugebauer (1938, p. 14) apparently had no problem with the text as it stands. If indeed this passage is misplaced, I see no reason why it cannot be here where it belongs mathematically, with the rest of the analemma exposition. This would make grammatical sense as well, since the ἐσθω can be picked up by the following clauses which lack verbs. The μέν/δέ configuration would then be off, but this is easily explained by the work of a later scribe trying to sort out the mess.
15. These are seasonal hours (see note 8 above).

16. Rome rightly adds A′, since its position is assumed in what follows (Rome 1923, p. 239, n. 1).
17. This passage is corrupt. Schöne reads καὶ τῇ ΞΩ περιφερείᾳ ὁμοία κείσθω ἡ A′B′, ἀπὸ δὲ τοῦ CA′ τετραγώνου κείσθω ἡ A′B′Z, and Rome reads καὶ τῇ ΞΩ περιφερείᾳ ὁμοία, ἀπὸ δὲ τοῦ A′, τετραγώνου κείσθω ἡ A′B′. Both of them translate τετράγωνος as 'quadrilateral', a meaning not elsewhere attested in the mathematical corpus (Schöne 1903, p. 303; Rome 1923, p. 240). I read the text as καὶ τῇ ΞΩ περιφερείᾳ ὁμοία κείσθω ἡ A′B′, ἀπὸ δὲ τοῦ EA′ τετραγώνου κείσθω ἡ A′B′ which allows us to take τετράγωνος to mean 'quadrant', a common meaning in the technical literature (Mugler 1959, pp. 419–420).
18. These final passages are corrupt and cannot be read as continuous text. My translation follows (Rome 1923, p. 241) with minor changes. The geometrical considerations underlying my reading are given below, page 245.
19. The text reads τὴν Γ. 'Point G' would be τὸ Γ. The τήν probably refers to an arc and a label may have dropped out.
20. I have replaced ΣB with ⟨E⟩B′ and B with B′. As Rome (1923, p. 245) points out, ΣB cannot be right because Σ is on the analemma, B on the sphere. Moreover, the goal of the entire construction is to find B′. Point B is irrelevant after the exposition.
21. Where necessary, I justify steps in the mathematical argument by reference to canonical texts in the Greek tradition. Hence, *Elem.* IV 6 denotes the sixth proposition of Book IV of Euclid's *Elements*; Theo. *Spher.* I 20, the 20th proposition of Book I of Theodosius' *Spherics* (Heiberg and Stamatis 1969; Heiberg 1927).
22. I reconstruct these calculations using the methods of ancient mathematics which I have called *metrical analysis* (see Sidoli 2004b). This is more laborious than using our current trigonometric practice, but staying in the ancient idiom saves one the trouble of having to translate back and forth between different mathematical styles. As is customary, the steps of the mathematical argument are justified by reference to Euclid's *Data*, despite the fact that metrical analysis is a later, arithmetical tradition (Menge 1896; Taisbak 2003).
23. In the rest of this section, the argument often depends on comparing and transferring objects between the hemisphere and the analemma. Objects in the hemisphere can be located in Figures 2a and 3; those on the analemma, in Figure 4. Due to the structure of the Greek and Roman alphabets, it was necessary to repeat some labels. In most cases, the named object only exists in one of the two figures.
24. This construction is often assumed in the literature on spherics (see Sidoli 2004a).
25. A number of these solutions are given by Pappus in his *Collection* (Hultsch 1876, vol. 1, pp. 272–288).
26. This step is not directly supported by a proposition of the *Data*; nevertheless, it is made possible by the use of a chord table. Taisbak (2003, p. 226) proposes a variant of *Data* 87 which justifies this step.
27. For example, Ptolemy uses the cognate noun ἐξέτασις in this way a number of times in the *Almagest* (Heiberg 1898, pt. 1, pp. 1, 47, 301, 402, 407, 473; pt. 2, p. 3; Toomer 1984). Indeed, in a few of these cases, the word can be suitably translated by *calculation*. An occurrence in Nicomachus' *Arithmetic* should be included among these. A passage in *Arith.* I 9, which reads τούτων δὲ κατὰ τὴν αὐτὴν ἀμοιβὴν καὶ ἐξέτασιν ὑποδιπλάσιον τὸ μέσον τῶν δύο ἄκρων συντεθέντων, should probably be translated 'but in this case, in accordance with the same reciprocal relationship and calculation the mean is half the sum of the two extremes' (Hoche 1866, p. 21).
28. My interpretation follows Rome (1923, pp. 244–247).
29. The mathematical aspects of the problem of finding the qibla are summarized by King (1979, reprinted in King 1993, sec. IX, pp. 1–18).
30. See note 5.
31. In fact, this simple conversion is only possible when equinoctial hours are used. For seasonal hours, as in our case, a precise conversion can be effected by means of the analemma.

32. See Ptolemy's *Analemma* 6 for evidence of this way of using the analemma model (Heiberg 1907, p. 197).
33. In terms of the analemma model, where east and west are defined as points on the great circle of the horizon, the nadir of Rome is southeast of the nadir of Alexandria, since Rome is northwest of Alexandria.
34. It should be noted that this configuration can be used to transform between ecliptic and equatorial coordinates. Hence, my claim (in Sidoli 2004b, p. 79) that the analemma cannot be used to effect such a transformation of coordinates is unfounded. In this case, in Figure 7, ADBC will be the solstitial colure, AB the diameter of the equator, and GF the diameter of the ecliptic. Assume point N is given in ecliptic coordinates, $N(\lambda, \beta)$. Then, the necessary arcs are given, $\overset{\frown}{BH} = 90° - \varepsilon$, $\overset{\frown}{GJ} = \beta$, and $\overset{\frown}{MN} = 360° - \lambda$. Hence, points P and R are determined and the equatorial coordinates of N are given, $\overset{\frown}{TC} = \alpha$ and $\overset{\frown}{AV} = \delta$.
35. Lewis (2001, pp. 53–56) gives a description of the *Dioptra*. A translation of most of the text, excluding *Dioptra* 35, is also provided (Lewis 2001, pp. 259–286).
36. Steele (2000, p. 10) points out that there are 'no firmly dated observations of penumbral eclipses from pretelescopic times'.
37. Steele (2000, pp. 100–102) shows that early Greek astronomers generally recorded the times of eclipse observations by the beginning of the eclipse. See Berggren and Jones (2001, pp. 29–30) for a discussion of the timing errors involved with the only other simultaneous eclipse observation reported from Greco-Roman antiquity.
38. A fourth possible eclipse should be mentioned, although it is much too late. The total lunar eclipse of −87 March 11 began at 5:11 in Alexandria.
39. The values for the times are based on the rough assumption that Alexandria is exactly 2 hours ahead of UTC.
40. The assumption that the local time is right for Rome gives no new possible eclipses because the error in the time difference is only about 50 min.
41. Heron *Dioptra* 35, Pliny *Nat. Hist.* II 180, and Ptolemy *Geog.* I 4 (Rackham 1949; Berggren and Jones 2001). For discussions of these, see Dicks (1960, pp. 121–122), Berggren and Jones (2001, pp. 29–30), and Swerdlow (2003, p. 316).