Drawing Diagrams and Making Arguments in Greek Geometry

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Opening Remarks

Our evidence for ancient Greek mathematical activity comes, almost exclusively, from texts that were passed down through the medieval period in contexts that were largely removed from mathematical activity and by individuals who, by and large, did not themselves produce original mathematics. Indeed in some sense, the majority of the texts we still posses may have been originally written as objects of transmission and hence meant to satisfy a different set of criteria than texts which might have been written to train mathematicians or to more directly bear on mathematical activity. Nevertheless, the individuals who originally composed the texts were engaged in a range of activities of which writing treatises was just one component. The evidence for this, however, is only rarely explicit. In this talk, I will make a number of more or less speculative attempts to try to uncover a broader range of mathematical activity – activity that would have included, for example, drawing accurate figures and constructing models for the sake of producing new knowledge, working with texts and figures in the course of mathematical study or research, making oral presentations both for the sake of passing on mathematical knowledge and in order to discuss new results.

In order to set the stage, let us consider the following tripartite division of mathematical activities for which we have evidence in Greco-Roman antiquity.

Oral practices: Mathematics as a cultural activity came to be practiced in schools that were, first and foremost, schools of philosophy and rhetoric. Mathematicians themselves were, by and large, taught by philosophers and they well understood how to present themselves in oral disputation so as to gain the rational consent of their audience. There is much evidence for these oral practices in texts such as Euclid's *Elements* or Theodosius' *Spherics*, which were clearly intended for an elementary audience. The format of the propositions lends itself to oral presentation and memorization. For example, the fact that earlier theorems are cited by a

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synopsis of the enunciation indicates that the listeners were probably expected to memorize the enunciations.

Literary practices: (A) On the one hand, Greek mathematicians were members of a small class of individuals in Greco-Roman society who produced literary works meant for posterity (Netz 2002). When they were not writing technical mathematics, they took pains to employ the convoluted constructions, intricate phrases and literary tropes that would secure their position among this group (Mansfeld 1998). Indeed, even the structure of mathematical prose shows borrowings from the poetic tradition (Netz 1999). Hence, they produced their treatises as objects of literary transmission. (B) On the other hand, literary practices involved the material production of written words, diagrams, tables and such in rolls or on tablets. Whereas the majority of those who studied Euclid or Ptolemy probably did not own a copy of the text, the small number of individuals who produced original mathematics probably belonged to the narrow class for whom it was possible to own such works, or they lived and worked in places where such treatises were available to them. Indeed, it is clear that the usage of texts as material objects changed throughout the period of Greco-Roman mathematical activity. While it is possible to do geometry in the style of Euclid's *Elements* by means of oral presentation and memorization, only a deranged mind would attempt to memorize Ptolemy's table of chords (*Almagest* I 11), without which it is not possible to carry out the calculations involved in ancient trigonometry.

Material practices: As well as writing tools, treatises and tables, Greco-Roman mathematicians employed a variety of instruments in the production and transmission of their field of study (Sidoli and Saito Forthcoming). As well as the basic straightedge and compass, they used instruments that were drawn from various artistic and scientific fields such as architecture, surveying, observational astronomy, sun dial construction, ornamental globe making, and so forth. Indeed, these disciplines largely drew their principles from mathematics. Moreover, it is clear from the commentators that ancient mathematicians designed special instruments in order to make accurate drawings of the objects they studied (Netz 2004).

It is clear that these three divisions are neither absolute nor exhaustive. Since the practitioners themselves carried out activities in all three categories, most of their practices fit into more than one place in the schema. Nevertheless, in what follows we will find this schema useful as a general guide in our inquiry into ancient Greek mathematical practice.

In order to show how the texts that are edited from the medieval manuscripts may be used to shed light on ancient mathematical practice, we will look at three specific examples. These examples have been selected because they each give evidence for a variety of different mathematical practices.

Archimedes' *Sphere & Cylinder* **II 7**

Archimedes' *Sphere & Cylinder* was clearly a literary production (Netz 2004). It was sent in the form of two letters by Archimedes, in Syracus, to Dositheus, in Alexandria, who, although not himself a mathematician, was at least sympathetic to mathematical works and might bring the treatise to the attention of someone who could really appreciate it. Hence, Archimedes expected that his potential audience would encounter the work as a written object.

Nevertheless, some propositions of *Sphere & Cylinder* can be taken as evidence for the oral practice of presenting arguments in a public format. In particular, we will consider *Sphere & Cylinder* II 7. This is an analyzed proposition with both an analysis and a synthesis, each of which has a separate diagram. In fact, however, for the purpose of reading and understanding the proposition a single diagram will do. The use of two diagrams, found in the majority of analyzed propositions from the Hellenistic period, probably derives from oral presentation, in which the mathematician would draw two different diagrams as he talked his audience through the solution of the problem. This would help the reader understand the distinction between the assumed solution that is analyzed and the actual solution that is constructed, to follow the actual steps of the construction and to see clearly the different approach of the two arguments.

Sphere & Cylinder II 7 shows how to cut a given sphere with a plane such that the section of the sphere standing on the plane has a given ratio to the cone under the same hight and on the same base.
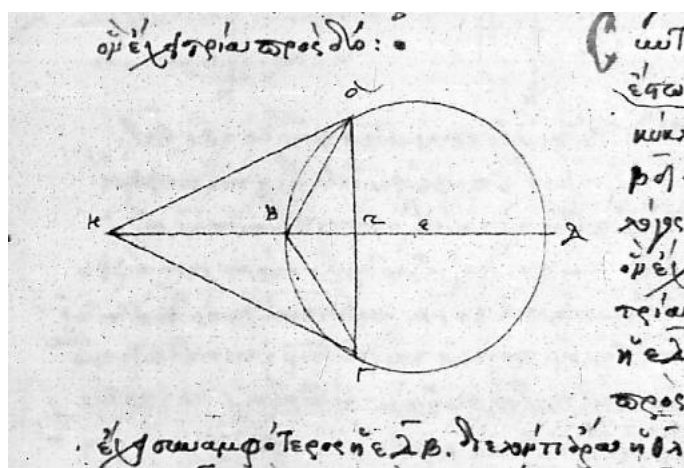

Figure 1: Marciana Library, Venice, Gr. 305: *Sphere & Cylinder* II 7, analysis

In the manuscripts, all the steps of the analysis are included in a single diagram, which comes at the end of the analysis. Nevertheless, in order to understand the oral context in which the practice of using multiple diagrams most likely originated, we will work through the proposition using a number of different diagrams in an attempt to

simulate the process of drawing the diagram in front of an audience and talking through the proof.
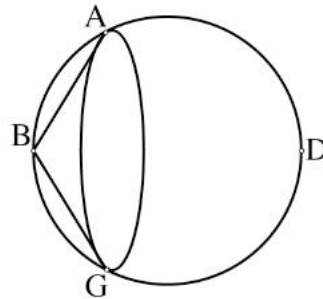

Let us begin by assuming that we have before us some given sphere, $ABGD$, which is cut by plane AG such that cone ABG has to spherical segment ABG some given ratio. We now introduce an auxiliary construction which will allow us to construct a cone equal to spherical segment ABG standing on the same base as cone ABG. This will transform the given ratio between the segment and the cone into one between two cones, which are more readily compared.
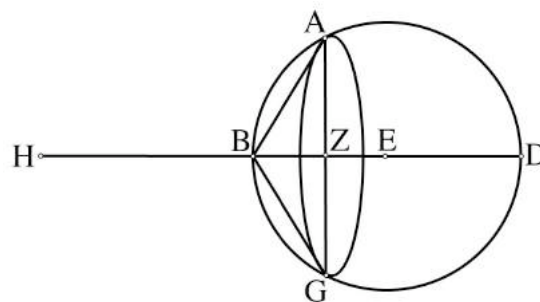
We join the center of the sphere, Z, with the vertex of the cone, B, and produce the line to point D and beyond the sphere. We cut off point H such that

$$(ED + DZ) : DZ = HZ : ZB.$$

Hence, by *Sphere & Cylinder* II 2, the cone under height HZ is equal to the spherical segment under height ZB.
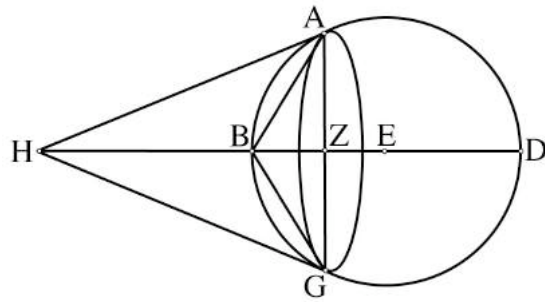
We complete cone AHG, so that the ratio of cone ABG to cone AHG is given. Hence, by *Elements* XII 4,

$$BZ : HZ \text{ is } given.$$

Therefore,

$$(ED + DZ) : DZ \text{ is } given,$$

and by *separation*

$$ED : DZ \text{ is } given.$$

Since line ED is the radius of a given sphere, line DZ is given, therefore line AG is given. Here, as generally in Greek geometrical analysis, to be given means to have been furnished at the start or constructible on this basis.
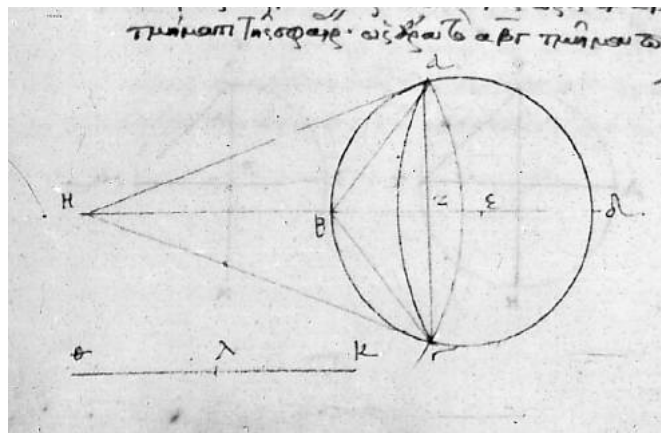

Figure 2: Marciana Library, Venice, Gr. 305: *Sphere & Cylinder* II 7, synthesis

For the synthesis, the manuscripts have a second figure, found at the end of the proposition. Again, however, we will work our way through the steps of the construction.

Since, in the analysis, the given ratio was set equal to $(ED + DZ) : DZ$, by the geometry of the sphere, the upper limit to the given ratio must be $3 : 2$. Hence, there is a limit to the solvability of the problem.
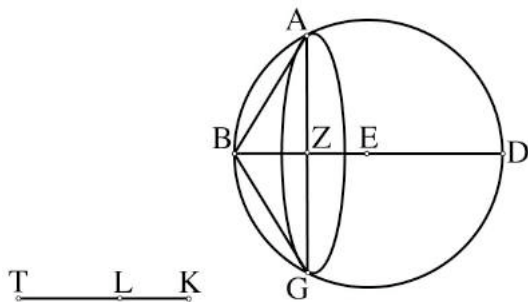
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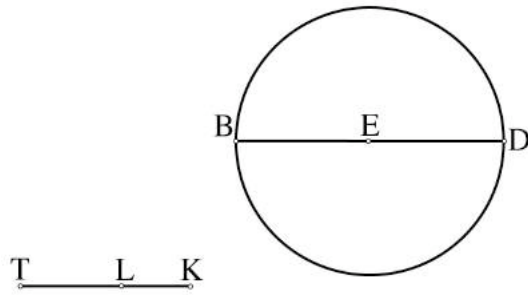
The synthesis begins from the assumption of an uncut sphere, AB about center E, and a given ratio, $TK : KL > 3 : 2$. This new set of starting points is made concrete in a new figure.
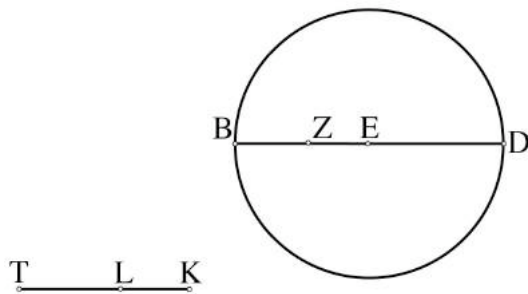

Point Z is constructed by setting

$$TL : LK = ED : DZ.$$

We pass a plane through point Z perpendicular to line BG and construct cone ABG on the plane through AZG. This plane solves the problem.

For the proof, we introduce the auxiliary construction used in the analysis. This is not necessary for the construction of the solution itself, but for the proof that the solution is valid.

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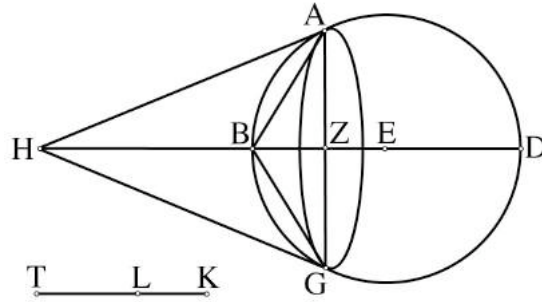
We extend diameter DB and find point H such that

$$(ED + DZ) : DZ = HZ : ZB.$$

Since, by construction

$$TL : KL = ED : DZ,$$

by *composition*

$$
\begin{aligned}
TK : LK &= (ED + DZ) : DZ \\
&= HZ : ZB \\
&= \text{cone } AHG : \text{cone } ABG.
\end{aligned}
$$

Cone AHG, however, is equal to spherical segment ABG. Therefore, plane AZG solves the problem.

In *Sphere & Cylinder* II 7 we see an example of a style of mathematics which almost certainly originated in the context of oral presentation or discussion. Since the use of two figures is not necessary for a written document, it must have originated in an oral context. Although Archimedes composed *Sphere & Cylinder* in the form of a letter, as a written document, he may have hoped that it would eventually reach an audience that would want to present the material in a public format, perhaps in a study group or lecture.

We turn now to an example of a mathematical text that, on the contrary, was probably written to be studied as a written document by someone from the very small group of individuals who sought to produce original mathematical results.

Apollonius *Cutting off a Ratio* 6.4

Apollonius' *Cutting off a Ratio* is a long, dry repetitive work exhaustively solving a simple problem that seems to have little inherent interest and was not fundamental to any part of ancient geometric theory. Apollonius' purpose in composting the treatise was presumably to train others in the problem solving art known as analysis by taking them

through the full details of a simple example. The work is full of long chains of mathematical operations that are difficult to follow if one does not work one's way through them by jotting down some notes, the structure of the argument is often convoluted and it is the only treatise of Hellenistic mathematics that we posses that makes repeated reference to other parts of itself by case and section number. All this leads to the conclusion that Apollonius wrote *Cutting off a Ratio* for individuals who would have access to the treatise as a material object while they worked their way through the argument. This is probably why it was preserved through the medieval period only in an Arabic translation, since during this time, while there were many people carrying out mathematical research in Arabic, there was almost no one doing so in Greek.

Cutting off a Ratio solves the following problem. Given two lines and two points on them and a point not on the given lines to draw a line through the independent given point such that it falls on the two given lines cutting segments adjacent to the two former points that have to one another a given ratio. The text solves this problem for all geometrically significant configurations of the given objects and for all possible ways in which the solution can fall.
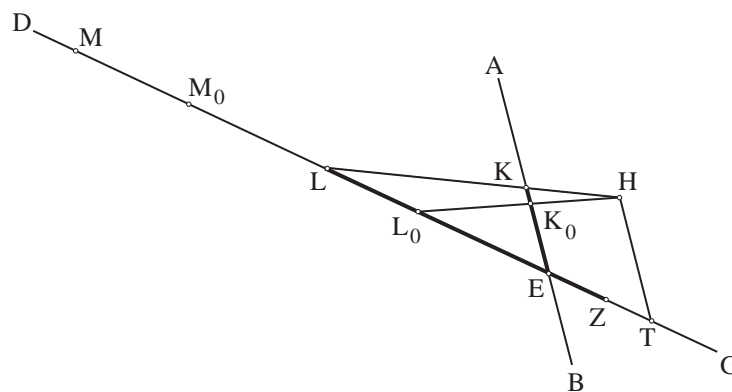

Figure 3: *Cutting off a Ratio* 6.4, overview

Cutting off a Ratio 6 handles the following case. In Figure 3, let the given lines be AB and GD, meeting at point E and let the given point not on the lines be H. Let the given points on the lines be the intersection E and some other point on GE, say Z. *Cutting off a Ratio* 6.4 solves the problem of constructing a line through H, say HL, falling on EA and ZD such that

$$EK : ZL = r,$$

where r is a given ratio.

In the manuscripts, there are three diagrams for this case of the problem. Since the auxiliary constructions in this case are quite rudimentary, it is likely that Apollonius did

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not expect his readers to redraw as they worked through the details of the argument but simply used the diagram to structure the overall argument in deference to the tradition of introducing a new diagram for each part of an analyzed proposition.
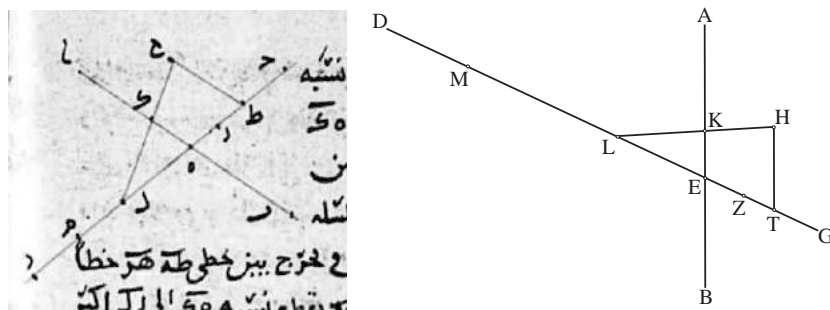

Figure 4: *Cutting off a Ratio* 6.4, analysis, Aya Sophia 4830 and a reconstructed figure

The analysis proceeds as follows. Where lines AB, GD and points H, E and Z are given, we assume that there exists some line, HL, cutting two segments from the original lines, EK and ZL, such that $EK : ZL$ is a given ratio. We draw HT parallel to line AB and take point M on ZD such that

$$TH : ZM = EK : ZL.$$

By *alteration*,
$$TH : EK[= TL : LE] = ZM : ZL,$$

and by *conversion*,
$$TL : TE = MZ : ML$$

therefore,
$$(MZ \times TE) = (TL \times LM). \tag{1}$$

Then, since both MZ and TE are given, a given rectangle, $(TL \times LM)$, has been applied to a given line TM, deficient by a square, therefore, by *Data* 58,

$$L \text{ is } given,$$

and hence

$$LH \text{ is } given.$$

Once again, given means constructible through the postulates and problems of elementary geometry.

The diorism, which we will not follow in full detail, is in three parts. In Figure 5, since, TE and MZ are given by the geometry of the initial configuration, it may
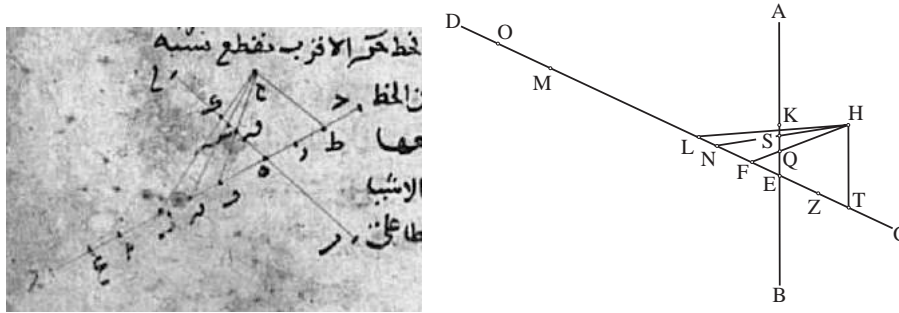
Figure 5: *Cutting off a Ratio* 6.4, diorism, Aya Sophia 4830 and a reconstructed figure

not always be possible to take a point L between E and M, such that $(MZ \times TE) = (TL \times LM)$. Moreover, since $(TL \times LM)$ will be greatest when $TL = LM$, this is a limit to the possible solutions. Hence, we have the following problem. How to find points L and M such that $EK : ZL = TH : ZM, (MZ \times TE) = (TL \times LM)$ and $TL = LM$. It should be noted that in the diorism, $EK : ZL$ is not the original given ratio but a special limiting ratio, whose properties are yet to be determined.

The problem is solved using a standard analyzed proposition. The analysis shows that if the three conditions are assumed, EL will be a mean proportional between lines TE and EZ, such that $TE : EL = EL : EZ$, and proceeds to show that this implies that both points L and M are given. The synthesis, then, constructs point L, straightforwardly, by setting $TE : EL = EL : EZ$, using *Elements* VI 13, and M by setting $TL = LM$, and uses ratio manipulation to show that $(MZ \times TE) = (TL \times LM)$ and $EK : ZL = TH : ZM$.

The next step of the diorism is to show that the limit obtained by setting $TL = LM$ is an upper limit. Apollonius shows this by drawing another line, say HN, and then showing that the ratio cut off by HN is less than that cut off by HL, that is $EK : ZL > ES : ZN$. This is also done with an analyzed proposition, but of a kind not generally found in Greek mathematical works. First, we assume that there must be some relation between the two ratios, that is

$$EK : ZL \ [?] \ ES : ZN.$$

We, then, use ratio manipulation and the geometry of the figure to convert this undetermined relation to one that we know. We argue as follows.

Since
$$EK : ZL = TH : ZM,$$

we have,
$$TH : ZM \ [?] \ ES : ZN,$$

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and, by *alteration*, we have,

$$TH : SE \ [?] \ ZM : ZN.$$

But,
$$TH : SE = TN : EN,$$

so we have,
$$TN : EN \ [?] \ ZM : ZN,$$

and *by conversion* this becomes,

$$TN : TE \ [?] \ ZM : NM,$$

so we have,
$$(ZM \times TE) \ [?] \ (TN \times NM).$$

But, by (1),
$$(ZM \times TE) = (TL \times LM),$$

so that we have,
$$(TL \times LM) \ [?] \ (TN \times NM).$$

Now, since point L is the midpoint of TM,

$$(TL \times LM) > (TN \times NM).$$

We can then use this relation to work backwards through the same set of steps to show that $EK : ZL > ES : ZN$, and this is exactly what Apollonius does.

The final stage of the diorism is to show that the ratios vary monotonically as the lines approach HL. This is done by showing that any other line, taken farther from HL than HN, as say HF, cuts off a still lesser ratio. Apollonius shows this by an analyzed proposition analogous to that we just saw. He assumes that there exists some relation $ES : ZN \ [?] \ EQ : ZF$, uses ratio manipulation and the geometry of the figure to reduce this to some known relation and then proceeds backwards through the same steps to show that $ES : ZN > EQ : ZF$.

The three parts of the diorism together show that when L is the midpoint of TM, HL is an upper limit to the possible solutions and solutions cutting off lesser ratios are arranged in pairs on either side of point L.

For the synthesis, Apollonius begins with a new figure, see Figure 6. Let ratio $N : S$, lines AB and GD and points E, Z and H be given. Then $N : S \gtreqless EK : EL$. Thus,

(1) where $N : S = EK : EL$, the problem is solved by HL alone,

(2) where $N : S > EK : EL$, there is no solution, and

(3) where $N : S > EK : EL$, the problem is solved as follows.
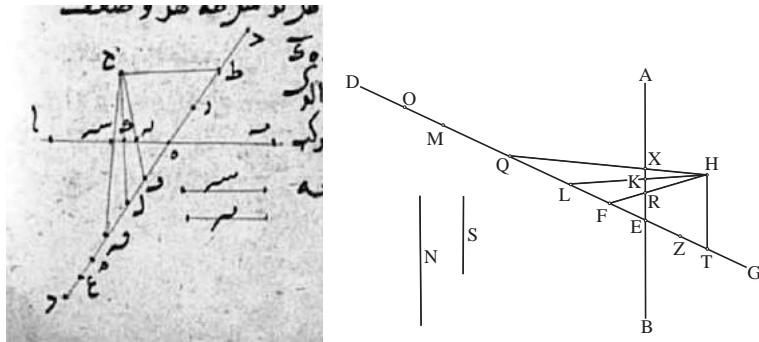
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Figure 6: *Cutting off a Ratio* 6.4, synthesis, Aya Sophia 4830 and a reconstructed figure

We set $TL = LM$, so that

$$(TL \times LM) = (TE \times ZM)$$

and

$$EK : ZL = TH : ZM.$$

Then, since

$$N : S < EK : ZL = TH : ZM,$$

we take point O such that

$$N : S = TH : ZO.$$

Then, since

$$
\begin{aligned}
(TL \times LM) &= (TL \times LO) - (TL \times MO) = \\
(TE \times ZM) &= (TE \times ZO) - (TE \times MO),
\end{aligned}
$$

while

$$(TL \times MO) > (TE \times MO),$$

therefore,

$$(TL \times LO) > (TE \times ZO).$$

Hence, it is possible to apply a rectangle equal to $(TE \times ZO)$ to line TO being deficient by a square at two points equally distant from the midpoint of TO, and thus on either side of point L. These points are constructed as F and Q. We then use ratio manipulation to show that if

$$(TE \times ZO) = (TF \times FO) = (TQ \times QO),$$

then

$$N : S = ER : ZF = EX : ZQ,$$

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so that both lines HF and HQ solve the problem.

Although this problem clearly belongs in the domain of geometry – and particularly to the geometry of position, in which Apollonius seems to have had a special interest – it is presented in such a way that it can best be understood by outlining the overall structure of the argument and then following through the details by actually writing out the steps of the proof. We have followed only one of the many chains of ratio manipulations in detail, but Apollonius gives them all in full and he must have intended his audience to work through them in detail. Here we have a clear example of a text that was written for a readership that was expected to experience the work as a literary object – that is to have the text in hand and be able to work through its contents with the aid of writing instruments.

Theodosius' *Spherics* II 15

We turn now to a text that was intended for a beginning audience and which by late antiquity was being taught as part of the field of astronomy. Theodosius' *Spherics* bears the marks of a work that was often presented in an oral context, probably as a series of lectures to audiences interested in basic mathematical astronomy. It is full of verbose explanations, and it cites references to its own propositions and to propositions of the *Elements* by summarizing or quoting the relevant enunciation. In general, it is modeled on the structure and presentation of Euclid's *Elements*.

Despite this focus on the oral practice of instruction, however, the *Spherics* also provides evidence for material practices. Specifically, for the practice of drawing figures on the surface of a real globe. All of the problems in this text are constructed in such a way that one can follow through the steps of the construction that lead to the solution by using the elementary tools of geometry, a straight edge and compass, and transferring between a globe and a plane surface.
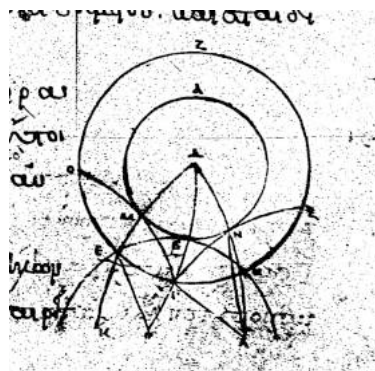

Figure 7: Vatican Library, Gr. 305: *Spherics* II 15

To develop a sense for how this might have been done before an audience, we will

look at *Spherics* II 15. This problem shows how to draw a great circle that passes through a given point such that it is tangent to a given lesser circle.

In the manuscripts, as is usual for a standard problem, there is one figure at the end of the proposition. The diagram is drawn in a style common to Greek texts in spherical geometry, in which the objects, although described as lying in the surface of a sphere are depicted as all laying in the plan of the figure. It may take modern readers some time to orient themselves to the format of these diagrams. For our purposes here, however, this will not be necessary, since we will follow the argument using perspective figures. This will simulate to some extent the experience that ancient audiences may have had of seeing the problem solved on the surface of a globe.
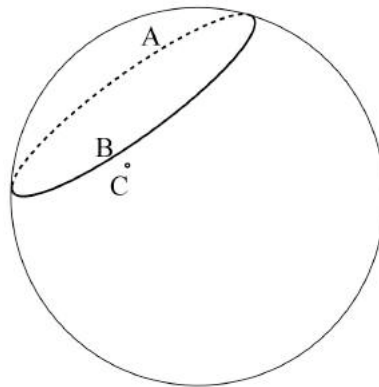

We begin with a given lesser circle, AB, and a given point, C, between AB and the circle equal and parallel to it.
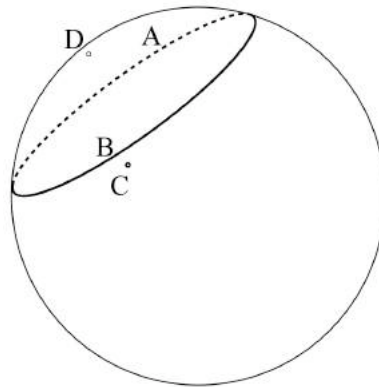

Using the constructions in *Spherics* I 21, we take the pole of circle AB, point D. This construction, can be carried out on a globe using a straight edge and a compass. (For the full details of these practical constructions, see Sidoli and Saito (Forthcoming).)

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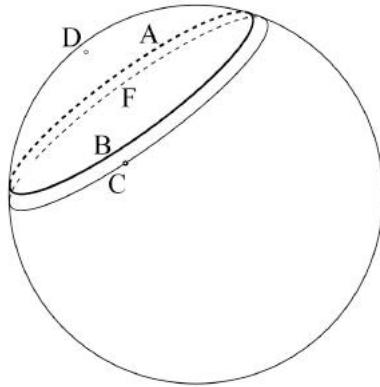
With D as a pole and DC as the pole-distance (the compass span), we draw lesser circle CF parallel to circle AB.
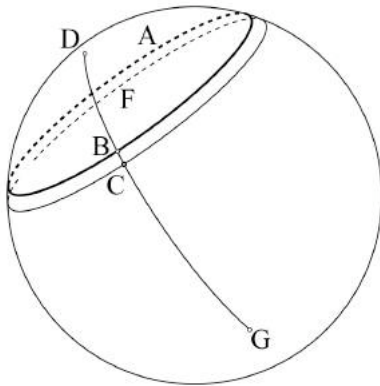

Using *Spherics* II 20, which can also be carried out with a straight edge and compass, we draw a great circle through points D and C. We, then, cut off point G such that arc BG is a quadrant.
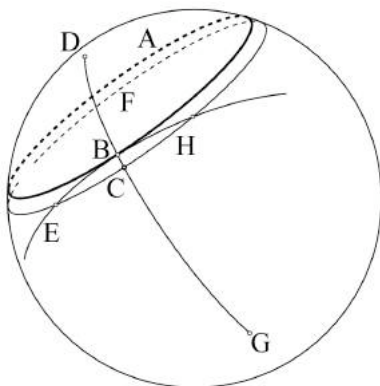

Since, *Spherics* I 16 has shown that the pole-distance of a great circle is the cord subtending a quadrant of a great circle, if we draw a circle about pole G with BG as the

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pole-distance, it will be a great circle. We draw it as great circle EBH, cutting lesser circle CF in points E and H.
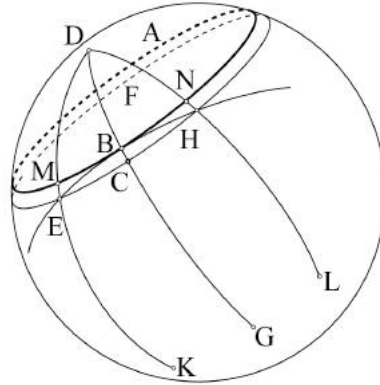
We draw a great circle through points D and H and cut off arc HL equal to arc CG; and another great circle through points D and E and cut off arc EK equal to CG.
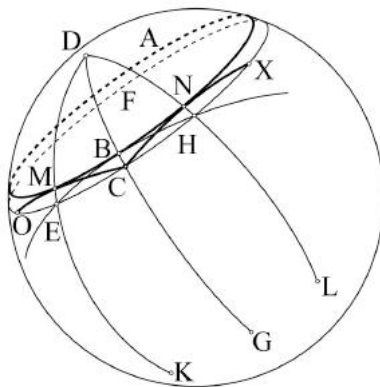
With L as pole and LN as pole-distance, we draw circle XNC. Again, with K as pole and KM as pole-distance, we draw circle OMC. This completes the construction.

It now remains to show that the two circles, XNC and OMC, are great circles and that they pass through the given point C.

In fact, for the proof, we will need to introduce further, auxiliary lines. For the proof of *Spherics* II 15, it is sufficient to draw lines KM, KC, GE, GH, LC, LN and to show that these are all equal.

Since these are lines internal to the sphere and could not have been drawn on a solid globe, it is likely that they were drawn on a flat surface, merely for the sake of the argument. In fact, a number of the constructions involved in this problem also involve making drawings on an accompanying plane. For example, in order to draw a great circle, one must first draw the diameter of the sphere outside the sphere in a plane. In fact, *Spherics* I 19 solves just this problem. Thus, the material practice of presenting
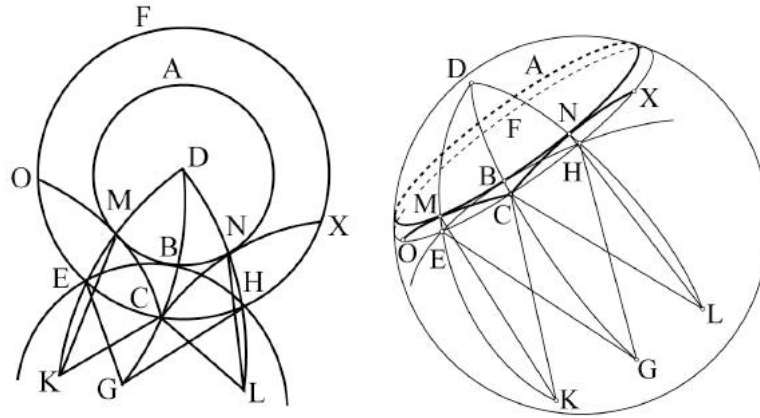
Figure 8: *Spherics* II 15, Vatican Gr. 204 and a reconstructed figure

spherical geometry appears to have involved the usual practice of making drawings in the plane as well as making drawings on the sphere, and transferring objects between the two.

Here we have seen an example of the way in which the texts can be used to elicit information about both the oral practices that the texts were meant to facilitate and the material practices that accompanied them.

Final remarks

In this talk, I have used examples drawn from the extent texts of Greek mathematics to argue that Greek mathematicians were engaged in a broader range of activity than the production of these texts themselves. Indeed, the texts were written by mathematicians, themselves members of a small group of literary scholars, with the deliberate intention of transmitting mathematical knowledge, practices and theories to future generations. They were transmitted through the long centuries by scholars who where largely not mathematicians, and who introduced countless changes to the texts. Nevertheless, the authors of these texts were writing for mathematicians – individuals who they must have hoped would also be engaged in a broad range of mathematical activities.

The mathematicians who composed these texts were probably well aware of the vagaries of manuscript transmission and reproduction. They probably deliberately structured the texts in ways that would have been less susceptible to alteration; they probably designed the diagrams that they believed would be the easiest to reproduce faithfully. That is to say, they designed their works to resist the alterations of fortune and to carry their ideas through the centuries to individuals who they believed would know how to read them – would know how to transform the ideas in the texts into a living practice of

mathematical teaching and research.

The three examples that we have seen demonstrate ways in which the texts carry information about mathematical practice that is more than simply the transmission of results and theories. For the attentive reader, they carry information about how to actually *do* geometry. That is, they tell us how to draw diagrams and make compelling arguments about them. From Archimedes' *Sphere & Cylinder*, for example, we not only learn a number of interesting results that Archimedes obtained, but also general ways in which results that are derived in theorems can be used in analyses to solve interesting and difficult problems. From Apollonius' *Cutting off a Ratio*, we learn how to apply the techniques of analysis systematically and exhaustively, so as to derive not only *a* solution to a given geometric problem, but all possible solutions and their geometric arrangement. From Theodosius' *Spherics* we not only learn the results that Theodosius collected and derived, but also how to draw diagrams on globes, so that we may ourselves investigate these objects in a more intuitively accessible way. Indeed, it is clear that contrary to the early modern belief, the ancient mathematicians, far from intentionally obscuring their methods, attempted to transmit this material to posterity. The fact that we fact that we now possess very few works that teach us how ancient mathematics was carried out is due more to the accidents of the channels of transmission than to a lack of effort on the part of the ancient mathematicians.

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