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Mathematical tables in Ptolemy's *Almagest*

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Abstract

This paper is a discussion of Ptolemy's use of mathematical tables in the *Almagest*. By focusing on Ptolemy's mathematical practice and terminology, I argue that Ptolemy used tables as part of an organized group of units of text, which I call the *table nexus*. In the context of this deductive structure, tables function in the *Almagest* in much the same way as theorems in a canonical work, such as the *Elements*, both as means of presenting acquired knowledge and as tools for producing further knowledge.

MSC: 01A20

Keywords: Ptolemy; Mathematical tables; *Almagest*; Greek mathematics; Greek astronomy; Mathematical functions

1. Introduction

By the time Claudius Ptolemy (early to mid-2nd CE) composed his works, the production and use of tables was an accepted part of Greek mathematical practice. Although Ptolemy makes certain programmatic statements about how he thinks tables ought to be presented and used, it is difficult to know if his articulation of best practices with regard to tabular methods was original or had its origin in the work of his predecessors, such as Hipparchus (ca. mid-2nd BCE) or Menelaus (ca. turn of 2nd CE). It does, however, seem clear that tabular methods entered Greek mathematical practice from Babylonian sources, particularly Babylonian mathematical astronomy, sometime during the mid to late Hellenistic period.

When we look at just those sorts of works from the early Hellenistic period where we might expect to find tables — such as the proto-trigonometric, astronomical *Sizes and Distances of the Sun and Moon* or *Sand Calculator* by Aristarchus (ca. early-3rd BCE) and Archimedes (280s–212), respectively — we find no evidence for tabular mathematics (Berggren and Sidoli, 2007; Van Brummelen, 2009, 20–32). Although there is some evidence for tabular displays and formats being used in harmonics and Pythagorean number theory — as found in the work of Nicomachus and Theon of Smyrna (both ca. early-2nd CE) — these traditions do not seem to have involved the use of tabular elements to encode mathematical relationships. Again, there was a long tradition of monumental tables, such as parapegmata and other inscriptions, which

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may have contributed to Greek interest in Babylonian astronomical tables, but these were either public resources or votive inscriptions and were not meant to be used in mathematical practice (Jones, 2006; Lehoux, 2007). By the Roman Imperial period, however, there was a wealth of numerical tables for astronomical and astrological purposes in circulation in Greco-Roman Egypt, including direct translations of substantial parts of Babylonian mathematical astronomy (Jones, 1999a, 1999b).

This is the context in which Ptolemy produced his amalgam of the deductive approach of classical geometry and the algorithmic procedures of tabular astronomy. The extent to which Ptolemy was original in this is uncertain, but it is clear is that his particular articulation of this strategy had a lasting effect on mathematical practice. This paper is an investigation of Ptolemy's approach to tables as a part of Greek mathematical practice.

1.1. *Tables in Ptolemy's works*

The presence of tables is a characteristic feature of nearly the entire Ptolemaic corpus. Some works, such as the *Geography* and the *Handy Tables*, consist largely of tables. In others, such as the *Harmonics* or the *Optics*, tables are less frequent, but still play an important role in the development and presentation of the thought.

Ptolemy's treatises have two basic types of tables. The first is a tabular presentation of non-mathematical information. Examples of this type are the tables in the *Geography* or the star catalog in the *Almagest*. These are presented in a tabular format in our sources, and the tabulation associates numbers with items in a set, such as stars or geographical features. The tables, however, do not represent mathematical relations and the method of associating the numbers with the items is, at least in principle, empirical. These sorts of tables will not be discussed in this paper.

The second type of table conveys mathematical information, and this mathematical relationship is the core idea that is presented in a tabular format. Indeed, although it would be cumbersome, this kind of mathematical table could simply be stated as a well-ordered list. For example, the first time Ptolemy gives the rising times for 10° arcs of the ecliptic for the terrestrial latitude of the equator, *Almagest* I.16, he gives a well-ordered list in normal prose (Heiberg, 1898–1903, I, 85).[1] The second time he presents this same information, however, it is found as part of the first column in the table of rising times, *Almagest* II.8. For Ptolemy, the visual arrangement is not essential to presenting this mathematical relationship on its own, but later becomes important for comparing this relationship to other, geometrically related relationships. Ptolemy's mathematical tables are, in principle, determined by mathematical criteria, and the arrangement of the elements in a tabular layout is meant to convey specific mathematical information, which must be understood in context. We can call these *mathematical tables*. The subject matter of this paper is Ptolemy's practice with regard to the mathematical tables in the *Almagest*.

1.2. *Tables and functions*

As O. Pedersen (1974, 32–33) first pointed out, although Ptolemy claims, at the beginning of Book III, that he has presented all of the mathematical prerequisites for reading the *Almagest*, his ideas about tables and their role as mathematical tools is never explicitly discussed, but must be drawn from an investigation of his practices. Indeed, as Pedersen went on to show, Ptolemy used tables in systematic ways that can be related to various aspects of the modern idea of a *function*. In this section, I expand on Pedersen's discussion, introducing my own terminology and attempting to distinguish between concept and practice (Pedersen 1974, 2010, 78–91).

[1] In his translation, Toomer (1984, 74) presents this list in a quasi-tabular format, which is not found in the manuscripts.

Although it is important to note that Ptolemy never explicitly introduces the ideas presented in this section, I will argue that his methodology in laying out and using tables, as well as his naming conventions, indicates that he had operational concepts similar to those I will discuss here. Hence, although this section introduces a number of modern categories that do not always correspond directly to Ptolemy's categories, these should be understood as abbreviations for a collection of consistently applied practices. At any rate, the terminology introduced here will help us discuss the various ways in which Ptolemy used tables.

One of our fundamental ideas about functions is that they serve to associate, or map, the members of one set of numbers individually with the members of another set of numbers.[2] Functions in this sense are clearly present in Ptolemy's works. Indeed, the most basic type of functional relationship that Ptolemy's mathematical tables convey is that of a correspondence of the members of one set of numbers to members of another. Many of Ptolemy's mathematical tables have the basic form

a_1	b_1
a_2	b_2
a_3	b_3
$\dots$	$\dots$
a_n	b_n

in which a set of numbers $\{a_1, a_2, a_3, \dots, a_n\}$ is mapped, individually, to another set of numbers $\{b_1, b_2, b_3, \dots, b_n\}$. I will call this aspect of Ptolemy's use of tables a *relational function*.

In a number of key cases, this mapping is a one-to-one correspondence, such that the table represents a pair of inverse functions, and one can enter into either column to determine a value from the other column. In such cases, both the computational use and labeling of the table shows us that the table is understood as a pair of inverse functions. For example, the chord table, *Almagest* I.11, which has the basic structure of the table above but in which the two columns are labeled "arcs" and "lines,"[3] is used as follows:

α_1	$Crd(\alpha_1)$
$\dots$	$\dots$
α_n	$Crd(\alpha_n)$

$Ang(x_1)$	x_1
$\dots$	$\dots$
$Ang(x_n)$	x_n

In other words, one can enter into either column as argument, resulting in a value in the other column, understood as dependent on the value of the argument.

Another core idea involved in functions is that each member of the domain is associated with a single member of the codomain, where the inverse is not necessarily the case. In some fields, this is developed with the idea of a simple graph, in which ordinates drawn to the x-axis pass through the graph only once, whereas ordinates drawn to the y-axis are not so constrained. Graphs of this kind, of course, are not necessarily pairs of inverse functions and we must have some convention to distinguish between the argument of the function and its result. For simple graphs, the conventions regarding the x- and y-axes serve this purpose. Ptolemy's naming practices indicate that he clearly understood the difference between these two types of relational functions. In those tables that produce inverse functions, the two columns are explicitly labeled, and one can enter the table in either column. In tables that represent a simple graph such as *Almagest* III.6, however, the domain is labeled "numbers" or "common numbers," meaning something like *argument*, and indicating

2 Pedersen (1974, 36) called this conception of a function "a general relation associating the elements of one set of numbers . . . with the elements of another set."

3 There is also a column for interpolation labeled "sixtieths," but that is irrelevant to my point here.

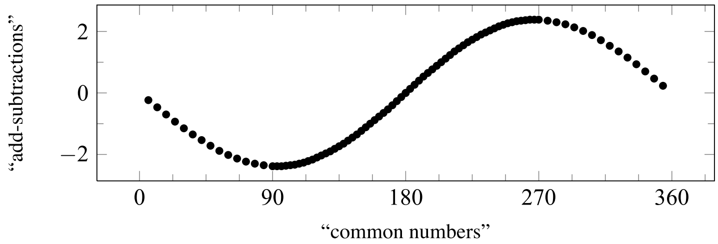

Figure 1. Graphical representation of the table of solar anomaly, *Almagest* III.6. In the table, only one set of absolute values is given, 0–180 and 360–180, which must be understood as either additive or subtractive corrections.

that one only enters the table through this column.[4] The "numbers" are always set down in the leftmost column and serve to relate the irregular changes of the tabulated functions to the regular progression of the 360° of a circle. That is, the "numbers" act as a conventional, and regular, domain against which we can clearly see how the tabulated function changes. In this sense, as I will argue at length below, Ptolemy's table play a similar role as a modern graph.

If we compare Figure 1 with Table 1, we can see how the table of solar anomaly, *Almagest* III.6, acts as a kind of graph. Because the function is symmetrical about 180° for positive and negative values, Ptolemy only tabulates half of the graph and points out, in the algorithm that follows the table, that the corrections are subtractive when the argument is between 0° and 180°, and additive when it is between 180° and 360°. Nevertheless, by looking at the tabulated values in the two columns, we can discern the fundamental characteristics of the function: it is perfectly symmetrical around 180°, it has a maximum near 93° around which it is nearly symmetrical, and its rate of change must be slightly faster towards the perigee, 180°, than the apogee, 0° or 360°.[5] Although we might prefer to convey this kind of quantitative information with a graph, Ptolemy's procedure was to design his tables so as to reveal such aspects of the mathematical functions with which he worked. I will call this aspect of Ptolemy's practice with regard to tables a *representational function*.

Finally, one of the most straightforward ways in which we use the term function is to describe some explicit set of rules that takes as input some numerical value, and, for each input, unambiguously outputs another numerical value. For example, on a computer, we can simply define a set of rules for associating an input value with an output value. It is clear that the algorithms that Ptolemy gives for working with his tables are functions in this sense.[6] Following each table he gives some clearly stated rule that allows us to convert a given number into another unique number. Although Ptolemy never proves that his tables function in this way, they are designed and labeled such that the argument column never contains more than a single occurrence of each member of the domain. Hence, if we bear in mind that the following symbols represent an explicit algorithm, or program, Ptolemy works with functions of the following forms:

$$f(x), \qquad f(x, y), \qquad f(x, y, z).$$

[4] The fact that the tables of mean motion, *Almagest* III.2, IV.4, IX.4, do not have this labeling, even though they can only be entered through the time column, indicates that Ptolemy thought of these tables as mathematically different from tables that represent changing functions. Indeed, the tables of mean motion do not tabulate a mathematically interesting function, but simply allow us to find mean motions through a simpler series of arithmetical operations than is possible from the mean motion stated as a rate.

[5] Ptolemy himself points out that he will include more entries in the table around perigee than around apogee because of this slight difference in the rate of change (see note 22, below).

[6] As Pedersen (1974, 35) says, Ptolemy's algorithms "may be called 'programmes,' similar in many respects to the programmes fed into a modern computer."

Although most of the functions in Ptolemy's works are functions of one variable, he also works with functions of two and three variables. I will call this aspect of Ptolemy's use of tables a *computational function*.

Although some readers may still be unconvinced that such modern concepts can be fruitfully applied to Ptolemy's writings, the bulk of this paper will be devoted to showing that Ptolemy worked with tables systematically so as to produce a type of tacit mathematical knowledge of such functional relations. In discussing the details of Ptolemy's tabular practices, below, I hope to show that although Ptolemy does not discuss these ideas explicitly, he understood the nature of mathematical tables in ways that can be legitimately summarized with such terminology.

The usual interpretation of Ptolemy's mathematical tables is that they are simply computational tools and are not integral to the presentation of the theory. Under this view, the *Handy Tables* have been construed as a later development in Ptolemy's thinking about tabular methods and all the changes he introduced have been understood as improvements. It is also possible, however, that Ptolemy thought of the two works as developing tabular methods towards different ends: the *Almagest* towards the articulation of a complete theory, and the *Handy Tables* towards supplying the needs of practical computation. This paper will argue that Ptolemy designed the mathematical tables in the *Almagest* so as to play a number of roles, of which computation was just one.

2. Tables in the *Almagest*

The *Almagest* is a systematic, theoretical treatise. It is a mathematical exposition of an ideal mathematical subject, the motion of the heavenly bodies, which Ptolemy viewed as both rational and containing to a fair extent the "harmonic faculty" (ἡ ἁρμονικὴ δύναμις).[7] The presentation is essentially a single argument, in which earlier sections provide necessary tools for the sections that follow. The goal of the *Almagest* is to construct moving, geometric models that are used to describe the apparent phenomena of the celestial bodies as reducible to motions that are both circular and have a regular angular velocity about some point. In *Almagest* III.1, Ptolemy tells us that the goal of the mathematician,[8] which is to show how the apparent motions of the celestial bodies are made up of regular circular motions, is best accomplished by the types of tables he will use, which separate mean motions from corrections; that is, the regular from the anomalous (Heiberg, 1898–1903, I, 202).[9] Then, before he sets out the solar model, which is the first that describes an anomalous motion, he states his general assumption that the celestial bodies move in their proper motions with regular angular velocities.[10]

As is suggested in the *Almagest*, and made obvious in his *Planetary Hypotheses*, these models are meant to be a reflection of the physical reality of the heavens, even if they function very differently from the objects

[7] As he states in *Harmonics* III.3, Ptolemy regarded astronomy and harmonics as the highest branches of mathematics, for which geometry and arithmetic acted as "indisputable instruments" (Düring, 1930, 91–94). In III.4, he goes on to state that the harmonic faculty is found in all things that are self-moving but especially in things that have a more rational nature, such as human souls and heavenly bodies.

[8] Ptolemy's categories of "mathematics" and "mathematician" were much broader than ours. Most importantly, he regarded mathematics as the study of rational objects, in which the harmonic faculty is revealed, such as celestial bodies and human souls, which are apprehended through the rational senses of sight and hearing. See Feke and Jones (2010), Bernard (2010, 502–512), and Lehoux (2012, 195–198).

[9] This text is translated on page 20, below.

[10] Ptolemy is very careful in his expression of this, see *Almagest* III.3 (Heiberg, 1898–1903, I, 216). It should be pointed out that Ptolemy expresses this as a principle that has to do with the mathematical properties of circular motion, and there is nothing incompatible between regular angular velocity and the irregular motions of the spheres in which the celestial bodies are carried, as in the case of the lunar or planetary models.

we experience on the earth.[11] As is clear in a number of passages, the tables in the *Almagest* are meant to exhibit this regularity by showing, numerically, how increasingly irregular motions can be composed of simpler, regular motions.

In the *Almagest*, the tables are meant to serve a double function: they both allow computation and they reveal the underlying mathematical pattern, usually by relating a set of numbers with changing differences to a set of numbers composed of regular differences of degrees of a circle, which acts as a sort of standard for regularity. The point of these tables is not to track the apparent motions of the celestial bodies, as was done in other tables then in circulation in Greco-Roman Egypt (Jones, 1997a, 1997b, 1999b), but rather to show that any apparently irregular motion is actually based on the periodic motion of the circle; that is, to reveal the underlying mathematical principle of regular circular motion.

Tables perform a number of functions in the deductive structure of the treatise. At the basic level, they are organized lists that catalog information, such as the list of the characteristics of parallels of latitude, *Almagest* II.6. They are also used to simplify calculations that can be carried out with simple arithmetic, by reducing the number of divisions or multiplications that must be carried out, such as the tables of mean motion, *Almagest* III.2, IV.4, IX.4. They can also be used to carry out a calculation that cannot, in principle, be done on the basis of a simple arithmetic or geometric calculation, such as the table of chords, *Almagest* I.11. They can represent a relational function between two or more magnitudes using numbers, such as *Almagest* I.11 or the table of rising-times, *Almagest* II.8. And they can form the basis of an algorithm that acts as a computational function, such as *Almagest* I.11, II.8, or the table of the full lunar anomaly, *Almagest* V.8. Indeed, an individual table usually fulfills more than one of these functions at the same time.

2.1. The place of tables in the overall structure

Ptolemy, as all Greek mathematicians, divided his argument up into different units of text. In the case of the exact sciences, we find a number of units of text that are not found in the treatises of pure mathematics. In particular, Ptolemy, uses *descriptions* to explain the relationship between a model and the natural phenomena it describes, or between some element in a model and the numbers in a table that tabulate the model's varying numeric values.[12] He uses *theorems* to demonstrate that certain facts hold for a particular model. He, occasionally, uses *problems* to show that certain objects can be constructed. He uses *metrical resolutions*, which are a type of analysis that show that if some components of a model have given numerical values, then the numerical values of other components are also given — that is, computable.[13] He uses *calculations* to produce a specific value given some starting values, using arithmetical operations, geometric theorems and tables. He uses *tables*, which act as relational and representational functions, exhibiting a mapping of one set of numbers to another. He uses *algorithms* to explain how to use given numbers to compute a sought number. When an algorithm is used with a table, the table acts as a computational

[11] Murschel (1995) describes the physical models of the *Planetary Hypotheses*. As for clear indications of physicality in the *Almagest*, in I.8 Ptolemy refers to the movement of the "spheres of the stars," in III.3 Ptolemy mentions the spheres "by means of which they make their motion," in IV.6 he refers to the "sphere of the moon," and in *Almagest* XIII.2 he discusses his belief that the celestial spheres and their elements can pass through one another and move in contrary directions without interference (Heiberg, 1898–1903, I, 27, 216, 301; II, 532–533). Jones (2005) also makes an argument for the physicality of Ptolemy's *Almagest* models.

[12] The names of units of mathematical text are my own, although some of them, such as *theorem*, *problem*, and *table*, were used as categories by ancient mathematical scholars as well. Examples of these types of texts will be given in the course of this paper.

[13] The term *metrical resolution* is my own. It comes from the terminology of H. Hankel (1874, 137–150), who used *resolution* to denote that part of an *analyzed proposition* that is composed of a "chain of givens," the justifications for each step of which can be supplied from Euclid's *Data* (Acerbi, 2007, 519–523). Because Heron and Ptolemy use this type of argument to determine metrical relations, in a way not clearly intended by the *Data*, I call it *metrical resolution*. Acerbi (2007, 512–519) discusses metrical resolution as analysis and synthesis in a metrical context.

function, since, entering with a certain number of the domain, the algorithm for operating with the table unambiguously produces a single number of the codomain. Very nearly every passage of Ptolemy's text that is a part of the mathematical argument can be assigned to one of these types.

In the *Almagest*, a table never appears on its own. It appears as part of an extended mathematical argument that relates it to the geometric model, shows how the entries of the table are related to one another, describes how the table is laid out, and provides algorithms for its use. I call this overall mathematical argument a *table nexus*. The table nexus is made up of three nodes: *derivation*, *representation*, and *evaluation*. This grouping of text and table is called a nexus because it almost always involves the use of other tables besides that in the representation node of the nexus. For example, the calculation of a table generally involves the use of the chord table, *Almagest* I.11, and the algorithm following a table often involves the use of two or more tables in conjunction, such as the combination of tables of mean motion and correction.

The derivation node of a single table nexus provides the mathematical relationship between the numbers in the table and the objects in the model. In particular, the derivation uses calculation or analysis to show how a given model and its parameters determine the numbers that are found in the table. This does not necessarily mean that the numbers in the table were actually computed by the methods provided in the derivation, and in a number of cases it has been shown that they were not (Newton, 1985; Van Brummelen, 1993, 1994). Rather, the derivation argues, more generally, that given a certain model, some parameters, and an input value, the corresponding value in the table is also fixed.

The tables themselves can be taken as a quantitative representation of the geometric model. The representation node consists of a description of how the table is laid out, followed by the table itself. The description gives an account of various graphical and mathematical features of the layout, such as the number of lines per page, the layout of the columns, and the relationship between components of the model and the entries in the table. The table itself can be understood as a sort of representational function, in which the members of one set of numbers are associated with the members of another set.

The tables are then used in various ways to carry out computations. The evaluation node consists of algorithms and calculations that refer to the tables. The algorithm describes how to perform calculations on the various entries in the tables in order to produce a definite result. The algorithm is not always exactly precise, so that there is sometimes ambiguity about how an interpolation should be carried out.[14] Nevertheless, the algorithm gives a method for producing a numerical value and points out how the table can be used to produce numerical values for the model. The combination of the algorithm and table can be understood as a computational function, which allows us to enter in with a given number of the domain and produce a single number of the codomain.

Ptolemy gives no general argument that his tabular methods are valid or effective. Presumably he thought that such an argument would be apparent from the mathematical structure of the table nexus. Indeed, the various components of the table nexus help us understand the relationship between the model and the table, upon which we must base our understanding of the tables. An appreciation of what each of the components of the table means, is based on having followed the derivation. In this way, we see that the table must numerically express a relational function between two, or more, components of the model. In the simpler table nexuses, the individual columns represent individual geometric objects, such as angles or lengths. Any claim that an algorithm actually produces the apparent motions that we are interested in, and properly relates these to the more regular motions of the model, will be based on an assessment of the relationship between the various components of the table nexus. Finally, an intuitive understanding of how the components of the model move is derived from studying the numbers in the tables, and the tables are designed in such a way as to try to make the nature of these movements as clear as possible.

[14] For example, when we write a simple computer program to carry out one of Ptolemy's algorithms, we often find ourselves in a position of having to make choices about how interpolation should be carried out, since this is usually not explicitly specified by Ptolemy.

2.2. *A model table nexus:* Almagest *III.5, III.6 and III.8*

In order to illustrate how the table nexus functions with an example, I take the table of anomaly for the solar model. This is not the first table nexus in the text, nor the most interesting, but since the solar model is the first that treats an irregular apparent motion and acts in many ways as a model for the more involved theories, it serves as a paradigmatic case. Moreover, it is in reference to this table nexus that Ptolemy makes two of his most programmatic statements about how he works with tables. It is here that he tells us, for the first time, that a primary goal of the *Almagest* tables is to model the motion of the heavenly bodies in such a way as to show that any apparent irregularities in their motion is actually composed of regular motions in the model. In *Almagest* III.1, Ptolemy puts this, somewhat obliquely, as follows:[15]

> As for the evaluation of the paths of the sun and the others, according to the occurrences of each of them, which the arrangement of the table framework, part by part, (ἡ σύνταξις τῆς κατὰ μέρος κανονοποιίας) is disposed to supply as handy, or rather explicit, we hold [1] that it must be set out as a purpose and an aim of the mathematician to show the accomplishment (ἀποτελούμενα) of all phenomena of the heavens by means of regular and circular motions, and [2a] that the table framework (κανονοποιίαν) most appropriate and suited to this purpose is the separation of the regular motion, part by part, from the apparent irregularity, following from the circular models, and then [2b] the exhibition of their apparent paths is from the mixing and combination of both of these.
>
> [Heiberg, 1898–1903, I, 208]

In other words, the goal of the mathematician[16] is to show that all the heavenly phenomena are brought about by regular, circular motions, and that the table framework[17] that best does this is one in which the mean motions, as functions of time, and anomalies, as functions of mean motion, are tabulated independently. In this way, the components of the tables will exhibit the numerical relationship that the various parts of the model have to the regular motion of a circle.

The table nexus of the solar anomaly fits into the overall development of the solar theory and it will be useful to sketch this development. The theory begins, in *Almagest* III.1, with a long argument that the length of the tropical solar year is constant, including a discussion of how it is measured, reports of ancient observations and the conclusion that Hipparchus was right about its value. The mean motion of the sun is then set out in a table, III.2, so that it can readily be used in the arguments that follow.[18] The solar mean motion table sets out the pattern of all the mean motion tables in the treatise. It tabulates 45 rows of eighteen-year periods, eighteen rows of one-year periods, twelve rows of 30-month periods, 30 rows of one-day periods, and 24 rows of one-hour periods — each in integer degrees with six sexagesimal-fractional places, in subcolumns.[19] *Almagest* III.3 sets out the simple epicycle and the eccentric models and proves their equivalence for the simple, solar model.[20] Hipparchus' solar eccentric model is then set out in III.4,

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[15] In order to pay close attention to Ptolemy's terminology with respect to tables, I have provided my own translations.

[16] See note 8, above.

[17] The term κανονοποιία, which I translate with "table framework," is used seven times in the *Almagest* (Heiberg, 1898–1903, I, 208, 240, 251; II, 218, 427). This word, which literally means something like "table construction," refers to the mathematical, or technical, design of the table.

[18] Ptolemy makes this point about why the table of mean motion is set out at the end of *Almagest* III.1, just after the quote above in which he introduces the structure of the table and explains why his table framework is split into a mean motion table and a table of anomaly (Heiberg, 1898–1903, I, 208).

[19] There is no mathematical difficulty involved in finding any of these values, because Ptolemy uses the Egyptian year, which always has 365 days, composed of twelve 30-day months, followed by five epagomenal days. Each day, in turn, contains 24 equinoctial hours.

[20] The argument for equivalence is given in a series of general arguments that Ptolemy notes will be backed up by numerical computation later on (Heiberg, 1898–1903, 222–229). Bowen (2012, 46, n. 28), apparently misreading a note by Toomer (1984,

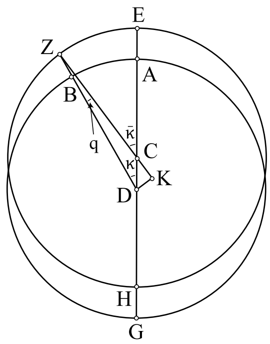

Figure 2. *Almagest* III.5.

and Ptolemy determines its parameters using "observations" that he claims to have made, but which are all off by about a day and agree with Hipparchus' erroneous year length. Using calculations and metrical resolutions, it is then shown, in III.5, that given the parameters of the model and one of either the mean motion from apogee, $\bar{\kappa}$, the apparent motion from apogee, κ, or the equation of anomaly, q, the other two can also be determined. *Almagest* III.6 is the table of anomaly, structured such that we can enter with the mean motion and produce the equation of anomaly, $q(\bar{\kappa})$. *Almagest* III.7 determines the mean position of the sun in the epoch of Nabonassar, Thoth 1 in the Egyptian calendar (−746, Feb 26), so that we can state the mean position relative to the model, $\bar{\kappa}$, as a mean position of longitude relative to the zodiac, $\bar{\lambda}$. Finally, in III.8, Ptolemy gives an algorithm that shows how to use the two tables, III.2 and III.6, along with the astronomical parameters of the model to state the apparent position of the sun, $\lambda_{\odot}$, for any time since epoch.

The theory combines more mathematical sections, which set out the general principles of the models and the tables, with more observational sections, which establish the spacial and temporal parameters of the models. The table nexus of the table of solar anomaly, III.5, III.6 and III.8, fits into a broader context of table nexuses that includes the chord table, I.11, and the table of solar mean motion, III.2. The architecture is designed to be modular so that the table nexus of the table of anomaly is necessarily embedded in this overall theory.

The derivation of the table of solar mean motion is given in *Almagest* III.5, followed by a discussion of the table framework that Ptolemy prefers. He begins with a general claim that one of the three principal arcs of the model is sufficient to determine the other two, stating,

> In order to make it possible to determine the motion of the anomaly, part by part, again for each of the models, we show how, with one of the stated arcs ($\bar{\kappa}$, κ, and q) being a given, we get the others. [Heiberg, 1898–1903, I, 240]

He proceeds to show, in Figure 2, that if we assume the eccenter model, such that $\angle ECZ = \bar{\kappa} = 30°$, then we can complete right triangle ZKD and use the geometry of the figure, along with the chord table, I.11, and the parameters of the model from III.4, namely $DC = 2;30^{\mathrm{p}}$ where $ZC = 60^{\mathrm{p}}$, to show that $\angle DZK = q = 1;9°$ and $\angle ADB = \kappa = 28;51°$. He then uses metrical resolution to show that, in general, if $\angle EDZ = \kappa$ is given, then both $\angle CZD = q$ and $\angle ECZ = \bar{\kappa}$ will be given, and that if $\angle CZD = q$ is given, then both $\angle EDZ = \kappa$ and $\angle ECZ = \bar{\kappa}$ will be given. This is followed by a repetition of these three

144, n. 32), claims that the equivalence of the eccentric and epicycle models is only "suggested" in *Almagest* III.3, but is not "discussed" until XII.1. Toomer's note, however — which uses the words "hinting" and "discuss" — refers to the fact that Ptolemy is making an obscure reference to a more elaborate type of eccentric model, which is equivalent to a different type of epicycle model, both of which are treated in XII.1. What makes the later model different, as Toomer notes, is that the two motions of the epicycle model are in the same direction.

arguments, a calculation and two metrical resolutions, now for the epicyclic model. Ptolemy then gives all six arguments, three for the eccentric and three for the epicyclic model, for the case where the mean motion, $\bar{\kappa}$, is 30° from the perigee, G. Since the geometry of the figure is somewhat different when Z is in the lower half of the solar orbit, he must have regarded this as a separate case. All of this serves to show that the geometry of the model is such that the various components are determinate.

The introduction to the table itself, III.5 contains a description of the table, as well as a programmatic description of the table framework. Ptolemy says,

> While, from these theorems, many possible table frameworks (κανονοποιίας) — with sections comprising the determination, from the irregularity, of the apparent paths — can be constructed, in order to immediately get the quantity of the corrections, part by part, that [table] having the regular arcs ($\bar{\kappa}$) laid out besides the equation of anomaly (q) pleases us best, both by its accordance in itself with the model and because it is simple and apt for the calculation of each.
>
> [Heiberg, 1898–1903, I, 251]

In other words, Ptolemy choses a table with the mean motion laid out next to the corresponding anomaly both to exhibit the functional relationship between them, and to be suitable for calculation.

Ptolemy then tells us that he calculated the table using chord-table trigonometry (διὰ τῶν γραμμῶν, literally "by means of lines"),[21] using the eccentric model. He states that, in general, the tables will have 3° intervals in the semicircle around the perigee and 6° intervals in that around the apogee, because the successive differences of the anomaly are greater around the perigee than the apogee.[22] This is followed by a description of the graphical layout of the table: it will have 45 rows and three columns, with the first two columns containing the mean motion from 0° to 360° and the third column containing the correction to be added or subtracted.

This detailed description is followed by the table itself; see Table 1. The table, as a whole, is labeled "the table of the sun's anomaly," and the two columns are individually labeled as "common numbers" and "add-subtractions" (meaning *additive or subtractive corrections*). In fact, the third column contains two subcolumns with a ruled line separating them, the first containing degrees and the second containing minutes. The two subcolumns are clearly marked in all the manuscripts, however, they have a single heading and are referred to in the text together as the "third column." The column heading "common numbers" reveals the symmetry of the function through the fact that the whole function is described with only 180°. Moreover, the minimum and maximum can be read directly off the table to be 93° and 267°, and this displacement of the maximum and minimum values of the anomaly from quadrature helps us form a quantitative understanding of Ptolemy's proof, in *Almagest* III.4, that the position of the extreme values of the anomaly are symmetrically placed between apogee and perigee, but displaced from quadrature. This helps us understand what Ptolemy means by saying that the table both corresponds to the model and is useful for calculation. The table was meant to serve much the same role as a graph: it reveals the changing numerical relationship that exists between $\angle ECZ = \bar{\kappa}$ and $\angle CZD = q$. By studying its numbers we can develop a

---

[21] Διὰ τῶν γραμμῶν is a technical expression in Ptolemy's writings, which means through the chord-table methods of trigonometry (Heiberg, 1898–1903, I, 32, 42, 251, 335, 349, 380, 383, 416, 449; II, 193, 198, 201, 210, 321, 426, 427, 429; Heiberg, 1907, 202, 203). It can either designate an actual calculation, or a metrical resolution, where the later is understood as showing that the former is, in principle, possible.

[22] This is due to the fact that the sun appears to move somewhat more rapidly from mean speed to greatest speed than from mean speed to least speed, as was demonstrated in III.4 and will be exhibited in III.6.

This slight difference in the apparent speed, however, seems hardly sufficient to justify tabulating the table at different intervals in the two semicircles. Probably, Ptolemy wanted tables of 45 rows to agree with his other tables, and devised this sectioning as a convenient way of obtaining them. In some of the later tables, such as *Almagest* V.8, discussed below, the difference in speed is more apparent. Hence, this division into intervals of 3° and 6° may have been decided for them.

Table 1
The table of solar equation, *Almagest* III.6. Formatting based on *Paris BnF gr. 2889*.

Table of the sun's anomaly

1st	2nd	3rd	
Common numbers		Add-subtractions	
6	354	0	14
12	348	0	28
18	342	0	42
24	336	0	56
30	330	1	9
36	324	1	21
42	318	1	32
48	312	1	43
54	306	1	53
60	300	2	1
66	294	2	8
72	288	2	14
78	282	2	18
84	276	2	21
90	270	2	23
93	267	2	23
96	264	2	23
99	261	2	22
102	258	2	21
105	255	2	20
108	252	2	18
111	249	2	16
114	246	2	13
117	243	2	10
120	240	2	6
123	237	2	2
126	234	1	58
129	231	1	54
132	228	1	49
135	225	1	44
138	222	1	39
141	219	1	33
144	216	1	27
147	213	1	21
150	210	1	14
153	207	1	7
156	204	1	0
159	201	0	53
162	198	0	46
165	195	0	39
168	192	0	32
171	189	0	24
174	186	0	16
177	183	0	8
180	180	0	0

more intuitive understanding of the way the speed of the sun appears to change throughout its orbit than we can discern in the geometrical diagram alone.

In *Almagest* III.8, Ptolemy provides the algorithm for using III.6 and III.2 to compute the apparent position of the sun at some given time, t. We begin by taking the time elapsed since epoch, Δt, and enter with this into the tables of mean motion, in divisions suitable to the table — eighteen year periods, years, months and so on — to find the change in mean longitude $\Delta\bar{\lambda}$. We then add this to 256;15° — which was the mean position of the sun from its apogee at epoch, $\bar{\kappa}_{\text{epoch}}$, determined in III.7 — and take this sum modulo 360 to find the mean position from apogee for the time in question, $\bar{\kappa}_t$. We count this off in the order of the signs from Gem 5;30° (65;30°), the zodiacal position of the sun's apogee, λ_A, determined in III.4, to find the mean position of the sun at our given time, $\bar{\lambda}_t$. We then enter with $\bar{\kappa}_t$ into the table of the

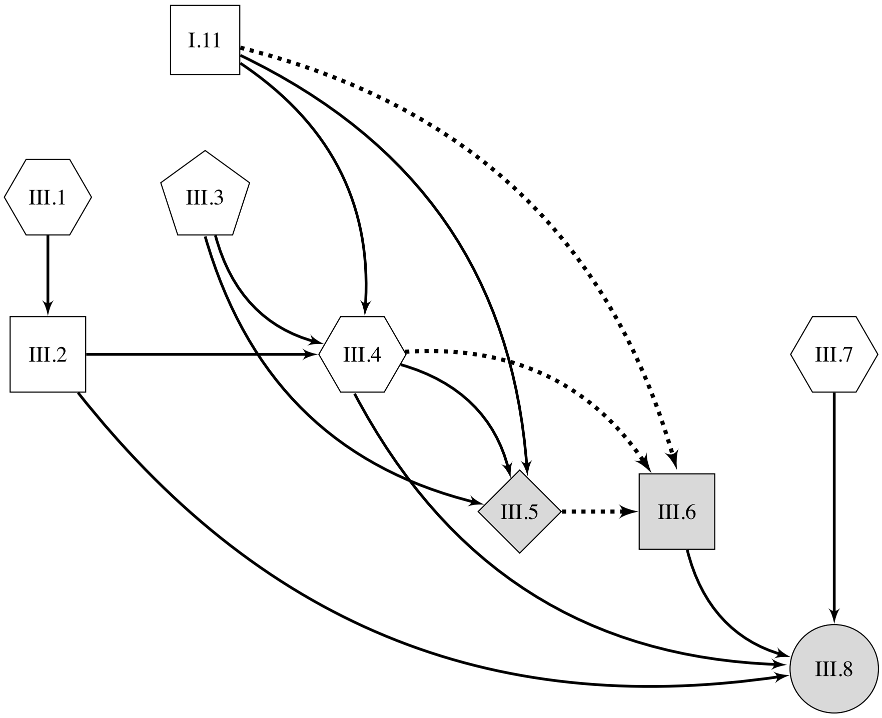

Figure 3. Flowchart of the table nexus for the solar anomaly table in the development of the solar theory, *Almagest* III. The square nodes are tables, the diamond is a table derivation using calculation, metrical resolution and description, the circle is an algorithm, the pentagon is a treatment of the model using description and theorems, the hexagons are derivations of the parameters of the model using calculation and astronomical observations.

solar anomaly, III.6, to find the correction to $\bar{\lambda}_t$, treated as subtractive when $0 > \bar{\kappa}_t > 180°$ and as additive when $180° > \bar{\kappa}_t > 360°$. This gives us the apparent position of the sun, $\lambda_t = \bar{\lambda}_t + q(\bar{\kappa}_t)$.

Having examined the structure of the nexus of the table of solar anomaly, it becomes clear that although this nexus itself treats the production, display and use of *Almagest* III.6, it is embedded in the context of a deductive argument that necessarily involves the use of various types of mathematical text, including other tables. Ptolemy calls the overall design of the tables, which is usually broken up into multiple tables having certain relationships to the geometrical diagram, the *kanonopoiïa*, the table construction, or table framework (see note 17, above). The introduction of the concept of a table framework highlights the fact that tables are developed in groups, and that the way in which they are related to the geometric model involves various technical, and mathematical, decisions.[23] This structure takes place at the conceptual level and is related to Ptolemy's idea of the proper relationship between the geometric model and the numerical tables. The table nexus, however, is a structure that takes place at the textual level. Not all of the tables in the *Almagest* have a complete table nexus, but most do. For example, the chord table, *Almagest* I.11, is proceeded by a full derivation and followed by an algorithm. The table of solar mean motion, on the other hand, is derived by simply noting that the tropical year can be multiplied and divided in various ways to give the differences found in the table, and there is no algorithm for the mean motion table itself, since its purpose is to be "useful and handy for the proofs themselves" (Heiberg, 1898–1903, I, 208–209).

We can use a flowchart to visualize how the table nexus of *Almagest* III.6 fits into the overall solar theory; see Figure 3. In this way, we can see that all of the sections of *Almagest* III work together to form a single, deductive theory, whose culmination is the table nexus, III.5, III.6 and III.8. The chord table, *Almagest* I.11,

[23] In the solar theory, the relationship between the columns of the table and objects in the model is essentially one-to-one. In more involved theories, however, some of the columns of the table are constructed using various simplifying assumptions.

has also been included in the diagram to remind us that the solar theory is not developed independently of the rest of the treatise.

2.3. *Details of the* table nexus

Now that we have seen the overall scope of the table nexus, we can focus on some of the particular features of the three nodes. Although *Almagest* III provides a convenient model of the table nexus, a number of the special features of the three nodes are better exemplified by more involved tables. Most of the examples that follow are drawn from the table nexuses of the complete lunar and planetary models.

2.3.1. *Derivation:* Almagest *XI.9*

Above, I claimed that the derivation provides a general argument that the phenomena, or mathematical relations, that the table expresses can be determined on the basis of the parameters of the model, rather than a set of instructions for actually calculating the table. A good way to substantiate this claim is by looking at the derivation for the tables of planetary anomalies, *Almagest* XI.9 as the derivation for XI.11, because these tables are constructed using simplifying assumptions such that the final four columns of the table no longer correspond directly to components of the diagram.[24] Hence, as Ptolemy makes clear, *Almagest* XI.9 shows how it would be possible to use exact, trigonometric methods (διὰ τῶν γραμμῶν) to find planetary positions that we will actually calculate using tabular methods (Heiberg, 1898–1903, I, 246).

The object of the derivation, *Almagest* XI.9, is to show that where the mean anomaly, $\bar{\alpha}$, and the mean longitude from apogee, which in the equant model is the angular velocity of the center of the epicycle around the equant point, $\bar{\kappa}$, are both given, then it is possible to calculate the value of the apparent position, λ, for any assumed model.

The bulk of this short section is simply the following metrical resolution; see Figure 4:

> Then, in the simple diagram of the eccenter and the epicycle, if we join ZBC and EBH, where the mean position in longitude is given — that is, angle AZB ($\bar{\kappa}$) — the angles AEB (κ) and EBZ — that is, HBC — will be given for both models[25] from the proceeding demonstrations,[26] and, likewise, the ratio of line EB to the radius of the epicycle.[27] And, assuming the argument (λόγου) of the star[28] is at point K of the epicycle, and with EK and BK joined, arc CK ($\bar{\alpha}$) is also given.[29] And if, rather than producing a perpendicular from the center B of the epicycle to EK, as in the inverse proof,[30] but instead from that on the star K to line EB — as KL, in this case — then, the whole of angle HBK (α) is given, and because of that, the ratio of KL and LB to both BK and, obviously, to EB.[31] Consequently, the whole of EBL to

---

24 Another example is *Almagest* V.6–8, discussed below.

25 As Toomer (1984, 544, n. 41) points out, it is unclear what is meant by "both models." His further statement that "both" probably refers to (1) the simple eccentric and (2) the full equant model seems unlikely since we begin with the mean motion of the epicycle's center and immediately derive its apparent motion.

26 The means of calculating these angles has been given for various planetary models in the preceding sections: *Almagest* X.4 for Venus, X.7 for Mars, XI.1 for Jupiter and XI.5 for Saturn. The point, however, is not to point to these specific derivations but simply to note that such a determination is generally available.

27 This ratio is calculated in *Almagest* IX.9 for Mercury, X.2 for Venus, X.8 for Mars, XI.2 for Jupiter and XI.6 for Saturn.

28 He means, of course, the planet. Since, however, planets were considered to be a type of star, I have decided to translate literally.

29 This is the mean anomaly, $\bar{\alpha}$, taken from the mean motion tables, *Almagest* IX.4.

30 This is the construction used in the calculations mentioned above.

31 That is (a) $KL : BK$ and $LB : BK$ are given, and also (b) $KL : EB$ and $LB : EB$. The first of these claims could be justified by *Data* 40 and the second by *Data* 8, but what Ptolemy means is that if $\angle LBK$ is given, then use of the chord table, *Almagest* I.11, in $\triangle LKB$ allows us to calculate values for the first pair of ratios, (a), and since $BK : EB$ was asserted as given, we can also calculate the values of the second pair of ratios (b).

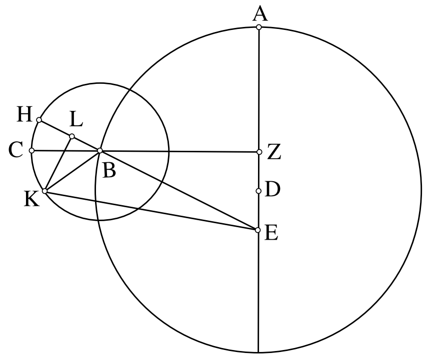

Figure 4. *Almagest* XI.9.

> LK will be given.[32] That is, angle LEK (q) will be given,[33] and the whole arc AEK ($\kappa + q$), which is the apparent distance of the star from the apogee, is calculated (συνῆχθαι) by us.
>
> [Heiberg, 1898–1903, II, 426–427]

What this metrical resolution shows is simply that, given the parameters of the model and the mean motions in longitude and anomaly, the apparent position will be mathematically determined. In the next section, *Almagest* XI.10, Ptolemy will go on to show how the table is actually constructed. This construction, in fact, does not follow the foregoing procedure. Hence, the determination serves as a sort of justification that the values calculated by means of the table are determinate, even though the computational method that employs the table will be approximate. The determination may be composed of metrical resolution, as a general argument that the value is determinate, or calculation, serving as an example showing how value can be determined. Ptolemy appears to use these interchangeably, and it is not clear that he regards one as more general than the other.

Although this section gives a mathematical argument that the values calculated in the table are determinate, it does not provide a full derivation of the actual columns of the table. For the table of planetary anomaly, just as the table of full lunar anomaly, the derivation is carried out in two steps: first a general argument that the model is determinate, and then a more detailed derivation, mixed in with the representation, which shows how the columns of the table can, in principle, be calculated. This more detailed derivation will be considered below.

2.3.2. Representation: Almagest *II.7–8, XI.10–11 and V.7–8*

The representation node is made up of both a description of how the table is constructed and a presentation of the table itself. Often the description is fairly simple, and may take up only the final few sentences of the proceeding section, but sometimes the description is involved and may have its own section containing internal derivations for certain columns. The goal of the representation is to exhibit the table such that it is both useful for calculation and also reveals the nature of the objects, or motions, in question.

Almagest *II.7–8* As a first example of the various functions of the representation node, I will use the table of rising-times of arcs of the ecliptic, *Almagest* II.7–8. The description is a simple statement of the dimensions and contents of the table. The table itself — one of the more impressive in the whole of the *Almagest* — tabulates the time-degees that arcs of the ecliptic take to rise over the horizon as a function of two variables, geographic latitude, φ, and ecliptic longitude, λ, from Ari 0° — that is, $\rho(\varphi, \lambda)$.

[32] This could be justified by an application of *Data* 6 and 8, but what Ptolemy means is that a value for $EL : LK$ can be calculated.

[33] This could be justified by *Data* 43, but Ptolemy means that we can apply the chord table in $\triangle LEK$ and calculate $\angle LEK$.

Table 2
Almagest II.8. The column of differences for the terrestrial equator.

9;10
9;15
9;25
9;40
9;58
10;16
10;34
10;47
10;55
10;55
10;47
10;34
10;16
9;58
9;40
9;25
9;15
9;10
9;10
9;15
9;25
9;40
9;58
10;16
10;34
10;47
10;55
10;55
10;47
10;34
10;16
9;58
9;40
9;25
9;15
9;10

The table is arranged in eleven columns for terrestrial latitudes with two subcolumns for accumulated time-degrees, and time-degrees for 10° intervals of the ecliptic. A third column acts as a list of headers for the thirty six rows, which tabulate entries at 10° intervals of longitude measured along the ecliptic. For the purposes of solving problems of spherical astronomy, one uses the column of accumulated time-degrees, with linear interpolation for individual degrees. To aid in this interpolation, the differences between successive entries are tabulated as well — that is, the time-degrees for each 10° interval. There is also mathematical evidence that Ptolemy used linear interpolation on his table of right ascension, *Almagest* II.8 col_1, to calculate his table for the equation of time in the *Handy Tables* (van Dalen, 1994, 129–131); however, this can hardly have been his sole reason for choosing this format for the *Almagest* table. As well as being useful for calculation, this column of differences serves as a representational function by displaying the changing behavior of the rising-times. By looking at the numbers in these columns, and comparing these columns with one another, we can see how the rising-times of arcs of the ecliptic change as one moves away from the equator.

For example, the column of differences for the terrestrial equator is given in Table 2. An inspection of this sequence of numbers shows that there are two pairwise equal minima and maxima, that the change is symmetric about these extrema, and that the rate of change is least in the vicinity of the extrema. Moreover, by comparing the column for the equator with the columns for the other terrestrial latitudes (Toomer, 1984, 100–103), we can see how this function changes as we move north — the absolute minimums at 0° become progressively lesser, the local minimums at 180° become greater, while the maximums become greater and move away from 90° and 270° towards 180°, but still are symmetrically situated about 180°. That is, nearly

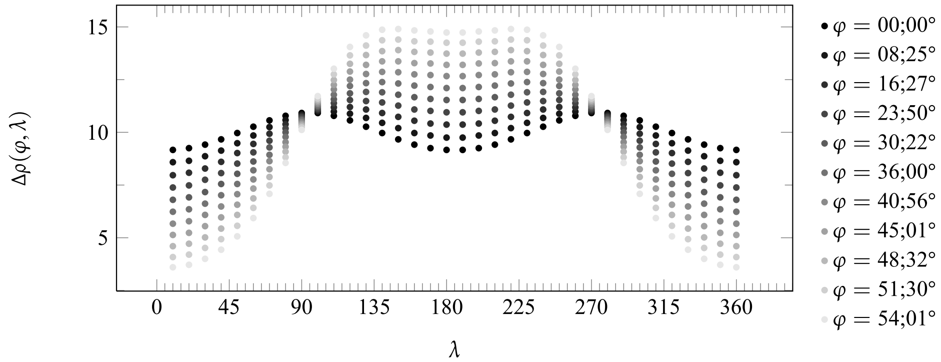

Figure 5. Graphical representation of the columns of 10° differences of time-degrees in the table of rising signs, *Almagest* II.8.

all of the information that we would encode visually, in a graph, as Figure 5, can also be seen in Ptolemy's presentation of the numbers.

Hence, like a modern graph, one of the primary representational goals of Ptolemy's tables, is to give a visual presentation of quantitative information, structured so that we may better understand the function that the numbers describe. This is especially clear when we compare the tables for rising-times in the *Almagest* with those in the *Handy Tables*. In the *Handy Tables*, for the equator, the values of the rising-times of arcs of the ecliptic have been tabulated from a new starting point, the interval of the rows has been set at 1°, and another column for the equation of time has been added (Tihon, 2011; Mercier, 2011; van Dalen, 1994). That is, since the latitude of the equator has a different norm, we can now no longer directly compare the rising times at the equator with those at the other latitudes; since the differences are not given, it is difficult to see how the functions change; and since the table spreads over a number of pages, it has been transformed into a computational tool that no longer allows us to visualize a mathematical relation. The result of these changes is that the tables are now easier to use, but the pattern of numbers no longer reveals the fundamental motion of the sphere of the cosmos against the sphere of the earth.

Almagest *XI.10–11* Another aspect of the representational character of tables is that they are meant to correspond to, and hence reveal, fundamental characteristics of the underlying models, which are geometric. A demonstration of this can be drawn from the tables of planetary anomalies, *Almagest* XI.10–11. These tables are somewhat involved, but for the moment we will only consider one aspect of their arrangement. As well as containing a number of features that allow the calculation of apparent position to be simplified using "tabular methods" — in the same way as the table of complete lunar anomaly, *Almagest* V.8, which will be discussed using the example of the lunar theory below — the tables of planetary anomaly also contain two columns that make the calculation slightly more complicated, but which reveal the geometrical function of the equant.[34]

As is well known, in Ptolemy's planetary models, the center of regular angular velocity is separated from that of the center of the motion itself — a feature which was deeply problematic for later readers who erroneously read Ptolemy as an Aristotelian and believed that he must hold that each sphere carrying a planet should also rotate with a regular motion. With respects to calculating the apparent position of the planets, the effect of this model is that the mean center of the epicycle, $\bar{\kappa}$, must be used to calculate the true position of the center of the epicycle, κ, before the effect of the epicyclic anomaly can be considered. Considering Figure 6, we will let T be the observer on earth; D the center of the circular path on which the center of the epicycle C moves; and E the center of a circle of projection about which the center of

[34] The expression "tabular methods" is actually used to describe the procedure for constructing the complete table of lunar anomaly, *Almagest* V.8, but since the method is essentially the same, the term is also applicable here. See page 30, below.

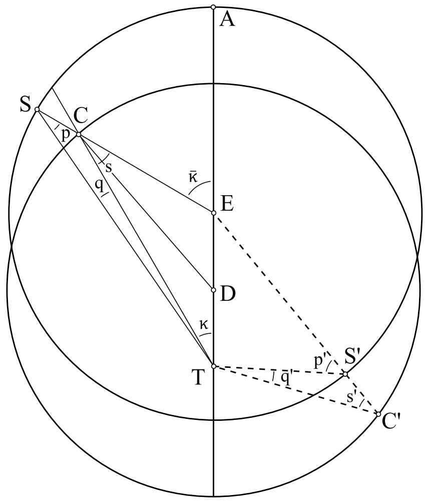

Figure 6. Simplified diagram for computing the location of the center of the epicycle given its mean motion, in the nexus for the tables of planetary anomaly, *Almagest*, XI.9–11.

the epicycle appears to move with a constant angular velocity. That is, as C moves about its circle, the projection of EC onto a circle of equal radius about E will fall on a point S, such that $\bar{\kappa}(t) = \angle AES$ sweeps out equal angles in equal times.

Thus, in order to find κ from $\bar{\kappa}$, it is sufficient to note that $\angle ECT = \angle EST + \angle CTS$, where $\angle CTS$ may be positive or negative, depending on whether TC falls before or behind ST, with respect to point A. That is, where q may be positive or negative,

$$s = p + q.$$

Clearly, the simplest way of tabulating this would be to diretly tabulate s for values between 0° and 180°. Nevertheless, in *Almagest* XI.11, Ptolemy tabulates p and q separately, as col_3 and col_4 for values between 0° and 180°, and then in the algorithm, *Almagest* XI.12, these are explicitly combined. O. Pedersen (2010, 292) thought that this roundabout procedure might have been a residue of the historical production of the text, but as G. Toomer (1984, 554, n. 48) has noted, Ptolemy explicitly states that this arrangement is in accordance with his theoretical goals. In *Almagest* XI.10, Ptolemy states:

> In this case, as it is in a systematic treatise (ἐν συντάξει), it were appropriate to make this separation of the zodiacal anomaly apparent, and, on account of this, to set out two columns; although, for its use, a single column, brought together by adding or subtracting both of them, will suffice.
>
> [Heiberg, 1898–1903, II, 429]

In other words, the goal of the tables in the *Almagest* is to model the components of the geometrical diagrams in such a way that the reader may follow, numerically, how the irregular motions that are apparent are actually composed out of more uniform motions, and ultimately related to the uniform motion of the reference circle. The contrast is made clear when we compare this with the practical goals of the *Handy Tables*. In the *Handy Tables*, a function equivalent to s is tabulated directly (Neugebauer, 1975, 1002), saving us one entry into the table and one calculation in the algorithm, but concealing what the table in the *Almagest* makes clear — namely, that the mathematically regular motion of S about E is first displaced by the displacement of the observer to T and then again by the displacement of the physical motion from S to C, centered on a new point D, midway between E and T.

Almagest *V.7–8* With respect to mathematical methods, one of the most impressive tables in the *Almagest* is that for the complete lunar anomaly, *Almagest* V.8. In this table, Ptolemy elegantly arranges the corrections for both of the lunar anomalies in a single table so that the "common numbers" can represent either double the mean elongation, later known as the *centrum*, $c = 2\bar{\eta} = 2(\bar{\lambda}_{☾} - \bar{\lambda}_{\odot})$, or the true anomaly, α, and all of the irregular motions of the model are displayed in one glance against the mathematically regular motion of the circle. This is the first table that incorporates a special method that Ptolemy used for tabulating a function of two variables in a single-entry table, called *Ptolemaic interpolation* by O. Pedersen (2010, 84–89). This table is also a good example of the claim I made above that the justification for the validity of the tabular methods must be based on a full assessment of the table nexus. In this section, I will present the description and layout of the table along with some discussion of the motivation, but in reading the *Almagest* the motivation for the different components of the table does not become fully apparent until we also take into consideration the algorithm in the following section.

The derivation node for the lunar theory, *Almagest* V.6, is simply a computation that shows that given the centrum and mean anomaly, c and $\bar{\alpha}$, the equation of anomaly, q, can be computed through chord-table trigonometry (διὰ τῶν γραμμῶν) (Heiberg, 1898–1903, I, 380). The description component of the representation node, however, is fairly involved and makes it clear that the entries in the table will not always correspond directly to geometric objects in the model. Indeed, one of the columns, V.8 col_6, will tabulate a fraction to be multiplied by col_5 as a function of c and its motivation is not provided by the general derivation. Ptolemy introduces the description, all of *Almagest* V.7, by noting that the corrections to the mean motions can also be computed "through tabular methods" (διὰ τῆς κανονικῆς, literally *by means of tabular things*) (Heiberg, 1898–1903, I, 383), then in the course of this description, he provides a derivation of col_6. Hence, in this case, the description of the overall table contains an internal derivation of a single column. As we will see, the "tabular methods" that Ptolemy discusses here are not simply a straightforward means of computing values for the geometrical objects in the model, but are motivated by a number of supplementary considerations.

At issue is the fact that the lunar model has compound anomalies, because (1a) the epicycle of the moon rotates around a deferent (1b) the center of which itself rotates around the observer on earth, while (2) the moon's motion on the epicycle is complicated by the fact that the mean apogee of the epicycle points towards a point which is not the center of either the ecliptic or the deferent.[35] From the perspective of trying to calculate the position of the moon on its epicycle, this means (1) that the apparent position of the moon on its epicycle is effected both by its angular location on the epicycle — its anomaly — and the distance of the center of the epicycle from the earth, and (2) that the mean anomaly, reckoned from the mean apogee of the epicycle, must be corrected to the true anomaly, as seen from the earth. The second of these issues is solved fairly simply by computing the correction to the mean anomaly as a function of the centrum, which is tabulated in the 3rd column of *Almagest* V.8, $\text{col}_3(c)$.

For the first, consider Figure 7(a), in which we let D be the center of the deferent on which the epicycle of the moon is carried, T the observer on earth, N a point on DT extended such that $DT = TN$, C the center of the epicycle seen from the earth under angle c, A_m the mean epicyclic apogee of the moon as seen from N, A_v the true epicyclic apogee of the moon as seen from the earth, $\bar{\alpha}$ the mean anomaly of the moon as measured from A_m, α the true anomaly of the moon as measured from A_v, and L the moon. Then the apparent position of the moon, which we seek, is given by correcting the mean longitude, $\bar{\lambda}$, by c which is, in turn, corrected by q.

[35] Ptolemy calls the orientation of the epicycle's apogee, its πρόσνευσις, which means "an inclination, or verging, towards." Whether or not this is a technical term of Ptolemy's lunar theory, it is almost certainly adapted from the standard mathematical νεῦσις, which is a line drawn under certain conditions such that when produced, it will go through a given point (Mugler, 1958, 296). Tommer (1984, 43, n. 38) gives a discussion of Ptolemy's various uses of *prosneusis*.

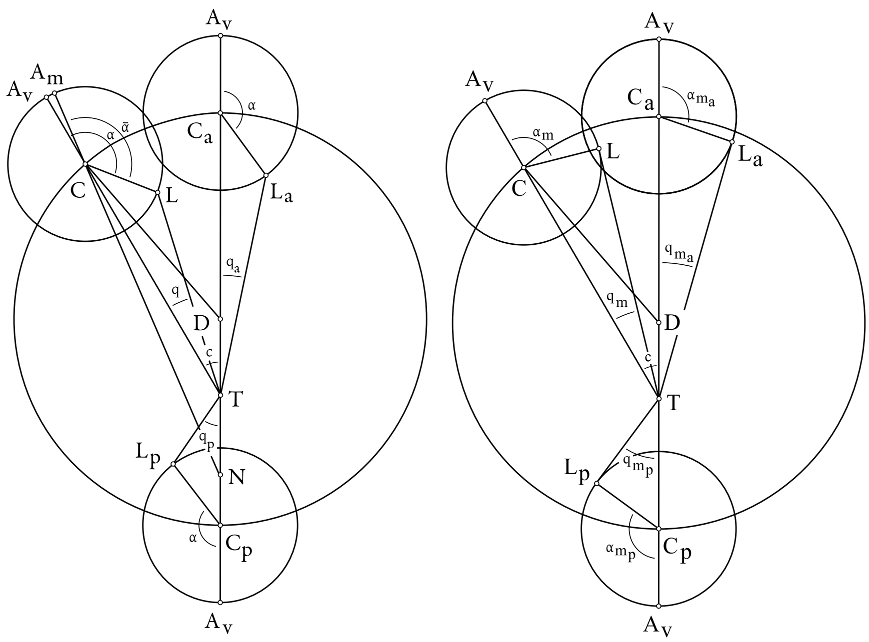

Figure 7. Simplified diagrams of *Almagest* V.6–8 for computing the apparent position of the moon as a function of two variables, angular distance from true apogee, α, and position of the center of the epicycle, c. (a) Three locations of the moon with a given anomaly α when the epicycle is at apogee, perigee and some place in between. (b) Three locations of the moon such that the equation of anomaly, q_m, is maximized — that is, TL is tangent to the epicycle — when the epicycle is at apogee, perigee and some place in between.

As the diagram makes clear, the magnitude of q is primarily determined by the true anomaly, α, but also somewhat determined by the location of the epicycle on its orbit, c. For example, in Figure 7(a), if we consider the situation where the moon is at the same location on its epicycle in a number of different positions of c, say at L and also at apogee and perigee, L_a and L_p, then it is clear that the primary variable is α, but that for a given α, there is a minor correction that varies from being least at apogee to being greatest at perigee, $q_a < q < p_p$. That is, the equation of anomaly is a function of two variables, $q(c, \alpha)$, where α is strong and c is weak. Although modern scholars have characterized the difference between Ptolemy's strong and weak variables analytically and graphically (Pedersen, 2010, 85–86; Van Brummelen, 1994, 299–301), it is likely Ptolemy's concept of this distinction was based on a mathematical investigation of the geometrical characteristics of the model.

In order to fully tabulate such a function as a double-entry table of two variables, using the same level of precision as found in the other tables, we would require a double-entry table of at least 45 rows and 45 columns, or 2025 entries, which would be cumbersome to compute and tedious to use. In order to get around this problem, Ptolemy has a way of exploiting the fact that c is a weak variable to produce a single-entry table for $q(c, \alpha)$. He uses three columns, *Almagest* V.8 col_4, col_5, and col_6, where the argument for col_4 and col_5 is the true anomaly, α, and that for col_6 is the centrum, c, but all are arranged in the same table, with arguments given as "common numbers." In this way, col_4 tabulates the value of the correction at the apogee, q_a, for 45 values of α between 6° and 180°, $\text{col}_4(a) = q(0°, \alpha)$; col_5 tabulates the value of the difference between correction at the perigee and the correction at apogee, $q_p - q_a$, for 45 values of α between 6° and 180°, $\text{col}_5(a) = q(180°, \alpha) - q(0°, \alpha)$. Hence, col_5 tabulates the greatest possible correction to q for each value α, which only applies in the case where the epicycle is a perigee. In all other cases, the correction will be some fraction of this.

In order to tabulate this fraction, Ptolemy simply assumes that the corrections will vary from q_a, through q to q_p in the same way for every value of α as it does for those values of α that maximize q — that is, when TL is tangent to the epicycle; see Figure 7(b). Ptolemy then gives a derivation of this fraction, by showing how it can be calculated for a single example. Using $c = 120°$ as an example, in *Almagest* V.7, Ptolemy calculates the excess of the maximum value of the equation of anomaly, q_m, at $c = 120°$ over its maximum value at apogee, q_{m_p}, as a fraction of its maximum value at perigee over its maximum at apogee, q_{m_a}, which he expresses in sixtieths. He states that this value will be put in col_6 corresponding to $c = 120°$ (Heiberg, 1898–1903, I, 385–388). He then says,

> In a similar way (ὡσαύτως),[36] the calculations of the remaining sections, the fraction (μέρη) of the difference of the two anomalies, in turn, obtained in the same way — we placed them, as sixtieths of each of the stated differences, besides the appropriate "numbers" ...
>
> [Heiberg, 1898–1903, I, 388]

Hence, col_6 tabulates the fraction by which each of the values in col_5, the differences, must be multiplied for each value of c, in minutes and seconds. In general,

$$\text{col}_6(c) = \frac{q_m(c) - q_m(0°)}{q_m(180°) - q_m(0°)} \cdot 60, \quad \text{or, in the case of the moon,}$$
$$= \frac{q_m(c) - 5;1}{2;39} \cdot 60.$$

As G. Van Brummelen (1994, 297–305) has shown, however, Ptolemy almost certainly did not calculate the 45 entries in V.8 col_6 from scratch each time using chord-table trigonometry. Instead, since the trigonometric calculation depends critically on a value for q_m, he appears to have first calculated a table of $q_m(c)$ at intervals of 12° and then used linear interpolation to calculate the intermediate values of q_m. Indeed, since the way in which the apparent size of the epicycle changes as the epicycle is carried on the deferent is a crucial characteristic of an epicycle model, it is not unlikely that Ptolemy was in the habit of constructing such tables to help himself understand how this variable changes under various assumptions for the parameters of the model. In this way, we see that tables played a key role not only in Ptolemy's presentation of his findings, but also in his research methods.

The final column, *Almagest* V.8 col_7, tabulates the correction to the mean latitude, calculated from the northernmost point of the moon's orbit. There is no need, at this juncture, to discuss it further.

The structure of the table of the compete lunar anomaly not only allows us to calculate a function of two variables with a single-entry table, it also allows us to see the irregular motions of the moon tabulated next to one another as a function of the mathematically regular motion of the circle. By studying the numbers in the table, we can see how the various functions are related to one another — where their extrema lie, and the relative rates of change. For example, we can clearly see that the correction to the mean anomaly has its maximum around 114°, its minimum around 246°, disappears at both apogee and perigee, and that the rate of change is greater around the perigee than around the apogee, which explains why the table has 6° intervals at the apogee and 3° intervals at the perigee.[37] Once again, an important goal of the table is a presentation of quantitative information such that we may apprehend the actual regularities underlying the apparently irregular motions of the moon.

From the perspective of quantitative information, a modern reader would probably prefer to see the numerical information found in the table of complete lunar anomaly presented in a graph, as Figure 8. It

[36] Toomer (1984, 237) translated this with "in exactly the same way," which is possible but not necessary. Since, as is discussed below, Ptolemy did not, in fact, calculate all of the values in this table in exactly the same way as his example, it seems more fitting to translate with the looser reading.

[37] Ptolemy explicitly states that the table of solar anomaly is so constructed for the same reason (Heiberg, 1898–1903, I, 251–252). See page 22, above.

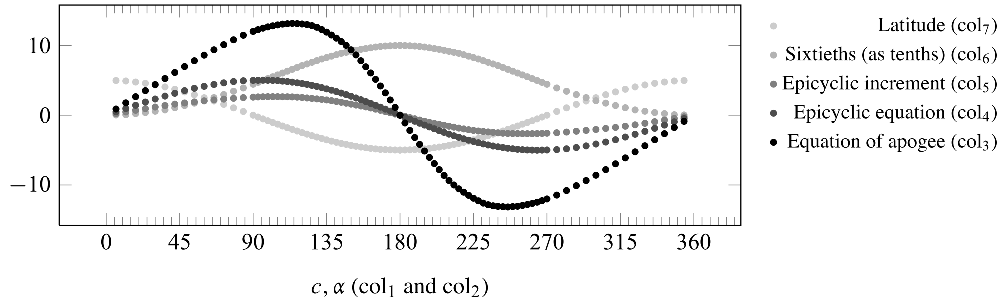

Figure 8. Graphical representation of *Almagest* V.8. In the table, only one set of absolute values is given, 0–180 and 360–180, which must be understood as either additive or subtractive corrections. The values in col_6 have been modified so that they now represent tenths. The values in col_7 are understood as either displacements to the north or to the south, with the limits noted at the top and bottom of the table.

should be pointed out, however, that a table format is better at presenting some information than a graph. For example, simply by looking at Figure 8 one might well think that the maximum of the epicyclic equation is at 90°, whereas it is clear in the table that it occurs at 96°. In the table, it is clear that the maximum of the increment occurs yet later, at 102°, although in the graph this maximum is difficult to see. Also, in order to graph the fractions that are multiplied by the increment, called "sixtieths" in the table, we need to multiply through by some factor. If we left them at their absolute value, they would simply appear in the graph as a straight line around 0, whereas if we graphed the sixtieths as integers, the y-axis of the graph would be so tall that the other curves would hardly be visible. In Figure 8, I have avoided these issues by representing these fractions as values between 0 and 10, so that we must now read the y-axis of the graph has representing both units and tenths.

As these examples make clear, the representation node of the table nexus simultaneously plays two essential roles. It both represents our developed knowledge of the mathematical objects, and it provides us with a computational tool for producing new knowledge about these objects.

2.3.3. Evaluation: Almagest *V.9*

In order to understand how the table represents the motions in the model, we must consider the diagram and the table together, mediated by a reading of the algorithm. The algorithm, as a general set of instructions for using the table, shows us how the individual columns of the table are computed with given values to produce sought values. We can write this out as a set of instructions, following Ptolemy's practice, or an equation, following modern practice. It does not seem to me to make much conceptual difference which we use, as long as we interpret the equation as a computational rule. As an example, I will use the algorithm for producing the apparent longitude of the moon, $\lambda_{☾}$, from the table of complete lunar anomaly, *Almagest* V.8.

In *Almagest* V.9, Ptolemy describes how to use *Almagest* V.8 along with values for change in mean longitude, $\Delta\bar{\lambda}$, mean elongation, $\bar{\eta} = \bar{\lambda}_{☾} - \bar{\lambda}_{\odot}$, mean anomaly, $\bar{\alpha}$, and mean argument of latitude, $\bar{\omega}'$, to calculate the apparent position of the moon, $L(\lambda, \beta)$. For the purposes of this example, we will neglect the computation of β, which is simple. The procedure for longitude is as follows (Heiberg, 1898–1903, I, 392–393):[38]

1. For the time in question, use the table of mean motions, *Almagest* IX.4, to calculate the change in mean latitude since epoch, $\Delta\bar{\lambda}$, and the mean anomaly, and elongation, reckoned from epoch, $\bar{\alpha}$ and $\bar{\eta}$, all as functions of time elapsed since epoch, Δt.

38 Examples of how to use the table to compute actual positions have been given by Neugebauer (1975, 96) and Toomer (1984, 682).

2. Double the mean elongation to determine the argument for col_3 and col_6, $c = 2\bar{\eta}$ mod 360.
3. Determine the true anomaly as the argument for col_4 and col_5 from the mean anomaly and col_3, as[39]

$$\alpha = \bar{\alpha} + \mathrm{col}_3(c) \quad \mathrm{mod}\ 360.$$

4. Determine the approximate equation of anomaly as a function of both anomaly and twice the mean elongation, as

$$q_{\mathrm{table}} = \mathrm{col}_4(\alpha) + \mathrm{col}(\alpha)_5\, \mathrm{col}_6(c).$$

5. Find the apparent longitude as reckoned from the mean longitude at epoch, $\lambda_{\mathrm{epoch}} = \mathrm{Tau}11;22°$ $(41;22°)$,[40] by adding to it the change in mean longitude and the approximate equation of anomaly,

$$\lambda = 41;22° + \Delta\bar{\lambda} + q_{\mathrm{table}} \quad \mathrm{mod}\ 360.$$

Without much change in meaning, we may also express this procedure as a single equation,

$$\lambda = 41;22° + \bar{\lambda} + \big(\mathrm{col}_4\big(\mathrm{col}_3(\bar{\alpha})\big) + \mathrm{col}_5\big(\mathrm{col}_3(\bar{\alpha})\big)\, \mathrm{col}_6(c)\big) \quad \mathrm{mod}\ 360$$

which is the tabular means of representing the following equation:

$$\lambda = 41;22° + \bar{\lambda} + \left(q\big(0°, \alpha(\bar{\alpha})\big) + \big(q\big(180°, \alpha(\bar{\alpha})\big) - q\big(0°, \alpha(\bar{\alpha})\big)\big)\frac{q_m(c)5;1}{2;39}60\right) \quad \mathrm{mod}\ 360.$$

By following through the algorithm and thinking about its implications — or by looking at the equation — we can get a better sense for Ptolemy's motivation in arranging *Almagest* V.8. Although I sketched some of the motivation in my discussion of the representation node of the table of compete lunar anomaly, above, Ptolemy himself makes relatively few remarks about why he proceeds as he does. Nevertheless, when we consider how the algorithm specifies the use of the individual columns, we see that, in Figure 7, the mean longitude is corrected first by the value of q when the epicycle is at its apogee, q_a, which is in turn corrected by the total difference of q at perigee and apogee, $p_p - q_a$, multiplied by a fraction that represents the effect of c on the value of q, namely $(q_m - q_{m_a})/(q_{m_p} - q_{m_a})$. Hence, it is clear that a full understanding of the table has to be composed from a synthesis of all the parts of its nexus: its derivation from the geometrical model, which is represented in certain columns, and the recombination of these columns in a set of rules that clarifies how the columns are related to the model.

3. Conclusion

With these discussions, it has become clear that tables were a crucial feature of Ptolemy's account of the celestial motions. Tables, in conjunction with the other parts of the table nexus, allow us to visualize the inner, and fundamental, regularity, of what are only apparently irregular motions. Indeed, in the *Almagest* tables — such as V.8 or XI.11 — all of the various anomalies are relational functions of the uniform progression of the circle. These tables allow us to grasp, in a single visual presentation,[41] how increasingly

[39] Here and below Ptolemy does not explicitly state that this is taken mod 360, but at this point it goes without saying.
[40] This was determined in *Almagest* IV.4.
[41] For Ptolemy's views of the importance of sight in apprehending rational objects, see *Harmonics* III.3 (Düring, 1930, 91–94).

anomalistic motions can be demonstrated to be in accordance with the principle of regular circular motion, which is the most basic, rational principle in operation.[42]

This is most clearly revealed when we contrast the tables in the *Almagest* with those in the *Handy Tables*.[43] The *Handy Tables*, were meant to codify the theoretical advances Ptolemy had made in mathematical astronomy in a format optimized for the tabular methods employed by individuals who were interested in calculating the positions of the celestial bodies but were largely uninterested in the underlying theory, such as professional astrologers.[44] They were also meant to show that purely tabular astronomy could be brought into conformity with what Ptolemy regarded as good astronomical practice. That is, they were meant to be contrasted with various other traditions of tables in Greco-Roman Egypt that were not based on regular circular motion — such as Greek translations of Babylonian mathematical astronomy or tables of solar or lunar motion that directly tabulated apparent anomalistic motion from epochs that were themselves calculated with a separate table (Jones, 1997a, 1997b, 1999b). The *Handy Tables* made it easier to calculate the rising and culminating signs and the positions of the celestial bodies, using computational and graphical methods, but they did this at the expense of spreading tables out over many pages to eliminate interpolation and combining different effects into a single column to eliminate excess calculation. Using the *Handy Tables*, it is no longer possible to perceive in one view how all of the compound motions are the result of regular circular motions.

In the *Almagest*, tables are one type of unit of mathematical text, which are used with other units, including other tables, as part of the overall deductive architecture of the treatise (see Section 2.2, especially Figure 3). Within this structure, tables function in a similar way as theorems in more canonical texts, such as the *Elements*. Some tables, such as the tables of mean motions, *Almagest* III.2, IV.4, IX.4, act more like lemmas — they are useful for further work, but are not intrinsically interesting. Other tables, such as the table of zenith distances and ecliptic angles, *Almagest* II.13, act analogously to certain culminating theorems in a theory — they set out a representational function as a goal of the investigation, although they may not be much used for further work. The most significant tables, however, such as the tables of solar anomaly, *Almagest* III.6, or the tables of rising times, II.8, are like an important theorem — they are both presentations of acquired mathematical knowledge and tools for producing new mathematical results.

It is clear that Ptolemy's articulation of tabular methods had a lasting effect on the Greek mathematical sciences. Although purely tabular methods, which appear to have been more popular and were more effective in astrology,[45] continued to be used, it was Ptolemy's *Almagest* and *Handy Tables* that received the most serious attention from expert scholars (Jones, 1999c, 160–168), and which were translated into Arabic and Latin. Since the *Almagest* became canonical in the curriculums of mathematics adopted by mathematical scholars like Pappus and Theon of Alexandria, and made explicit by the Arabic collection known as the *Middle Books*, Ptolemy's table nexus and tabular ways of working with functions formed a core part

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42 Ptolemy's claim that regular circular motion — understood now in a mathematical, as opposed to physical or theological sense — is the most basic principle is not fully developed until *Almagest* IX.2 (Heiberg, 1898–1903, II, 208–213).

43 A technical overview of the *Handy Tables* has been given by Neugebauer (1975, 969–1028). There is still no complete critical edition of the *Handy Tables*. The first edition of the tables was made by N.B. Halma (1822, 1823–1825), on the basis of a late manuscript. A study, and translation, of the tables in *Vatican gr. 1291* was made by Stahlman (1960) in his Ph.D. dissertation. Tihon (2011) and Mercier (2011) have completed the first part of a critical edition, starting with the tables of rising times, which Ptolemy tells us were the first tables in his work. Ptolemy's introduction to the *Handy Tables*, which was often transmitted independently, has been edited by Heiberg (1907, 159–185).

44 The claim by Bernard (2010) that professional astrologers made up the main readership of the *Almagest* should be treated with skepticism. There is almost nothing of use for astrologers in this text and it is rather heavy going for anyone who is not interested in mathematical demonstrations of the underlying theory. It seems, rather, that Ptolemy composed his *Mathematical Treatise in Four Books* and his *Handy Tables* to be of more interest to astrologers.

45 The popularity of purely tabular methods can be drawn from a consideration of the astronomical papyri, especially those from Oxyrhynchus (Jones, 1999a, 1999c).

of the mathematical education of countless scholars from the late ancient to the early modern periods. In order to understand how these mathematicians thought about the relationship between geometric objects, or motions, and the relational functions made up of sets of numbers that represented these, we can do well to start with a reading of Ptolemy's use of tables in the *Almagest*.

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References

Ancient mathematical works referenced by title

Almagest, Ptolemy: Heiberg (1898–1903), Toomer (1984).
Data, Euclid. Menge (1896), Taisbak (2003).

Medieval manuscripts

Paris BnF gr. 2889: Parisinus graecus 2389. Bibliothèque nationale de France (BnF), mostly uncial, 9th century.
Vatican gr. 1291: Vaticanus graecus 1291, uncial, late-8th–9th century, with a few later additions.

Modern scholarship

Acerbi, F., 2007. Euclide, Tutte le opere. Bompiani, Milano.
Berggren, J.L., Sidoli, N., 2007. Aristarchus's On the Sizes and Distances of the Sun and the Moon: Greek and Arabic texts. Arch. Hist. Exact Sci. 61, 213–254.
Bernard, A., 2010. The significance of Ptolemy's Almagest for its early readers. Rev. Synth. 131, 495–521.
Bowen, A., 2012. Simplicius on the Planets and Their Motions. Brill, Leiden.
van Dalen, B., 1994. On Ptolemy's table for the equation of time. Centaurus 37, 97–153.
Düring, I., 1930. Die Harmonielehre des Klaudios Ptolemaios. Elanders, Götenborg.
Feke, J., Jones, A., 2010. Ptolemy. In: Gerson, L.P. (Ed.), The Cambridge History of Philosophy in Late Antiquity, vol. 1. Cambridge University Press, Cambridge, pp. 197–209.
Halma, N.B., 1822. Commentaire de Théon d'Alexandrie, sur les tables manuelles astronomiques de Ptolemée. Merlin, Paris.
Halma, N.B., 1823–1825. Tables manuelles astronomiques de Ptolemée et de Théon, 2 vols. Bobee et Eberhart, Paris.
Hankel, H., 1874. Zur Geschichte der Mathematik in Altertum und Mittelalter. Teubner, Leipzig. Reprinted: Olms, Hildesheim, 1965.
Heiberg, J.L., 1898–1903. Claudii Ptolemaei Syntaxis mathematica. Teubner, Leipzig.
Heiberg, J.L., 1907. Claudii Ptolemaei Opera astronomica minora. Teubner, Leipzig.
Jones, A., 1997a. Studies in the astronomy of the Roman period: I. The standard Lunar scheme. Centaurus 39, 1–36.
Jones, A., 1997b. Studies in the astronomy of the Roman period: II. Tables for Solar longitude. Centaurus 39, 211–229.
Jones, A., 1999a. Astronomical Papyri from Oxyrhynchus. American Philosophical Society, Philadelphia.
Jones, A., 1999b. A classification of astronomical tables on papyrus. In: Swerdlow, N.M. (Ed.), Ancient Astronomy and Celestial Divination. MIT Press, Cambridge, MA, pp. 299–340.
Jones, A., 1999c. Uses and users of astronomical commentaries in antiquity. In: Most, G.W. (Ed.), Commentaries – Kommentare. Vandenhoeck & Ruprecht, Göttingen, pp. 163–172.

Jones, A., 2005. Ptolemy's mathematical models and their meaning. In: Kinyon, M., Van Brummelen, G. (Eds.), Mathematics and the Historian's Craft. Springer, New York, pp. 23–42.

Jones, A., 2006. The Keskintos astronomical inscription: Text and interpretations. SCIAMVS 7, 3–41.

Lehoux, D., 2007. Astronomy, Weather, and Calendars. Cambridge University Press, Cambridge.

Lehoux, D., 2012. What Did the Romans Know? Chicago University Press, Chicago.

Menge, H., 1896. Euclidis Data cum commentario Marini et scholiis antiquis. Teubner, Leipzig.

Mercier, R., 2011. Ptolemy's Handy Tables, Volume 1b, Tables A1–A2. Institute Orientaliste, Lourvain.

Mugler, C., 1958. Dictionnaire historique de la terminologie géométrique des grecs. C. Klinckseick, Paris.

Murschel, A., 1995. The structure and function of Ptolemy's physical hypotheses of planetary motion. J. Hist. Astron. 24, 33–61.

Neugebauer, O., 1975. A History of Ancient Mathematical Astronomy. Springer, New York.

Newton, R.R., 1985. The Origins of Ptolemy's Astronomical Tables. Johns Hopkins University, Baltimore.

Pedersen, O., 1974. Logistics and the theory of functions: An essay in the history of Greek mathematics. Arch. Int. Hist. Sci. 24, 29–50.

Pedersen, O., 2010. A Survey of the Almagest, with Annotation and a New Commentary. Edited by A. Jones. Springer, New York. Reprint with additions of the 1975 edition.

Stahlman, W.D., 1960. The Astronomical Tables of Codex Vaticanus graecus 1291. Ph.D. Thesis. Brown University.

Taisbak, C.M., 2003. Euclid's Data, Or the Importance of Being Given. Museum Tusculanum Press, Copenhagen.

Tihon, A., 2011. Les Tables Faciles de Ptolémée, Volume 1a, Tables A1–A2. Institute Orientaliste, Lourvain.

Toomer, G.J., 1984. Ptolemy's Almagest. Duckworth, London. Reprinted: Princeton University Press, Princeton, 1998.

Van Brummelen, G., 1993. Mathematical Tables in Ptolemy's Almagest. Ph.D. Thesis. Simon Fraser University, Department of Mathematics and Statistics.

Van Brummelen, G., 1994. Lunar and planetary interpolation tables in Ptolemy's Almagest. J. Hist. Astron. 25, 297–311.

Van Brummelen, G., 2009. The Mathematics of the Heavens and the Earth. Princeton University Press, Princeton.