CHAPTER 26

Mathematics Education

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It is difficult to say much with real certainty about mathematics education in the ancient Greco-Roman world. When it comes to discussing their education, the mathematicians themselves are all but silent; when discussing the place of mathematics in education in general, philosophical and rhetorical authors are frustratingly vague; and the papyri related to mathematics education have not yet received the same type of overview studies that the papyri related to literary education have (Cribiore 1996, 2001; Morgan 1998). Nevertheless, this chapter provides a survey of what we can say about ancient mathematics education, on the basis of the evidence from the papyrological and literary sources, and guided by analogies from what we now know about literary education.

Although it was once commonly held that the mathematical sciences made up a standard part of a fairly regular, liberal arts curriculum (Marrou 1948; Clarke 1971), this has been shown to be largely a fanciful characterization (Hadot 2005: 436–443, 252–253), and in the case of mathematics it is difficult to be certain about precisely what was learned at what age, and to what end. Indeed, the diversity of mathematical practices and cultures that we find represented in the sources gives the impression that mathematical education was even more private and individualized than literary education.

In the Greco-Roman period, the range of activities that was designated by the word *mathēmatikē* was not identical to those denoted by our understanding of the word *mathematics*. Although *mathēmatikē* was originally associated with any type of learning, it came to mean those literary disciplines that used mathematical techniques or that investigated mathematical objects—actual or ideal—such as geometry, mechanics, optics, astronomy and astrology, number theory (*arithmetikē*), harmonics, computational methods (*logistikē*), spherics (*sphairikos*), sphere making (*sphairopoiïa*), sundial theory (*gnōmonikos*), and so on. In this chapter, I consider mathematics education to be training in any of these mathematical sciences (*mathēmatikai*) or in the methods they employ.

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While many educated authors extolled the virtue of mathematics education, their remarks do not give us a concrete sense of what this entailed. For example, Quintilian, in the 1st century CE, recommends that a prospective orator supplement his reading in literature with studies of mathematics (*geometria*) and music (*Inst.* 1.10). He indicates that mathematics was studied under a geometer (*geometres*), but his description of the course of this study is much less detailed than his treatment of literary education. After claiming that one primarily studies mathematics in order to build character and sharpen the mind, he goes on to describe a few more specific benefits. In his opinion, the most significant of these is that mathematics teaches one about the idea of proof (*probatio*) by teaching the most powerful form of demonstration: geometrical proof (*grammikē apodeixis*). The only example of mathematics of this form that he specifically discusses, however, is how to produce an equilateral triangle on a given line, *Elements* 1.1 (*Inst.* 1.10.3). The longest description of actual mathematics in this section is given over to discussing the elementary fact that the perimeters and areas of plane figures do not vary proportionally (*Inst.* 1.10.40–45). Finally, he points out that through mathematics we learn a bit about astronomy, the cosmos and the natural causation of eclipses, and concludes by repeating the claim that geometrical proof (*linearis probatio*) can produce results on matters that are otherwise intractable. Not surprisingly, the general impression is that the amount of mathematics education regarded by Quintilian as sufficient to become a successful rhetorician was slight: basic calculation, including finger reckoning; practical geometry, including calculation of areas and perimeters; modest coverage of theoretical geometry, including at least a few proofs from *Elements* 1; basic spherics, and perhaps some proofs in this regard; basic cosmology, such as the geometry of eclipse theory, and so forth. The important thing was that the educated person should know that mathematics was a powerful and erudite discipline and that its practitioners could achieve a level of certainty that others could not.

In place of these sorts of vague claims about the virtues of mathematics education that we often find in literary and philosophical writings, it may be more useful to think of a three-stage division, modeled on the three stages of literary education (Cribiore 2001). (1) The primary stage involved instruction in basic numeracy, corresponding to basic literacy, and was carried out under a primary teacher (*didaskalos*, *grammatistēs*, *grammatodidaskalos*). (2) Secondary mathematical education corresponded to the school of the grammarian in literary education and was undertaken with either a grammarian (*grammatikos*, *grammatistēs*) or a specialized mathematics teacher, sometimes called a geometer (*geometres*). (3) The advanced stage, pursued by relatively few students, can be divided into two curricula: (a) professional and (b) philosophical. This division of tertiary education corresponds to the primary division of mathematical cultures into those which were more practical and those which were more theoretical (Asper 2009).

The articulation of this three-stage process is, however, simply a schematic, since the actual educational paths pursued by ancient learners must have varied considerably based on socioeconomic, geographical, and temporal differences, as well as the goals of the student (Kaster 1983). In smaller towns, it is likely that often all available stages of learning took place under a single teacher with students of all ages together in one setting. At the secondary level, although in smaller towns grammarians may have carried out mathematical instruction, in larger cities, at least by the Imperial period, there appear to have been specialized instructors in mathematics, known as geometers (Quint. *Inst.* 1.10.4; *Dig.* 50.13.1). While, for the more involved practice required by a surveyor, clerk, and accountant, mathematics could probably have been studied in professional scribal schools, many novice professionals, such as architects and astrologers, probably studied mathematics in more private settings directly from their masters, or from specialized mathematics teachers. In the philosophical schools, mathematics education would have varied depending on the school's attitude toward mathematics and the abilities and interests of the philosophers themselves. While education in the advanced mathematical subjects appears to have never been formalized to the same extent as rhetoric and philosophy, in the imperial and late-ancient periods mathematics education became increasingly canonical and institutionalized, first to meet the needs of empire and then by being incorporated into the late-ancient philosophical schools (Cuomo 2000: 16–55; Pingree 1994; Watts 2006; Riedlberger 2013: 32–41). Since the expectations of a mathematical education were more diverse than those of a literary education, we should expect that there would have been increasing diversity as a student progressed through their own, individual course of study. Hence, when considering the evidence from the papyri, it is often a matter of interpretive choice to assign any particular piece of evidence to one of the stages rather than another.

Since few people would need any mathematics beyond that taught in the first, or second stages, the availability of papyrological evidence is strongly slanted toward elementary education. Moreover, while individual mathematics curricula diverged more as students progressed through their studies, it appears that the content of the more advanced subjects also diverged over time. While the more elementary and practical subjects appear to have remained fairly constant over the centuries, the advanced, philosophical subjects changed as progress was made in these fields.

1. Elementary Education

The mathematics taught in primary education does not appear to have gone much beyond basic numeracy and was probably taught by the same teacher who taught basic literacy (Cribiore 2001: 180–183). Along with Augustine's statement that schoolchildren monotonously recited sums (*Conf.* 1.13), we have a fair bit of papyrological evidence for this early stage of numerical education. A list of numbers and a table of squares appear in a 3rd century BCE compilation of materials to be used in elementary education (MPER N.S. 1.23; Guéraud and Jouguet 1938). There are lists of numbers in schoolhands (ex. MPER N.S. 15.143–145, P.Laur. 4.150), tables of additions (ex. MPER N.S. 15.150–151; P.Köln 9.371; P.Berol.inv. 21346r), and many tables of multiplications and parts (ex. P.Berol.inv. 21310; P.Gen. 3.121; P.Mich. 15.686; P.Harrauer 3; MPER N.S. 15.152–157; P.Mich. 3.147; P.Berol.inv. 21303, T.Varie 4–5; MPER N.S. 15.159–171). Although some of the well-written tables may have been ready reckoners for use by professional scribes or calculators, the majority of them were probably written by, or for, schoolchildren (Fowler 1990: 234–240). A number of the papyri related to metrological conversions probably also arose in the context of primary education (ex. P.Köln 8.352). Finally, basic arithmetical operations were applied to the solution of simple word problems, such as calculations of profits or interest (P.Michael 62; T.Kellis

G 90; T.Tebt.inv. 3033). It appears that this stage of mathematics education was fairly constant through much of Greco-Roman antiquity. For example, fractional parts were always handled using tables of Egyptian fractions—what we call unit fractions—and there is no indication that the more sophisticated sexagesimal fractions that Greek mathematicians adopted from Mesopotamia for use in the astral sciences were ever taught in elementary education. Indeed, explicit discussions of arithmetical operations involving sexagesimal fractions in an educational context are found in Theon of Alexandria's *Commentary to Ptolemy's* Almagest, which indicates that this was a topic of the advanced philosophical curriculum.

2. Secondary Education

The secondary stage involved a transition to topics in geometry, although still at a fairly elementary level. This stage was probably taught by the grammarian in more rural settings, but by the Imperial period, in larger cities, seems to have been carried out by specialized teachers. A number of examples of the types of problems students would have encountered in their secondary education are preserved in the papyri written by schoolhands. We find a problem dealing with calculating the number of seats in a stadium (PSI 3.186), and three problems dealing with calculating areas and other features of plots of land (P.Col.inv. 157r; Bakker 2007). In one case, a set of elementary geometrical problems appears together with other matters pertaining to the secondary level, such as a passage from Homer and metrological problems, in what is probably a teacher's hand (MPER N.S. 15.178; Bruins, Sijpesteijn, and Worp 1974; Friberg 2005: 196–199). There is some indication that, as well as collections of various matters related to teaching, there were treatises comprising mathematical problems that acted as textbooks for mathematics education at the secondary level. In a letter from Hellenistic Egypt, Sarapion, a student presumably at the secondary level, asks his friend Ptolemaios to send him a papyrus roll that they had previously discussed, so as to help him, as he says, with "some geometrical [problems] assigned to me" (*tisi tithemenais moi geometrikais* (sic.), SB 111.7268). There are some candidates for such treatises among the extant papyri. An example is a collection of elementary exercises related to the calculation of areas, in a decent hand and apparently organized as a series of problems with solutions (P.Chic. 3; Goodspeed 1898).

While the question of what advanced, or theoretical, topics were taught in secondary education is not easily answered, it also seems clear that astronomy was approached through Aratus' *Phenomena* (Maass 1898: 80, 342). Because it is verse, the text would have been suitable for use by the grammarians; and the fact that it was read by nearly all educated people in their youth would have contributed to the well-established popularity of this work in antiquity. Although the commentaries written by Attalus and Hipparchus are not directed at an elementary audience, one of the other commentators of the text appears to have been a grammarian (Maass 1898: 91, 95). This indicates that in mathematical courses, just as in literary studies, students returned again to classical texts at various stages in their education, gradually appreciating them in greater depth. The same is probably true of other elementary theoretical texts, such as Euclid's *Elements* or Theodosius' *Spherics*. Although the introductory material, such as the definitions and the first few theorems, was probably introduced by some teachers, it is likely that few students approached these texts in any depth until later in their studies. The fact that Proclus' *Commentary to the First Book of Euclid's Elements* was addressed to students working through an advanced course in philosophy supports this hypothesis.

Hence, some of the material collected in Pseudo-Heron's *Definitions* and *Geometry*, such as definitions of various mathematical terms and simple geometry problems, probably originated in texts meant for secondary education, while other portions of it may have come from texts produced for training professionals or elementary texts for the philosophical curriculum. In the case of secondary education, it seems that much of the material taught remained fairly constant throughout the ancient period, such as simple geometry problems; however, new material appears to have been introduced after it became canonical, such as Aratus' poem, or Euclid's definitions.

3. Advanced Technical and Philosophical Curricula

More advanced mathematics education was probably quite diverse. Nevertheless, ~~it~~ in broad terms we may divide it into the practical mathematics that was taught to professionals who used mathematics in their work and the theoretical mathematics that was taught to individuals with more abstract interests, either philosophical or technical. The practical tradition of Greek mathematics is much less studied than the theoretical, in part because there are so few sources. This material is only known from a few handfuls of papyri and the compilations of problems and discursive material that are included in the Heronian corpus. What is clear, however, is that these texts are closely related to similar material in Mesopotamian and Egyptian sources (Friberg 2005). Hence, the methods used in the practical tradition were highly traditional, almost conservative, often neglecting new, more efficient methods that were developed in the theoretical tradition. Although it is not possible to be sure if all of the papyri from this tradition originated in an educational context, it seems likely that most of them did. By the Imperial period, mathematical education at this level appears to have been carried out by specialists, calculators or geometers (Cuomo 2000: 16–30). The ambiguity in our sources about whether these and similar terms apply to working professionals, such as accountants or surveyors, or to teachers of mathematical subjects is probably due to a number of factors, such as professionals engaging in teaching to supplement their income and novices learning their profession by studying with a practitioner.

A late papyrus from the 6th–7th century CE that preserves a genre of mathematical text common in Mesopotamian and Egyptian sources may have originated in the context of professional scribal education and demonstrates the continuity of this tradition over many centuries (P.Cair.Cat. 10758; Baillet 1982; Friberg 2005: 208–214). The content consists of a seemingly random selection of topics related to arithmetic, probably gathered together by a teacher or a student training to become a practitioner, such as tables of Egyptian fractions, problems involving prices and metrology, problems involving the manipulation of fractions, and what we would call systems of linear equations. There is no general discussion of the problems or their solutions, which supports the claim that the original context for this source was educational, since the teacher could have orally explained how to approach the material. Another text that may have been used in scribal classrooms, or those of perspective architects and surveyors, is a fragmentary treatise that deals with various solids (MPER 1.1; Fowler 1990: 254–259; Friberg 2005: 234–242). This text begins with some general remarks about measures and then gives a series of problems, in question-and-answer format, that could be used to teach rules for calculating the volumes of prisms, cylinders, pyramids, and truncated cones. These types of texts were probably also used in the education of accountants and other officials who made extensive use of computation (*logistēs*, *katholkos*, *calculator*, *tabularium*, *numerarius*). A more advanced text that was probably also related to this kind of education is a fragmentary treatise, preserved in a 2nd century CE papyri, that sets out a tabular method for solving systems of linear equations in two, three, and four unknowns, through the method of false position (P.Mich. 3.144; Robbins 1929; Friberg 2005: 200–208). The fact that this text contains problems whose conditions are unlikely to have been encountered in the course of ordinary work indicate that it has an educational character. That is, the purpose of working through these kinds of problems was to develop general problem-solving skills and ways of thinking, not merely to build a repertoire of methods that could be applied directly.

A number of papyri preserve treatises that may have been used in the education of architects, surveyors, and other professionals tasked with measuring land and its products (*agrimensor*, *mensor*, *gromaticus*, *geometres*). One of these, in a papyrus from the second century CE, is a series of problems, in question-and-answer format, which solve for features of right triangles, using rules equivalent to the solution of a linear-quadratic system of two equations in two unknowns (P.Gen. 3.124; Sesiano 1999; Friberg 2005: 220–221). Another, from the same period and also in question-and-answer format, finds features related to irregular rectilinear figures through the application of rules equivalent to the solution of two linear equations in two unknowns (P.Cornell.inv. 69; Jones 2012; Friberg 2005: 226–233). In both cases, the rules used can also be found in much older sources, going as far back as the Old Babylonian period. Moreover, again we find problems that do not correspond to anything that would have been encountered in actual work and whose purpose must have been to exercise general problem-solving abilities. Hence, the goal of mathematics education in the practical traditions must have been to produce individuals who were capable of solving new and unanticipated problems through the application of various rules that were learned through processes of repeated exposure and application.

Not surprisingly, almost all of the papyri of a clearly educational nature from the philosophical curriculum relate to Euclid's *Elements*. Of course, some of the many tables of numbers related to astronomy and astrology may have pertained to educational contexts, but it is difficult to be certain. There are also a number of papyri that either contain commentaries on mathematical works or works that include some mathematics, but these will be considered later. Aside from one fragment of the text itself (P.Fay 9; Fowler 1990: 212–214, pl. 3), the handful of papyri relating to the *Elements* almost certainly originated in an educational context. The first ten definitions of *Elements* 1 is preserved on a scrap of papyrus that was not part of a roll and was probably a private extract (P.Mich. 3.143; Turner et al. 1985). It is in a decent hand and was probably written by a teacher or a student who had advanced beyond the basic stages of education. Two papyri preserve an interesting type of digest, and prove that early propositions of the *Elements*, at least, were memorized in educational contexts. Both of these are series of enunciations of propositions, accompanied by purely symbolic, unlettered diagrams, and hence must have served as aids to memorizing the mathematical content of the propositions, as opposed to their argument. One of them preserves enunciations and symbolic diagrams for *Elements* 1.8–10 (P.Berol. 17469; Brashear 1994: 29–30, Abb. 16), while the other preserves those for *Elements* 2.4,5 (P.Oxy. 1.29; Fowler 1990: 208–212, pl. 2). These types of texts would be useful for students who were expected to master the elementary theorems so as to understand how they were used later in the text, or in other mathematical contexts. A final series of sources should be mentioned in this regard. This is a set of ostraka containing notes on solid geometry that make some reference to *Elements* 13 but contain a diagram unlike anything found in the manuscript tradition of Greek mathematical texts (Mau and Müller 1962). Here we find a rare piece of evidence of someone working through a more advanced text, or, less likely, using an advanced text to do some original mathematical work.

While the discussion so far has focused on elementary education, with some treatment of professional education and a passing reference to studying Euclid, some of the most interesting questions one might have about Greco-Roman mathematics education concern the educational role of advanced, theoretical mathematics. For example: Was there ever anything like a liberal arts education that culminated in the four mathematical fields of arithmetic, geometry, harmonics, and astronomy? What was the role of mathematics education at the various philosophical schools? How did people like Euclid and Ptolemy learn their mathematics? Unfortunately, none of these questions can be fully answered on the basis of our sources. The only approach we have to such questions is to consider broadly the social context of the mathematicians themselves and the nature and contents of the texts that they wrote. One thing that is clear, however, is that the canon of texts that were studied changed over time as new texts were written, although some texts remained classics. The incorporation into education of texts that were almost certainly not written for an educational purpose, such as Euclid's *Elements*, Aratus' *Phenomena*, or Ptolemy's *Almagest*, is one of the defining features of the philosophical curriculum of mathematics education. Indeed, a development from established results, including the incorporation of new methods, is characteristic of the theoretical tradition. Whereas in practical mathematics, outmoded methods and superseded parameters continued to be used long after simpler methods or better parameters had been derived, the theoretical tradition was progressive and cumulative, at least over sufficiently long periods of time. For example, in the theoretical tradition, for centuries after Euclid wrote his *Elements*, or Ptolemy his *Almagest*, no mathematical scholar could be taken seriously who did not know these works and respond to them, whereas in the practical tradition, knowing the latest value of π, or using precise, trigonometric methods, was not regarded essential. For these reasons, the theoretical tradition can be considered scientific, whereas the practical material is sometimes called "sub-scientific" (Høyrup 1990).

Although the scanty nature of our evidence makes it difficult to describe in detail the social circumstances in which Greek mathematicians worked, we can nevertheless paint a picture in broad strokes. In order to assess the social setting, we read discussions of mathematics and mathematicians in literary authors and philosophers, make inferences based on the few places where Greek mathematicians make personal comments in their writings, and make some guesses regarding the role of higher mathematics in education on the basis of the writings of mathematicians, philosophers, and other intellectuals.

It is clear is that, in Greco-Roman culture, unlike professionals who used mathematics in their work, theoretical mathematicians did not form a professional group that had been educated in a standardized way and earned their living through developing and teaching their mathematical skills, although some mathematicians apparently did earn a living through teaching and writing mathematics. Rather, we find a broad array of backgrounds: Archytas was a statesman and a general (Diog. Laert. 8.79), Hippocrates of Chios was a wealthy merchant (Philoponus *In Phys.* A2, 185a 16), Eudoxus was a respected legislator and a philosopher with many students (Diog. Laert. 8.89–91), Eratosthenes was the head of the library of Alexandria (P.Oxy. 10.1241), Archimedes was associated with the royal court of Syracuse (Plut. *Vit. Marc.* 14–19; Polyb. 8.5–8), and Hypatia was the daughter of a mathematical scholar and herself taught philosophy to youths of the Alexandrian social and political elite (Dzielska 1995: 27–46). The one thing that these individuals all share is that they had privileged lives and were participants in the type of high culture that revolved around literary and philosophical pursuits. We know none of the details of their education, but they came from backgrounds that could have provided them the kind of education that resources and leisure allow. They could have had private tutors when they were young, and when they were grown, they could have traveled to those cities where philosophy and rhetoric flourished. Since mathematics had no special institutional settings in the ancient period, and since mathematicians were members of the literate elite, they probably undertook the bulk of their advanced education in the same schools as other intellectuals such as sophists, philosophers, and poets.

Although it is uncertain precisely what role theoretical mathematics had in education, it is clear that this role increased throughout the ancient period. From early times, most schools of higher education were centered around a specific philosophical tradition, and mathematics education would have depended on the importance of mathematics within the school's thought. Of the mathematical curriculum of the most famous school of antiquity, Plato's Academy, we know almost nothing (Zhmud 1998). Whereas one could probably study mathematics, and even the history of mathematics at the Lyceum, it is unlikely that much, if any, mathematics was taught at Epicurus' Garden. Throughout the Hellenistic and Imperial periods, the teaching of higher mathematics appears to have become more established, at least in certain times and places. We are told by Pappus, for example, that Apollonius studied under the pupils of Euclid at Alexandria (*Coll.* 7.35). Apollonius says that *Conics* V will be useful for the "student" (*ṭālib*) of analysis (Toomer 1990: 5), and he appears to have organized whole treatises for use in teaching the techniques of analysis and synthesis in geometry (Saito and Sidoli 2010: 596, n. 43). The fact that some people did study treatises of theoretical mathematics is shown by the Euclidean papyri discussed earlier. By the late-ancient period, teachers, like Pappus and Theon of Alexandria, or Eutocius of Ascalon, were organizing treatises into canons for study, producing new editions of the classics, writing commentaries on important works, and producing text-based studies of specialized fields. Nevertheless, although it is clear that mathematics was occasionally taught, it is not clear if there were any general patterns to the teacher–student relationship. Although in some schools, most students may have listened to lectures on elementary geometry and spherics, it is unlikely that many progressed on to more advanced topics, such as geometrical analysis or advanced, theoretical arithmetic related to our modern number theory, or algebra. Most working mathematicians would have been lucky to have one or two really advanced students in their lifetime, and in many cases, the "students" referred to in our texts may have been rhetorical students whom the author hoped posterity would furnish.

Throughout the course of the ancient period, it seems likely that Greek mathematicians generally worked alone and not in research groups or schools (Netz 2002). Of course, there are some clear exceptions to this. In Athens, during the Classical period, there were small groups of mathematicians who worked together, or at least on the same set of problems. Some of these, such as Eudoxus, then returned to their homes along the eastern Mediterranean and founded schools of mathematical and philosophical instruction (Diog. Laert. 4.29). During this period, Athens was the main center of mathematical activity, but there were also peripheral nexuses, of which a striking example was the group at Cyzicus (Sedley 1976). Another exception is that of Alexandria during the Hellenistic and Imperial periods. Starting from the time of Euclid, there were almost continuously a few mathematicians working in Alexandria, perhaps associated with the museum and library. Archimedes publicized most of his work by sending it to mathematicians working in Alexandria, but it does not seem that there were more than two or three working there at any time for whom he had much regard (Heiberg 1910–1915: vol. 1, 2–4; vol. 2, 426–430). As mentioned earlier, Apollonius was said to have studied with Euclid's students in Alexandria. Hypsicles tells us that when a certain Basilides of Tyre was in Alexandria, he and Hypsicles' father spent much of their time discussing a mathematical work by Apollonius (Vitrac and Djebbar 2011: 53, 89). Thus, Athens and then Alexandria acted as centers that attracted talented young mathematicians from the peripheries, while in other cases, they were cultural nexuses where people studied and disseminated the works of important mathematicians, such as Eudoxus and Archimedes, who chose to live in peripheral locations.

It seems that it was in the Imperial period that theoretical mathematics education began to become standardized. In fact, a number of treatises that were clearly written for educational purposes—such as Geminus' *Introduction to the Phenomena* and Theon of Smyrna's *Mathematics Useful for Reading Plato*—were composed during this period. To get a concrete sense of one particular example of mathematics education during the Imperial period, we may consider the autobiographical testimony of the second-century physician Galen, who speaks of himself at every available opportunity. Although Galen cannot be taken as representative, because of the diversity of educational experiences there is little sense in hoping for a typical example. Nevertheless, Galen makes it clear that there were certain subjects and works that he regarded as canonical, usually by way of pointing out that his intellectual opponents lacked a secure foundation in just those matters. Galen had received his mathematical instruction from his father, Nikon, a man whom he held in high regard, and who "had trained in geometry, number theory, architecture and astronomy" (Galen *De aff. dig.* 7.1–4). Under this tutelage, Galen learned to harbor a general disdain for those "unpracticed in the deductive method, nor in the other mathematical sciences, which hone the soul, such as geometry, number theory, computation, architecture and astronomy" (Galen *De pecc. dig.* 2.1–8). In his *Commentary on Hippocrates' Airs, Waters, Places*, which survives only in medieval translations, Galen criticizes the Roman astrologers because of their lack of mathematical knowledge. We are told that although most of the Romans studied geometry in their youth, they did so only superficially: some studied only Euclid's *Elements* and *Data*, a few of them studied spherics, but almost none of them studied conic theory (*fī ashkāl al-bayḍah wa-al-ṣanawbar*) (Toomer 1985: 199). From this, and a few other passages, we may infer the order of instruction along which Galen had been led. He states that general training with numbers is followed by computation and number theory (*De pecc. dig.* 2.1–2), that the *Elements* is followed by spherics (*sphairikos theōrēmatos*), with the implication that this is followed by the theories of conics (*kōnikos*) and sundials (*De pecc. dig.* 1.4). Although he does not clearly say that analysis was taught last, this may be inferred from the fact that he spends the latter half of his *Diagnosis and Cure of the Soul's Errors* discussing the importance of the analytical method (*analytikē methodos*) for any discipline that seeks to produce knowledge about the world (*De pecc. dig.* 5). That his idea of analysis and synthesis derives from his mathematical, as opposed to philosophical, studies is made clear both by his derision, in this text, of discursive, school-based philosophy and by the fact that all his examples are mathematical, such as dividing a line into a proposed number of parts (*Elements* 6.9), circumscribing a regular figure with a circle (*Elements* 4), or marking lines on a sundial or water clock (*De pecc. dig.* 3). It is not certain when each of the stages of his mathematical education took place, but since he continued to live in his father's house at the age of fourteen when he began to listen to the lectures of the local philosophers (*De aff. dig.* 8.3), it is likely that he continued to study the exact sciences under his father while he began his philosophical studies. From Galen's remarks, we learn that by the Imperial period there was a fairly standard course of philosophical mathematics education, which he regarded as important both for the habits of mind that it imparted and for the benefits that it offered to those who mastered it in proposing solutions to problems encountered in the real world.

Our knowledge of the substantial texts of Greek mathematics comes through the filter of the scholarship of the mathematicians of late antiquity, most of whom were associated with schools of philosophy and regarded mathematics as an important part of a broader cultural and educational project centered around philosophy and religious activities. The mathematical texts of the earlier periods were edited and commented upon by these mathematical scholars, and this process acted as an informal process of selection, in so far as texts which did not receive attention had a dramatically reduced chance of being passed down.

These late-ancient scholars were primarily responsible for creating the image of theoretical mathematics that was transmitted to the various cultures around the Mediterranean in the medieval and early modern periods. Through their teaching and scholarship, they established various canons of the great works of the past, arranged courses of study through select topics, reinforced a sound and lasting architecture by shoring up arguments and making justifications explicit, and, finally, they secured their place in this tradition by intermingling their work with that of their predecessors and situating the whole project in contemporary modes of philosophic discourse.

One of the most mathematically minded of these scholars was Pappus of Alexandria, who was a competent mathematician, a gifted teacher, and made important strides in associating mathematics with areas of interest in philosophy by constantly arguing for the relevance of mathematics to other aspects of intellectual life. Pappus worked in many areas of the exact sciences, wrote commentaries on canonical works, such as the *Elements* 10 and the *Almagest*, and produced a series of short studies that were later gathered together into the *Mathematical Collection*. It is clear from Pappus' writing that he was part of an extended community of mathematicians and students who had regard for his work and interest in his teaching. The fact that his commentary and explanation of spherics, *Collection* 6, is followed by that on analysis, including the *Conics*, *Collection* 7, may be an indication that by this time an advanced geometry curriculum proceeding from spherics to analysis had become canonical.

The other mathematical scholars of the late-ancient period were also involved in teaching and expounding the classics, and hence mostly worked through the medium of commentaries. Theon of Alexandria edited works by Euclid and wrote commentaries to Ptolemy's *Almagest* and *Handy Tables.* Hypatia, his daughter, collaborated with her father on various projects and wrote commentaries to Apollonius and Diophantus. Proclus of Athens wrote a commentary on the *Elements* that was meant to be preparatory for students pursuing his advanced lectures in philosophy. Eutocius of Ascalon edited works by Archimedes and Apollonius, and wrote commentaries to them.

This work was a continuation of a tradition of commentating and editing that began in the Imperial period. The scholars of this period paid particular attention to issues of logical completeness, formal structure, and readability. They produced fuller texts with more explicit arguments, wrote auxiliary lemmas, introduced internal references to other parts of the canon, restructured the treatises and individual elements of the text, added introductions and conclusions, advocated explicit classifications, rewrote theories from new perspectives, and summarized long works for the purposes of study (Netz 1998). The goal of much of this work was educational, in that it paved the way for larger numbers of students to access these sometimes obscure classics (Bernard 2003).

All of this was part of a broad trend, begun in the Imperial period by authors such as Geminus, Heron, and Ptolemy, to incorporate the mathematical sciences into philosophical traditions (Feke and Jones 2010). Although in the Classical and early Hellenistic periods, philosophers showed interest in mathematical approaches, there is little indication that mathematicians had a similar regard for philosophy. The mathematicians of the late-ancient period, however, were concerned that mathematics be part of an education in philosophy and rhetoric (Bernard 2003; Riedlberger 2013: 34–38). Their texts show a combination of modes of thought from the traditions of pure mathematics with those from the various exact sciences, and a mixture of philosophical concerns with mathematical issues. Their project, situated as it was in the philosophical schools, argued both explicitly and implicitly for the value of an advanced mathematical education.

It remains to discuss briefly what texts and other aids were used in this advanced philosophical curriculum. It would seem that as the mathematical sciences developed, and as significant texts were produced, more and more of this material was presented in educational settings. Probably most mathematics education in the philosophical curriculum focused on reading standard texts—such as Euclid's *Elements*, Theodosius' *Spherics*, and Ptolemy's *Almagest*—and the commentaries that were written about them in order to help a growing numbers of student progress through this sometimes difficult material. The fact that such texts later came to be used in education, however, should not compel us to believe that they were originally written with education in mind. These texts were written as treatises (*pragmateia*, *suntaxis*) expounding a mathematical field on a structured foundation; they were not originally meant to guide beginners. Just as literary education focused on works that were composed for adults, so technical education was focused on reading and understanding the achievements of past masters. There was nothing quite equivalent to a modern "textbook" in ancient mathematics education.

Nevertheless, mathematical scholars were interested in guiding readers through their works (Mansfeld 1998); a number of authors produced introductory texts (*eisogōge*, *encheiridion*), and in some cases the exact sciences were taught with physical aids (Cicero *Tusc.* 5.64, 113; Geminus *Elem. Astron.* 5.69). Examples of introductory texts are Nichomachus' *Introduction to Arithmetic* and *Introduction to Harmonics*, which later received their own commentaries and epitomes; Geminus' *Introduction to the Phenomena*; and Pappus' *Introduction to Mechanics*, which became Book 8 of his *Collection*. These works, however, were not exclusively directed at students and were also of interest to educated adults and scholars. Mathematical sciences were also discussed at an elementary level in the course of studies on philosophy more broadly, such as in commentaries on Plato's *Theatetus* or lectures on Stoic cosmology (Diels and Schubart 1905; Bowen and Todd 2004). In the late-ancient period, the primary approach to mathematical scholarship was through commentaries, which although apparently directed at students were also a way for scholars to make their own contributions to the mathematical sciences. As discussed earlier, almost all the scholars of this period produced commentaries on past mathematical works. Although there is no physical evidence for teaching aids other than papyri and writing instruments, there is considerable evidence that Greek mathematicians made various instruments to model the mathematical objects with which they worked, and many of these were probably used in educational settings (Evans and Berggren 2006: 51–53; Sidoli and Saito 2009: 605–607).

This survey of the available evidence suggests that although there was never anything like a stable curriculum in what later came to be known as the *quadrivium*, mathematics education at the lower, and professional, levels was fairly constant throughout Greco-Roman antiquity, while students who had an interest in theoretical mathematics pursued these studies according to their own abilities and means. As significant work was done in the mathematical sciences, it was commented upon and organized into more systematic curricula. By the late-ancient period, advanced theoretical mathematics education came to be subsumed within the theological curricula of the late-platonic schools, but this seems not to have affected mathematics education at the elementary levels, or in professional contexts.

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