Arch. Hist. Exact Sci. (2018) 72:353–402
https://doi.org/10.1007/s00407-018-0211-5

The concept of *given* in Greek mathematics

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Received: 26 February 2018 / Published online: 14 June 2018

Abstract This paper is a contribution to our understanding of the technical concept of *given* in Greek mathematical texts. By working through mathematical arguments by Menaechmus, Euclid, Apollonius, Heron and Ptolemy, I elucidate the meaning of *given* in various mathematical practices. I next show how the concept of *given* is related to the terms discussed by Marinus in his philosophical discussion of Euclid's *Data*. I will argue that what is *given* does not simply exist, but can be unproblematically assumed or produced through some effective procedure. Arguments by *givens* are shown to be general claims about constructibility and computability. The claim that an object is *given* is related to our concept of an assignment—what is *given* is available in some uniquely determined, or determinable, way for future mathematical work.

1 Introduction

This paper is a contribution to our understanding of the technical concepts of *given* in Greek mathematical texts. The linguistic practices surrounding this and related concepts have recently been the subject of an excellent study by Acerbi (2011a), from whom I have learned much. In this paper, I focus on mathematical practice, hoping to elucidate how Greek mathematicians made use of the concepts of *given* in solving problems and in pursuing mathematical research.

Communicated by Jeremy Gray.

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The terminology of *given* in Greek mathematical texts needs to be distinguished from the general usage of the same term. In the first place, the ancient mathematical usage appears to be fairly restricted. Whereas we might say "a theorem shows that, given some configuration of objects…" or "given that a square number is equal to a given number…" and so on, we do not find such statements in Greek mathematical texts.[1] Indeed the terminology of *givens* does not show up in Greek theorems, except those directly related to the analytical activity of problem-solving, such as those of Euclid's *Data*, or in the lost treatises on loci and porisms.[2] As I will argue at length below, the notion of *given* specifies the status of certain objects that we work with in completing problems.[3] In the second place, the status of being *given* can apply to an object at any stage of a problem. It is not that case that some objects are given and then some other objects result, or are produced, from these. As we will see below in reading the ancient texts, objects are also said to be *given* when they can be produced from *given* objects through an effective procedure—that is, one that can actually be carried out in a finite number of steps.[4] In order to adhere to this usage, I will use *given*, in italics, to indicate the technical usage, and given, in roman, in the few cases where I indicate more general conceptions.

The concept of *given* was denoted by various forms of the verb "to give" (δίδωμι) and its participles. Although the nominal forms are most common, a range of verbal forms are also found. In terms of mathematical practice, I cannot find any distinction between these various forms, and will denote them all with *given*.[5] There appear to have been a number of related terms that were used in mathematical texts as synonyms of *given*, and which are discussed in the late fifth century CE by the Platonic philosopher Marinus of Neapolis. Since Marinus apparently took his terminology from the readings of the mathematical texts themselves, however, I will postpone any discussion of the various terms he introduces until after we have seen how the concept of *given* functions in a number of the mathematical authors whom he mentions.

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[1] Greek philosophical texts, on the other hand, do speak in this more general way. See, for example, Proclus' discussion of the parts of a proposition (Friedlein 1873, 203–205).

[2] I do not follow Heath (1921, I.422) in his claim that propositions in the *Data* can be either theorems or problems depending on how they are used. In this I agree with Acerbi (2011a, 126).

The contents of the lost ancient treatises on loci and porisms have been summarized by Jones (1986, 547–602).

[3] Here I mean "problem" as a unit of mathematical text that can be contrasted with a "theorem." In Greek mathematical discourse, solving a problem generally means doing what one set out to do, and this is usually expressed by saying that the problem has been done, or produced.

[4] This way of talking has struck many readers as strange, and has resulted in modern scholars repeatedly mistranslating the ancient and medieval texts. Taisbak (2003, 13–14) mentions K. Manitius and G.J. Toomer, but see also Hughes (1981, 127 ff.) and Lewis (2001, 271–273).

[5] Fournarakis and Christianidis (2006)—contrary to almost all other scholars—argue that there is a meaningful distinction between the perfect and aorist participles of δίδωμι that has both philosophical and mathematical implications. I do not see that this difference has, however, any meaning for the mathematical practice. Acerbi points out that, in the *Data*, finite verb forms are also used with what appears to be the same meaning, and he gives statistics for the occurrences of the various forms of the particle in canonical authors (Acerbi 2011a, 122, 127). In other authors, we also have a range of usage of finite verbs and particles. I cannot see any mathematically meaningful distinction between the different forms and I am dubious of our ability to propose emendations to the received manuscripts along these lines.

I will take the position that Greek mathematicians were deliberate in their choice of language and that they used the terminology of *given* in a wide range of mathematical contexts because they meant to designate a certain similarity of status among all those objects that they called *given*. In this paper, by reading a variety of different texts that treat *given* objects, I will seek to develop a general notion of the concept of *given* that is applicable in all of these different situations.

After a discussion of some historiographical considerations, the introduction of the mathematical notation that I will use, and the proposal of a functional definition of *given*, I will discuss the role of *given* in a number of mathematical contexts: a problem presented as an analysis–synthesis pair attributed to Menaechmus, Euclid's *Data*, an analysis–synthesis pair from Apollonius' *Conics*, and arguments by *givens* in Heron and Ptolemy.[6] I conclude by looking for common aspects of the concept of *given* in all of these various contexts.

1.1 History and narrative

Due to the overall lack of sources, and the uncertainty surrounding our extant sources, it is difficult, if not impossible, to write the history of Greek mathematics in anything like the sense in which history is usually understood. Some of the sources that I examine in this paper are subject to historical criticism to the extent that we can reasonably doubt whether they were even written by the people to whom I attribute them, or in the period in which I date them.

For example, the first source that I will examine was purportedly written by Menaechmus, a contemporary of Plato's of whom we know little beyond the tantalizing fragments of his mathematics and ideas reported in much later authors. The text we will read is attributed to Menaechmus by Eutocius around the turn of the sixth century CE, probably by way of Eudemus in the late fourth century BCE. It contains terminology that is regarded as being much later than Menaechmus, and it is possible that key ideas were also imported into the text by later writers.

The core text that we will read, the *Data*, was probably composed by Euclid in the early third century BCE on the basis of some set of theorems that had been established by his predecessors. It is believed to have been added to by later authors, and we can be fairly certain about certain additions made by Theon of Alexandria in late fourth century CE. There is no objective way to be certain by whom, and at what time in the classical or Hellenistic periods, the theorems that we will read in this paper were produced.

We do not know when Heron wrote, although it may have been around, or at least at some time after, the mid-first century CE (Souffrin 2000; Sidoli 2011; Masià 2015). With Ptolemy we are on firmer ground and are confident that he worked in the second century CE—and no one has ever called into doubt his authorship of the passages of the *Almagest* that we will read.

[6] Diophantus' use of *given* and the related concepts of *posited*, *sought*, *found*, *determined* and so on, in both *Arithmetics* and *Polygonal Numbers* is rather involved and, consequently, I will put off examining these texts for another time.

Nevertheless, despite these caveats, I will present the texts in what I believe is their historical order, and I will argue that there is a broad historical narrative. In particular, I will argue that the concept of *given*, in its technical sense, began in the constructive context of geometric problem-solving in the classical and early Hellenistic periods, and was then reconstrued, or repurposed, to apply also to the computational context of the measuration of geometric objects in the exact sciences of the late Hellenistic and imperial periods.

This is not, however, the most important issue. My main purpose in this paper is to convince the reader that the concepts of *given* with which ancient mathematicians worked were different from the concepts of given that developed from them in later periods, although they are closely related. Furthermore, for readers who come to this paper interested in the way these texts were read by later mathematicians in the medieval and early modern periods—when the ideas and methods in them might still have been thought of as relevant—such periodization of the ancient sources is probably irrelevant. It is unlikely that many, if any, readers of Greek mathematical texts in the medieval and early modern periods were at all concerned with the original context of discovery, so that once it became clear that the Euclidean texts could be subjected to arithmetical and computational readings, these readings were taken to be inherent to the mathematical ideas articulated in the sources. In particular, there is no indication that any reader in the late-ancient or medieval periods thought that there was anything unexpected about making an arithmetical or computational reading of the theorems of the *Data*.

1.2 Notation

In order to facilitate discussions of the mathematical concept of *given*, I introduce the following notational conventions.[7]

Primitive objects: I denote geometric objects such as points, and occasionally lines, with the same letter-name as the text under discussion, in italic type, using the distinction between capital and lowercase letters to mark the distinction between *given* and non-*given* objects, such that A denotes a general point, while a_p denotes the same point when it is *given*—that is, *given in position*. Hence, we can denote a general line as AB, and the same line as AB_p when it is *given in position* but none of its points are *given*, AB_m when it is *given in magnitude* but none of its points are *given*,[8] or $ab_{p,m}$ when both of its endpoints are *given*. In this way, aB is the name of any line passing through the *given* point a_p, while aB_p is the name of a certain line *given in position* that has *given* point a_p as an endpoint—the label B, however, is just part of the name of the line and does not indicate any particular point. Since the texts dealt with in this paper sometimes use a single letter-name for a point and sometimes for a line, when such an object is *given* I will always

[7] This notation is based on that introduced by Dijksterhuis (1987, 51–52) in the 1938 Dutch original of his work on Archimedes, and uses some aspects of a notation briefly employed by Taisbak (1991).

[8] For example, a line that falls at a *given* angle on two parallels *given in position*.

designate it with a subscript, such that a_m is a line *given in magnitude*, while a_p is a point *given in position*.

Figures: I denote rectilinear figures with bold type, such that a general, rectilinear figure, constructed from points A, B, C, ... is denoted as $\mathbf{F}(ABC\ldots)$, a triangle as $\mathbf{T}(ABC)$, a square as $\mathbf{S}(ABCD)$ or $\mathbf{S}(AB)$, a rectangle as $\mathbf{R}(ABCD)$ or $\mathbf{R}(AB, BC)$, and so on. A figure can be *given in magnitude*, $\mathbf{F}(ABC\ldots)_m$, *in form*, $\mathbf{F}(ABC\ldots)_f$, and so on. I also use some more specific names for rectilinear figures, the meaning of which I hope will be obvious in context.

Ratios: A general ratio between two magnitudes A and B is denoted $(A : B)$. A *given* ratio is denoted $(A : B)_r$.

Equations: Another convention that I will use is to put the object that was originally known, or assumed, to be *given* on the right-hand side of an equation and the object that is shown to be *given* on this basis on the lefthand side. In this way,

$$B = a_m$$

means that b_m is *given in magnitude* because it is equal to a_m, which was previously taken, or shown, to be *given in magnitude*.

Chord-table trigonometry: In order to discuss Ptolemy's chord-table trigonometry symbolically, I introduce the following special notation. In his plane trigonometry, Ptolemy often switches between various measures of length, so that he can operate with his chord table in such a way that the hypotenuse of the right triangle at issue is always set to 120^{p}. That is, if $\mathbf{T}(ABC)$ has $\angle BAC = 90°$, then in order to operate with the chord table using $\angle ABC_m$, $\angle ACB_m$, $(AB : BC)_r$, or $(AC : BC)_r$, we must first set $BC_m := 120^{\mathrm{p}}$. In this way, we have $(AB_m : 120^{\mathrm{p}})_r$ and $(AC_m : 120^{\mathrm{p}})_r$ as two given ratios. The AB_m and AC_m of these expressions are, however, not *given* absolutely, but only when we set $BC_m := 120^{\mathrm{p}}$. Hence, I will denote them as $AB_{m(BC)}$ and $AC_{m(BC)}$—which expressions denote the numerical values measuring AB and AC when $BC_m := 120^{\mathrm{p}}$. This somewhat pedantic notation keeps us mindful of the fact that these measures of length with which Ptolemy computes are stated as pairs of numbers, that is as ratios—as is always made explicit in Ptolemy's arguments by *givens*.

1.3 A functional definition of *given*

As we will see below in reading through a number of Greek mathematical texts, there are three primary ways in which an object can be *given*. An object is said to be *given* when it

($\mathbf{G_1}$) is stipulated as there in some fixed and determinate way at the beginning of the argumentative discourse—what we would call the data of the problem, or what is asserted as *given* in the enunciation of a theorem of the *Data*,

($\mathbf{G_2}$) is assumed at the mathematician's discretion, or is arbitrary in the sense that an arbitrary object can be chosen at the mathematician's discretion, or

($\mathbf{G_3}$) can be determined by an argument starting from ($\mathbf{G_1}$) or ($\mathbf{G_2}$) and employing constructions and chains of inferences calling on either synthetic theorems (say, from the *Elements*, *Conics*, and so forth), or propositions concerning *givens* (say, from the *Data*).

This tripartite definition serves to describe the way the concept of *given* is used in Greek mathematical texts. As I will point out throughout the rest of this paper, all of the various usages of *given* fall into one of these three types. It remains, however, to say what being *given* actually means. I hope to develop an understanding of the essential meaning of this term over the course of this study by considering how the concept functions in a range of different sources.

Philosophically minded readers may notice that from a logical, and perhaps ontological, point of view there is no difference between ($\mathbf{G_1}$) and ($\mathbf{G_2}$), since they both serve as essentially arbitrary starting points for the reasoning.[9] Nevertheless, from an epistemological point of view, and from the perspective of problem-solving practice—which is the proper arena of *given* objects in Greek mathematics—there is a difference. Objects *given* in the sense of ($\mathbf{G_1}$), are *given* by the problem-setter, or by the context of the problem, whereas those *given* in the sense of ($\mathbf{G_2}$) are *given* by the problem-solver, or in the course of establising the solution—although this may, of course, accidentally be the same individual. Moreover, objects *given* in the sense of ($\mathbf{G_1}$) may be *given* by various inherent mathematical constraints, such as through the transformation of one problem into another, the division of a problem into parts, the geometry of the figure, and so on. Furthermore, the types of objects that were assumed in the sense of ($\mathbf{G_2}$) were, in practice, quite limited—such as an example number *given* in value, in a numerical computation, or a point or line *given in position*, in a geometrical construction. Finally, in the exact sciences, to which we will see the concept of *given* applied, objects, or values, which are *given* by the properties of the model that we will use to address the problem are *given* in the sense of ($\mathbf{G_1}$), whereas those that are ostensibly *given* empirically, or which we will use for an example calculation are *given* in the sense of ($\mathbf{G_2}$).

An important feature of the concept of *given* is that it is always local and depends on the method of production, so an object that is not *given* by one procedure may, once arrived at by some other procedure, be treated as *given* for a following procedure.[10]

[9] For example, Marinus states that those who believe that what is *given by hypothesis*—which he also calls "the *exposited* (ἐκτιθέμενον) in the hypothesis of the problem"—is the *given* are mistaken (Menge 1896, 248, 236). He may be talking to his teacher in this passage. Proclus, in his discussion of the parts of a proposition, claims that in *Elem.* IV.10 nothing is *given*, because nothing is stated to be *given* in the enunciation and there is no exposition (Friedlein 1873, 203–204). But Proclus is wrong here. Although nothing is stated to be *given*, the construction of *Elem.* IV.10 begins by setting out an arbitrary line, AB, and an understanding of the practice in the *Elements* and *Data* makes it clear that when an arbitrary line segment is set out in a construction, it is set out as *given in position* and *in magnitude*. Hence, *Elem.* IV.10, in fact, shows us how to construct an isosceles triangle that has as one of its equal sides as a *given* line, ($\mathbf{G_2}$). That is, although the *Elem.* IV.10 makes no mention of an object *given* in the sense of ($\mathbf{G_1}$), there is an object *given* in the sense of ($\mathbf{G_2}$). Thus, although Marinus is correct that there is no logical difference between ($\mathbf{G_1}$) and ($\mathbf{G_2}$), we could ask for an isosceles triangle whose base is a *given* line, ($\mathbf{G_1}$), which would be a different problem. This highlights the practical difference between ($\mathbf{G_1}$) and ($\mathbf{G_2}$)—namely the enunciation of *Elem.* IV.10 leaves us free to set out a *given* line as either a leg or the base, so that the constraints of the problem, ($\mathbf{G_1}$), leave us free in our choice of the line *given* in the sense of ($\mathbf{G_2}$). Acerbi (2011a, 121), following Marinus, takes ($\mathbf{G_1}$) and ($\mathbf{G_2}$) to be the same.

[10] A clear example of this is Ptolemy's claim that the theorems he sets out in *Alm.* I.10 can be used to compute the entries in the chord table "by means of lines" (διὰ τῶν γραμμῶν) despite his later claim that

Indeed, as we will see below, *given* means little more than assumed or produced by the permitted operations.

2 An early analysis–synthesis pair

In this section, we look at the construction of two mean proportionals to two *given* lines proposed by Menaechmus in the mid-fourth century BCE as a solution to the transformation of the so-called Delian problem—to double a given cube. The text of this solution comes from Eutocius' *Commentary to Archimedes' Sphere and Cylinder I*, written over eight centuries later and containing what are believed to be various anachronisms in its presentation.

For our purposes here, we need not be concerned with addressing all of the many interesting historical and mathematical questions that arise when reading this passage from Eutocius.[11] Instead, we will simply focus on the way that the term *given* is used, and try to understand how *given* objects function in the argument. In the translation itself, I have not provided justifications for each of the steps in the argument from propositions in the *Elements* and the *Data*, because this argument was supposedly produced before those treatises were composed. Nevertheless, as I will make clear in the commentary that follows, all of the steps of the argument by *givens* can be justified by theorems of the *Data*, and all of the steps of the proof can be justified by theorems of the *Elements*. Furthermore, most of the construction steps can be justified by problems of the *Elements*, with the exception of the constructions of the conic sections themselves, which appear to be direct pointwise productions of the curves as the loci of points satisfying certain conditions (Knorr 1986, 63–66).[12] Although I have separated the argument according the traditional divisions of an analysis–synthesis pair—*transformation*, *resolution*, *construction*, and *demonstration*—it should be noted that in this problem, transformation steps, and resolutions steps, as well as construction steps and demonstration steps, are intermingled, so that, as often, the division into parts is only a loose characterization of the actual structure.[13]

Footnote 10 continued
Crd(1/2°) "is not *given* in any way by means of lines" (διὰ τῶν γραμμῶν οὐ δίδοταί πως), because it is Crd(11/2° ÷ 3), where Crd$(11/2°)_m$ is *given* (Heiberg 1898–1903, I.31–32, I.42). Indeed, in the derivation of the chord table, *Alm.* I.10, the metrical analyses treat Crd(1/2°) as *given*, despite the fact that he has claimed that it is not *given* "by means of lines." That is, once the value of this chord has been determined as *given*, ($\mathbf{G_3}$), through his approximation technique it can be taken as *given*, ($\mathbf{G_2}$), in a new procedure.

Διὰ τῶν γραμμῶν is a technical expression in Ptolemy's writings (Heiberg 1898–1903, I.32, 42, 251, 335, 380, 383, 416, 449; II.193, 198, 201, 210, 321, 426, 427, 429; Heiberg 1907, 202, 203). It designates either a metrical analysis or an actual calculation—either through elementary geometry or chord-table trigonometry.

[11] Useful overviews of the mathematics have been provided by Health (1921, I.253–254) and Knorr (1986, 61–62).

[12] The construction of the conic sections in both the analysis and the synthesis appears to produce the curves using a locus definition, for which we have no corresponding problem in the extant elementary treatises, including the *Conics*.

[13] For discussions of the structure of an analyzed proposition, see Hankel (1874, 137–150), Hintikka and Remes (1974, 22–26), Berggren and Van Brummelen (2000, 5–16), Fournarakis and Christianidis (2006, 49–50), Saito and Sidoli (2010, 583–588), and Acerbi (2011a, 138–141).

Eutocius' presentation of Menaechmus' argument reads as follows (see Fig. 1):

[*Exposition*:]
Let there be two *given* lines, A and E. [*Specification*:] Now, it is necessary to find two means proportionals to A and E.

[*Analysis*:]
Let it be done (γεγονέτω); and let them be B and G. [*Transformation*:] And let there be set forth a line [*given*] *in position*, DH, terminating at D; and from D let DZ be laid down equal to G; and let ZQ be produced upright, and let ZQ be laid down equal to B.

[*Resolution*:]
Since, then, three lines, A, B and G, are proportional, the rectangle between A and G is equal to the square on B—therefore, the rectangle between G and the *given* A, that is DZ, is equal to the square on B, that is to the square on ZQ. Therefore, Q is on a parabola drawn through D. Let parallels, QK and DK, be produced. And since the rectangle between B and G is *given*—for it is equal to the rectangle between A and E—therefore, the rectangle between KQ and QZ is also *given*. Therefore, Q is on a hyperbola in asymptotes, KD and DZ. Therefore, Q is *given*, and so also is Z.

[*Synthesis*:]
Now, it will be put together (συντεθήσεται) as follows. [*Construction*:] Let the *given* lines be A and E; and line DH is [*given*] *in position*, terminating at D. And let a parabola be drawn through D, whose axis is DH, and the upright side of the figure is A; and let the ordinates dropped perpendicular upon DH be equal in square (δυνάσθωσαν) to the surface applied to A having a side cut off from them toward the point D. And let it be drawn, and let it be DQ. And let DK be upright, and in asymptotes KD, DZ let a hyperbola be drawn, such that the lines produced parallel to KD and DZ will make a surface equal to the rectangle between A and E. Now, it cuts the parabola. Let it cut at Q, and let perpendiculars, QK and QZ, be produced .

[*Demonstration*:]
Now, since the square on ZQ is equal to the rectangle between A and DZ, it is that as A to ZQ, QZ to ZD. Again, since the rectangle between A and E is

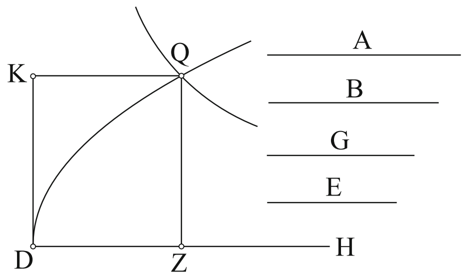

Fig. 1 Menaechmus' production of two means

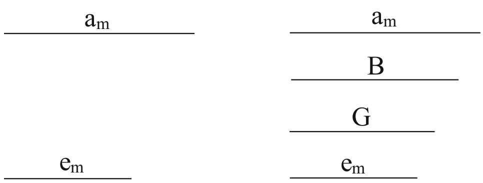

Fig. 2 Menaechmus' production of two means: Enunciation and analytical assumption

equal to the rectangle between QZ and ZD, it is that as A to ZQ, ZD to E. But, as A to ZQ, ZQ to ZD, therefore as A to ZQ, ZQ to ZD and ZD to E. Let B be laid down equal to QZ, and G equal to DZ. Therefore, it is that as A to B, B to G and G to E. Therefore, A, B, G and E are as a continued proportion. Which was to be found.

(Heiberg and Stamatis 1972, III.78–80)

In order to understand how the *given* objects function in this argument, it may help to work through some of the details. The first thing to notice is that there are different levels of construction.[14] Considering Fig. 2 (left), we begin, in the enunciation, simply by assuming that there are two lines, a_m and e_m, *given in magnitude*, $(\mathbf{G_1})$.[15]

The analysis begins with the analytical assumption, in Fig. 2 (right), that there are two other lines B and G—such that $(a_m : B) = (B : G) = (G : e_m)$. Then begins the first constructive process, generally known as the transformation, which must be understood, at this stage in the argument, to have a purely hypothetical character.[16] It starts, in Fig. 3 (left), with the assumption that line dH_p is *given in position*, with endpoint d_p *given in position*, $(\mathbf{G_2})$.[17] In fact, this assumption will also serve as the starting point of the resolution, as explained below.

The transformation then proceeds by a series of constructions, which, once the analytical assumption has been made, can be carried out straightforwardly. Namely, in Fig. 3 (right), we cut off $dZ = G$, say by *Elem.* I.3, and set up $ZQ \perp dH_p$, say by *Elem.* I.11, such that $ZQ = B$, by *Elem.* I.3. We do not at this point know whether or not B and G can really be produced, nor are we even assuming that they are *given*—we are simply interested in exploring what types of constructions might be performed *if*

[14] The function of these different constructions will become clearer in Sect. 4.

[15] These lines may also be *given in position* as well, but that is not essential to the argument.

[16] Acerbi (2011a, 139) also emphasizes this point.

[17] In the text, we are simply told that the line is laid out "in position" (θέσει), but this is the standard expression by ellipsis for *given in position*, often used in Euclid's *Data* and probably in Apollonius' *Cutting off a Ratio*. As just one example, see *Data* 43 , Sect. 3.3.

Furthermore, the point D is not explicitly said to be *given in position*, but DH_p is said to terminate at D, and will we see in the argument that follows that we do, in fact, assume d_p as *given in position*. Hence, the claim that line DH_p is terminating at D must be read to mean that point d_p is *given*. This is essentially the same as the initial constructive step of *Elem.* I.22.

It is a common practice to assume points and lines as given in position—see, for example, *Data* 28, 40–43, 55.

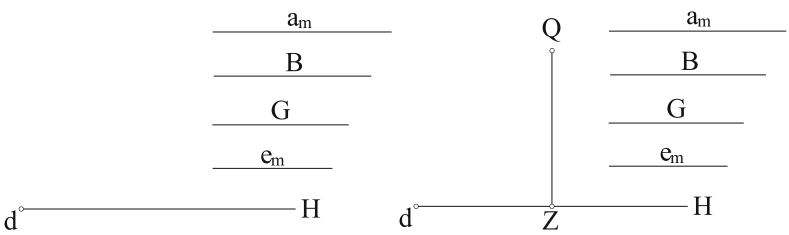

Fig. 3 Menaechmus' production of two means: transformation

they are assumed to exist.[18] Hence, all of the constructions in the transformation have a sort of hypothetical status.

The second stage of the argument, known as the resolution and involving a different sort of constructive process, proceeds to show that if certain objects are assumed to be *given* in the sense of ($\mathbf{G_1}$) and ($\mathbf{G_2}$), then the objects that will complete the problem can also be shown to be *given* in the sense of ($\mathbf{G_3}$). It is important to note that we do not in this process assume that B and G are *given*, although we will show that lines equal to them are *given*.[19] The argument proceeds as follows.

We begin, in Fig. 4 (left), by assuming a line *given in position* and with a *given* endpoint, dH_p, ($\mathbf{G_2}$). That is, it is taken for granted that we can assume points and lines *given in position*.[20] Since, $ZQ = B$, $dZ_p = G$, and $(a_m : B) = (B : G)$, by an argument similar to *Elem.* VI.16, we have

$$\mathbf{R}(a_m, dZ_p) = \mathbf{S}(ZQ),$$

[18] Some scholars want to avoid the claim that we must assume the existence of these objects by arguing that we are simply assuming a potential configuration. In this case, however, it is only their existence as lines having a certain mathematical constraint that is of any importance—their configuration is irrelevant.

[19] This "directionality" of the argument by *givens* is crucial. It begins with the objects asserted as *given* in the enunciation of the problem itself, ($\mathbf{G_1}$), not with the objects assumed in the analytical assumption—unless those later objects are assumed to be equal to the originally *given* objects. This makes it clear that Acerbi (2011a, 139) is incorrect in his assessment of the starting point of the resolution of *Conics* II.50. He claims that $\angle \mathrm{B}\Delta\Gamma$ is *given* because it is equal to another angle, and that it is irrelevant that this other angle is *given*. But the fact that this other angle is *given* is absolutely essential. In fact, $\angle \mathrm{B}\Delta\Gamma = \angle \mathrm{EZH}_m$ is a key part of the analytical assumption, while $\angle \mathrm{EZH}_m$ being *given in magnitude* is one of the conditions of the problem. That is, what it means to assume that $\angle \mathrm{EZH}_m$ is *given* in the statement of the problem is that we assume that another angle, say $\angle \mathrm{B}\Delta\Gamma$, can be produced equal to it. Then, $\angle \mathrm{B}\Delta\Gamma_m$ is also *given* by *Data* Def. 1, because we can construct another angle equal to it—namely any angle equal to $\angle \mathrm{EZH}_m$. (See the discussion of *Data* Def. 1, Sect. 3.1.) The location of $\angle \mathrm{B}\Delta\Gamma_m$ is not yet relevant, but it will shortly be proven to be *given in position*. In order to start the resolution, however, all we need is to know that we can produce somewhere on line $\mathrm{B}\Delta$ an angle equal to $\angle \mathrm{EZH}_m$—which, of course, can actually be done with *Elem.* I.23. The logical force of this "directionality" is also why the false arguments advanced by Berggren and Van Brummelen (2000, 25–26) that use *Data* 2 to show that the side of a square that is equal to a *given* circle is *given*, or that the third of a *given* angle is *given*, are not successful as analytical resolutions.

[20] It is clear from the problems of the *Elements* and the theorems of the *Data*, that the assumption of points and lines as *given* was taken for granted. For points, see, for example, *Elem.* I.9, 11, 23, *Data* 32, 33, 37, 38. The case for lines can be seen in *Elem.* I.22, *Data* 39–43. That the points and lines set out in the problem-constructions of the *Elements* must be considered as *given* can be shown from the fact that they serve as the basis for further constructions that are performed through problems that themselves assume as *given* these very points and lines.

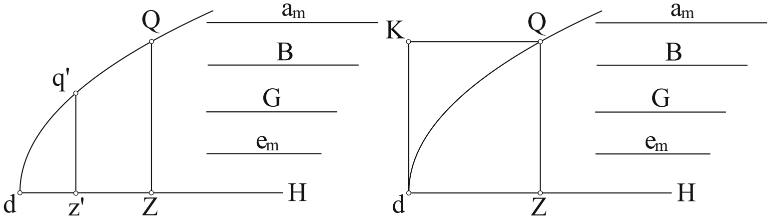

Fig. 4 Menaechmus' production of two means: resolution I

and where ZQ is an ordinate and dZ_p an abscissa, this is the *symptoma* (σύμπτωμα) of a parabola, *Conics* I.11—which can also serve as a locus description of the curve.[21] At this stage in the argument, points Z and Q are not *given*, because lines B and G are not *given*, so that the curve is defined for all of the possible pairs of Z and Q, Z' and Q', and so on. Nevertheless, although the text does not explicitly say so, this curve itself is *given in position*, say $\mathcal{P}_p$—as, indeed, it must be where d_p is *given in position* and its upright side, a_m, is *given in magnitude*, since for any point z'_p that we locally assume as *given*, ($\mathbf{G_2}$), point q'_p can be shown to be *given*, ($\mathbf{G_3}$).[22] Moreover, the fact that $\mathcal{P}_p$ is *given in position* is also entailed by the locus definition of the curve, where $\angle dZQ_m$ is right.

Next, in Fig. 4 (right), we produce $QK \parallel dZ$ and $dK \parallel ZQ$, say by *Elem.* I.31. These constructions are also hypothetical at this stage, because QK must be produced passing through Q, which is not yet located—because B and G are not *given*. That is, this construction, although it comes in the middle of the resolution argument, is, in fact, a transformation step and it could have been carried out in the transformation with no change to the force of the argument. Then, since $\angle KdZ = \angle QZH_m$, being right, line dK_p is *given in position*, by *Data* 29.

Next in Fig. 5 (left), since, by assumption $(a_m : B) = (G : e_m)$, then by *Elem.* VI.16, $\mathbf{R}(B, G)_m = \mathbf{R}(a_m, e_m)_m$. That is, since $\mathbf{R}(a_m, e_m)_m$ is *given in magnitude*, by *Data* 52, we have

$$\mathbf{R}(dZ, ZQ)_m = \mathbf{R}(a_m, e_m)_m,$$

which is the locus description of a hyperbola passing through point Q, between asymptotes dZ_p and dK_p, both *given in position*. It is not known how Menaechmus related this locus to the originating cone, but the property can be seen in *Conics* II.12. Indeed, Menaechmus may have simply thought of this property as defining the curve directly, as a locus (Knorr 1986, 63–66). Whatever the case, as before, where $\angle dZQ_m$ is *given in magnitude* and dZ_p is *given in position*, the hyperbola is *given in position*, say

21 See Dijksterhuis (1987, 58) for a derivation of the *symptoma* from what is believed to be the pre-Apollonian construction of the conic section.

22 That is, taking z'_p as *given*, ($\mathbf{G_2}$), we would have $\mathbf{R}(a_m, dz'_m)_m$ as *given*, by *Data* 26 and 52, so that $\mathbf{S}(zQ)_m$ is *given* by *Data* Def. 1. Then, by *Data* 55, line zQ'_m is *given*. And since $\angle dz'Q'$ is right on line dH_p *given in position*, locally ($\mathbf{G_1}$), line $z'Q'_p$ is *given in position*, by *Data* 29, so that point q'_p is *given*, ($\mathbf{G_3}$), by *Data* 27.

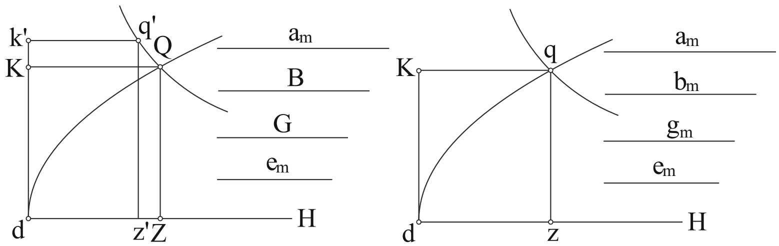

Fig. 5 Menaechmus' production of two means: resolution II

$\mathcal{H}_p$—since for any since for any point z'_p that we locally assume as *given*, (**G**$_{\mathbf{2}}$), point q'_p can be shown to be *given*, (**G**$_{\mathbf{3}}$).[23] The locus definition of the curve also makes it clear that where $\angle KdZ$ is right and dZ_p and dK_p are *given in position*, the curve must be *given in position*.

Finally, in Fig. 5 (right), since $\mathcal{P}_p$ and $\mathcal{H}_p$ intersect, by *Data* 25, their intersection is *given in position*, q_p. Therefore, by *Data* 30, qZ_p is *given in position*, so that z_p is *given in position*, by *Data* 25. The final step of the argument is omitted as obvious—namely, $G = zd_{p,m}$ and $B = zq_{p,m}$ are fully given, both by *Data* 26, so that g_m and b_m are *given in magnitude*, by *Data* Def. 1.[24]

That is, we have shown that starting from the assumptions that a_m, e_m, (**G**$_{\mathbf{1}}$), and dH_p, (**G**$_{\mathbf{2}}$), are *given*, and introducing the locus production of a parabola and a hyperbola using only these *given* objects, we can constructively produce two straight lines such that $(a_m : zq_m) = (zq_m : zd_m) = (zd_m : e_m)$. The resolution, then, acts as a general argument for the constructibility of these lines as uniquely determined, that is *given in magnitude*. The argument is general in the sense that we can designate any line as the line *given in position*, (**G**$_2$), upon which the construction will be based.

Following this, the synthesis begins with another construction, which includes a construction of the two conic sections—which in terms of later work could be carried out with *Conics* I.52, II.4.[25] The demonstration then unpacks the definitions of the curves to show that the problem has been completed by setting out two lines equal to the ordinate and abscissa of the parabola at the point of intersection.

There are a number of features of this problem that will be of use to our study of the concept of *given*. The first is that the contrast between *given* and non-*given* objects plays an essential role in the definition and, indeed, production of the two conic sections. Although the characterization through *givens* can be justified by the relationship between the conic sections and the originating cone in Apollonius' *Conics* I.11 and

[23] That is, taking z'_p as *given*, (**G**$_{\mathbf{2}}$), we would have dz'_m as *given*, by *Data* 26, so that $z'q'_m$ is *given*, by *Data* 57. And, again, And since $\angle dz'Q'$ is right on line dH_p *given in position*, locally (**G**$_{\mathbf{1}}$), line $z'Q'_p$ is *given in position*, by *Data* 29, so that point q'_p is *given*, (**G**$_{\mathbf{3}}$), by *Data* 27.

[24] See Sect. 3.

[25] In fact, there are substantial issues with *Conics* II.4, which seems simply to reduce the problem to a different problem. Also, notice, that the parabola required for Menaechmus' problem is produced by *Conics* I.52, but not by *Conics* VI.31, which may help us understand the difference between those two problems; see also the discussion of this issue by Fried and Unguru (2001, 264–268).

II.12, because these propositions are theorems, the notion of *given* is not there explicitly invoked. In Menaechmus' problem, however, the curves are essentially determined by a relation made up of both *given* and non-*given* objects—for example, point Q is non-*given* in the locus description of either curve; it is only *given* at their intersection. Menaechmus has no need to appeal to the cone and we do not need any propositions of the *Conics* to follow this proposition. The objects which are non-*given*—such as point Q in the locus descriptions of the curves—are, however, not completely arbitrary—they are constrained by their relations to the *given* objects, in particular the points and lines *given in position*. This was the general characteristic of objects treated in locus theorems and porisms. In fact, this may have been a characteristic of Menaechmus' mathematical style.

We are told by Proclus that the "mathematicians around Menaechmus' considered all propositions to be problems, which are divided into "those providing (πορίσασθαι) the sought, and those seeing that something is, or how something is, or that something obtains or has a certain relation to something else" (Friedlein 1873, 78).[26] The incorporation of the notion of *given* into the production of the curves themselves, and its applicability in addressing some of the issues handled in this second class of problems, may have been a feature of the problem-based approach of Menaechmus' conic theory.

By seeing how Menaechmus, or his compiler, works with *given* objects, we can confirm the various functions that they play in the resolution. The goal of the resolution is to show that, starting from the objects that are assumed to be *given* in the enunciation of the problem, a_m and e_m in the sense of ($\mathbf{G_1}$), and introducing only objects that can unproblematically be assumed to be given, such as points and lines given in position, d_p and dH_p in the sense of ($\mathbf{G_2}$), then the objects that complete the problem are also *given*, $G = zd_{p,m}$ and $B = zq_{p,m}$, in the sense of ($\mathbf{G_3}$). That is, the resolution constitutes an argument for general constructibility. The types of construction allowed, of course, vary from problem to problem. In this case, conic sections are allowed. That is, we have shown that under the assumption that such conic sections can be introduced the objects that complete the problem are *given*—that is, fixed or determined. Hence, in the construction that follows in the synthesis, the conic sections discussed in the resolution are produced.

Finally, the resolution as presented in Menaechmus' treatment is rather abbreviated, and it is only by providing a full commentary that we can see how the reasoning by *givens* proceeds deductively. The fact that we may have difficulty following Menaechmus' argument, however, should not lead us to believe that it presented any difficulty to ancient mathematicians who were familiar with such analytical procedures. As I will argue in the next section, we can understand Euclid's *Data* as providing justifications for the steps of such resolutions in an attempt to bring clarity and organization to the analytical presentation of problems.

[26] Vitrac (2005, 40–42) gives a discussion of the context in which Menaechmus and his circle argued for the primacy of problems.

3 Euclid's *Data*

Euclid's *Data* is the only systematic treatise that has come down to us that is devoted to developing theories of the different ways in which geometric objects can be *given*. It was probably composed to show how—by justifying the steps of the resolution in an analysis, such as that we read in the previous section—the material of *Elements* I–VI can be put to use within geometric problem-solving.[27] The text itself begins with a series of definitions and theorems that, in themselves, may strike some readers as vague to the point of obscurity. Before reading through the core definitions and a number of the fundamental theorems of the text, it may be helpful to sketch a path backwards toward these from the types of justifications that are required for an analysis–synthesis pair like that in Menaechmus' problem, above. Although I do not claim that this is the approach that Euclid actually took, it nevertheless helps us understand the motivation for the development and structure of the *Data*.

In order to justify all of the steps in Menaechmus' resolution, we need proofs of the following claims: (1) the intersection of two lines *given in position* is *given*, (2) the line joining two *given* points is *given in position* and *in magnitude*, (3) the line passing through a *given* point and meeting a line *given* in position at a *given* angle is *given*, (4) and the rectangle contained between two lines *given in magnitude* is *given in magnitude*.

The first three of these claims are all inherent in the concept of *given in position* and Euclid demonstrates them indirectly, in *Data* 25, 26 and 30, as an immediate consequence of assuming that his definition of *given in position*, *Data* Def. 4, does not hold.[28] Indeed, the definition *Data* Def. 4 was probably specifically formulated so that it could be developed into the sequence of theorems *Data* 25–31, which flesh out the concept of *given in position* and are some of the most often used theorems in the *Data*—applied both in the *Data* itself and in other analytical works, such as Apollonius' *Conics* and *Cutting off a Ratio*.

The fourth claim appears to be obvious and could have been shown most simply by exhibiting the construction of a rectangle from a pair of lines *given in magnitude*. But this is not the approach that Euclid takes—nor is there any corresponding problem in the *Elements*. Instead, he shows a geometric generalization of this claim—namely, that if a figure has *given* angles and the ratios of its sides are *given*—that is, it is *given in form*—and it has one side *given in magnitude*, then the figure itself is *given in magnitude* (*Data* 52). In order to show this, he needs to show, among other things, that two triangles, or figures *given in form* that are standing on the same line have to one another a *given* ratio (*Data* 48, 49). These, in turn, depend on showing that a triangle that has *given* angles also has the ratios of its sides *given* (*Data* 40). In this way, he works his way back to making arguments about the relationship between objects

27 Very nearly every step in the *Data* can be justified by a proposition, definition or postulate, either from earlier in the *Data* itself, or from *Elements* I–VI. The few exceptions require lemmas that can themselves be demonstrated using only *Elements* I–VI and the *Data*, such as in *Data* 24 and 67, or are corollaries of a definition, such as *Data* 25–30 (see note 28).

28 *Data* 25–29 are corollaries of *Data* Def. 4, the proofs of which simply involve pointing out that if they do not hold the consequence is contrary to the definition. *Data* 30 shows a contradiction with *Elem.* I.16, but it also depends on the assumption that *Data* Def. 4 does not hold.

that are *given in position* (*Data* 24–30) and *given in form* (*Data* 40–43), and making general proofs about *given* magnitudes and ratios. For example, for these arguments he needs a proposition that shows that if a ratio is *given* and one of its terms is *given*, then the other term is *given* (*Data* 2), and another that shows that if two *given* ratios share a term then the ratio between the other two terms is also *given* (*Data* 8).

It is important to keep in mind the fact that in all of these arguments about *givens* the original motivation has to do with settling the general question of constructibility—namely, to show that *given* some initial configuration of objects assumed in the enunciation of a problem, ($\mathbf{G_1}$), and introducing only points and lines *given in position*, ($\mathbf{G_2}$), the configuration satisfying the specification of the problem is also *given* through some set of permissible operations, ($\mathbf{G_3}$). Hence, when we read the definitions of the *Data* and the first set of propositions, which concern magnitudes in general, we must be mindful that these are introduced to cover their uses in the later propositions, which are, in turn, introduced to demonstrate constructibility in an analytical resolution.

3.1 *Data* Defs. 1–4

The text of the *Data* begins with a series of definitions, the first two of which refer explicitly to the constructive processes involved in problem-solving. The text reads,

> [*Data* Def. 1] Regions (χωρία), [straight and curved] lines (γραμμαί), and angles are called *given in magnitude*, of which we are able to provide (πορίσασθαι) equals.
>
> [*Data* Def. 2] A ratio is called *given*, of which we are able to provide the same.
> (Menge 1896, 2)

The use of the first person plural that we find in these two definitions is somewhat unusual in Greek mathematical prose and should be taken as deliberate. That is, the issue involved in the notion of *given* is not primarily about the existence of the object, but rather about what we, as mathematicians, are able to do.[29]

Both of these definitions invoke the action of being able "*to provide*" as a sort of primitive concept using the verb πορεῖν, meaning "to give" or "to supply." In the early propositions of the *Data*, as we will see, these definitions are used to directly introduce new objects into the discourse—either magnitudes equal to *given* magnitudes, *Data* Def. 1, or a magnitude that has a *given* ratio to a *given* magnitude, *Data* Def. 2. Nevertheless, as we will see in the later theorems, these propositions are also meant to stand in for all of the different types of constructions that can produce geometric magnitudes. Since, in the Greek mathematical texts, different verbs are used for different kinds of constructions, this verb probably simply stands in for all of the various types of constructions that might be introduced in a problem—the elementary constructions of the *Elements*, of course, but also conic sections, neusis constructions, loci, and so

[29] The alternate opinion that *given* is a predicate asserting the existence of an object is claimed by Taisbak (2003, 24–30) and Acerbi (2011a, 123).

on. In the context of the *Data*, however, we simple assume that these constructions can be carried out.

That is, *Data* Def. 1 states that when we say that an object is *given in magnitude*, what we mean is that there is some effective procedure through which we can produce another object of the same kind that is the same size as the *given* object. And if there is such an effective procedure, then the object is *given in magnitude*.

Data Def. 2 asserts that what it means for a ratio to be *given* is that there is a constructive procedure that we can use to produce an instantiation of it—namely, two geometric magnitudes, of the same kind, that have to one another the *given* ratio. And if there is such an effective procedure, then the ratio is *given*. As we will see below, when this definition is used in the text, it is often used as an introduction rule that allows us to set out a magnitude having a *given* ratio to a *given* magnitude—that is, as an introduction rule for a fourth proportional.[30]

It should be noted that a ratio can initially be *given* even if no pair of objects having that ratio are themselves *given*. While it is fairly clear what this would mean from an arithmetical perspective—since Greek texts speak of the half, the triple, the hemiolic, and so on—it may take a moment's reflection to see what this might mean in a geometrical context. Suppose two non-parallel lines are *given in position* and another line falls on them at *given* angles, then the sides of the triangle so formed will have *given* ratios, despite the fact that none its sides are *given in magnitude*.[31] Such a configuration, and others of this kind, might arise in geometrical analysis.

Since these are definitions, we are not concerned at this stage with stating what these constructive procedures are, because we will apply the term *given* to objects that we assume and set out, as well as to objects that we construct—indeed objects constructed in problems are necessarily produced by some constructive procedure, which can only be detailed by assumptions about constructions and problems built up from these.[32] For the early propositions of the *Data*, the procedure used to introduce a *given* object, or two objects in a *given* ratio, is unspecified, but it must be some procedure that, in a well-defined way, produces the object. Of course, the constructions used in problems in the *Elements*, involving postulates and previously established problems, are procedures of this sort, but perhaps there are others involving different constructive methods—such as conic sections, neusis lines, and so on. What is *given* is simply an assumption about what is, or what can be, constructed. That is, every constructed object is *given*, but not every *given* object is constructed.

The goal of the *Data* is to show how we can use the problems and theorems of *Elements* I–VI to make inferences from things that are assumed to be *given* to other things that can then be constructively shown to be *given*. Of course, in these deductions, we also use theorems of the *Data*, but as I will argue below, these are themselves argued constructively using either problems of the *Elements* or definitions of the *Data*, which themselves make assumptions about what kinds of objects can be constructively

30 Taisbak (2003, 32–33) points out that *Data* Def. 2 must often be read to mean that both of the terms of the instantiated ratio are themselves *given*.

31 See the commentary to *Data* 2, Sect. 3.2.

32 This restriction does not apply to objects introduced in theorems and in the demonstrations of problems, which do not need to be constructible through an effective procedure.

produced. The theorems of the *Data* are not themselves concerned with the ultimate origin of the objects under discussion, but only with what can be inferred under the assumption that they are *given*.

There are two more fundamental definitions:

> [*Data* Def. 3] Rectilinear figures (σχήματα) are said to be *given in form*, of which each of the angles are *given* [*in magnitude*] and the ratios of the sides to one another are *given*.
>
> [*Data* Def. 4] Points, lines and angles are said to be *given in position*, which always keep the same position.
>
> (Menge 1896, 2)

Data Def. 3, which is one of the most frequently used definitions in the text, is, in fact, an extension of *Data* Defs. 1 and 2 to rectilinear figures. According to this reading, a rectilinear figure is *given in form* if there is some procedure for producing other angles, equal to its angles, and other lines, having the same ratio as its sides. The first time this definition is applied, in *Data* 39, we see that this means that if a figure is *given in form*, then there is an effective procedure for producing a *similar* figure, *Elem.* VI.def.1, and likewise having produced a similar figure through an effective procedure, it is possible to assert that the original figure is *given in form*. Hence, since this definition falls back on *Data* Defs. 1 and 2, it is also essentially constructive. Indeed, we see in *Data* 39–43, when *Data* Def. 3 is first applied, that the status of a figure being *given in form* is secured by constructing another, similar figure.

The final definition—concerning *given in position*—is harder to explain without reference to the propositions in which it is used, but it is usually characterized as dealing with the concept of uniqueness (Taisbak 2003, 95; Acerbi 2011a, 146–148). Indeed, its application in *Data* 25–30,[33] which are essentially corollary statements of the definition, shows that uniqueness is a core concern. As is clear from the practice of both the *Elements* and the *Data*, however, a geometric object that is *given in position* is available for use—the endpoints of a *given* line may serve as the center or distance of a circle,[34] a *given* point may serve as the endpoint of a line, a *given* point may serve as the distance of a circle, or the endpoint of a line; a *given* line may be extended, a point taken on it, and so on—as for example in *Elem.* I.1, 1.2, I.9, I.11, I.44, *Data* 39, 33, 38, and so on. Once again, we see that there is a constructive aspect at play—an object *given in position* does not simply exist, but is fixed in location in some definite way, so as to be available to the mathematician for constructive procedures.

Starting already in the ancient period, as we will see below, people appear to have read the theorems of the *Data* as justifying both computational and geometrical problem-solving practices. Indeed, the enunciations of certain theorems of the text are susceptible to such a reading.[35] It is much less clear, however, that the text

33 See note 28, above.

34 Here, I use "distance" as a translation of the Greek term διάστημα, the span with which a circle is drawn (Sidoli 2004b).

35 Friberg (2007, 211–234) gives a recent reading of the enunciations of select theorems of the text along these lines.

was originally composed with such an interpretation in mind, and various aspects of the way the argument is advanced make this reading rather strained. Certainly, the articulation of *Data* Defs. 1, 3 and 4 are explicitly addressed to a geometrical reading, there is no discussion of what it would mean for a number to be *given*, and there is no systematic treatment of the arithmetic operations. In the following, we will read through a number of propositions of the text, both to flesh out Euclid's articulation of the notion of *given* and to see to what extent the underlying mathematics of the text is compatible with a reading centered on metrical analysis, or numerical computation.

3.2 *Data* 2

We will begin by examining *Data* 2, which comes from the opening section of the text, dealing with *given* magnitudes. It is sometimes maintained that these magnitudes are general quantities—that is, either the abstract sizes of geometric objects or numbers—but various aspects of the text, including the precise statement of *Data* Def. 1, make this unlikely. As we will see, both the internal constraints of the demonstrations of these propositions, as well as their application in the later propositions of the text, imply that we should understand the notion of *magnitude* in the *Data* as a hypernym for all those geometric objects that have size—in particular regions, lines, or angles—that is, as the same sorts of magnitudes as are treated in *Elements* V.[36]

Data 2 shows that if a magnitude is *given*, and its ratio to a second magnitude is *given*, then the second magnitude is also *given*—that is,

$$(a_m : B\,)_r \Rightarrow b_m.$$

Computationally, this would be a justification of the rule-of-three, but, as I will argue below, it is more natural to read this proposition as having originally been composed so as to be applicable to geometric construction. The text of *Data* 2 reads as follows (Fig. 6):

> If a *given* magnitude have to another magnitude a *given* ratio, the latter is *given in magnitude*.
>
> For let a magnitude, A, have a *given* ratio to some magnitude, B. I say that B is also *given in magnitude*.

[36] Doubt has been expressed about whether or not angles can be considered to be magnitudes—largely because the definition of ratio in the *Elements*, *Elem.* V.def.4, cannot apply to the conception of angles in Greek geometrical texts as always being less than 180°. Among modern scholars, see for example Artmann (1999, 123) and Taisbak (2003, 23, 30). Rashed (2015) has provided a thorough study of various approaches taken to this topic by ancient and medieval scholars. Whatever the reservations that might be advanced, however, some of the propositions in the first part of the *Data* must be used to justify claims that angles are *given in magnitude* in the rest of the treatise, so angles must be included in this general treatment of *given* magnitudes. Moreover, Greek mathematicians worked with ratios between angles in practice, and applied theorems from *Elements* V to proportions resulting from such ratios. Hence, they were either willing to overlook the failure of *Elem.* V.def.4 to apply to angles or they regarded this definition as of use to the local foundational issues of *Elements* V but not as sufficient to invalidate the clear utility of working with ratios and proportions among angles—particularly, when represented by numerical values.

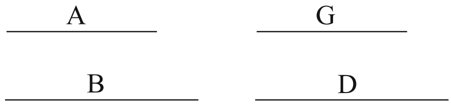

Fig. 6 Diagram for *Data* 2

> For, since A is *given*, it is possible to provide an equal to it.[37] Let it have been provided, and let it be G. And since the ratio A to B is *given*, for that was assumed, it is possible to provide the same as it.[38] And let it have been provided, and let it be the ratio G to D. And since as A to B so G to D, therefore, alternately, as A to G so B to D.[39] But, A is equal to G, therefore B is equal to D. Therefore, magnitude B is *given*, for an equal to it, D, has been provided.[40]
>
> (Menge 1896, 6)

As in many of the early propositions of the *Data*, the core of the proof of *Data* 2 is a direct application of the definitions. In this case, we apply *Data* Defs. 1 and 2 to "provide" magnitudes and ratios—but the text gives us no indication of how we should actually go about "providing" these things. Moreover, the way in which the definitions are applied makes it clear that here *providing* is a primitive concept. Nevertheless, by remembering that the purpose of this proposition is to serve in the demonstration of later, geometric propositions and geometric analytical resolutions, we can stay mindful of the fact that these definitions stand in for geometric constructions that can actually be carried out with various types of magnitudes.

Furthermore, the details of the proof support my claims that the magnitudes at issue are geometric objects with size and that the definitions concern operations, or procedures, that we can perform.

Since a_m is *given in magnitude*, ($\mathbf{G_1}$), *Data* Def. 1 can be applied to find another magnitude, say G, such that $g_m = a_m$, where g_m is also *given*—because we have just set it out by some procedure. If we are working with a geometrical configuration, there is some sense in this operation. The original *given* magnitude might be *given* by the constraints of the initial configuration as a line or a rectilinear figure, and so on, which has a *given* ratio to another by the very fact that they both occur, already there, in the configuration with which the discourse begins. *Data* Def. 1 assures us that in order to work with this initially *given* object, the geometer can set out another, equal to it, in a different part of the configuration.

The next step of the argument is operational and makes clear the constructive meaning of *Data* Def. 2—the text explicitly says that we can set $(g_m : D) = (a_m : B)_r$. Since g_m is *given*, this is the production of a fourth proportional, namely D—which, since we have produced it, by *Data* Def. 1, must be *given*, d_m, because another equal to it could be produced by the same procedure. We then argue that this implies

37 *Data* Def. 1.

38 *Data* Def. 2.

39 *Elem.* V.16.

40 *Data* Def. 1.

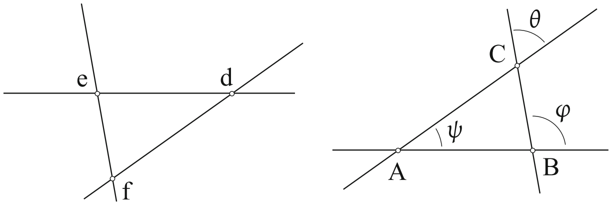

Fig. 7 Application of *Data* 2 to a set of similar triangles

that $b_m = d_m$, ($\mathbf{G_3}$), by *Data* Def. 1. This shows how we can use *Data* Def. 1 to show that a magnitude is *given*—we exhibit a procedure for setting out an equal.

In this argument, the operational meaning of Def. 2 is made clear: When a ratio is *given*, we can produce a magnitude that has this *given* ratio to some other *given* magnitude. That is, it is used to introduce a fourth proportional—and hence serves as an introduction rule for this object, which is thus itself *given in magnitude*, by *Data* Def. 1. In the case of *Data* 2, the *given* fourth proportional that we produce through *Data* Def. 2 happens to also be equal to B, the magnitude whose status we are investigating—which it must be for this proof, so that we can apply *Data* Def. 1.

If we try to read this argument as applying to abstract quantities, including numerical values, Def. 1 would imply that if we have a *given* value, say 5, $\sqrt{2}$, or r, then we can set out another value equal to it—but there is no such other equal value. Moreover, when we wish to apply this sort of proposition to numbers, using the rule-of-three, we generally have a *given* number, say 5, and a ratio *given* as a pair of two *other* numbers, say (3 : 4). It would be senseless to apply this proposition to a situation where we began with a *given* number and the *given* ratio stated in terms of the very same *given* number, say 5 and (5 : 6 2/3).

On the other hand, both the argument and the application of this proposition is explicable in terms of geometric objects. It is possible to imagine a number of situations in which *Data* 2 could be applied to geometric configurations, but we can use a set of similar triangles as a simple example.

Let us consider three bundles of parallel lines intersecting at *given* angles, θ_m, φ_m and ψ_m, then we have a set of triangles *given in form*, $\mathbf{T}(ABC)_f$, such that the ratios of the sides are *given*, ($\mathbf{G_1}$), by *Data* 40, but no other elements are *given* (Fig. 7). Then, if we suppose one of the sides of the triangle, say AB_m, to be *given* in magnitude, ($\mathbf{G_1}$) or ($\mathbf{G_2}$), the argument of *Data* 2 shows that the other sides of the triangle are also *given in magnitude*, ($\mathbf{G_3}$). If we assume some point d_p to be *given*, ($\mathbf{G_2}$), and draw line dE parallel to AB, *Elem.* I.31, then dE_p is *given in position*, ($\mathbf{G_3}$), *Data* 28. Then, since AB_m is *given in magnitude*, ($\mathbf{G_1}$) or ($\mathbf{G_2}$), we can produce an equal to it, say $de_{p,m}$, *given in position* and *in magnitude*, ($\mathbf{G_3}$), *Data* 27, 26. Then, if lines are produced through d_p and e_p parallel to lines AC and BC, *Elem.* I.31, the triangle $\mathbf{T}(def)_{f,m,p}$ will be fully *given*, ($\mathbf{G_3}$), *Data* 28, 25. Then, since $(de : ef)_r = (AB_m : BC)_r$, where $de_m = AB_m$, BC_m will be *given in magnitude*, ($\mathbf{G_3}$), since ef_m, equal to it, has been provided, *Data* Def. 1.

This argument, of course, applies only to this specific geometric situation. The argument given in *Data* 2, however, is meant to serve for any geometric magnitude—so that it is both more general, but also more vague.

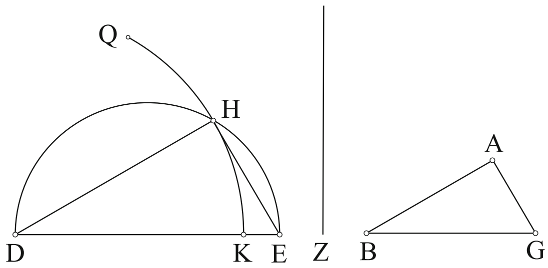

Fig. 8 Diagram for *Data* 43

3.3 *Data* 43

The next passage of the *Data* that we will read comes from the opening propositions of Euclid's theory of *given in form*—dealing with the conditions under which a triangle is *given in form*. We will read *Data* 43, which is a key theorem in the metrical analysis employed by Ptolemy and probably others, for the investigation of problems to be solved using chord-table trigonometry. *Data* 43 shows that, in a right triangle, if the ratio of two sides about an acute angle is *given*, the triangle is *given in form*—that is,

$$\mathbf{T}(ABG), \angle BAG = 90^\circ, (AB : BG)_r \text{ or } (AG : GB)_r \Rightarrow \mathbf{T}(ABG)_f.$$

If the enunciation is read computationally, it suggests that this theorem might be used to evaluate the angles of a right triangle, where the ratio between a leg and the hypotenuse of the triangle is *given*. The text of *Data* 43 reads as follows (Fig. 8):

> If the sides about one of the acute angles of a right triangle have a *given* ratio, the triangle is *given in form*.
>
> For, of a right triangle, ABG, having right angle BAG, let the sides, GB, BA, about one of its acute angles have to one another a *given* ratio. I say that triangle ABG is *given in form*.
>
> For let a straight line *given in magnitude*, DE, be set forth [*given*] *in position*.[41] And let a semicircle, DHE, be drawn on DE.[42] Therefore, semicircle DHE is [*given*] *in position*.[43] And, since the ratio of GB to BA is *given*, let DE to Z have become the same as it.[44] Therefore, the ratio DE to Z is also *given*.[45] But DE is *given*, therefore Z is also *given*.[46] And GB is greater than BA,[47]

[41] That is, $de_{m,p}$ is *assumed* to be *given in magnitude* and *in position*, $(\mathbf{G_2})$.

[42] *Elem.* III.33.

[43] *Data* Def. 8. Of course, there are two semicircles that could have been produced on $DE_{m,p}$, but we are only concerned with the one that is actually produced.

[44] This is the production of Z as a fourth proportional through *Data* Def. 2.

[45] *Data* Def. 2, since there is a procedure for producing lines in the same ratio.

[46] *Data* 2.

[47] GB is the hypotenuse.

> therefore ED is greater than Z.[48] Let DH be inserted equal to Z,[49] and let HE be joined,[50] and with center D and distance DH let a circle, QHK, be drawn.[51] Therefore, circle QHK is [*given*] *in position*, for its center is *given in position* and is radius is *given in magnitude*.[52] But semicircle DHE is [*given*] *in position*, therefore point H is [*given*] *in position*.[53] But each of D and E are also *given*, therefore each of HD, DE, and EH are *given in magnitude* and *in position*.[54] Therefore, triangle HDE is *given in form*.[55]
>
> Now, since there are two triangles, ABG, DEH, having one angle equal to one angle, angle BAG to angle DHE, and the sides about the other angles, angles GBA and EDH, proportional, and at the same time the remaining angles BGA and DEH are each less than a right [angle], therefore triangle ABG is similar to triangle DEH.[56] But, triangle DEH is *given in form*, therefore triangle ABG is also *given in form*.[57]
>
> (Menge 1896, 76–78)

Although the enunciation of this theorem makes it appear to be ideally suited for the chord-table methods in which Ptolemy will use it, the mechanics of the proof itself show how far Euclid's approach is from any metrical considerations. The key to the proof is that we set out an arbitrary *given* line, $de_{m,p}$, ($\mathbf{G_2}$), and then use the constructive techniques of the *Elements* to make a new triangle, $\mathbf{T}(deh)_{f,m,p}$, on the line which we set out. We can think of the construction of this triangle, which is also at the mathematician's discretion, as the production of further objects that are given in the sense of ($\mathbf{G_2}$). Finally, the demonstration shows $\mathbf{T}(ABG) \sim \mathbf{T}(deh)_{f,m,p}$, so that $\mathbf{T}(ABG)_f$ is *given in form*, ($\mathbf{G_3}$).

The initial construction is justified through *Data* Def. 2, which, as we saw above, asserts that we can produce some line Z, such that $(Z : de_m)_r = (BA : BG)_r$, so that z_m, by *Data* Def. 1.[58] We then use this newly provided line segment to complete

48 *Elem.* V.def.5.

49 *Elem.* IV.1. As Taisbak (2003, 128) points out, the construction for inserting line DH involves producing the same circle about center D with distance DH that will be explicitly introduced two steps later. But this "repeated construction" follows the conventions of constructions in Euclid's problems. Whenever an object in invoked using a previously established problem, only the object itself is produced. Any auxiliary objects that may have been involved in its construction are not available for further work and have to be produced independently if they are needed—as is done here.

50 *Elem.* I.post.1.

51 *Elem.* I.post.3. See note 34, above, for my choice to use "distance."

52 *Data* Def. 6.

53 *Data* 25.

54 *Data* 26.

55 *Data* 1, Def. 1 and Def. 3.

56 *Elem.* VI.7.

57 *Data* Def. 3.

58 It might appear that this can be done through *Elem.* VI.12, but this problem requires that we have three *given* lines, not merely one *given* line and a *given* ratio. If we consider the bundles of parallel lines discussed in Sect. 3.2, above, we can see an example of how this construction could actually be carried out with only one *given* line—all of which is simply taken as a primitive concept in *Data* Def. 2.

$\mathbf{T}(deh)_{f,m,p}$, which is fully *given* because we constructed it using the methods of *Elements* I–VI—thereby establishing that we have an effective procedure for producing all of its elements.[59] The final phase of this argument shows us one of the key implications of *Data* Def. 3—namely, a figure is *given in form* when we can construct another figure similar to it.

The explicit use of constructive processes in this theorem shows that the theorems of the *Data* allow us to expand to scope of objects that can be *given* in the sense of ($\mathbf{G_2}$) from points *given in position*, as at the beginning of a *Data* construction, to any object that can be introduced through the problems of the *Elements*. By following the argument of the geometrical theorems of the *Data*, we come to see that any object that can be constructed from *given* objects is fully *given*.

Such a proof, however, relying as it does on geometric construction, provides no way of dealing with the *given* objects as numerical values. Even if we have the *given* ratio as a value, or rather as a pair of *given* values, the proof of the theorem, working through *Data* Def. 2 and then constructing a similar triangle, gives us no way to calculate the value of the triangle's angles from these *given* values. But it is this computation that the methods of chord-table trigonometry require.

3.4 *Data* 52

The final example that we will consider comes from the opening proposition of Euclid's theory of figures *given in magnitude*, and can be read as one of the cornerstones of his theory of the measurement of rectilinear figures. *Data* 52 shows that if a figure *given in form* stands on a line *given in magnitude*, then the figure is also *given in magnitude*. That is,

$$\mathbf{F}(ABGDE\ldots)_f,\ AB_m \Rightarrow \mathbf{F}(ABGDE\ldots)_m.$$

From this, since every right angle is *given*,[60] we have the following immediate corollaries,

$$a_m \Rightarrow \mathbf{S}(a_m)_m,$$

and

$$a_m, b_m \Rightarrow \mathbf{R}(a_m, b_m)_m,$$

which are often used in metrical analysis to justify the claim that a square on a *given* side is *given*, or that the rectangle formed from two *given* lengths is *given*. For readers

59 Notice that the only theorems of the *Data* that are applied in the argument that $\mathbf{T}(deh)_{f,m,p}$ is fully *given* are *Data* 25 and 26, which are simply reiterations of *Data* Def. 4. That is, $\mathbf{T}(deh)_{f,m,p}$ is fully *given* because it was so constructed, by the definition, and implications, of the concept of *given in position*. Hence, a figure whose points are *given in position* is fully *given*.

60 This is clear from the way right angles are treated in the *Data*. Acerbi (2011a, 123) gives an explanation for why this is so, based on the construction of right angles.

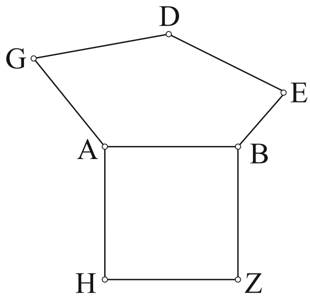

Fig. 9 Diagram for *Data* 52

who took the propositions of the *Data* to justify computations, as I will argue below was done by Heron and Ptolemy, this theorem is the only candidate for a justification that the product of two *given* numbers is *given*. That is, computationally this theorem can be regarded as a justification of the operation of the multiplication of two—and only two—terms.[61] The text of *Data* 52 reads as follows (Fig. 9):

> If a form (εἶδος) *given in form* is erected upon a line *given in magnitude*, the erected [form] is *given in magnitude*.[62]
>
> For, let a form *given in form*, $AGDEB$, be erected on a straight line *given in magnitude*, AB. I say that $AGDEB$ is *given in magnitude*.
> For, let a square, AZ, be erected on AB.[63] Therefore, the square on AB is *given in form* and *in magnitude*.[64] And since, two rectilinear figures *given in form*,

[61] Proponents of the strong form of the geometrical algebra interpretation of *Elements* II, VI and *Data* 57–60 and 84–85—namely, the claim that certain propositions in these texts were motivated by a desire to justify arithmetical problem-solving techniques—must contend with the fact that this highly geometrical, and frankly rather peculiar, proposition is the only candidate in the text for propositions demonstrating that the product of two *given* numbers is *given*. This strong interpretation of the hypothesis of geometrical algebra is the claim GA2 set out by Blåsjö (2016, 326). His GA1 is simply another way talking about the theory of the application of areas, which is not in any dispute. For example, Dijksterhuis (1987, 51–52), who developed a notation for expressing the theory of the "Application of Areas" in a symbolic form that is both true to the original conception and reflects the abstract operational nature of the theory, refers to it also as "Geometrical Algebra" only the first time he introduces his terminology in order to alert the reader to the fact that he will use these symbols to treat what Zeuthen (1885, 7) had called geometrical algebra. Dijksterhuis (1987, 7) presumably preferred his own symbolism and the terminology "application of areas" because, as he says, "in a representation of Greek proofs in the symbolism of modern algebra it is often precisely the most characteristic qualities of the classical argument that are lost."

[62] Taisbak (2003, 139, n. 105) has pointed out the strangeness of speaking of a "form *given in form*" (δεδομένον τῷ εἴδει εἶδος). *Data* Def. 3 construes *given in form* as applying to rectilinear figures, and the proof of this proposition makes it clear that the "form" we are dealing with here is, in fact, a rectilinear figure.

[63] *Elem.* I.46.

[64] *Data* Def. 3 and 1. The argument for this is detailed below. Taisbak (2003, 139, n. 106, 151, n. 110, 90–91), suggests that there is some circularity in this argument with *Data* 55 and, through two notes, refers the reader back to his commentary to *Data* 24 for this step, where he shows, for example, that $a_m, b_m \Rightarrow \mathbf{R}(a_m, b_m)_m$ and $\mathbf{S}(A)_m \Rightarrow a_m$. But the inference required for this proposition is $a_m \Rightarrow \mathbf{S}(a_m)_m$, which, he correctly notes, follows directly from *Elem.* I.46, *Data* Def. 3 and 1—as I will flesh out below.

> $AGDEB$ and AZ, have been erected on the same straight line, AB, therefore the ratio of $AGDEB$ to AZ is *given*,[65] But, AZ is *given in magnitude*, therefore $AGDEB$ is also *given in magnitude*.[66]
>
> (Menge 1896, 94)

Once again, the demonstration of this proposition is grounded in the constructive procedures of the *Elements*, and it is difficult to see how it gives us any insight into how to carry out the related computations necessary to metrical, or arithmetic, considerations.

The key to the proof of *Data* 52 is the application of *Data* 49, which shows that two figures *given in form* standing on the same line have to one another a *given* ratio. In order to apply this proposition, however, all that is necessary is that the new figure that we introduce, $\mathbf{F}(BH)_f$, be *given in form*. Hence, any figure will do—an equilateral triangle, a square, a figure similar to $\mathbf{F}(AGDEB)_f$, indeed, any arbitrary figure that we construct *given in form*. Because the possible the constructions that will straightforwardly lead to a proof of this theorem are so varied, we might justifiably ask why the proof begins with the construction of a square, *Elem.* I.46.

The most likely explanation is that, because this theorem has to do with determining the magnitude of a figure *given* a side, and hence addresses the issue of measurement, Euclid begins by constructing a square on the *given* side, in order to explicitly invoke the core concept used in the measuration of regions in Greek geometrical texts—quadrature. That is, figure $\mathbf{F}(ABGDE\ldots)_f$ will be shown to have a *given* ratio to a *given* square.[67]

The fact that $\mathbf{S}(AB_m)_m$ is *given* is, again, a direct consequence of the constructive articulation of *given in magnitude* stipulated in *Data* Def. 1. That is, since AB_m is *given*, and square $\mathbf{S}(AB_m)$ has been constructed on it, using *Elem.* I.46, there is clearly an effective procedure for producing a square equal to $\mathbf{S}(AB_m)$—namely the procedure set out in *Elem.* I.46. So that, by *Data* Def. 1, $\mathbf{S}(AB_m)_m$ is *given in magnitude*.

Furthermore, the argument itself hinges on *Data* 49—figures *given in form* that share a side have to one another a *given* ratio—which in turn requires *Data* 48—triangles *given in form* that share a side have to one another a *given* ratio. When we unpack these propositions by reading their proofs, however, we see that they involve reducing $\mathbf{F}(ABGDE\ldots)_f$ to triangles *given in form*, and then relating the *given* ratio of these to one another through the rectangles under the same heights. Finally, relating the bases of the two triangles under the same height involves applying *Data* 40, which, in turn, shows that the sides of a triangle with *given* angles have *given* ratios through the same constructive approach that we saw at work in *Data* 43—namely, the construction of another, similar triangle on a line set out as *given in magnitude* and *in position*, $(\mathbf{G_2})$. Hence, *Data* 49 and 48 rely only on the notion of *given in form*, as set out in *Data*

65 *Data* 49.

66 *Data* 2.

67 As Taisbak (2003, 139) has observed, this theorem could have been based directly on *Data* Def. 1, by constructing another figure equal to $\mathbf{F}(ABGDE\ldots)_f$. This construction, however, would be no simpler than that of the square, and it would not explicitly relate the figure to a known square, which was presumably a goal of Euclid's proof.

40, and do not involve any lines *given in magnitude*—so that they do not give us any insight into computational procedures. Indeed, these are the same, purely geometric, concepts that underlie the theory of area developed in *Elements* I and VI.

Finally, even if we know the sides of a certain figure—and even if this figure is a square or a rectangle—this proposition, in the way that it is expressed in the *Data*, would not show us how to calculate its area. The only way this theorem could have any use to us in metrical, or arithmetic, problem-solving, is if we already knew, through independent considerations, what sorts of arithmetic operations to carry out, but were interested in an unrelated geometric approach as a purely theoretical justification of these operations.

3.5 The goal of the *Data*

By reading through the proceeding propositions of the *Data*, we have seen that the core theorems of this work are founded upon, and demonstrated through, the constructive geometric approach of *Elements* I–VI. Although these types of procedures have only a weak relationship with metrical, or algebraic, problem-solving they are closely related to geometric problem-solving as it was developed in the classical and Hellenistic periods. Indeed, the theorems of the *Data* can be used to form chains of deductive inference from certain objects assumed as *given*, ($\mathbf{G_1}$), to other objects or properties of objects that must also be *given*, ($\mathbf{G_3}$), introducing as *given*, ($\mathbf{G_2}$), only points *given in position* and using the theorems and problems of *Elements* I–VI along with the theorems of the *Data*, which are, in turn, shown through the constructive methods of *Elements* I–VI. The effect of this deductive work is to expand the set of objects that can be introduced as *given* by the mathematician in the sense of ($\mathbf{G_2}$) in actual problem-solving practice.[68] We have already seen how this sort of inference was apparently used by Maneachmus in his problem-solving activities; we will now turn to another example from Apollonius' *Conics*.

4 *Givens* in geometric problem-solving

In order to observe the constructive reasoning of the *Data* at work in the resolution of a purely geometric analysis–synthesis pair, we will read *Conics* II.44—to find the diameter of a *given* conic section.[69] The language of this proposition, as most of the problems in the *Conics*, has some archaic elements (Federspiel 2000, 367), so that it may predate the work of Apollonius, going back to the time of Euclid or his predecessors. Hence, it is probably a good example of the sort of geometric problem-solving that the theorems of the *Data* were meant to facilitate. The text reads as follows (Fig. 10):

68 We will see an example of this in Sect. 4.

69 This problem is also discussed by Zheng (2012, 140–142), but he does not fully articulate the use of the propositions of the *Data* in the resolution—perhaps because he follows the translation of Decorps-Foulquier and Federspiel (2008–2010, 2.3.93), who do not provide justifications for the steps of the resolution of this analysis–synthesis pair.

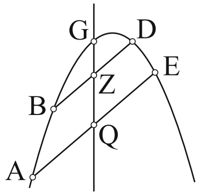

Fig. 10 Diagram for *Conics* II.44

[*Enunciation*:]
To find the diameter of a *given* section of a cone.

[*Exposition*:]
Let the *given* section of the cone be that on points A, B, G, D, E. [*Specification*:] Now, it is necessary to find its diameter.

[*Analysis*:]
Let it be done, and let it be GQ.[70] [*Transformation*:] If DZ, EQ are produced ordinatewise,[71] and extended,[72] DZ will be equal to ZB, and EQ to QA.[73]

[*Resolution*:]
Then, if we fix (τάξωμεν) BD, AE [*given*] *in position* as being parallels,[74] the points Q, Z will be *given*,[75] so that [line] QZG will be [*given*] *in position*.[76]

[*Synthesis*:]
Now, it will be put together (συντεθήσεται) as follows. [*Exposition*:]
Let the *given* section of the cone be that on points A, B, G, D, E. [*Construction*:]
And let parallels, BD, AE, be produced,[77] and let them be bisected at Z, Q.[78]
[*Demonstration*:] And a joining [line], ZQ, will be a diameter of the section.[79]
In the same way, we will also find countless diameters.

(Heiberg 1891–1893, 264–266)

In order to see how the reasoning by *givens* functions, and consequently the role of the theorems of the *Data*, we will go through the details of the argument. The analysis, Fig. 11, begins by taking the conic section $ABDE$ as *given* and making the analytical

70 This is the analytical assumption.

71 This is a purely hypothetical construction, because there is no problem in the *Conics* that shows how to produce ordinates—that is, parallel lines bisected by the diameter. See note 81.

72 *Elem.* 1.post.2.

73 *Conics* I.def.4.

74 These lines are simply assumed as *given* parallels, ($\mathbf{G_2}$). As often, however, this can be reduced the assumption of points *given in position*. This is fleshed out in the commentary below.

75 *Data* 25, 26, 7 and 27 (see commentary below).

76 *Data* 26.

77 *Elem.* I.post.1, I.31.

78 *Elem.* I.10, twice.

79 *Elem.* I.posts.1, *Conics* I.def.4.

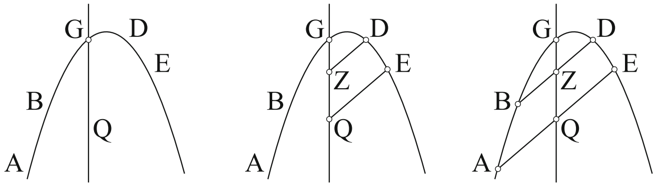

Fig. 11 *Conics* II.44, transformation

assumption that the diameter GQ exists in some way—but is not *given*. Notice that at this stage in the discourse, the letter-names A, B, D, E and Q simply denote as-of-yet unspecified points on the conic section and its diameter, they do not designate specific points.[80]

The transformation proceeds with the hypothetical claim that, if DZ and EQ are produced as ordinates, and then extended to meet the section again at A and B, perhaps using *Elem.* I.post.2, then $DZ = ZB$ and $EQ = QA$, by *Conics* I.def.4. The initial construction of the ordinates must be purely hypothetical because there is no problem in the *Conics* that tells us how to find an ordinate to a diameter—moreover, the diameter is not yet shown to be *given*.[81] Nevertheless, if such a configuration were possible, it would complete the problem.

The brief resolution, starting in Fig. 12, then argues that such a configuration will, indeed, be *given*—starting only with objects asserted to be *given* in the enunciation of the problem, ($\mathbf{G_1}$), and introducing only objects that can unproblematically be taken as *given*, ($\mathbf{G_2}$). The text reads "if we assign" (ἐάν ... τάξωμεν) BD, EA as parallels, using a verb related to one of the core meanings of *given*, and expressed in the first person.[82] This personal expression—unusual, in Greek mathematical texts—makes it explicit that we, as mathematicians, are setting these two lines out *as given*, ($\mathbf{G_2}$). As the text puts it, we simply assume these lines as *given* ordinates. But this assumption can, in fact, be reduced to the assumption of points *given in position*. First we take bD_p as *given in position*, ($\mathbf{G_2}$).[83] Then we take an arbitrary point, a_p, as *given in position*, ($\mathbf{G_2}$), on the *given* conic section, ($\mathbf{G_1}$),[84] and we draw aE through a_p parallel to bD_p, using *Elem.* I.31. Then aE_p is also *given in position*, ($\mathbf{G_3}$), by *Data* 34. That is, the ordinates that we can use *Data* methods to show are *given*, ($\mathbf{G_3}$), from points assumed

[80] Point G is specified as the intersection of the conic section and the diameter simply by the fact that it occurs in both of these names.

[81] In fact, ordinates would most simply be produced by drawing parallels to a tangent to the conic section, but tangents are produced in *Conics* II.49–51 and 53, which have yet to be established.

[82] Acerbi (2011a, 130–133) discusses the cognates of τάσσω in relation to the concept of *given*. Federspiel (2008, 347–349) covers the usage of personal verbs in Greek mathematics and especially the Apollonian corpus.

[83] That a line intersecting a *given* object, ($\mathbf{G_1}$), can be taken as *given in position*, ($\mathbf{G_2}$), is made clear in a number of propositions, such as *Elem.* III.1, *Data* 24, and 39, but see also the discussion by Taisbak (2003, 25). This is, of course, reducible to the assumption of two points, b_p and d_p, *given in position*, ($\mathbf{G_2}$), joined by a line, *Elem.* I.post.1.

[84] That an arbitrary point can be taken as *given*, ($\mathbf{G_2}$), on a line *given in position*, ($\mathbf{G_1}$), is seen in its implicit use in *Elem.* I.9, I.11, *Data* 32–33, and 37–38, and so on.

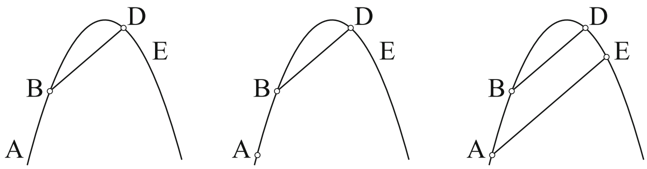

Fig. 12 *Conics* II.44, resolution (first part)

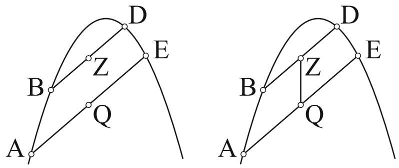

Fig. 13 *Conics* II.44, resolution (second part)

as *given in position*, (**G2**), are simply introduced by Apollonius as *given*, (**G2**). Here, we see clearly how the propositions of the *Data* serve to expand the range of objects that can be introduced by the mathematician without discussion in the sense of (**G2**).

Next, the resolution continues, in Fig. 13, by simply stating that points z_p and q_p will also be *given*, (**G3**), but a full argument can be made for this using the propositions of the *Data*. Since lines bD_p and aE_p are *given in position*, (**G2**), and meet conic section $ABDE_p$ *given in position*, (**G1**), points a_p, b_p, d_p and e_p will all be *given*, by *Data* 25. Hence, lines $bd_{p,m}$ and $ae_{p,m}$ are *given in magnitude*, by *Data* 26. Then, since dZ and eQ are half of bd_m and ae_m, respectively, dZ_m and eQ_m are also *given in magnitude*, by *Data* 7. Hence, points z_p and q_p are also *given*, by *Data* 27. Finally, although unstated, zq_p is *given in position*, since it passes through two *given* points, by *Data* 26, (**G3**).

The resolution contains its own constructive processes, which have a different status than those of the transformation. Whereas the constructions of the transformation are purely hypothetical, those of the resolution are used to demonstrate real constructibility, using the constructive methods of *Elements* I–VI, and starting only with the objects assumed as *given* in the enunciation of the problem, (**G1**)—in this case, the conic section—and introducing as *given* only *given* points, or objects demonstrably *given* from these by construction, (**G2**).

Notice that it is not important to the resolution that z_p and q_p bisect lines $bd_{p,m}$ and $ae_{p,m}$—since any *given* ratio will do. Indeed, it is a general characteristic of an analysis that the information needed for the demonstration in the synthesis is found in the hypothetical construction of the transformation, which supplies the actual solution to the problem, not in the resolution.[85] The resolution itself simply concerns the *status* of the geometric objects, namely, that they are given, (**G3**)—that is, constructible through the permitted operations.

Finally, a comparison of the truncated form of the argument in the *Conics* and its full articulation in my commentary makes it clear that, for those who had mastered

[85] Heath (1921, I.423) makes a similar point in his summary of the *Data*.

the theorems of the *Data*, such arguments could be used to rapidly confirm whether or not a proposed solution would work.

As this example shows, the resolution was meant to be a general articulation of an effective procedure for producing the sought configuration where some initial configuration is assumed as *given*, ($\mathbf{G_1}$)—and the theorems of *Data* were presumably organized to help produce such an argument. That is, the theorems of the *Data* can be used to help us investigate what objects are constructible through effective procedures in such a way as to be available for further construction, ($\mathbf{G_2}$), on the basis of some initial configuration of geometric objects, ($\mathbf{G_1}$)—where what we mean by an effective procedure is one that can actually be carried out in a finite number of steps, as established, say by the postulates and problems of *Elements* I–VI.

5 *Given* in metrical analysis

Whatever Euclid's original intention had been in composing the *Data*, it seems clear that by the imperial period, and probably for some time earlier, the theorems of the *Data* were being used to justify and to facilitate the computation of numerical values that stood as measures of the magnitudes of geometric objects—specifically lengths, areas, and angles, precisely those geometric objects for which the *Data* defines the concept of *given in magnitude*. We find clear examples of this in the writings of Heron and Ptolemy—particularly in *Measurements*, *Dioptra*, *Analemma*, and *Almagest*—and a less straightforward indication of this practice in a passage from Theon of Smyrna that goes back to Adrastus, and probably to Hipparchus.[86] These authors provide arguments consisting of a series of *givens*, associated with a calculation that starts with certain numbers, as measures, and computes other numbers, as measures, and which follows the same route as the deduction by *givens*. Following Heron, we can call this type of argument an analysis, and in order to differentiate it from the strongly geometrical arguments that we read above, we can designate it *metrical analysis*—since it concerns the numerical measuration of geometrical objects.[87]

Because neither Heron nor Ptolemy explicitly mention the enunciations of theorems of the *Data* in their metrical analyses, it might be thought that they are not referring to the *Data*, but to some general idea of the concept of *given* not articulated in a mathematical text, or, less likely, to some lost treatise that justified computations using propositions analogous to those in the *Data*, but which subsequently vanished without a trace. But there are a few pieces of ancient and medieval

[86] Acerbi (2007, 512–519) has discussed this type of reasoning in texts by Heron, Ptolemy and Diophantus. I have elsewhere analyzed the metrical analysis by Theon of Smyrna (Sidoli Forthcoming a).

[87] This type of argument is called an "analysis" by Heron throughout his *Measurements*, and by Pappus in his commentary on Ptolemy's *Almagest* V (Rome 1931–1943, 35). I have not found a passage where Ptolemy himself refers to this type of argument as an "analysis." In fact, Ptolemy refers to his articulation of metrical analysis in the style of Heron as a "theorem" (Heiberg 1898–1903, 38, 40).

I originally introduced the term *metrical analysis* as a category to discuss Ptolemy's practice and simply noted in passing the similarities between these types of arguments in Heron and Ptolemy (Sidoli 2004a, 17–19). I have elsewhere discussed the role of this type of argument with respect to mathematical tables in the *Almagest* (Sidoli 2014, 25–26). See also the discussion by Acerbi (2012, 201–208).

evidence suggesting that it was, indeed, the *Data* upon which their practice was based.

In the first place, both Heron and Ptolemy appear to be using the theorems of the *Data* in their deductions from *givens* implicitly—that is, as so well known as not to require any explicit citation, which is the same way that they generally use the propositions of the *Elements*. Hence, if Heron and Ptolemy are, in fact, implicitly referring to the theorems of the *Data*, it must have been a standard mathematical text in the imperial period. And, indeed, when Galen, a contemporary of Ptolemy's, is discussing what he considers to be the mathematical deficiencies of the Roman astrologers, he lists their mathematical abilities in a threefold division: Some of them have studied basic geometry and number theory, some spherics, but very few geometry in its entirety, including conic theory (Toomer 1985, 196, 199). In the category of basic mathematical knowledge, he includes a work "called the *Dādūmanā*, that is, the *Given* (المعطى)" (Toomer 1985, 196). This means that Galen, who had a high regard for his own mathematical education, considered the *Data* to be so elementary that anyone with even the most rudimentary mathematical education could have been expected to have mastered it—and this is precisely the way that Heron and Ptolemy are implicitly relying on it.

Another piece of ancient evidence that metrical analysis was understood to be justified by theorems of the *Data* can be drawn from the scholia to the *Almagest*. One of the scholia treating a metrical analysis in *Alm.* I.13, explicitly mentions the *Data* and states the enunciation of *Data* 7, which is the standard format for an ancient citation of a known proposition (Acerbi 2017, 213). Hence, by the time these scholia were composed, and going back at least as far as late antiquity, the metrical analyses in Ptolemy's *Almagest* were read as relying on the theorems of the *Data*.

Finally, we may add the circumstantial evidence that every step in the ancient metrical analyses can, in fact, be justified from the propositions of the *Data*, and the fact that Ptolemy's metrical analyses make efficient use of the theorems of the *Data*, but, as we will see, are somewhat removed from the trigonometric computations that they serve to justify (Acerbi 2012, 208). As we will see in Sects. 5.1 and 5.2, every step of the metrical analyses we will read can be justified by a theorem in the *Data*, but the context makes it clear that the argument is directly related to, and appears to justify, a computational procedure. After reading examples of metrical analyses from Heron and Ptolemy, we will discuss the significance of this type of argument.

5.1 Metrical analysis in Heron

In Heron's *Measurements*, metrical analyses are explicitly called "analyses" and the computations that follow are called "syntheses." It seems that Heron thought of metrical analysis as similar to the resolution of a geometrical analysis–synthesis pair, so that the analysis is meant to provide a justification, and perhaps motivation, for the calculation that follows, just as the resolution provides a justification for the construction, but was probably regarded as logically unnecessary, just as the resolution, once the

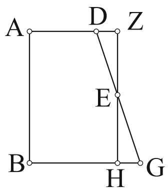

Fig. 14 Diagram for *Meas.* I.10

computation was actually carried through.[88] Another reason for including the metrical analysis is that the analysis, when read as describing an arithmetical computation as we will see below, provides a general statement of the algorithm, whereas the synthesis that follows gives only an example calculation with certain actual numbers.

To get a sense of metrical analysis in Heron, we will read part of the analysis of *Meas.* I.10, which shows that if three sides of a trapezoid having two right angles are *given*, then both the area and the fourth side are also *given*.[89] Considering Fig. 14, assuming **Trap**$(ABGD)$ where $\angle A = \angle B = 90°$, $(\mathbf{G_1})$, we let AD be 6 units, BG be 11 units, and AB be 12 units—that is, we take them to have certain known measures, as numerical values, $AD_m := 6^{\mathrm{u}}$, $BG_m := 11^{\mathrm{u}}$ and $AB_m := 12^{\mathrm{u}}$. The analysis then argues that where we assume AB_m, BG_m, and AD_m are *given*, $(\mathbf{G_2})$, then **Trap**$(ABGD)_m$ can be shown to *given*, $(\mathbf{G_3})$. The text of the first part of the analysis of *Meas.* I.10 reads as follows:

> Let GD be bisected at E,[90] and let ZEH be produced through E, parallel to AB,[91] and let AD be extended to Z.[92] Since DE is equal to EG, therefore DZ is equal to HG.[93] Let a common, AD [+] BH, be adjoined,[94] therefore the sum AZ [+] BH is equal to the sum AD [+] BG.[95] But the sum AD [+] BG, is *given*, since each of them is,[96] therefore, also the sum AZ [+] BH is *given*,[97] that is, twice BH.[98] Therefore, BH is also *given*.[99] But so is AB,[100]

[88] Acerbi and Vitrac (2014, 363–409) provide a complete analysis of Heron's use of arguments by "chains of givens" and their relationship to computational algorithms. In this section, I give one example and develop a symbolic representation of Heron's practice.

[89] For our purposes here, we will ignore the argument concerning the fourth side.

[90] *Elem.* I.10.

[91] *Elem.* I.31.

[92] *Elem.* I.post.2.

[93] *Elem.* I.15 and 26.

[94] That is, $DZ + (AD + BH) = HG + (AD + BH)$.

[95] *Elem.* I.c.n.2, since $AZ = DZ + AD$ and $BG = HG + BH$.

[96] *Data* 3, and by assumption, $(\mathbf{G_2})$.

[97] *Data* Def. 1.

[98] *Elem.* I.33.

[99] *Data* 7.

[100] By assumption, $(\mathbf{G_2})$.

> therefore, the parallelogram[101] $ABZH$ is *given*.[102] And since triangle DEZ is equal to EHG, let a common, pentagon $ABHED$, be adjoined,[103] therefore the whole parallelogram $ABZH$ is equal to the whole trapezoid $ABGD$.[104] But, the parallelogram was shown to be *given*, therefore, trapezoid $ABGD$ is also *given*.[105]
>
> (Schöne 1903, 28–30; Acerbi and Vitrac 2014, 172)

Following this passage, and a short metrical analysis showing that the fourth side is *given*, Heron proceeds to give the computation that corresponds to this analysis. The text reads:

> In conformity (ἀκολούθως) with the analysis, this is put together (συντεθήσεται) as follows.
>
> Combine the 6 and the 11; 17 results. And of this, the half; 8 1/2 results. This by the 12, 102 results. Therefore, of so many is the area (ἐμβαδόν).
>
> (Schöne 1903, 30; Acerbi and Vitrac 2014, 172)

This brief overview of the calculation, which we are told conforms to the analysis, provides insight into how we should read the claims of the metrical analysis. Because the values are simply set out as some number of units, and then in the metrical analysis these are simply assumed to be *given*, when Heron tells us, in the metrical analysis, that a geometric object is *given* he means both that it is *given in magnitude*, as stated in the theorems of the *Data* to which he implicitly appeals, and also that it is *given in numerical value*, as is made clear by the synthesis-computation, and by the fact that he introduces the *given* numbers with the definite article. The metrical analysis, then, serves as a justification and an explicit generalization of this computation—just as we saw a resolution serves as a justification and generalization of a construction—and hence operates on two different levels.

On the one hand, based on our reading of the *Data*, we can follow each of the steps of the metrical analysis with a justification from the *Data*, so that we have a claim that where the geometry of $\mathbf{Trap}(ABGD)$ *given*, ($\mathbf{G_1}$), and AB_m, BG_m, and AD_m are taken as *given*, ($\mathbf{G_2}$), then the geometrical region $\mathbf{Trap}(ABGD)_m$ is *given*, ($\mathbf{G_3}$). Since the *Data* propositions that are involved in this deduction treat the concept of *given in magnitude*,[106] we can drop the qualification and speak simply of *given*—just as is done in the *Data* itself when the context makes clear the mode in which the objects are *given*, *Data* 57–59, 60 (the gnomon), 84–86. But, the core of this geometric approach is an appeal to *Data* 52, which, as we saw above, is a fully geometric argument that if a

101 Heron is using παραλληλόγραμμον to mean *rectangle*. This is not uncommon in Greek mathematical texts. For example, Archimedes also employs the term in this way (Heiberg and Stamatis 1972, II.418, 426, 428).

102 *Data* 52.

103 That is, $\mathbf{T}(DEZ) + \mathbf{Pent}(ABHED) = \mathbf{T}(EHG) + \mathbf{Pent}(ABHED)$.

104 *Elem.* I.c.n.2.

105 *Data* Def. 1.

106 One of the conditions of *Data* 52 also stipulates that the figure must be *given in form*, but, of course, $\mathbf{Trap}(ABGD)$ meets this condition.

figure *given in form*—that is, we can construct other angles equal to its angles and other lengths having the ratios of its sides—stands on a line *given in magnitude*, then the figure is also *given in magnitude*. The proof of *Data* 52 relies on constructing a square on one of its sides, and comparing this to the figure *given in form*, by deconstructing the figure into triangles *known in form*, and ultimately comparing of rectangles under the same height, using *Data* 49 and 48. That is, the concept of *given in magnitude*, in the *Data* approach, reduces to the ability to construct a rectilinear figure—in fact, a square—whose size can be geometrically related to the figure we are interested in measuring.

But, the mathematical context of Heron's metrical analysis—being proceeded by an explicit claim that certain lengths, which in the analysis are asserted to be *given*, are certain actual numbers of units, and followed by a synthesis-computation in which we straightforwardly compute with these numbers—indicates that, in fact, the more significant reading of the metrical analysis is as a general justification for the computation that immediately follows.[107]

We can make this explicit as follows. Assigning the values of the *given* objects as $AB_m := q$, $AD_m := r$, and $BG_m := s$, and following through the chain of *givens* in the text, we have

$$AD_m + BG_m := r + s, \qquad (Data\ 3)$$

so that, since $AZ = DZ + AD_m$ and $BH = BG_m - HG$, while $DZ = HG$,

$$AZ + BH = 2BH = AD_m + BG_m := r + s,$$

so that

$$BH_m := \frac{(r + s)}{2}. \qquad (Data\ 7)$$

Therefore,

$$\mathbf{Trap}(ABGD)_m = \mathbf{R}(AB_m, BH_m)_m := q \times \frac{(r + s)}{2}. \qquad (Data\ 52)$$

When considered in this light, it becomes clear that the step that could be justified by *Data* 3 is an *addition*, that which could be justified by *Data* 7 is a *division*, and that which could be justified by *Data* 52 is a *multiplication*—as is made explicit in the synthesis-computation. In this way, it becomes clear that Heron is using arguments by *givens* to justify a series of straightforward arithmetical operations.

My use of symbols—as opposed to the numbers Heron uses in his example computation—in order to represent this argument by *givens* seeks to capture the sense in which the argument by *givens* is general, applicable to any *given* numbers, ($\mathbf{G_2}$). Moreover, by organizing the argument in this way, we can see at a glance that at each step some

[107] Heron's use of this type of reasoning is also discussed by Acerbi (2011a, 143–144; 2012, 201–204).

newly considered geometrical element, on the left, is assigned to some computed *given* value, on the right. In this way, we see that claims about what is *given* are not equations in which there are non-*given* objects on both sides of an equation—such as we sometimes encounter in algebraic problem-solving, as in the premodern algebra of Diophantus.[108]

It may also be useful to consider this argument in terms of the symbolism developed by Ritter (1989; 1995, 50–52) to describe the form of computational algorithms in Egyptian and Mesopotamian sources.

	AD_m	D_1
	BG_m	D_2
	AB_m	D_3
(1)	$AD_m + BG_m$	$D_1 + D_2$
(2)	$2BH = AD_m + BG_m$	$(1) \div 2$
(3)	$\mathbf{Trap}(ABGD)_m = \mathbf{R}(AB_m, BH_m)_m$	$(2) \times D_3$

We see that this use of symbols in the right-hand column, which corresponds to Heron's computation, although it accurately describes the computational process and makes it clear that each operation is performed on a value obtained in the operation directly proceeding it, overly obscures the geometrical relations that are articulated in the metrical analysis.

There is, however, one significant way in which the first symbolic representation can be deceptive. When we look at the final line of the first symbolic derivation, we see at a glance the structure of the initially *given* objects, ($\mathbf{G_1}$) and ($\mathbf{G_2}$), in the sought object, ($\mathbf{G_3}$). This structure is, however, obscured in the computation, which operates at each step with the actual number produced in the foregoing step.[109] The metrical analysis, on the other hand, while it provides a context of justification for the computations, based in the geometry of the figure, also obscures the computational structure of the sought object, ($\mathbf{G_3}$). Nevertheless, the metrical analysis, although not explicitly stating all of the calculations, contains a series of geometrical inferences that includes information about structure that is closely related to that which we perceive immediately in the final equation of the first symbolic representation.

5.2 Metrical analysis in Ptolemy

A number of the metrical analyses in Ptolemy's writings—particularly those in the derivation of the chord table—are of precisely the same type as those in Heron's *Measurements*. The majority, however, must be characterized somewhat differently because of their particular use of *given* ratios, due to their role in chord-table

108 See Christianidis and Oaks (2013) for study of Diophantus' problem-solving techniques.

109 This is not always the case in the computations that correspond to metrical analysis. Sometimes a computed number can be taken up some steps later—as we will see below in the example drawn from Ptolemy.

trigonometry. Moreover, Ptolemy almost never performs a metrical analysis for the exact same problem for which he performs a calculation, and he does not refer to an argument by *givens* as an analysis. Indeed, if a metrical analysis is used either to establish general computability, or as a heuristic technique to look for a computable solution, then a metrical analysis, once the reader understands the difference between values *given* in the sense of ($\mathbf{G_1}$) and those *given* in the sense of ($\mathbf{G_2}$), would only need to be included for the same problem as a calculation for didactic or rhetorical reasons—much like the geometrical analysis in a analysis–synthesis pair (Netz 2000). Hence, late ancient mathematical scholars sometimes provided the metrical analysis corresponding to a computation—for example, Pappus in his *Commentary to Almagest V*, gives a metrical analysis corresponding to one of Ptolemy's computations, which he introduces by saying "we will analyze…" (Rome 1931–1943, 35–37).

In order to see how Ptolemy uses metrical analysis as a complement to chord-table trigonometry, we will read a passage from the development of his solar theory. In *Alm.* III.5.2,[110] Ptolemy uses metrical analysis to argue that, where the geometric configuration and parameters of the solar model are *given*, ($\mathbf{G_1}$), if, the apparent position of the sun, κ, is *given*, ($\mathbf{G_2}$), then both the equation of anomaly, α, and the mean position of the sun, $\overline{\kappa}$, will also be *given*, ($\mathbf{G_3}$).

Because of the context of this argument, immediately following and closely correlated to a computation through chord-table trigonometry that begins with the explicit statement that the value with which we begin the computation is *given* as 30°, *Alm.* III.5.1,[111] it is clear that for Ptolemy *given* means *known as a numerical value*—a degree, or a pair of numbers in ratio (Heiberg 1898–1903, I.241). Considering Fig. 15, *Alm.* III.5.2 shows that, assuming the eccentricity of the model, $(DQ : QZ)_r := (e : r)_r$ as *given*, ($\mathbf{G_1}$), and taking the apparent motion, arc $AB_m = \angle ADB_m := \kappa$ as *given*, ($\mathbf{G_2}$), then both equation of anomaly, $\angle QZL_m := \alpha$, and the mean motion, arc $EZ = \angle EQZ_m := \overline{\kappa}$, are also *given*, ($\mathbf{G_3}$). The text of *Alm.* III.5.2 reads as follows:

> That, if another one of the angles is *given*, the remaining [ones] will be *given* is immediately clear, with a perpendicular, QL, being produced in the same diagram from Q to ZD.[112] For if we suppose the arc AB of the zodiac—that is, angle QDL—to be *given*,[113] because of that, the ratio DQ to QL will also be *given*.[114] But, DQ to QZ being *given*,[115] QZ to QL will be *given*.[116] But

[110] The *Almagest* can be divided into units of text based on the types of mathematical argument involved. *Alm.* III.5.1 is a chord-table trigonometric computation that is followed by *Alm.* III.5.2 (I.241.14–242.24) and *Alm.* III.5.3 (I.243.1–243.15), which are both metrical analyses.

[111] *Alm.* III.5.1 (I.241.1–242.13) uses chord-table trigonometry to compute $\alpha = 1;9°$ and $\kappa = 28;51°$, where $\bar{\kappa}$ is explicitly asserted to be *given* as 30°, and $(e{:}r)_r$ is treated as the pair of numbers 2; 30^{p} and 60^{p}.

[112] *Elem.* I.12.

[113] That is, arc $AB_m := \kappa$.

[114] *Data* 40, Def. 3.

[115] Namely $(DQ : QZ)_r := (e : r,)_r = (2; 30^{\mathrm{P}} : 60^{\mathrm{P}})$, as shown in *Alm.* III.4.

[116] *Data* 8.

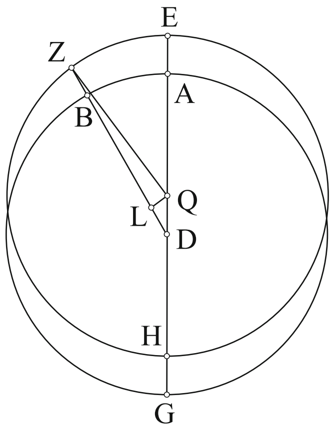

Fig. 15 *Almagest* III.5.2

through this we will have the angle QZL *given*,[117] that is the one corresponding to the difference of nonuniformity,[118] and angle EQZ [will be *given*], that is the arc of the eccenter [circle].[119]

(Heiberg 1898–1903, I.242–243)

Again, this analysis can be read on two levels. The first is that of the purely geometric operations involved in the proofs of *Data* 40 and 43, as required in the argument by *givens*. But as we saw in our reading of *Data* 43, these propositions proceed by geometric construction.[120] Hence, following the arguments in these propositions, what it would mean for ratio $(DQ : QL)_r$ to be *given* is that we could construct another triangle similar to $\mathbf{T}(QDL)$, thus providing another, constructed instantiation of the ratio, as in *Data* 40. And what it would mean for $\angle QZL_m$ to be *given* is that we could construct another triangle similar to $\mathbf{T}(QZL)$ including an angle equal to $\angle QZL_m$, as in *Data* 43.

But this constructive approach is of little or no use for the computations that Ptolemy intends to justify. Hence, just as with Heron's metrical analysis above, and as clearly indicated by the context in which the argument occurs, the more significant reading of this passage must be as the justification of a computation—now also involving chord-table trigonometry.[121]

Hence, the second, and probably more significant, level on which we can read Ptolemy's metrical analysis is as a justification of a computation. Assigning the *given* objects as $(DQ : QZ)_r := (e : r)_r$, $(\mathbf{G_1})$, and $\angle ADB_m := \kappa$, $(\mathbf{G_2})$, the relationship between the metrical analysis and the computation to which it corresponds can be

117 *Data* 43.

118 That is, the equation of anomaly, $\angle QZL_m := \alpha$.

119 That is arc $EZ_m := \bar{\kappa}$, by *Elem.* I.32 and *Data* 4. or *Data* 3

120 The strategy of the proof of *Data* 40 is essentially the same as that for *Data* 43. See Sect. 3.3, above.

121 Ptolemy's use of this type of reasoning is also discussed by Acerbi (2012, 204–208).

detailed as follows. Entering into a chord table with κ, ($\mathbf{G_2}$), in $\mathbf{T}(QDL)$, where we set $DQ_m \coloneqq 120^{\mathrm{p}}$, ($\mathbf{G_2}$),[122] we have

$$(DQ : QL)_r = (120^{\mathrm{p}} : QL_{m(DQ)})_r \coloneqq (120^{\mathrm{p}}, \mathrm{Crd}(\kappa)_m)_r.^{123} \qquad (\textit{Data } 40)$$

And, since $(DQ : QZ)_r = (120^{\mathrm{p}} : QZ_{m(DQ)})_r \coloneqq (e : r)_r$, ($\mathbf{G_1}$), we have

$$(QZ : QL)_r = (QZ_{m(DQ)} : QL_{m(DQ)})_r. \qquad (\textit{Data } 8)$$

Hence, we can set $(120^{\mathrm{p}} : QL_{m(QZ)})_r = (QZ_{m(DQ)} : QL_{m(DQ)})_r$, and enter into a chord table with $QL_{m(QZ)}$, so that

$$\angle QZL_m \coloneqq \alpha = \frac{\mathrm{Arc}(QL_{m(QZ)})_m}{2}. \qquad (\textit{Data } 43)$$

Finally, from the geometry of the figure, we also have

$$\angle EQZ_m = \overline{\kappa} \coloneqq 180^\circ \pm \left((90^\circ - \kappa) \pm \left(90^\circ - \frac{\mathrm{Arc}(QL_{m(QZ)})_m}{2}\right)\right).$$
(*Elem.* I.32, *Data* 4 or *Data* 3)

When we read the text in this way, it becomes clear that the step that can be justified by *Data* 4 or *Data* 3 is a *subtraction* or an *addition*, that justified by *Data* 8 is the *elimination of equal terms* from two known ratios, that justified by *Data* 40 is *entering a chord table with an angle*, and that justified by *Data* 43 is *entering a chord table with a ratio*[124] between a leg and the hypotenuse of a right triangle.[125]

Because of the structure of Ptolemy's chord table, in doing plane trigonometry we continually switch between different measures of length, and hence generally work with *given* ratios, not with lengths *given* strictly.[126] Hence, when we write $QL_{m(QZ)}$ we do not that mean QL_m *given* strictly, but rather QL as *given*, ($\mathbf{G_3}$), where we set $LZ_m = 120^{\mathrm{p}}$, ($\mathbf{G_2}$)—that is, where the ratio $(QL : LZ)_r$ is *given*, ($\mathbf{G_1}$), and LZ_m is assumed as *given*, ($\mathbf{G_2}$).[127] For this reason, and because of the need to enter into a chord table, sometimes multiple times, it is not possible to look at the final lines of this

[122] In fact, we could say that this is either *given* in the sense of ($\mathbf{G_2}$), because we can set it to whatever value we like, or given in the sense of ($\mathbf{G_1}$), because the use of Ptolemy's chord table, *Alm.* I.11, always determines this value. This, again, shows the ambiguity between ($\mathbf{G_1}$) and ($\mathbf{G_2}$), and reinforces the claim that the function of *given* is always local.

[124] Or rather with a sort of pseudo-ratio—namely, a length *given* in terms of another length.

[125] Or rather, entering a chord table with the value of the leg when the hypotenuse has an assumed value.

[126] When we enter into an ancient chord table with a *given* angle we produce a length *given* in terms of the hypotenuse—that is, as a pair of numbers, a sort of pseudo-ratio—as is made clear from the fact that Ptolemy always expresses these lengths in terms of a certain hypotenuse. The 120th part of the diameter of the chord table is not understood as a unit in the normal sense, because we divide as many diameters into 120 parts as are required to solve the problem. For this reason, we should not follow Acerbi (2012, 204–208) in rewriting Ptolemy's text. See also note 142.

[127] Once again we see that the difference between ($\mathbf{G_1}$) and ($\mathbf{G_2}$) is merely local.

symbolic representation and see at a glance the relationships that the sought objects have to the *given* values.

Again, it may be useful to compare this symbolic summary of Ptolemy's computation with a formal description of the arithmetical operations.

	$\angle ADB_m := \kappa$	D_1
	$(DQ : QZ)_r := (e : r)_r$	D_2, D_3
(1)	$(DQ : QL)_r = (120^{\mathrm{p}} : QL_{m(DQ)})_r$	$\mathrm{Crd}(D_1)$
(2)	$QZ_{m(DQ)} = (QZ : DQ)_r \cdot 120^{\mathrm{p}}$	$D_3 \div D_2 \times 120^{\mathrm{p}}$
(3)	$QL_{m(QZ)} = (QL_{m(DQ)} : QZ_{m(DQ)})_r \cdot 120^{\mathrm{p}}$	$(1) \div (2) \times 120^{\mathrm{p}}$
(4)	$\angle QZL_m$ in $\mathbf{T}(QZL)$	$\mathrm{Arc}((3)) \div 2$
(5)	$\angle EQZ_m = 180^\circ \pm (\angle ZQL_m \pm \angle LQD_m)$	$180^\circ \pm ((90^\circ - D_1) \pm (90^\circ - (4)))$

Here we see that while the symbolism in the right-hand column describes the operations to be performed as a sort of recipe that shows that values may be stored and operated on later, it is less successful than the metrical analysis in articulating the justifications for these operations.

When we look at the first symbolic description of Ptolemy's metrical analysis, however, we observe the overall pattern that at each stage of the argument some newly considered element, on the right, is said to be *given* in terms of some previously established value, or pair of values, on the left. Moreover, the first symbolic representation is faithful to Ptolemy's use of metrical analysis as a general argument that if some values are assumed to be *given*, $(\mathbf{G_1})$ and $(\mathbf{G_2})$, then other values can be shown to be computably *given*, $(\mathbf{G_3})$. Moreover, for those familiar with this style of mathematics, the metrical analysis also indicates how to actually carry through the computation, although the use of the chord table means that the steps of the metrical analysis are somewhat farther removed from the steps of the computation than is the case with metrical analyses not involving a chord table, both in Heron and Ptolemy (Acerbi 2012, 208). This looseness is, however, a function of the mathematical constraints involved, not some carelessness on Ptolemy's part.

Finally, the metrical analysis of a configuration amenable to chord-table trigonometry is generally is much faster than carrying through the corresponding computation by hand.[128] Hence, for mathematicians skilled in the methods of the *Data*, metrical analysis probably functioned as a sort of heuristic technique that could be used to explore the feasibility of making calculations without the labor of actually carrying them through.

[128] Compare, for example, the metrical analysis above with the calculation that proceeds it in *Alm.* III.5.1. The computation takes 39 lines of Heiberg's text whereas the metrical analysis takes 9 (Heiberg 1898–1903, 241–243). Moreover, for someone familiar with reasoning by inferences from the *Data* the metrical analysis given above is trivial and rapid, whereas the computational effort involved in entering even one time into the chord table is rather time consuming for anyone not able to perform sexagesimal computations in their head.

5.3 Characteristics of metrical analysis

In reading through these passages of metrical analysis, some, but tellingly not all, modern readers perceive a tension between the computational procedures that the process seeks to justify and the geometrical theorems of the *Data* that the claims made in the argument implicitly call upon. It is important to note, however, that this tension was either not evident to Heron and Ptolemy, or, if it was, they passed over it in silence. In fact, there is no indication that they thought of their use of metrical analysis as in any way new, or innovative.[129]

Heron and Ptolemy probably thought of their metrical analyses as addressing issues dealing with measuring geometric objects with numerical values, as opposed to numerical computability taken strictly. That is, a metrical analysis, with its possible grounding in the geometrical theorems of the *Data*, was able to make a claim that, say, the diagonal of a square, or the chord of an arbitrary arc, is *given* in terms of the side of the square or the diameter of the circle, *Data* 52 and 87—even if we do not have a method for computing these ratios exactly. Indeed, in most of our ancient sources, metrical analysis was carried out in a fundamentally geometric context.[130] Nevertheless, as we made explicit in the symbolic representations above, the steps of a metrical analysis are computable through some well-known process, as well as being supported by theorems of the *Data*. In fact, it is likely that being computable was more crucial than being supported through theorems in the *Data*. For example, Ptolemy's derivation of the chord table, *Alm.* I.10, which employs theorems, computations, and metrical analyses, seems to prefer metrical analyses that are suitable for computation over others that correspond to the simplest argument made possible by the theorems of the *Data*.[131]

In this way, metrical analysis, although based on underlying geometrical objects, was first and foremost a technique for addressing questions of computability. What is *given* at each step is some actual number. It need not be a perfectly accurate measure of the object in question—for example, the value of the chord subtending $1/2°$, or the diagonal of a square in the same measure as its side—but it must be a number that we can compute, and which we can then use in further computations. This is made clear by looking at the symbolic representations of metrical analyses above, in which we see that the right-hand side always assigns a computed number to some geometric object. Indeed, a statement that something is *given* occurring in a metrical analysis is a claim it can be assigned some definite, computable value.

A metrical analysis need not proceed in such a way that each step computes with the value just assigned in the previous step—although some simple metrical analyses do work this way, as in the example from *Meas.* I.10. Once a value has been assigned

129 At least by the classical Islamicate period, and probably from much earlier, this blending of the geometrical and arithmetical readings of geometrical books of Euclidean works was commonplace. Thābit ibn Qurra, who knew Euclid's work as well as anyone, tells us in his *Composition of Ratios* that although Euclid only defined "quantity" or "magnitude" (مقدار) to refer to geometric objects that have "extent" (مساحة), in his actual works the meaning of the term is broader and refers also to angles, numbers, movements, and so on; and that whenever we read *quantity* we should also understand *number* (Lorch 2001, 170; Rashed 2009, 431).

130 A possible exception is the analysis in the final theorem of Diophantus' *Polygonal Numbers*.

131 I will flesh out the details of this in separate paper, focusing on Ptolemy's use of metrical analysis.

to an object, that value remains available for future computation at any later point in the analysis, as in the example from *Alm.* III.5.2. Hence, the chain of *givens* need not be direct—in the sense that values can be set aside and taken up at some later point for further computation.

In texts containing metrical analysis the basic computational operations—the four arithmetical operations, taking roots, eliminating an equal term from *given* ratios, entering a chord table, and so on—are neither postulated nor discussed. They are simply assumed unproblematically to be possible, and executed without discussion in a computation. This approach, however, is similar to that taken by Euclid in his number theory, *Elements* VII–IX, or by Diophantus in his premodern algebra, *Arithmetics*. In both of these works, basic computational operations are simply assumed with no formal introduction and little, or no, discussion. Indeed, a metrical analysis is not an explicit arithmetical algorithm, since no arithmetical operations are ever mentioned. A metrical analysis deals then, not explicitly with the mechanics of any operation, but rather with the possibility of computability more generally.

There are a number of fundamental differences between the methods of metrical analysis and those of premodern algebra—found, for example, in Diophantus' *Arithmetics*. For one thing, Diophantus' procedures are, at least in principle, purely numerical, and do not rely on any underlying geometric conception.

Next, metrical analysis has no special nomenclature for designating objects to be sought, as is introduced in premodern algebra. In ancient analysis, we deal only with geometric objects and with those same objects when they are *given* in various ways—hence, there is no notational, or conceptual, device that allows us to set up a relation containing both *given* elements and explicitly sought elements. A problem in premodern algebra, however, begins with an instantiation of the stated problem using a certain equation, in which terms to be sought are set into relation with actual numbers that are stated to be *given*.[132]

Finally, although metrical analysis provides an algorithm for computing a definite value, it does so in a different manner than premodern algebra. Diophantus' procedure for setting up the equation with which he will work, allows him to set known and sought values on both sides of his equation, so that he can apply operations mathematically equivalent to the arithmetical operations without regard to the epistemological, or ontological, status of any of the objects in his equation.[133] Of course, in metrical analysis we also find equations, and proportions, with non-*given* objects on both sides of the equation, which are then subjected to arithmetical operations and ratio manipulations,[134] but this is done only as an intermediate step before immediately asserting

[132] This section of a Diophantine problem has been called the *invention* and the *set up of the equation* (Christianidis 2007, 296–298; Christianidis and Oaks 2013, 132–134).

It should be noted that, in general, premodern algebra does not employ the concept of *given* as part of its problem-solving procedure, but rather uses the practice of assigning actual numbers to stand in as examples of the values that are asserted to be *given* in the enunciation.

[133] In fact, Diophantus' three primary operations for working with equations are not expressed by him as arithmetical operations, but, of course, they can be so expressed (Tannery 1893–1895, I.14; Sesiano 1982, 88).

[134] See, for example, the second line of the symbolic representation of the example from Heron above, Section 31.

that some geometrical object is *given* in terms of some computable number. In metrical analysis, it is this series of claims about what is *given* that functions as the primary problem-solving procedure.

As well as showing the possibility of producing a computation, and providing a general language in which to state an algorithm, metrical analysis was probably used by ancient mathematicians because in many cases it would have been faster, and more convenient, to look for a solution by metrical analysis then to carry through an actual computation. Once the metrical analysis had assured the mathematician that a certain value was, in fact, *given*, then the computation itself could actually be carried through—and the metrical analysis would point out the general direction of the computation. Hence, in this restricted sense metrical analysis could play a similar role in ancient problem-solving as our use of a purely symbolic equation, although the strengths of the two methods are rather different.

6 Conclusion

Although from a mathematical perspective it is possible to claim that that the arguments about *givens* in the *Data* were produced from the beginning to address the dual questions of general constructibility and general computability,[135] on balance I think that this is less likely than the claim the *Data* was originally composed to address the needs of geometrical problem-solving and was then later repurposed as a means to justify and generalize metrical arguments.

In the first place, I am aware of no evidence—direct, indirect, or circumstantial—for the use of arguments by *givens* in the classical or early Hellenistic periods in computing numerical values, despite the fact that we have a number of direct and indirect sources for these periods that do carry out computations; while, on the other hand, there is clear and direct evidence that Greek geometers of these periods used the concept of *given* extensively in their geometric problem-solving activities.

Moreover, the articulation of the *Data* itself is fully consistent with a purely geometrical reading. For example, as we saw, the core definitions are explicitly geometrical, and the key theorems are demonstrated using geometrical construction. If we want to argue that the *Data* itself was originally written to justify numerical operations, however, we must confront a number of inconsistencies. While *Data* 3 and 4 could be taken to justify *adding* and *subtracting* for general quantities, for *multiplying and squaring*, *taking square roots* and *dividing*, we must turn to *Data* 52, 55 and 57—but, as we saw, the proof of these theorems rely on the geometric construction of a square and a similar triangle, since *Data* 55 depends on *Data* 52, and *Data* 57 depends on *Data* 40. And moreover, they only allow for multiplying or dividing by one value—since the resulting object, being geometrical, would have to be transformed, in some

[135] Blåsjö (2016) has recently made a related argument in regards to the interpretation of certain theorems of Euclid's *Elements* as so-called geometrical algebra. He argues, essentially, that this reading has not been definitively refuted. This is, of course, true, but it misses the point. Almost no one would argue that it is not possible to make a reading of *Elements* I and VI as motivated by and justifying computational problem-solving. The question is rather whether such a reading, or that through the theory of the application of areas, is more broadly successful in explicating the ancient sources.

unspecified way, before the operation could be performed again. At the very least, we must accept that if, in fact, Euclid had devised his text in order to justify arithmetic procedures, he did so in a rather disorganized way. Finally, it should be added that there is no clear evidence that Greek mathematicians thought that the basic arithmetical operations needed to be justified, so there is no reason for us to believe that Euclid felt the need to engage in such a project.

Hence, it is more likely that mathematicians in the late Hellenistic period, such as Hipparchus and perhaps others, adopted the theorems of the *Data* as a structure for justifying their computations. They probably did this because the *Data* was the only text available that provided such justifications and they probably felt warranted in doing so because their computations were, after all, meant to measure the sizes of various geometric objects. Whatever the case, metrical analysis, as it was practiced by Heron and Ptolemy sets geometry in a truly fundamental position, so that only computations that measure geometric objects are, in fact, justified. With few exceptions, metrical analysis is not used in purely numeric problem-solving in our sources.[136] This realization leads us to see the analytical approach of Jordanus of Nemore, in his *Given Numbers*, as all the more innovative (Hughes 1981; Zheng 2012).

In order to elucidate and summarize the various uses of *given* that we have seen, we will return to Marinus' discussion of the various ways that ancient mathematicians talked about objects being *given* (Menge 1896, 238–242). Instead of closely following Marinus' own discussion of these terms, we will try to understand them on the basis of the mathematical passages we have read.

The first term that Marinus introduces is *fixed*, *assigned*, or *ordered* (τεταγμένον), which he attributes to Apollonius, and which he tells us is that which remains always the same in regard to whatever it is *fixed* (Menge 1896, 234, 238). As Acerbi (2011a, 130–133) has argued, the most common meaning of this term in the mathematical corpus is "unequivocally determined." We saw above that Apollonius used it to introduce as *given*, ($\mathbf{G_2}$), the ordinates of a conic section assumed as *given*, ($\mathbf{G_1}$), and, in fact, Apollonius' terminology for the ordinates themselves is a cognate of this word (Acerbi 2011a, 131–132). This way of talking about *given* emphasizes that what is *given* is fixed, or assigned, as opposed to what is free, or variable. This distinction between what is fixed and what is variable is especially drawn out in the study of loci and porisms (Acerbi 2011a, 137–138). Moreover, it has important overlaps with our understanding of the difference between the given and variable terms of a Cartesian equation. For example, if we compare the Cartesian equations with the locus definitions of the conic sections in Manaechmus' problem discussed above, namely

$$\begin{aligned} ax &= y^2 \quad \text{with} \quad \mathbf{R}(a_m, dZ_p) = \mathbf{S}(ZQ), \text{ and} \\ xy &= ae \quad \text{with} \quad \mathbf{R}(dZ_p, ZQ)_m = \mathbf{R}(a_m, e_m)_m, \end{aligned}$$

where a and e, on the left, and a_m and e_m, on the right, are simply *given in magnitude*, ($\mathbf{G_1}$), while x, on the left, and dZ_p, on the right, are non-*given* segments of a line

[136] A possible exception is Diophantus' *Polygonal Numbers*, which is essentially arithmetical despite an outward veneer of geometrical language and concepts. (Heath (1921, II.516), for example, states that "the method of proof is strictly geometrical.")

given in position with a *given* endpoint, ($\mathbf{G_2}$), it is clear that the notion of *given* is here closely related to our conception of fixed, or assigned, as opposed to variable. It is unfortunate that the ancient works on loci problems and porisms have been lost, but it is to be hoped that a careful investigation of the fragmentary evidence for this material, especially in Pappus' *Collection* VII, will further shed light on this aspect of the ancient use of *given* as opposed to unassigned objects.[137]

Marinus' second term is *known* (γνώριμον), which he attributes to Diodorus, who apparently said that rays and angles are *given* in this way (Menge 1896, 234, 238–240). Since Diodorus is known to have worked in sundial theory, and since Ptolemy's *Analemma* 9 and 10 use metrical analysis to determine the angles that specify the position of the solar ray, it is likely that Diodorus used *known* to describe the status of the *given* objects in metrical analyses that he included in his now lost *Analemma*.[138] This is supported by the observation of Acerbi (2011a, 134) that an analytical argument in the Euclidian *Optics* 18, version A, uses the term *known* as perfectly synonymous with *given*. Indeed, it is quite possible that Diodorus wrote metrical analyses similar to those now found in Ptolemy's *Analemma* 9 and 10 using *known* in place of *given*.

At any rate, the notion of *known* appears to be closely related to the usage of *given* in metrical analysis—namely, it designates a value that can be assigned in the beginning and then computed at each stage through an effective procedure, in such a way that each step is also justified by a theorem of the *Data*. In both, metrical analysis and premodern algebra, we start out with the essentially arbitrary assignment of certain values to be operated on in the procedure. The contrast, here, between what is *given* and what is not is that between the terms to which we can assign a definite, known value, and those whose values are sought, or depend in some essential way upon the *given* values. In metrical analysis, however, unlike premodern algebra, we have no special terminology for the values we seek, or any values we may determine along the way. Moreover, as discussed above, metrical analysis does not lend itself to working with unknown values on both sides of an equation. Finally, in Diophantus' *Arithmetics*, we do not find analyses by *givens*, and the terminology is used only for the arbitrary values that are used to state the problem or assumed in completing the problem, ($\mathbf{G_1}$) and ($\mathbf{G_2}$). This may be because in Diophantus' problem-solving procedure the relationships between the various terms is crucial to choosing appropriate *given* values to complete the problem, whereas in metrical analysis these underlying relationships are obscured by a chain of *givens* that results in the simple assertion that the sought object is *given*. It is also possible that Diophantus thought of arguments by *givens* as justified by the *Data* and considered that the geometrical context of the *Data* bares

137 The vital role of loci and porisms in ancient analysis is elucidated by Knorr (1986) in his brilliant study of ancient geometric problem-solving. See Jones (1986, 547–602) for a summary of the mathematical contents of the lost treatises on porisms and loci. Acerbi (2011a, 137–138, 146–448) discusses the importance of locus theorems and porisms with respect to the language of *givens*. See also the reconstruction of Euclid's *Porisms* by Simpson, which although speculative, can give us a sense for the mathematical concepts involved (Tweddle 2000).

138 See Edwards (1984, 152–182) for a discussion of the evidence concerning Diodorus. Ptolemy's use of metrical analysis in *Analemma* 9 and 10 is described by Edwards in the notes to his translation of these passages.

little relation to the purely arithmetic context of his problem-solving approach in *Arithmetics*.

Marinus' next term is *provided*, *found* or *calculated* (πόριμον), which, although not associated with any particular name, given the context and the terminology, he must have ~~been~~ taken from Euclid's usage (Menge 1896, 240). This is the meaning of *given* used in the problems of the *Elements*, the theorems of the *Data*, and the resolutions of geometrical analyses. Here, what is *given* is whatever objects are there available to work with in the beginning of the mathematical discourse, whatever we introduce in some well-defined way, and whatever we can produce from these through an effective procedure. In terms of the elementary Euclidean texts, these procedures were confined to the postulates and the problems, but in more advanced works objects could be considered *given* so long as they were introduced by some well-defined procedure, such as a locus description or a mechanical construction—certainly, both Menaechmus and Apollonius considered it possible to introduce conic sections *given in position*. The term is also used by Heron in *Dioptra* 13–14, 25–30, for values that can be computed through some effective procedure (Acerbi 2011a, 135). This way of treating *given* objects as *provided* or *found* gives us a general way of treating the effective procedures used to produce or locate these objects. An argument by *givens* in this sense does not necessarily actually produce any object, it simply establishes that it is possible to do so. Hence, a problem, since it produces, or finds, ~~and~~ actual object, need not be accompanied by an argument by *givens*, since the possibility of producing or finding the object is established as a corollary of having actually done so.

Marinus' final term is *stated*, *specified* or *expressible* (ῥητόν), which he seems to put forward with some hesitation (Menge 1896, 234, 240–242).[139] In the first place, he attributes it rather obliquely to Ptolemy and then in his own discussion of the term gives no more than the most basic definition of the term, with no attempt to explain how it is related to the concept of *given*. When Marinus first introduces the term he says, "… some declare it to be expressible (ῥητόν), as it seems Ptolemy, when he calls *given* those things whose measure is known either accurately, or approximately" (Menge 1896, 234). This expression implies, however, that Marinus did not actually find a passage in Ptolemy's text that makes this assertion, but rather he inferred it indirectly from something else that Ptolemy says. Indeed, I have not found any passages in Ptolemy that directly assert that what is *given* is *expressible*. There are a few places where Ptolemy speaks of the precision of his calculations, but he generally does not mention what is *given* in this regard.[140] Perhaps Marinus is referring to Ptolemy's general practice of using metrical analysis to compute values that we know are not necessarily *expressible*. For example, in the course of the derivation of this chord table, Ptolemy

[139] As is well known, ῥητόν refers to what is precisely expressible in terms of some arbitrary measure, and hence relates to the distinction between what is commensurable and incommensurable—as, for example, in *Elements* X. Hence, the concept is mathematically related to our notion of a *rational* number, and the word is sometimes translated in this way. There is a long literature on this material that need not concern us here.

[140] For example, Taisbak (2003, 243) refers this passage in Marinus to *Almagest* I.10.1, where Ptolemy introduces the system of sexagesimal fractions, but Ptolemy does not say anything about what is *given* in this passage (Heiberg 1898–1903, 32).

tells us that if a chord is taken as *given*, (**G$_2$**), then the chord of the supplementary arc is *given*, (**G$_3$**), since the sum of their squares equals the square of the diameter, which is tacitly assumed to be *given*, locally (**G$_1$**). He then gives an example calculation and states that the chord is "nearly" (ἔγγιστα) a certain sexagesimal value (Heiberg 1898–1903, 35–36). This is fairly plainly an assertion that even though we cannot compute this value exactly, it is still *given*. I do not see, however, that this amounts to a claim that what is *given* is *expressible*, in the technical sense of the latter term. Indeed, this seems to be equivalent to the claim that Marinus attributed to Diodorus—namely, that what is *given* is "anything that arrives to some knowledge, even if it is not *expressible*" —because Ptolemy must have been aware of proofs that some of the values computed in this way are *irrational* (ἄλογος), or non-*expressible* (Menge 1896, 134). Since I cannot find any statement by an ancient mathematician claiming that what is *given* is what is *expressible*, and since Marinus himself does not seem to be interested in discussing the subject at any length, it is likely that the only reason he brought it up is that someone in his circle had, incorrectly, attributed this interpretation to Ptolemy. Whatever the case, since we do not find this in Ptolemy, or any other mathematical author, we may, like Marinus, dismiss this reading as irrelevant.

Although Marinus introduces these alternative terms as definitions, or explanations, of the *given* (Menge 1896, 134), the first three terms, or their cognates, are all found used in Greek mathematical texts as synonyms for *given*.[141] Indeed, considering the general lack of metamathematical discussion in Greek mathematical texts, it is likely that synonyms such as these are the only sources for Marinus' terminology and that his discussion has little more to teach us than what we can already learn from the mathematical texts themselves.

From these texts we see that an object that is designated as *given* is one which is directly accessible to the mathematician and is available for further mathematical work. What is *given* is uniquely determined in such a way that it is available for a constructive or computational procedure, and what results from this procedure then becomes *given*—what we might call *assigned*. In the texts that have come down to us, once an object has been assigned in a certain mode of *given*, it cannot be reassigned.[142] Of course, a newly assigned object often depends on a previously assigned object, and it is just this dependency that an argument by *givens* seeks to establish. Arguments and claims about *givens* do not concern themselves with how the original objects are assigned—it is simply assumed that this can be done. There does not appear to have

141 For example, the verb τάσσω is used by Apollonius, *Conics* II.44 to set out a line *given in position*, as seen above, Sect. 4; γνώριμον is used as a synonym of *given* in Euclid's *Optics* A 18 (Heiberg 1895, 28); and Heron uses πορίζω in *Dioptra* 13, 14 and elsewhere to mean *given* (Schöne 1903, 234–236). (The use of ποιέω by Apollonius in the enunciation and exposition of a number of theorems of the *Conics* is unrelated to the concept of *given*. The discussion by Acerbi (2011a, 135) of Apollonius' usage is obscure to me—it does not introduce something which is "provided," nor is it used in any problems. It introduces a constructive assumption without which the theorem would not hold—what we can call a contrivance.)

142 Perhaps this is why Ptolemy, in the metrical analyses of his chord-table trigonometry generally does not speak of a length being *given in magnitude*, because this would involve him in assigning the same line multiple times. Acerbi (2012, 208) observes this problem of multiple assignments. The end of the metrical analyses in *Analemma* 9 and 10 support this assessment. In each case Ptolemy first sets the radius of the analemma as *given in magnitude* and then all the other lengths that he has previously shown to be *given in ratio* to the radius are all *given in magnitude*—so that each length has only one assignment.

been any restriction on the objects that can be assigned in setting out a problem. We see, however, from reading the texts that the objects that can be assigned arbitrarily in the course of an argument are restricted to points and lines in position, magnitudes, numerical values, and certain well-defined constructions.

Arguments by *givens* establish the constructibility, or computability, of a mathematical object—that is, they establish the validity of an effective procedure. One of the primary goals of an effective procedure is to produce a mathematical object in such a way that it may be of use in further mathematical work—so that it may enter into new procedures. This means that a problem must produce a new object that will be as concrete and available to the mathematician as whatever we started with. For example, in *Elem.* I.1 an equilateral triangle is constructed on a *given* line, then in *Elem.* I.2, this constructive operation is called in to produce an equilateral triangle that is, in turn, operated on—its sides are extended, circles are drawn about its vertices and so on. Hence, the triangle that is produced using *Elem.* I.1 must be *given* in the same sense as any other mathematical object that we introduce in a controlled manner. It is in light of this that it makes sense to say that whatever we arrive to at the end of a productive mathematical procedure—an ancient problem—is also *given*.

Acknowledgements The core ideas of this paper go back some years now to my dissertation, and I thank Alexander Jones and Jan Hogendijk for their comments on that work. I presented an overview of this argument at a conference of the SAW Project, under the direction of Karine Chemla. The discussion following this presentation helped me to clarify some of my thinking. During the time that I was a guest of the SAW Project in Paris, 2015, some of the research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007–2013) / ERC Grant Agreement No. 269804. Ken Saito read an earlier draft of this paper and made a number of valuable suggestions. This paper has benefited considerably from the extensive notes made by Karine Chemla and Matthieu Husson.

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