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# Uses of construction in problems and theorems in Euclid's *Elements* I–VI

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**Abstract** In this paper, I present an interpretation of the use of constructions in both the problems and theorems of *Elements* I–VI, in light of the concept of *given* as developed in the *Data*, that makes a distinction between the way that constructions are used in problems, *problem-constructions*, and the way that they are used in theorems and in the proofs of problems, *proof-constructions*. I begin by showing that the general structure of a problem is slightly different from that stated by Proclus in his commentary on the *Elements*. I then give a reading of all five postulates, *Elem.* I.post.1–5, in terms of the concept of *given*. This is followed by a detailed exhibition of the syntax of *problem-constructions*, which shows that these are not practical instructions for using a straightedge and compass, but rather demonstrations of the existence of an effective procedure for introducing geometric objects, which procedure is reducible to operations of the postulates but not directly stated in terms of the postulates. Finally, I argue that theorems and the proofs of problems employ a wider range of constructive and semi- and non-constructive assumptions that those made possible by problems.

## 1 Introduction

This paper is a treatment of the role of construction, and constructive thinking, in Euclid's *Elements* I–VI, based on a reading of both the *Elements* itself and the *Data*, which has a close relationship to *Elements* I–VI and is a formal treatment—in the

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ancient sense—of constructibility.[1] It is now widely recognized that the predicate logic used in the early twentieth century axiomatizations of Euclidean geometry, and by Mueller (1981) in his treatment of the logical structure of the *Elements* itself, is insufficient to axiomatize geometric constructions (Pambuccian 2008). Starting from the 1960s, however, logicians began to produce axioms for operations, as is required for the axiomatization of geometric construction. Although I have been influenced by this work and have used some of it to help clarify what I think that Euclid is doing,[2] I do not give a formal treatment of Euclid's practice and I have avoided formalization as much as possible. Instead, I have attempted to keep things intuitive and grounded in concepts and procedures that we can find in the ancient texts, because ancient geometry is a study of geometric objects themselves, using a fairly simple logic articulated in natural language (Panza 2012; Acerbi 2011a).

Hence, in reading the *Elements*, I endeavor to provide an interpretation of obscure, or uncertain, passages that is both consistent with other passages in the *Elements* itself and with the *Data*. That is, I make the methodological assumption that the author of the *Elements* was not simply a compiler—although he doubtless did work with and assimilate previously available material—but rather, intentionally composed and organized *Elements* I–VI and the *Data*, both to be read together and to express a coherent project of plane geometry.[3] In this way, I hope to present an integrated view Euclid's constructive geometry that can be placed in the context of ancient debates about the nature of geometry even if it is sometimes not the way that we would proceed.

In this paper, I utilize a number of concepts from computer science—or rather from the philosophy of computer science. This is not because I believe that ancient geometers were thinking along similar lines but rather because these concepts help us to discuss constructive procedures more precisely. For example, the concept of *function* introduced in this paper is simply that of a type of procedure or routine that performs a specific task—namely, it operates on some specified object, or objects, and returns some definite object. I use the term *routine* for a procedure that is explicitly stated as a finite series of steps, each one of which has already been fully articulated. Finally, I use *subroutine* for a procedure that acts as a unit in other routines. In fact, all of these could simply be called procedures, or rules, but the terminology introduced will help us be more specific about various aspects of the practice that we find in the Euclidean texts.

### 1.1 Constructions in problems and in theorems

In order to discuss the role of constructions in the *Elements*, it is useful to point out that constructions are employed in both problems and theorems, and to distinguish

[1] Of course, there are no explicit constructions in *Elements* V. Nevertheless, I take *Elements* V to be necessary to the development of *Elements* VI, which includes a number of important problems and involves considerably use of construction. Furthermore, it is possible that construction is implicitly involved in the essential definitions of ratio and proportion, *Elem.* V.def.4 and V.def.5.

[2] I have particularly found useful the approaches of Mäenpää and von Plato (1990) and Beeson (2010).

[3] In fact, the *Data* is more of a compilation than the *Elements* and it is really only the first half to three quarters of the text that can be read as articulating a single program.

between problems, which are a type of assertion, and constructions, which are a way of introducing, or perhaps producing, objects. Constructions are used to introduce new objects into both theorems and in problems—and in problems this is done in two ways, or rather to two ends, as we will see.

The terminology and practice of the *Elements* clearly divide propositions between theorems and problems (Heath 1908, 124–129; Caveing in Vitrac 1990–2001, I.133–137). The first six books of the *Elements* present a blend of theorems and problems—most books are mixed; *Elements* IV has no theorems, *Elements* V no problems. Since the term *problem* is used in discussions of Greek mathematics in a way that differs from that normal in current usage, it may be helpful to state that every proposition of the *Elements* can be categorized as follows:

> Problem: Given some set of initial objects, a *problem* shows how to do something (say, how *to find*, *to produce*, *to construct*, *to set out*, and so on) and then demonstrates that what has been done is satisfactory. ("To do such-and-such…")
>
> Theorem: A *theorem* asserts some property that holds for certain objects that are not asserted to be given. ("If… then…," "Such-and-such an object has such-and-such property…")

The distinction between these two types of propositions is discussed by Proclus, who also tells us that in the classical period there was a lively debate over which type of proposition should be regarded as primary (Friedlein 1873, 77–81). Since at least as far back as Proclus, however, it has been more common for scholars of the text to regard theorems as primary and to construe the role of problems as auxiliary (Zeuthen 1896; Harari 2003). This view has been challenged by Mueller (1981, 15–41), who points out that a desire to produce problems could account for much of *Elements* I, and by Knorr (1983) who makes it clear that in other mathematical texts, including other texts by Euclid himself, problems play a fundamental role. Nevertheless, because of the weight placed on theorems, there seems to be still a general tendency, when reading the *Elements*, to see problems as auxiliary to theorems and hence to conflate problems with the constructions used in proving theorems.

In fact, in the *Elements*, constructions are used in both theorems and in problems—and in problems in two different ways. A clear distinction between problems and constructions has already been delineated on the basis of a reading of Theodosius' *Spherics* (Sidoli and Saito 2009), and I will argue in this paper that this distinction must be maintained for the *Elements* as well.

A problem is completed by producing a specific geometric object that meets certain conditions. A problem (a) shows how to produce the object using an explicitly articulated routine of postulates and previously established problems and then, (b) through deductive argumentation, using first principles—of three different types—and previously established theorems, shows why this object is the one that was to be produced. This deductive argumentation sometimes requires the introduction of new objects. That is, the geometric objects that complete the problem, together with the initial objects stipulated in the enunciation of the problem itself, are sometimes not sufficient to show that the constructed object satisfies the requirements of the problem. In such cases, we must introduce new objects for the proof—which, just as with theorems, can be done using a variety of different means, including, but not limited to postulates and previously established constructions.

In a theorem, on the other hand, while it is sometimes the case that the properties of the objects stated in the enunciation are sufficient to demonstrate the proposition, more often than not we must introduce new objects and use their properties in the course of the demonstration. These new objects are introduced using constructions that can usually be carried out using the postulates, or previously demonstrated problems. Euclid, however, also demonstrates theorems by introducing, or supposing the existence of, objects whose production cannot be so explained—to be discussed in Sect. 5. In such cases, we may call the introduction of these objects hypothetical, in the sense that no effective procedure has been, or in some cases can be, stated for their production.[4]

Indeed, whereas the introduction of objects in a theorem, or in the proof of a problem, is sometimes hypothetical, the use of construction in problems is strictly reducible to the postulates—as has long been known. Nevertheless, the way that constructions function in the production of problems has never been accurately described in detail. Indeed, since most scholarly work on the *Elements* has gone into understanding the theorems, the way that previous problems are employed in the constructions of problems has often been overlooked. For example, Heiberg (1883) often notes a fairly minimal set of justifications for the production of a problem in his Latin translation and in this he has been followed by Heath (1908). More recently, Vitrac (1990–2001), Joyce (Online), and Fitzpatrick (2008) have made reference to more construction steps, but I have often found that the list of justifications that they provide in their text is also incomplete.[5] Hence, in order to fully understand the way that previous problems function in the construction of new problems, we must work through every step from the beginning of the text.[6]

[4] These semi-constructive, or non-constructive introduction assumptions have the same status as the analytical assumption of an ancient geometrical analysis–synthesis pair; see Sidoli (2018, Sects. 2, 4).

[5] For example, for the construction of *Elem.* III.33—to describe a segment of a circle on a given line admitting a given angle—Heiberg (1883, I.251–255) makes reference to only one problem, *Elem.* I.23, and Heath (1908, I.67–70) cites no problems, while Vitrac (1990–2001, I.455–457), Joyce (Online, III Proposition 33), and Fitzpatrick (2008, 101–103) cite all of the problems that are employed, omitting only the postulates—presumably because they are taken to be obvious. (Note that Joyce incorrectly cites *Elem.* I.12 in place of I.11.) Finally, Vitrac (1990–2001, I.514–517) gives the full application of all of the postulates in his tables for *Elements* I–IV.

[6] I have worked through all of the problems for *Elements* I with paper and pencil and all of the problems for *Elements* I–VI using Alain Matthes' tkz-euclide, which allows us to emulate constructions according to the methods stipulated in the text (see Sect. 3, below). If we move through the text in order, we see that all of the *problem-constructions* are fully specified by the text, with the following exceptions: (1) *Elem.* III.16.corol. states that the line drawn at right angles to the diameter of a circle is tangent to the circle. This would allow us to produce a tangent to a circle at a given point with *Elem.* III.1, I.post.1, and I.11. Such a construction is implicitly used in *Elem.* III.34, IV.2, IV.3, IV.7, and IV.12. (2) *Elem.* IV.1 and IV.6 begin by producing the diameter to a given circle. This can be done with *Elem.* III.1, I.post.1 and I.post.2. (3) In *Elem.* IV.16 an equilateral triangle is inscribed in a given circle. In order to do this, we will have to apply *Elem.* I.1 to an arbitrary line before we can apply IV.2. The model for this is *Elem.* IV.10 and IV.11. (4) In *Elem.* VI.13—to find a mean proportional between two given lines—the given lines first have to be set up in the necessary configuration. This can be done with *Elem.* I.post.2 and I.3. Also, in order to draw the semicircle in this problem, a right angle must be produced, *Elem.* I.11, before the semicircle can be drawn

I will argue in this paper that whereas all of the objects introduced in the constructions of problems can be unproblematically produced by applications of the postulates and previously established problems, the ways in which objects can be introduced in theorems and also in the proofs of problems are not subject to this constraint. Hence, Euclid's problems cannot be understood as simply auxiliary to theorems, but constitute a mathematical project in their own right.

### 1.2 Given objects in problems

Problems in the *Elements* involve the concept of *given*, which plays a much more restricted role in Greek mathematical texts than in our mathematical discourse.[7] Problems concern various constructive mathematical processes that can be carried out on objects that are asserted to be given,[8] while this terminology does not enter into the theorems of the *Elements*.[9] From a functional perspective, an object is said to be *given* when it is either ($\mathbf{G_1}$) assumed at the beginning of the argumentative discourse, in the enunciation, or ($\mathbf{G_2}$) assumed in the course of setting out the solution to the problem, at the mathematicians discretion, or ($\mathbf{G_3}$) is determined by an argument starting from ($\mathbf{G_1}$) or ($\mathbf{G_2}$) and employing constructions made up of postulates and problems and chains of inferences using theorems (Sidoli 2018).

Although from a logical perspective there is no difference between ($\mathbf{G_1}$) and ($\mathbf{G_2}$), in any particular problem objects given in the sense of ($\mathbf{G_1}$) are part of the problem itself and are expressed in the enunciation and exposition, whereas those given in the sense of ($\mathbf{G_2}$) are part of the solution and are expressed in the construction. That is, objects given in the sense of ($\mathbf{G_2}$) are locally arbitrary.[10] Objects given in the sense of ($\mathbf{G_3}$) are not explicitly treated in the *Elements*, nevertheless, as we will see below, the propositions of the *Data* can be used to make claims about whether or not other objects, besides those set out in the enunciation, are also given.

From a more essential perspective, I will argue that a given object is one that can possibly enter into or result from a routine, or an effective procedure—which is understood as a constructive process that can be carried out using a definite series of well-defined procedures. That is, a given object is one that can, but need not, actually be constructed. Since, as we will see below, *problem-constructions* are routines composed of postulates or previously established problems, each of which starts with certain

Footnote 6 continued
with *Elem.* III.33. In these four cases as well, however, we can easily see how these constructions can be reduced to the postulates and previously established problems, as I have made clear.

7 For a recent discussion the terminology of *givens* in Greek mathematical sources see Acerbi (2011b). The discussion here is a short summary of my recent study of the ancient concept of *given* (Sidoli 2018).

8 *Elem.* IV.10, which does not assert any object to be given in the enunciation, appears to be an exception, but a line is set out assumed as given in the beginning of the construction and the proposition could be rewritten along these lines. See note 29, below, and Sidoli (2018, n. 9).

9 Greek philosophers such as Proclus, on the other hand, speaking rather more loosely, do refer to the objects that we start with in a theorem as being given (for example, Friedlein 1873, 203–205).

10 In fact, the difference between ($\mathbf{G_1}$) and ($\mathbf{G_1}$) can only be stated locally, as is discussed in detail in (Sidoli 2018).

given objects, this implies that each must also result in certain given objects—which can then serve as a given starting point for further construction operations.

One apparent difference between the way that given objects are dealt with in the *Elements* and in the *Data* is that in the *Elements* objects are referred to as being given without qualification, whereas in the *Data* they are often qualified as being given *in magnitude*, *in position*, and *in form*—which expressions are all defined in the text and treated by theorems grouped into theories. One might be tempted to propose that *given* means something different in the *Elements* than in the *Data*, but there is a good reason to believe that this is not the case. The theorems of the *Data* are used to justify the steps of the part of an analysis for a problem called the resolution, in which the argument proceeds from the objects stated to be given in the enunciation of the problem and shows that the object that completes the problem is also given.[11] The resolution is a straightforward, synthetic deduction that takes as its starting point the givenness of certain objects asserted as given in the enunciation. If the givenness of those objects was somehow different than the givenness of objects treated in the *Data*, the theorems of the *Data* could not be applied in the analysis of a problem. Of course, there are no analyses in the *Elements*, but the presence or lack of an analysis for a problem is a purely rhetorical difference—no problem is logically required to have an analysis, and any problem could be supplied with one.

Hence, the different modes of being given defined and articulated in the *Data* do not denote a different status for given objects than that in the *Elements*, but are simply different modes in which objects can be given. Indeed, it is clear that they were read this way in antiquity, since Proclus in his *Commentary on Euclid's Elements I* first tells us that there are four ways (τρόποι) in which an object can be given—in position, in ratio, in magnitude, and in form, just as in the *Data*[12]—and then shortly after that the student should state for each proposition of the *Elements*, in how many ways (ποσαχῶς) the given is given (Friedlein 1873, 205, 210).

In fact, using the definitions and theorems of the *Data*, it is clear that these qualifications can be used to classify the ways in which objects are given in the *Elements* as well. In general, if an object is said to be given without qualification in *Elements* I–VI, it is simplest to understand it to be fully given—that is, given such that all of its components are given in position. Indeed, because position is the primary characteristic of constructed geometric objects, all of the objects that are said to be given in *Elements* I–VI are at least given in position. For angles and figures this already means that they are fully given—since specifying the position of two intersecting lines implies that their angles are given in magnitude, *Data* Defs. 4 and 1, and specifying the position of all of the points of a figure implies that it is also given in magnitude and in form, *Data* Defs. 1 and 3. If a finite line is said to be given simply, such as in *Elem.* I.1, it is

[11] For the details of this type of argument and the use of *Data* for this purpose, see Sidoli (2018, Sects. 2–4).

[12] Proclus states that given "in ratio" is a mode of being given, whereas *Data* Def. 2 simply defines "given ratio."

given in magnitude and in position and its endpoints are given in position.[13] Indeed, if we do not assume that the endpoints of the initially given line are given, then it would not be possible to show that the constructed triangle is also given, which is necessary for the solution of the problem to be satisfactory.[14] If an unspecified or unbounded line is said to be given, such as in *Elem.* I.11 and I.12, it is only given in position.[15] Some cases are potentially ambiguous, such as *Elem.* I.22. Here we have three lines given without qualification, and we must understand them to be given in both position and in magnitude, with their endpoints given. Of course, it is only their magnitude that is essential to the problem, since they could be moved around arbitrarily, using *Elem.* I.2, before the construction begins; but, nevertheless, the construction provided in *Elem.* I.22 cannot proceed if the three given lines are not eventually given in some position with their endpoints given.

In many of the enunciations, interpreted in this way, we are presented with a claim that a single object—a line, an angle, a circle, and so on—is given in position. Since it is difficult for us to imagine what it would mean for a single object to be given in position, this may strike modern readers as peculiar. Nevertheless, it is made explicit in a number of propositions in the *Data* that in Euclid's practice a single object can be assumed as given in position, *Data* 39–43. Hence, it clear that Euclid's notion of given in position is different from ours. Probably due to our familiarity with analytical geometry, we have a tendency to think that given in position means known in terms of some framework, but this is not how Euclid treats the concept. Indeed, the definition in the *Data* simply reads, "Points, lines and angles are said to be *given in position*, which always keep the same position" (Menge 1896, 2). There is nothing here about determining where these things are, only a claim that they do not undergo any transformation. In the *Elements* and the *Data*, what it means for an object to be given in position is simply that it will not undergo any transformation with respect to any other objects that are also given in position. That is, because there is no framework in Euclid's practice, the first object that is introduced as given in position acts as the framework for everything else that is introduced, and it is because the concept of given is used both in the sense of assumed and also in the sense of determined on the basis of what is assumed that Euclid can

[13] The claim that the endpoints are given in position may sound pedantic, but the *Data* also deals with lines given in position and magnitude whose endpoints are not themselves given (consider the implication of *Data* 27).

[14] See Sidoli and Isahaya (2018, 16–17) for a reconstruction of such an argument. If we only assume that the line is given in magnitude it is only possible to show that the other two lines of the equilateral triangle are given in magnitude, but not that the point which completes the triangle is given.

Furthermore, given in magnitude simply is a more geometrically involved constraint for a segment than given in position and in magnitude. To be given in position and in magnitude involves the two endpoints being given in position, but being given in magnitude simply involves something like the segment of a line falling at a given angle between two parallel lines given in position, or being the radius of a given circle. That is, the description of a segment given in magnitude only involves objects that are themselves not elements of the segment.

[15] Since there is no segment given on this line, the only alternative to the claim that this line is given in position would be the idea that given has a non-technical meaning in the *Elements* that is unrelated to its detailed treatment in the *Data*. But such an ad hoc assumption is doubtful and unnecessary.

designate this initial object as given.[16] Hence, the initially given object is whatever object is assumed, designated or constructed as that object in terms of which all other objects will be described.

Finally, although any actually given object is particular and unique, a procedure described in terms of given objects, or an argument about given objects, is fully general. This is because the initially given objects are simply whatever objects we assume, or set out, at the beginning of the discourse. For example, *Elem.* I.1 shows how to construct an equilateral triangle on a given line, where that line is simply any line designated, or set out, as given. Hence, the proposition sets out a procedure for constructing an equilateral triangle on the segment joining any two points that have been assigned. That is, Euclid's concept of given, just as ours, refers both to any particular given object and also to any object that may be taken as given.[17] In this way, Euclid is able to set out procedures and arguments about given objects that can be read as general claims about constructibility.[18]

## 2 The structure of a problem

Structure plays a role in shaping the deductive force of an ancient Greek proposition. Since at least as far back as Proclus, it has been traditional to divide a Euclidean proposition into six parts (Friedlein 1873, 203–205): *enunciation* (πρότασις), *exposition* (ἔκθεσις), *specification* (διορισμός), *construction* (κατασκευή), *demonstration* (ἀπόδειξις), and *conclusion* (συμπέρασμα).[19] Although Proclus' schema is suitable for many of the simple problems of *Elements* I, however, it fails to capture all of the possible components of a problem in the *Elements*. To see this, we will read through the somewhat unusual problem that begins *Elements* III.

### 2.1 Example: *Elem.* III.1

The first proposition of *Elements* III proposes to find the center of a given circle. What is at stake here is to find (εὑρεῖν) something, which must—given the definition of a circle, *Elem.* I.def.15—already be assumed to exist. In fact, the other problems which propose to find an object, in *Elements* VII, VIII and X, also seek an object which must, given the definitions of the concepts involved, already exist. This makes it clear

[16] It may be helpful to think of an analogy from analytical geometry. When we consider a Cartesian plane we tacitly assume the origin, $(0, 0)$, as the reference point by which all other points will be determined. In Euclid's sense of the term we are assuming this point as given in position—it does not undergo any transformation. That is, the position of everything that comes later in the discourse will be set out against this as the framework.

[17] Again, it may be helpful to consider the analogy with analytical geometry. Euclid's concept of a given point refers both to a particular point, as say $(2, 3)$, and also to any point that may be taken as given, as say $(a, b)$.

[18] In the later Imperial period, as I have argued, Greco-Roman mathematicians utilized related strategies to make general arguments about computational procedures (Sidoli 2018, Sect. 5).

[19] See Netz (1999b) for a discussion of Proclus' division of a proposition. The use of this structure as a deductive framework, focusing on the *Elements*, but also using other texts, is addressed by Acerbi (2011a, 1–117).

that *Elem.* III.1 has nothing to do with a demonstration of existence. In fact, in the definition of a circle given in magnitude and position, *Data* Def.6, we read that "a circle is said to be given in position and in magnitude, of which the center is given in position and the line from the center [is given] in magnitude" (Menge 1896, 2). Hence, in *Elem.* III.1 we are seeking a point which both exists and is given. Indeed, *Elem.* III.1 is a demonstration that whenever a circle appears, no matter how it has been introduced, there is an effective procedure for locating its center.

Of course, it is difficult for us to imagine how a circle could have been constructively introduced without first introducing its center, but, as we will see below in Sect. 4.3, Euclid's practice in his *problem-constructions* is consistent in introducing objects whose production is described by a problem as fully formed, without any auxiliary constructions that may have been used in the original *problem-construction*. Hence, if in the course of a *problem-construction* a circle is introduced as passing through three given points, $\mathbf{C}(abg)_{m,p}$—as, for example, in *Elem.* IV.4 or IV.5, or, in a solid context, as passing through three given points on the surface of a given sphere, and so on—then the circle's center will not have been introduced at the same time. Now, of course, the *problem-construction* of such a circle will first involve finding its center, but since this operates as a subroutine, its center is not immediately available when the circle is called in. Hence, if its center is required, it must be introduced by its own introduction rule.[20]

Finally, since the notion of given also applies to objects that are simply assumed at the beginning of the discourse, Euclid seems to be unconcerned with the ultimate origin of such objects. Since in both the *Elements* and the *Data*, a proposition often begins with the assumption of a given circle, or segment of a circle, whose center must then be found for use in the following *problem-construction*, Euclid requires this proposition in order to set out the construction as a routine. Hence, a possible reason for the existence of this problem is simply that it is required by propositions of both the *Elements* and the *Data*.

The parts of *Elem.* III.1 with which we are concerned, read as follows (Heiberg 1883, I.166):

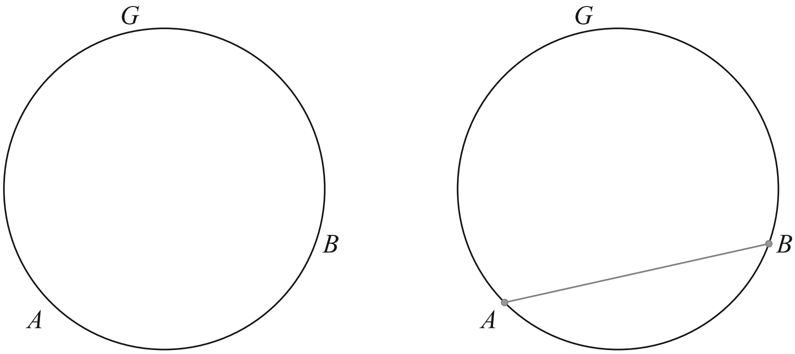
> [*Enunciation*:]
> To find the center of a given circle.
>
> [*Exposition*:]
> [1] Let there be the given circle, $ABG$.
>
> [*Problem-specification*:]
> Then, it is necessary to find the center of circle $ABG$.
>
> [*Problem-construction*:]
> [2] Let some line, $AB$, be produced through it, at random, and [3] let it be bisected at point $D$. And, [4] at $D$ let $DG$ be produced upright to $AB$, and [5] produced through to $E$. And, [6] let $GE$ be bisected at $Z$.

20 See Sect. 4.3 for the full argument for this claim.

[*Proof-specification*:]
I say that $Z$ is the center of $ABG$.

[*Proof-construction*:]
For, otherwise, if possible, [7] let it be $H$. And, [8] let $HA$, $HD$ and $HB$ be joined.

[*demonstration*:]
And, since …



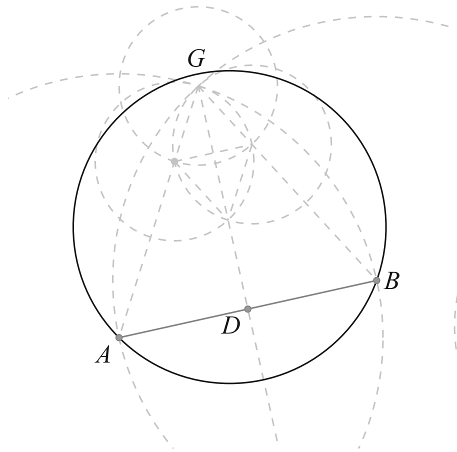

*Elem.* III.1: [1] Given configuration [2] Unpostulated

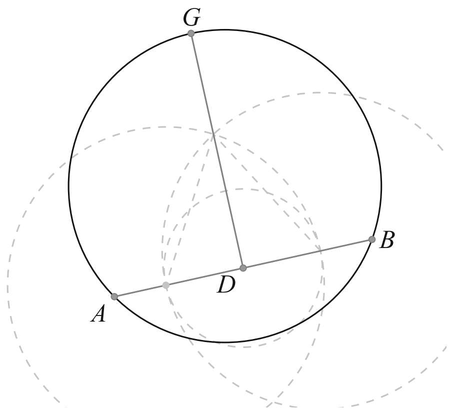
In order to follow the constructive process in detail, we will read through the details of the constructive steps, omitting the demonstration, which is unnecessary for the current discussion.

In the exposition, in [1], whatever circle is given is set out as assumed and is assigned the arbitrary name $ABG$, which is made explicit in the Greek through apposition (ὁ δοθεὶς κύκλος ὁ ΑΒΓ) —say, $\mathbf{C}(ABG)_{m,p}$ (Acerbi 2011b, 125–128). At this stage of the process, $A$, $B$ and $G$ are unspecified as points and simply act as labels that name the circle itself. As we will see, they will soon be assigned to specific points that will be produced in the course of the construction—after which they will serve the double function of naming individual points and the figures to which those points belong.

Following the *problem-specification*, which states what is to be done, the *problem-construction* begins, in [2], by assuming that some arbitrary line—which is thus given in the sense of taken at the mathematician's discretion—has been produced in the given circle—line $ab_{m,p}$. Consideration of the problems in the *Elements*, as well as the proofs in the *Data*, makes it clear that Euclid held that it is possible to set out arbitrary points and lines as given without appeal to any postulate.[21] Now, labels $A$ and $B$ are assigned to the endpoints of this line, assumed as given.

[21] That points can be set out as given may be seen from, for example, *Elem.* I.11, I.23, *Data* 32, 33, 37, 38. The case for lines can be seen in *Elem.* I.22, *Data* 39–43. That the points and lines set out in the *problem-constructions* of the *Elements* must be considered as *given* can be shown from the fact that they then serve as the basis for further constructions that are performed through problems that themselves assume *as given* these very points and lines. Of course, the assumption of a given line can generally be reduced to the assumption of two given points and an application of *Elem.* I.post.1 or I.post.2 (see Sect. 4.1).



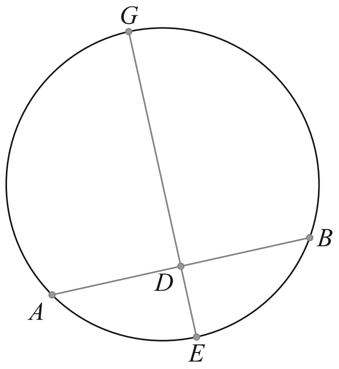

*Elem.* III.1: [3] *Elem.* I.10 [4] *Elem.* I.11

The next stage of the *problem-construction* is a series of operations performed on this assumed line, carried out through postulates and previously established problems. In step [3], $AB$ is bisected, which may be done through *Elem.* I.10—to bisect a given finite straight line. The fact that we can apply this problem confirms that $ab_{m,p}$ was meant to be assigned as given, so that it can enter into the effective procedure of *Elem.* I.10 as the given line on which that problem operates—that is, since *Elem.* I.10 takes as its starting point a given line, and since $AB$ here serves as the starting point for the *problem-construction* set out in *Elem.* I.10, we must understand that $ab_{m,p}$ was understood to be set out as given. In the diagram for step [3], in gray dotted lines, we see all of the subroutines that are used to carry out *Elem.* I.10—that is, *Elem.* I.10 relies on *Elem.* I.1 and *Elem.* I.9, which in turn relies on *Elem.* I.1, taking a random point, I.3, I.1, and I.post.1. Although this is not the optimal procedure for carrying out this construction, this is the routine specified in the *problem-construction* of *Elem.* I.10 and hence is the only routine that we are justified in supposing. None of these subroutines or auxiliary constructions, however, appear in the diagram for *Elem.* III.1.[22] This is significant, because the next object to be produced in the *problem-construction*—the perpendicular to $AB$—is already there in the auxiliary objects. The *problem-construction*, however, does not use this auxiliary object, but rather produces the perpendicular anew, in its own construction step. I will argue below, this serves two purposes. In the first place, it allows the *problem-construction* to proceed as a routine made up of postulates and previously established problems, and it allows the demonstration to forgo proving that the auxiliary objects produced in the *problem-construction* have certain properties—they are simply introduced as the objects having those properties. That is, since *Elem.* I.10 has already demonstrated that there is an

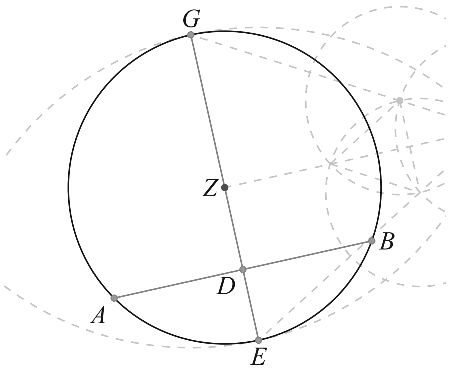
[22] The fact that the auxiliary objects do not appear in the diagram is not simply an accident of the manuscript transmission. The absence of such auxiliary objects is a characteristic of all *problem-constructions* in *Elements* I–VI in all of the primary manuscripts of the Greek, Arabic and Latin transmissions. As I will argue below, this a characteristic of the algorithmic practice of *problem-constructions*.

effective procedure for bisecting a given line, the point of bisection simply appears fully formed in the figure as $d_p$.

The next step, [4], applies *Elem.* I.11—to produce a straight line upright on a given line at a given point on it—from which we see that $d_p$ must be taken as given, and hence, must have been produced as given. Once again, in the diagram for step [4], we see the subroutines of *Elem.* I.11—in this case, taking a random point, *Elem.* I.post.3, I.1, and I.posts.1 and 2. Again, this is not an optimal construction, since, as pointed out above, the line that produced point $D$ as part of the subroutine used to introduce that point is already the perpendicular line introduced here. Once again, however, those auxiliary objects are apparently not available for use, so that the perpendicular line must be introduced separately—presumably using the routine set out out in *Elem.* I.11. As before, the subroutines that go into the production of the perpendicular line do not appear in the figure for *Elem.* III.1. Line $dg_{m,p}$ simply appears in the figure as a result of the effective procedure that produces it.[23]




[5] *Elem.* III.1: *Elem.* I.post.2




[6] *Elem.* I.10

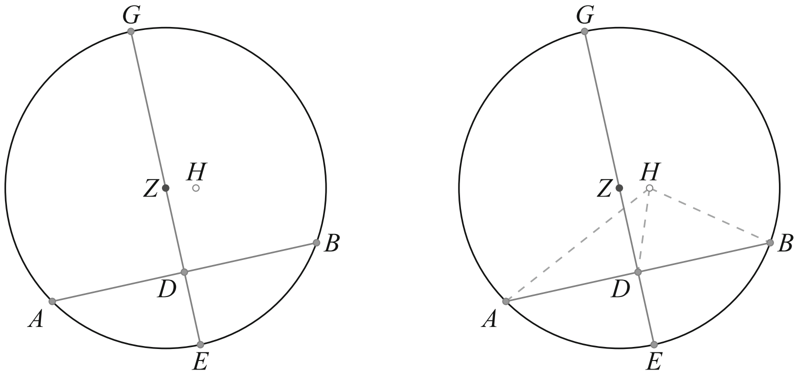
In step [5], line $dg_{m,p}$ is extended to meet the circle at $e_p$. Then $de_{m,p}$ is taken to be given. We can provide two arguments for this claim. The first is purely textual. Since the next step of the *problem-construction* bisects this line, using *Elem.* I.10, which applies to a given line, this line, $DE$, must be taken as given, $de_{m,p}$, in order to apply *Elem.* I.10. The second argument is mathematical, and relies on propositions of the *Data* to show that $de_{m,p}$ is given. Namely, since line $ab_{m,p}$ and point $d_p$ are given, by *Data* 29, line $dg_{m,p}$ is given, and since $\mathbf{C}(ABG)_{m,p}$ is given, by *Data* 25, points $g_p$ and $e_p$ are both given, so that by *Data* 26, $ge_{m,p}$ is given.[24] In fact, however, there is no need to show that this line is given using the theorems of the *Data*, because any object that is constructed is necessarily given.

In step [6], *Elem.* I.10 is applied once again, this time to line $ge_{m,p}$. As before, although the subroutines are depicted in the diagram for [6], in the manuscript sources and in the ancient practice, point $z_p$ simply appears in the figure (Saito 2011, 52).

[23] We will see below $dg_{m,p}$ must be understood to be given.

[24] Line $dg_{m,p}$ is given for the same reasons.

This completes the procedure that produces the center of $\mathbf{C}(ABG)_{m,p}$. As will be shown below, $z_p$ is the circle's center, and if it were not necessary to prove this, the problem would require no further constructions or introductions. Indeed, following the *problem-construction*, the proposition has a second specification, the *proof-specification*, which states what is to be shown and has the same linguistic structure as the specifications of theorems in *Elements* I.[25]




*Elem.* III.1: [7] Hypothetical [8] *Elem.* I.post.1 (3 times)

The proposition then proceeds with a series of assumptions and constructions introducing objects that will be used in the argument to demonstrate that point $z_p$ is, in fact, the center of the circle. This begins, in step [7], with the counterfactual assumption that some other point, say $H$, is the center. This assumption is part of an indirect argument, so it is clearly counterfactual, but it is worth stressing that there is no effective procedure that can produce any other center for the given circle besides $z_p$—as will be shown in the demonstration. In fact, the text makes it clear that some other point is simply *assumed* to be the center of the circle, so that no constructive assumptions are required.[26] This means that the assumption of $H$ as a center is both non-constructive and of a purely hypothetical nature. Hence, the introduction of point $H$ into the argument *as a center* is fundamentally different in nature than the construction steps of the *problem-construction*, which are synthetic routines based on, and reducible to, the postulates.[27] Nevertheless, since the introduction occurs in the section of the argument called the construction, we can refer to it as a construction in this limited sense.

In step [8], lines $aH$, $dH$, and $bH$ are joined, using *Elem.* I.post.1. Once the point $H$ is assumed to be the center, these lines can be produced straightforwardly, so that these constructions need not be regarded as purely hypothetical in the same way as that in step [7]. Nevertheless, these lines are unnecessary for finding the center, and hence their only role is to serve as auxiliaries to the demonstration by introducing new objects, and hence new information drawn from the definitions, into the argument. Hence, we

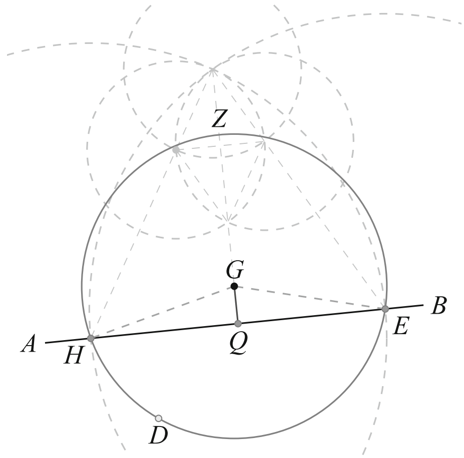
25 See Acerbi (2011a, 57–65) for a discussion of the stylistic format of the two types of specification.

26 The definition of the circle, *Elem.* I.def.15, rules against the possibility of considering points outside the circle in the indirect argument—although the argument will work for any other point in the plane.

27 This counterfactual assumption is discussed again in Sect. 5.2.

will categorize these constructions as *proof-constructions*—being necessary for the argument provided, but playing no role in producing the object to be set out in the problem.

The indirect structure of the argument in *Elem.* III.1 requires that the two different types of construction be separated from each other in the presentation of the argument, since it would make little sense to introduce another hypothetical center, $H$, prior to producing $Z$ and then asserting that it is the center. Nevertheless, it is possible to differentiate these two types of constructions and to categorize them as such, even if they are not clearly indicated by the structure of the argument. In *Elements* I, which served as the basis of Proclus' introduction of the canonical structure of a proposition, the majority of the problems are so simple that no auxiliary *proof-constructions* are required to introduce objects beyond those produced in the course of the *problem-construction* itself.




*Elem.* I.12

An exception is *Elem.* I.12—to produce a straight line as a perpendicular to a given unlimited straight line, from a given point, which is not on the given line. In this proposition, just before the proof-specification, we read "… and let straight lines $GH$, $GQ$, $GE$ have been joined" (Heiberg 1883, I.34).[28] Of these lines, only $GQ$, the perpendicular itself, is part of the *problem-construction*. The other two, produced in the same phrase as $GQ$, are, in fact, auxiliary lines introduced for the sake of the demonstration and serve no role in the production of the perpendicular. In this proposition, because lines $GH$ and $GE$ are not counterfactual, there is no reason why they cannot be produced at the same time as $GQ$. Nevertheless, it is also clear that they could have been produced after the proof-specification, just as the auxiliary lines in *Elem.* III.1. As is often the case in a Greek proposition, the overall structure acts as a sort of guideline that was not always applied strictly.

[28] Other exceptions in *Elements* I–VI are in *Elem.* II.14, III.30, IV.10, IV.12, VI.9, VI.10, VI.13, VI.28 and VI.29.

### 2.2 The complete, potential structure of a problem

With these preliminaries, we can set out a schema for the complete, potential structure of a problem in *Elements* I–VI.

Enunciation: A *general statement* of what is to be *done*, usually in terms of given objects.[29]

Exposition: A *statement* of *what is given*, usually using specific letter-names.

Problem-Specification: A *specific statement* in terms of letter-names of *what is to be done*, often with qualifications.

Problem-Construction: A *construction* of the geometric object that satisfies the requirements of the *problem*, using the postulates, or previously completed problems.

Proof-Specification: A *specific statement* of *what is to be shown*, usually using letter-names.

Proof-Construction: A *constructive assumption* or *introduction*—usually using postulates and previously established problems, but sometimes purely hypothetical—of any new objects, or configurations, that may be necessary to the proof.

Demonstration: An *argument* that the object produced by the *problem-construction* meets the requirements of the proposition; if necessary, using the objects newly introduced in the *proof-construction*.

Conclusion: A *restatement*, in general terms, of what has been done.[30]

As is generally the case with structure in Greek mathematical texts, this is simply a suggestive template that is useful for navigating, and perhaps for originally composing, the argument, but should not be taken as normative.

In *Elements* I–IV, all problems that are not simply sketches—such as some of the problems toward the end of *Elements* IV—have an enunciation and a proof-construction, but all of the other parts are sometimes absent. For example, as just noted, *Elem.* IV.10 is missing the exposition and the problem-specification, *Elem.* IV.16 is missing the demonstration, many problems require no proof-construction, and the conclusion is often abbreviated or omitted altogether.

The majority of the problems in *Elements* I–IV, 70%, have no proof-specification, so it is simplest to list those that do: *Elem.* I.9–12, I.22, II.11, III.1, III.17, IV.15. A smaller set of problems have no initial proof-specification, but have a sort of second proof-specification, introducing a second part of the demonstration with the usual

[29] An exception is *Elem.* IV.10, which does not mention any given objects in the enunciation, as remarked upon by Proclus (Friedlein 1873, 204–205), who incorrectly believed that there are no given objects and that this forced the exposition and specification to be missing. In fact, however, as the argument makes clear, line $AB$ is taken as given, and the proposition could be rewritten such that this given line is mentioned in the enunciation and set out in the exposition. This proposition is probably abbreviated as it is because it simply serves as lemma to the following problem, *Elem.* IV.11.

[30] This is very often abbreviated or absent.

locution for a proof-specification (*Elem.* I.46, IV.6, IV.7, IV.11, IV.12). One problem, *Elem.* IV.15, has both.

Although only *Elem.* III.1—apparently required by its indirect argument—has a proof-construction clearly delineated from the rest of the argument, a number of other problems employ *proof-constructions* set out in the course of the *problem-construction*: *Elem.* I.12, I.14, III.30, IV.10, IV.12, VI.9, VI.10, VI.13, VI.28 and VI.29.

Finally, a number of constructions intermix the demonstration with the *problem-* and *proof-constructions*: *Elem.* III.25, III.33, IV.4, IV.5, IV.8, VI.18. An interesting example of this sort of exceptional structure is the important problem *Elem.* I.44—to apply a parallelogram equal to a given triangle to a given line in a given rectilinear angle—which intermingles constructions with the demonstration and uses *Elem.* I.post.5 constructively.[31]

## 3 The syntax of *problem-constructions*

In this section, using the example already presented and the well-known problems *Elem.* I.1–3, I give a detailed presentation of the way that Euclid organizes *problem-constructions* in the *Elements*. It will be seen that this is not—as is often claimed—most readily understood as series of operations to be carried out with a straightedge and a collapsing compass, or directly using the postulates. Before this, however, I will discuss the five postulates and show that they all directly or indirectly serve a role in the problems of *Elements* I–VI.

### 3.1 Postulates: *Elem.* I.posts.1–5

As is well known, the first three postulates are directly related to *problem-constructions*. In this section, I will argue, moreover, that all of the postulates can be related to *problem-constructions* and can be fruitfully explicated though the concept of *given*.[32] In Euclid's texts, the wording of the introductory material is often vague and it is sometimes only possible to form a clear idea of what his rather philosophical starting points mean by seeing how they are applied. This is certainly the case with the postulates.

*Elem. I.post.1:* The text of the first postulate reads, "Let it be demanded to produce a straight line from any point to any point" (Heiberg 1883, I.8).[33] Since the wording of

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[31] The details of this proposition will be taken up in Sect. 3.1, below.

[32] My approach differs from the recent reading of Schneider (2015, 20), who interprets the postulates as extensions of the definitions and as making assertions about the fundamental nature of the objects involved. My thinking about the Euclidean postulates had been much clarified by the work of modern logicians, such as Mäenpää and von Plato (1990) and Beeson (2010). Whereas these scholars have used the *Elements* as a motivation for producing a well-founded logic of construction, I have been interested in using ideas from modern logical studies to explicate the received text of the *Elements*.

[33] The word that I have translated with "to produce" (ἀγαγεῖν) literally means "to lead," which is how Vitrac (1990–2001, I.167) translates it (*mener*). I use "to produce," following Fitzpatrick (2008, 7), because

this text is subject to various interpretations, we will look to its use in the problems to see what this postulate actually means. Here we find that the postulate is used to join a line between two given points. For example, in *Elem.* I.1, it is used to join the endpoints of a given line, which must be given, with the intersection of two given circles, which must also be given, by *Data* Def.6 and *Data* 25. In *Elem.* I.2, it is used to join a given point with the endpoint of a given line, which must also be given. Indeed, in every case in which the postulate is used to join a line in *Elements* I–VI, we can make an argument based on the *Data* that the two points that the line joins are given.

Hence, we can assert that *Elem.* I.post.1 defines a function on two given points that results in a line, having the two given points as endpoints, and, hence, being given in position and in magnitude, *Data* 26. That is, where $a_p$ and $b_p$ are two points given in position, we have the function

$$\frac{a_p \qquad b_p}{ab_{m,p}}$$

which produces a new type of object that we will call line $ab_{m,p}$.[34]

It is often remarked that Euclid does not explicitly assert the uniqueness of $ab_{m,p}$, but this functional interpretation of *Elem.* I.post.1, based as it is on objects given in position, which appears to have been Euclid's approach to the concept of uniqueness, makes such an explicit assertion unnecessary (Vitrac 1990–2001, I.167–168).[35]

It has long been held that *Elem.* I.post.1 is an abstraction of working with a straightedge. It is important to recognize, however, just how abstract the operation has become. The postulate allows us to join any two given points, no matter where they may occur, but no straightedge can perform such an operation. Of course, one could posit that we are dealing here with a straightedge that is indefinitely long—but, of course, such a straightedge, as an object of human manufacture, does not exist. Hence, the analogy with a straightedge is rudimentary. The postulate specifies a purely mathematical function, which, although it may have its conceptual basis in manipulating a straightedge, now serves as an introduction rule for a mathematical object.

*Elem. I.post.2:* The text continues with the second postulate, "And to extend a finite straight line continually in a straight line" (Heiberg 1883, I.8). Once again, to see what this means in detail, we will look at some of its applications in the early problems of *Elements* I. In *Elem.* I.2, the postulate is used to extend a line that results from

Footnote 33 continued
"to produce" has an overlapping meaning with "to lead" as a kind of abstraction. In any case, it does not mean "to draw," as it is often translated.

The word translated by "any" (πᾶς) literally means "every," but it is often used in Greek to imply a certain generality which in English we more naturally convey with "any" (Heath 1908, I.195). The same applies to *Elem.* I.post.3, below.

[34] I formulate such functions using Martin-Löf's intuitionistic type theory, following Mäenpää and von Plato (1990).

[35] For discussions of the relationship between the notion of given in position and uniqueness, see Taisbak (2003, 95) and Acerbi (2011b, 146–148).

*Elem.* I.1, and is hence given, in a straight line such that the extended length is longer than a line given in magnitude. In *Elem.* I.31, it is used to extend a line that results from the production of a triangle, through *Elem.* I.23 and I.22, and is hence given, some indeterminate length.[36] It is used in a similar way, to extend a given line some indeterminate length in *Elem.* IV.3. In *Elem.* I.44, II.11, II.14, III.1, III.25.1, IV.15, VI.11, and VI.28 it is used to extend a line whose endpoints are given by a length that is determined by the geometry of the figure, and hence given in magnitude.[37] Whenever this postulate is applied in the context of a problem, it involves extending a line whose endpoints are given by another line of arbitrary length or whose length is given by the geometry of the figure—and hence given in magnitude in the sense in which this concept is treated in the *Data*.

In this way, *Elem.* I.post.2 specifies a function that takes a line whose endpoints are given and returns a line that is extended an arbitrary length in one direction. That is, where $ab_{p,m}$ is a line whose endpoints are given in position and $\ell_m$ is a line given in magnitude in that sense that it is taken at a length suitable to the situation at hand, we have the function

$$\frac{ab_{m,p} \qquad \ell_m}{abC_p}$$

which produces an object of the same type that we will call $abC_p$.[38]

The claim that $\ell_m$ is given must be understood in the ancient sense—that is, that we are able to construct an equal length. In fact, however, we may not know the length of $\ell_m$ before the construction has been carried out, because on its own there is a certain indeterminacy involved in the procedure of applying *Elem.* I.post.2 (Lassalle Casanave and Panza 2015, 150–152). For example, when the postulate is applied in *Elem.* I.2, it must be applied such that $\ell_m$ is equal to or greater than some line that is already present in the configuration of given and constructed objects. In this case, we can determine $\ell_m$ straightforwardly before the procedure is carried out. On the other hand, in *Elem.* II.14, $\ell_m$ will be determined by the construction itself, and hence we cannot be certain that we know its length before we apply the postulate. Nevertheless, an arbitrary length that we can be certain will result in the intersection we need can be found in the given objects—such as the radius of the circle. In both cases, however, $\ell_m$ is determined locally by some aspect of the assumed configuration of geometric objects. Even in a case in which is was not clear that a length equal to or greater than the length to be produced is extant in the figure, we can still apply this postulate as an additive iteration with some length found in the given configuration of geometric objects. Hence, so long as we understand the initial line given in magnitude, $\ell_m$, to simply be an arbitrary length

[36] Following Heiberg (1883, I.77), this use of *Elem.* I.post.2 is generally not noted in modern translations (Heath 1908, 316; Vitrac 1990–2001, I.254; Fitzpatrick 2008, 34). Notice, however, that its use in *Elem.* I.31 is included in the table provided by Vitrac (1990–2001, I.514) and in the text of Joyce (Online, Book I, Proposition 31).

[37] It is perhaps also implicitly used in setting up *Elem.* VI.13. See note 6, above.

[38] We can use *Data* 26 to show that where $\ell_m$ is an arbitrary given length, then $c_p$ will also be given, but the postulate is not used in this way. It is simply used to make lines as long as we like, while points along this extended line are determined by further constructions.

that we are able to provide an equal to, then we can understand *Elem.* I.post.2 as a function that produces a unique object for each application.

Finally, an important application, or perhaps development of *Elem.* I.post.2, may be in *Elem.* I.post.5. Here, as will be discussed below, the later postulate appears to require the former. In this case, in terms of the discussion above, *Elem.* I.post.2 would have to be applied iteratively until an intersection is obtained. As we will see, this is not the only other postulate that is necessary for formulating *Elem.* I.post.5.

For modern treatments of Euclidean constructions, this postulate is unnecessary, because we clearly distinguish between lines, rays and segments—such that two points determine a line, a segment and two rays, the latter three of which are subsets of the first. Hence, if we regard *Elem.* I.post.1 as allowing us to introduce the line through two points, we can perform all of the constructions in the *Elements* by defining the intersection of such a line with other objects.[39] Indeed, even in the *Elements* the postulate sometimes plays a purely aesthetic, or illustrative, role, such as its use in *Elem.* I.31, noted above. In this problem, since the parallel line has already been constructed before *Elem.* I.post.2 is employed and since the line it produces plays no role in the demonstration, it simply serves to make the diagram more recognizable.

Once again, *Elem.* I.post.2 must be recognized to be a purely mathematical function, since no actual straightedge can perform all of the operations for which it is required.

*Elem. I.post.3:* The text continues, "And to describe a circle with any center and distance" (Heiberg 1883, I.8). In the early uses of this postulate, such as *Elem.* I.1, I.2, and I.3, it is used to produce a circle which passes through one given endpoint of a line such that the other given endpoint is the circle's center. The use of the postulate in *Elem.* I.12, as seen above, however, makes it clear that this line, which appears as the circle's radius in the earlier problems, is purely accidental.[40] All that matters is that the two points be given. Indeed, the use of the term "distance" or "interval" (διάστημα), in place of line, makes it clear that we are talking about a span, or interval that need not be occupied (Sidoli 2004). Both in the postulate itself, and also when it is used, circles are produced with this potentially empty interval which is determined because the two points defining it are given in position.

Thus, *Elem.* I.post.3 specifies a function that takes two points given in position and returns a circle given in position and in magnitude.[41] That is, where $a_p$ and $b_p$ are the two points given in position, we have the function

[39] If we read through the constructions provided by Martin (1998, Chapter 1)—and especially if we do the exercises to translate these into the notation that he develops for constructions—we see that all of the constructions can be carried out using lines and intersections. For example, if we compare Martin's account of the constructions for *Elem.* I.2, II.11 and II.14 with those in the *Elements*, we see that the applications of *Elem.* I.post.2 are avoided by simply considering the intersections of previously introduced objects (Martin 1998, 8, 12–13).

[40] This is the reason why my formulation of the function below differs from that of Mäenpää and von Plato (1990, 285), since they view the function as operating on a point and a line.

[41] It would also be possible to argue that *Elem.* I.post.3 defines a function that takes a given point and a given line having the given point as one endpoint, and then claiming that *Elem.* I.12 simply neglects to join this line. But this requires us to read material into *Elem.* I.12 and is not the most straightforward explanation of the text.

$$\frac{a_p \quad b_p}{\mathbf{C}(a_p, b_p)_{m,p}}$$

which produces a new type of object that we call circle $\mathbf{C}(a_p, b_p)_{m,p}$, in which $a_p$ is the center and $b_p$ a point on the circumference of the circle.

Again, it has long been held that this postulate is an abstraction of working with a compass. But the function specified by Euclid's postulate is at once more powerful and more restricted that the operation that can be performed with a compass. As with *Elem.* I.post.1 and I.post.2, the two points can be separated by any arbitrary distance, but every actual compass has a greatest span. Furthermore, *Elem.* I.post.3 can only produce a circle about a given point, passing through another point already given in the configuration—it cannot make a circle with a preassigned distance, for which we require *Elem.* I.2. Hence, we read of collapsing compasses, or compasses that close "the moment they cease to touch the paper" (De Morgan in Heath 1908, 246; Greenberg 2008, 47). But, once again, there is no such thing as a collapsing compass, and every actual compass can be set at a given radius and can be used to carry a length.[42] Indeed, Euclid's postulate is a purely mathematical function, organized so as to operate on the simplest starting points of a geometric construction—points assumed as given in position. Although perhaps originally based on the manipulations of a compass, the postulate now serves as the primitive introduction rule for a circle.

*Elem. I.post.4:* The postulate reads, "And all right angles are equal to one another" (Heiberg 1883, I.8). In this case, we are not dealing with an introduction rule, but with a claim about the nature of right angles. This postulate is generally read as concerning the magnitude of right angles, and hence the measuration of angles, but it is possible to give the postulate a constructive reading based on the concept of given (Acerbi 2011b, 123). In particular, by the definition of a right angle, *Elem.* I.def.10, once we have an effective procedure for producing a right angle, *Elem.* I.11 and I.12, the two angles so produced are both given in magnitude, by *Data* Def.1. That is, the two right angles at any particular point are both given.

This construction, however, does not guarantee that pairs of right angles constructed at different points of the same line are also equal (Schneider 2015, 17). In fact, as pointed out by Proclus, in the context of ancient mathematical methods, right angles at different points can straightforwardly be shown to be equal by superposition (Friedlein 1873, 188–819; Mueller 1981, 22). This proof, as all proofs by superposition, however, requires that the angles be subjected to some kind of transformation. As we see in *Data* 25–30, however, the fundamental property of points given in position is that they cannot be subjected to a transformation.[43] Hence, the argument by superposition cannot apply to right angles at two given points—that is, at points separated by a given distance, *Data* 26. For this reason we need this postulate to assert that any time a right angle is constructed, it is equal to any other right angle that has been constructed—which is the

[42] Drafters use, or used to use, a compass with two pins, known as a divider or a drafting compass, to carry length more accurately than a compass with a drawing end, but the design is essentially the same—in fact, many modern designs accommodate both functions (Martin 1998, 6).

[43] For my interpretation of Euclid's proofs by superposition, see Sect. 5.3, below.

full meaning of the assertion that all right angles are given in magnitude. Understood in this way, the postulate is a constructive assertion that all straight lines are everywhere straight in the same way, because the notion of given in magnitude is fundamentally constructive.[44]

This articulation of *Elem.* I.post.4 as treating the situation in which two right angles are separated by a given distance appears to be directly related to the situation required by *Elem.* I.post.5, which requires I.post.4.[45] Since *Elem.* I.post.5 is straightforwardly constructive, its reliance on *Elem.* I.post.4 further emphasizes my claim that all of the postulates are constructively grounded in Euclid's concept of given.

*Elem. I.post.5:* The most famous of Euclid's postulates reads, "And if a straight line falling on two straight lines makes the interior angles on the same side less than two right [angles], the two straight lines being extended indefinitely, they will meet on the side on which there are [angles] less than two right [angles]" (Heiberg 1883, I.8). This postulate is clearly constructive in the sense in which is it articulated.[46] It acts as an introduction rule for a point as the intersection of two given lines. It is well known that Euclid assumes the intersection of two circles and a circle with a line without a special postulate (Heath 1908, 234–240), but one could argue that there are fairly clear introduction rules for these intersections, because some points will be inside, and other points outside, any given circle—as specified in the definition, *Elem.* I.def.15. Hence, Euclid could, famously, be certain without discussion that the two circles of *Elem.* I.1 would intersect.[47]

The constructive determination of whether or not two lines will intersect, however, is not so straightforward, and nothing in the definition of a line, *Elem.* I.def.2, directly sheds light on this question. This postulate, however, gives us a constructive way to determine whether or not two given lines will have an intersection.

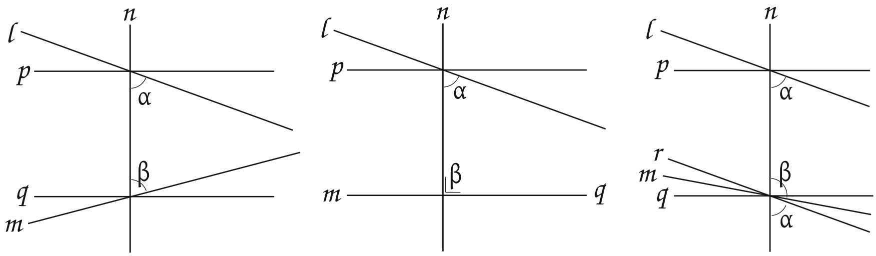
We proceed as follows. With two lines given in position, $\ell_p$ and $m_p$, let an arbitrary line, $n_p$, fall on them, making on one side angles $\alpha_m$ and $\beta_m$ given in magnitude. Then, using *Elem.* I.11, set up perpendiculars at the intersections, $p_p$ and $q_p$, which will themselves be given in position, *Data* 29.

[44] The constructive aspect of the notion of given in magnitude is explicit in the articulation of *Data* Def. 1. See also the discussion in Sidoli (2018, Sect. 3).

[45] In its extant articulation, the demonstration for *Elem.* I.46—to construct a square on a given line—requires *Elem.* I.post.4, because one angle is constructed equal to a right angle, say $\mathbf{rAngle}_1$, and then that angle with an equal angle is shown to be equal to two right angles, say $2\mathbf{rAngle}_2$. But if we do not have *Elem.* I.post.4 to assure us that $\mathbf{rAngle}_1 = \mathbf{rAngle}_2$, the proof will not follow. This argument depending on *Elem.* I.post.4, however, can be changed to one depending on *Elem.* I.def.10 alone by producing the external right angle and then arguing by *Elem.* I.29, so that it is not enough, on its own to require the formulation of *Elem.* I.post.4. (The uses of *Elem.* I.post.4 in *Elem.* I.13–15 can be reduced to the definition because the right angles in question are at a single point.)

[46] This is especially clear when we compare it to many of the alternative postulates that make direct existence claims (Heath 1908, 220).

[47] If the center of a circle falls on the circumference of another, it will clearly have some points inside and some points outside the other circle. Hence, the two circles will intersect.




Then there are three cases for intersection. If (1) $\alpha_m < 90°$ and $\beta_m < 90°$ (or $\alpha_m > 90°$ and $\beta_m > 90°$), (2) $\alpha_m < 90°$ and $\beta_m = 90°$ (or $\alpha_m > 90°$ and $\beta_m = 90°$), or (3) $\beta - 90° < 90° - \alpha$ (or $\beta - 90° > 90° - \alpha$),[48] then the two lines will meet in the direction of angles $\alpha$ and $\beta$ (or the opposite), say at $s_p$, which will be given in position, *Data* 25. In this way, we can state constructive criteria for the introduction of a given point, $a_p$, so long as we can be certain that the right angles about lines $n$ and $p$ are equal to those about $n$ and $q$—as is guaranteed by *Elem.* I.post.4.

That is, *Elem.* I.post.5 can be taken as specifying a function under certain conditions, such that where $\ell_p$, $m_p$ are lines given in position, and any other line taken as given in position, say $n_p$, falls on them such that angle $\alpha$ is formed between $\ell_p$ and $n_p$ and angle $\beta$ is formed between $m_p$ and $n_p$ in the same direction, such that $\alpha + \beta <$ 2 right angles, then, using *Elem.* I.post.2 iteratively, $\ell_p$ and $m_p$ can be extended without limit, producing an introduction rule for a point as the function

$$\frac{\ell_p \quad m_p}{s_p}$$

where $s_p$ is the point common to both $\ell_p$ and $m_p$, which is given in position, by *Data* 25.

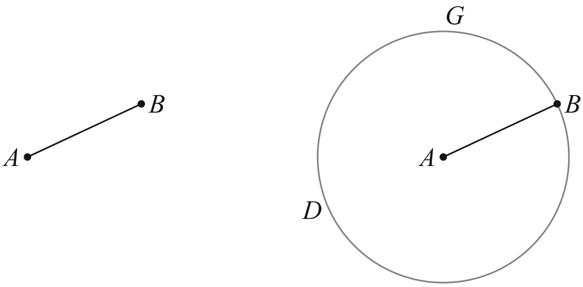
The first application of *Elem.* I.post.5 is in a theorem, *Elem.* I.29, where it is invoked as part of a counterfactual, and hence purely hypothetical, construction or constructive assumption—which will be discussed below. Its next application, however, is in the course of the *problem-construction* of a problem, *Elem.* I.44, where it is used as an introduction rule to produce a point as the intersection of two lines given in position. Furthermore, we are told by Proclus that this problem, laying the foundation of the theory of the application of areas, was considered to be one of the great achievements of the geometers of the classical period, particularly those known as the Pythagoreans (Friedlein 1873, 419–420). We cannot now know whether or not the pre-Euclidean approach to the theorem utilized *Elem.* I.post.5, but it is clear that in the context of the effective procedures developed in Euclid's problems, such an articulation of the postulate is necessary. In the course of the *problem-construction* of *Elem.* I.44, a line is produced from one of the intersections of a line that falls on a pair of parallel lines,

[48] This case can be seen constructively by producing $r_p \parallel \ell_p$, which will be given in position by *Data* 28.

making an angle less than that of the parallel line. The third of the constructive criteria set out above can be used to confirm that the line so produced must meet the other parallel line.[49] In this way, we can see that *Elem.* I.post.5 plays a clearly articulated role in Euclid's development of the theory of the application of areas—one of the most powerful tools of ancient geometric problem-solving.

### 3.2 Examples: *Elem.* I.1–3

In order to explicate Euclid's procedure for implementing postulates and previously established problems in the *problem-constructions* of *Elements* I–VI, we will follow through with a close reading of the first three propositions of the text, which are well known and simple enough to make his practice perfectly clear. In each problem, certain objects are stated to be given in the exposition, and then in the *problem-construction*, certain operations are performed, each of which utilizes either a postulate or a previously completed problem.




*Elem.* I.1: [1] Given objects [2] *Elem.* I.post.3

In *Elem.* I.1, [1] a line is assumed as fully given—that is, the line is given in magnitude and position, with both endpoints given, $ab_{m,p}$.

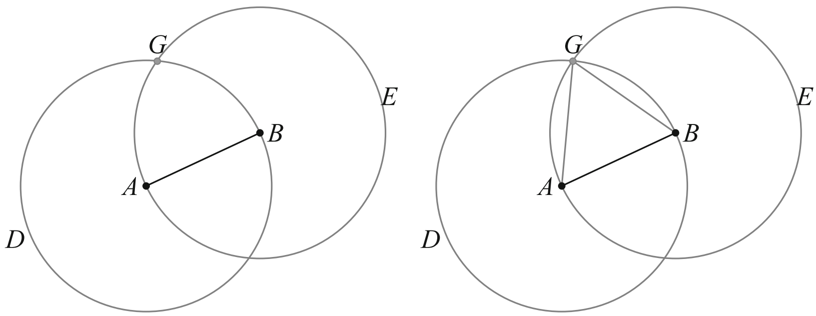
The *problem-construction*, then, reads, "[2] With center $A$ and distance $AB$ let a circle, $BGD$, be described" (Heiberg 1883, I.10–11). This is an application of *Elem.* I.post.3, which makes it explicit that $AB$ is being treated not as a line, but as a fixed span between two points, which must be given. At this point, $G$ and $D$ do not name specific points, but are simply parts of the name of the circle itself.

The text continues, "And [3] again with center $B$ and distance $BA$, let a circle, $AGE$, be described," which is a second application of *Elem.* I.post.3, differing only in the center and the order in which the points of the distance are named (Heiberg 1883, I.11). As noted in the discussion of *Elem.* I.post.5, above, the fact that the definition of the circle, *Elem.* I.def.15, specifies that a circle has an inside can be used to confirm that the two circles must intersect at a point not on the segment joining their centers—

49 For the sake of comparing the figure given here with that in the Greek text, we would have $\Theta\mathrm{Z} := n$, $\mathrm{KZ} := \ell$, $\Theta\Lambda := m$ and $\Theta\mathrm{B} := r$ (Heiberg 1883, I.102–103).

because neither will be entirely inside the other.[50] In the case of *Elem.* I.1, since the centers of the two circles fall on the each other's circumference, they will both have some points inside and some points outside the other. Hence, they must meet at some point not on the line joining their centers. Since $G$ is found in the letter-names of both circles, it now specifies one of the intersections of the circles and names this point.

Finally, the *problem-construction* concludes with, "And [4] from point $G$, at which the circles cut one another, let lines $GA$ and $GB$ be joined" (Heiberg 1883, I.11), which is the assertion of two applications of *Elem.* I.post.1 at the same time. The verb used to invoke the postulate is different than that used to state the postulate, which exemplifies a general principle—although the verb of invocation is often the same as that used to state a postulate or a problem, this is not essential. It is the mathematical operation itself which is most important, not the verb used to articulate it.




*Elem.* I.1: [3] *Elem.* I.post.3 [4] *Elem.* I.post.1 (2 times)

With these constructions the *problem-construction* is complete, and in this case no further objects are necessary for the proof, which follows from an immediate application of the definition of the new objects produced. In this case, all of the constructions are performed with postulates, so that we see every auxiliary object in the final diagram. In fact, this is the only problem in *Elements* I–VI that is of this kind. All other problems rely on at least one previously established problem,[51] and as seen already in *Elem.* III.1 and as we will see again below, the auxiliary operations used in previously established problems do not appear in the final diagram of the problem currently being completed, so that if any of them are required, they must be introduced through their own introduction rules.

Hence, we can think of the *problem-construction* of *Elem.* I.1 as specifying a function that operates on a given line and produces a given equilateral triangle. That is, where $ab_{m,p}$ is the given line, we have the function

[50] That is, the segment joining the centers will meet one circle or both circles as a radius, as per *Elem.* I.def.15—which determines whether it is entirely inside or also has some points outside the circle.

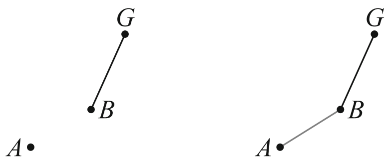
[51] A near exception is *Elem.* IV.15.

$$\frac{ab_{m,p}}{\mathbf{eqT}(abc)_{m,f,p}},$$

which returns a new type of object, an equilateral triangle, $\mathbf{eqT}(abc)_{m,f,p}$, which is fully given because $c_p$ can be shown to be given by *Data* Def.6 and 25.

It is sometimes remarked that two triangles can be produced by this method, one on either side of the given line. For the Euclid's purposes, however, this is irrelevant, since he is only interested in showing that there is an effective procedure for producing at least one such triangle. The choice of which triangle to use can be made in the context in which the problem is invoked—in most cases it will not matter much, for example, in *Elem.* I.2, but in some cases we may want the equilateral triangle to face a certain direction, for example, in *Elem.* I.9, since otherwise if the vertex of the triangle is close to the vertex of the given angle, it will be practically difficult to perform the final construction.[52] If we were writing this algorithm for a machine, we could write it such that if we are given $ab_{m,p}$, then $c_p$ will lie on one side of the line, while if we are given $ba_{m,p}$, it will lie on the other side. Euclid, however, is not writing his effective procedures for a machine, but for a human geometer, who can choose whichever direction suits the problem at hand.[53]

*Elem.* I.2, which uses *Elem.* I.1, proceeds as follows. It begins, in the enunciation, with [1] a given line and a given point—that is, the point is given in position, $a_p$, and the line is given in magnitude and in position with its endpoints given, $bg_{m,p}$.

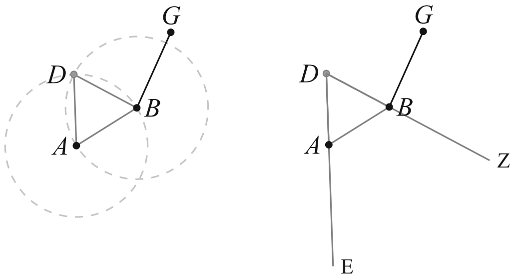



*Elem.* I.2: [1] Given objects [2] *Elem.* I.post.1

The *problem-construction* begins, "For, [2] let a line, $AB$, have been joined from point $A$ to point $B$" (Heiberg 1883, I.12). This is a straightforward application of *Elem.* I.post.1, joining given point $a_p$ with given point $b_p$, using the same verb as was used in *Elem.* I.1.

[52] Indeed, if they should happen to coincide, the algorithm for *Elem.* I.9 will fail, and the simplest way to avoid this is to construct the triangle facing away from the given angle.

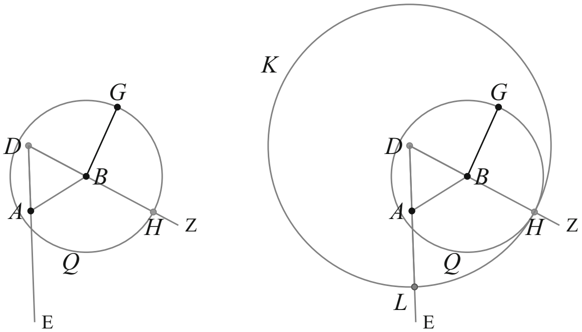
[53] The issue of the uniqueness of the triangle produced in *Elem.* I.1, and generally of the objects produced by a problem, has been discussed by Manders (2008, 100–103). Although my reading of the text has been influenced by his approach, I believe the ancient geometers would have addressed uniqueness through the concept of given (Taisbak 2003, 95; Acerbi 2011b, 146–148), where what it means for a point to be given is fundamentally that its position is unique.




*Elem.* I.2: [3] *Elem.* I.1 [4] *Elem.* I.post.2 (twice)

The text proceeds with, "And [3] upon it, let an equilateral triangle, $DAB$, be put together" (Heiberg 1883, I.12). This is an application of *Elem.* I.1, using the same verb as used to enunciate that problem. In the figure, the auxiliary constructions used to produce the triangle have been drawn in gray dotted lines, to remind ourselves that this triangle is actually produced by a series of operations of the postulates. In fact, however, these auxiliary constructions do not appear in Euclid's diagram. That is, *Elem.* I.1 is used directly to call in the triangle fully formed with no indication in this context of how that is done. At this stage in the constructive process there is no problem in assuming that we actually draw that triangle from scratch using the postulates, but as I will argue below, as we progress farther into the problems, it becomes less and less tenable that Euclid would have believed a human geometer would actually follow all of the constructive steps in exactly the way that he sets them out.

The text continues, "And [4] let lines $AE$ and $BZ$ have been extended from lines $DA$ and $DB$" (Heiberg 1883, I.12–13), which is a double application of *Elem.* I.post.2, using the same verb as was used to state the postulate. The length to which we should extend these lines is unspecified at this point (Lassalle Casanave and Panza 2015, 149–152). There are two ways that we can think of applying the postulate to this circumstance, both of which depend on the local configuration of this proposition. The first is that we use $BG_m$ or a line arbitrarily longer than this as the length by which to extend $DA$ and $DB$, which we know will work because of the argument that follows. The second is to extend $DA$ and $DB$ by some other arbitrary given line in the figure, such as $AB_m$, and then produce the circles with *Elem.* I.post.3 and seeing if intersections are produced. If not, we can iteratively extend $DA$ and $DB$ until intersections are produced. All that matters is that we can extend these lines farther than any assumed length until they meet the circles—which, because the original segments are inside the circles, they must do.




*Elem.* I.2: [5] *Elem.* I.post.3 [6] *Elem.* I.post.3

Finally, we have "And [5] with center $B$ and distance $BG$ let a circle, $GHQ$, be described. And [6] again with center $D$ and distance $DH$ let a circle, $HKL$, be described" (Heiberg 1883, I.14). This a double application of *Elem.* I.post.3—probably asserted separately because the grammatical construction is slightly more involved. Again, the way that the postulate is invoked, with the explicit statement of the interval, makes it clear that it is the span between the two points that is given, not the radius—although in this case they are the same.

This completes the *problem-construction*, and in this proposition, again, no special *proof-construction* is needed. In this case, all but one of the constructions are performed with postulates. Hence, in the final figure, we see all the lines and circles produced by postulates, as well as the equilateral triangle, which was produced with *Elem.* I.1, without appeal to the postulates. We do not see the auxiliary objects of *Elem.* I.1, because these are hidden within the direct invocation of *Elem.* I.1, which produces the triangle fully formed.

The *problem-construction* of *Elem.* I.2 establishes a function that operates on a given point and a line whose endpoints are given, and returns a certain line equal to the given line, one of whose endpoints is the given point. That is, where $a_p$ is the given point and $bg_{m,p}$ is the given line, we have the function

$$\frac{a_p \quad bg_{m,p}}{al_{m,p} = bg_{m,p}}$$

where $al_{m,p}$ is a line given in magnitude and in position, one of whose endpoints is the originally given point.

As is well known, this problem can be used to produce a circle about a given center with a given length, which is the primary use of the proposition in the text. Hence, *Elem.* I.2 establishes, as a corollary, the function

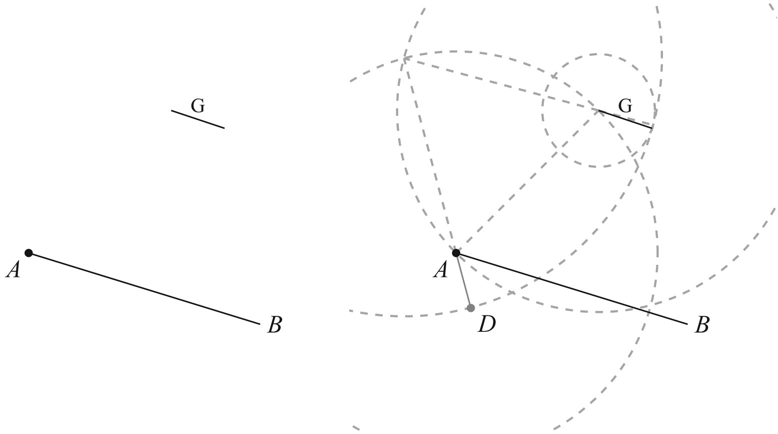
$$\frac{a_p \quad BG_m}{\mathbf{C}(a_p, BG_m)_{m,p}},$$

where $\mathbf{C}(a_p, BG_m)_{m,p}$ is a circle about a given point, $a_p$, with a given radius, $BG_m$.[54] Of course, for the construction to actually be carried out, both of the endpoints of $BG_m$

[54] Beeson (2010, 8) gives a similar account of *Elem.* I.2.

must also be given in position, $bg_{m,p}$, but this is a secondary consideration to its being given in magnitude, which satisfies the main goal of this problem.

It is sometimes argued that this problem shows that the use of a normal compass is compatible with the postulates, since a normal compass produces a circle around a given point with a given radius (Greenberg 2008, 47; Meskens and Tytgat 2017, 28). Stated as an issue of compatibility, this claim is, of course, correct. In order to see if Euclid treats this problem as the abstraction of the use of a normal compass, however, we must turn to how he applies it in *Elem.* I.3.



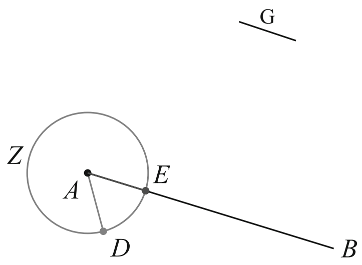

*Elem.* I.3: [1] Given objects [2] *Elem.* I.2

The exposition of *Elem.* I.3 sets out [1] a given line and a greater line given in position with one endpoint given—that is, $g_m$ and $aB_p$. Notice that no points on $g_m$ are given, and on $aB_p$ only point $a$ is given.

The *problem-construction* of *Elem.* I.3 is quite brief. It begins, "[2] Let $AD$ have been set out equal to line $G$ from point $A$" (Heiberg 1883, I.14). This is an application of *Elem.* I.2. In order to actually perform this construction, according to the only method specified in this text, we would have to apply all of the auxiliary constructions of *Elem.* I.2, including *Elem.* I.1 and its auxiliary constructions. All of these constructions are noted in the diagram, [2], in gray dotted lines. In fact, however, none of these auxiliary objects appear in the manuscript sources. As always, the object produced by *Elem.* I.2 simply appears, apparently fully formed, in the diagram, and the auxiliary constructions are not immediately available.[55]

[55] Indeed, in the manuscript diagrams, line $G$ generally appears as a vertical line and $AD$ is drawn as an equal, randomly skew line whose inclination does not depend on an operation of *Elem.* I.2, so that $AD$ must have been produced by purely graphical techniques that are unrelated to the specifications of the *problem-constructions* (Saito 2006, 99). This is the first indication of a general pattern. The diagrams produced in the manuscripts—and, indeed, the diagrams produced by any normal human geometer, diverge more and more from the algorithms laid out in the problems as we progress through the text.

There are two issues here. The first is that the manuscript diagrams appear to have been produced by individuals, or at least by graphic techniques, that were not much concerned with the underlying mathematics to be depicted. This is an accident of transmission. The second issue is more fundamental. Human geometers construct diagrams with drafting tools, or with the postulates themselves, so that they will rarely follow the algorithms set out in the text, and almost never for the later problems, as we will see below.




*Elem.* I.3: [3] *Elem.* I.post.3

The text continues, "And [3] with center $A$ and distance $AD$ let a circle, $DEZ$, have been described" (Heiberg 1883, I.14). This is an application of *Elem.* I.post.3, and completes the *problem-construction*. As usual, the final diagram contains only objects produced by the postulates and previous problems—no auxiliary objects are present.

This problem can be taken as providing an effective procedure that results in an operation on two unequal given lines and results in a line given in magnitude. That is, where $\mathfrak{g}_m$ and $aB_p$ are the two lines, we have

$$\frac{aB_p \quad \mathfrak{g}_m}{ae_{m,p} = \mathfrak{g}_m},$$

where $e_p$ lies between $a_p$ and $B$. Again, for the procedure specified in the proposition to actually be carried out, $\mathfrak{g}_m$ will also have to be given in position, but this is not essential for the problem, since $\mathfrak{g}_m$ can be moved to any position before the procedure begins.

As we have seen, this problem uses *Elem.* I.2 to transform the position of a line, $bg_m$, but it does so through the direct use of the proposition, which is rather different from the way that we carry a length, or span, with a compass. That is, when *Elem.* I.2 is invoked, the line that it produces, $ad_{m,p}$, appears in the figure, and we then work with the endpoints of this line to produce a circle. The actual operation of a compass, however, would not require the production of this line. Hence, it is clear that Euclid's *problem-constructions* work directly with the postulates as stated and do no try to emulate the possible actions of a compass and straightedge.

## 4 Constructions in problems

Using the propositions that we have looked at so far, *Elem.* I.1–3, I.12, and III.1, we are now in position to give a general description of the way that constructions function in the problems of the *Elements* I–VI. The basic tenet of this description is that we accept that each construction is carried out according to the algorithms specified by the text. Hence, each *problem-construction* is carried out by a series of steps, each of which can be performed by either a postulate or a previously established problem. In the event that a step is carried out by a previously established problem, we look to the *problem-construction* of the problem in question in order to see how it is carried out—which, for the later problems produces an iterative process. In this way, every *problem-construction* is reducible to postulates, although, with the exception of *Elem.* I.1, any individual *problem-construction* does not itself exhibit this reduction. That is, with the exception of *Elem.* I.1, no *problem-construction* in the *Elements* shows, on its own, how to produce the objects it constructs with the postulates.

### 4.1 Unpostulated constructions

As we saw in the discussion of *Elem.* III.1, and I.12, as well as introducing lines and circles through *Elem.* I.post.1–3, there are also some unpostulated constructions—such as setting out given points or lines. Since what we call segments may be regarded as introduced by *Elem.* I.post.1, these can be considered reducible to the introduction of given points. In *Elem.* I.22 a line with one endpoint given—what we would call a ray—is set out in the *problem-construction*. Since only one point is given, it is not clear that this is reducible to the introduction of given points and an application of a postulate. Points may be introduced as given either somewhere in the plane, as in *Elem.* I.12,[56] or on a given object, as in *Elem.* I.9, I.11 and I.31. A given line can be introduced by specifying two given points and then joining them using *Elem.* I.post.1, as in *Elem.* III.1. As we will see below, a wider range of unpostulated introductions, or constructive hypotheses, are allowed in *proof-constructions*.

### 4.2 Problems as demonstrations of an effective procedure

There have been a number of articulations of Euclid's approach to constructions in problems, which, although conceptually viable, and often mathematically equivalent to what Euclid does, are not the same as the algorithms stated by the text. In order to get a clear sense of the *problem-constructions* as Euclid presents them, it may be useful to articulate what they are not.

Euclid's *problem-constructions* are not practical instructions for the use of a compass and straightedge—despite the fact that this is the most common way of explaining Euclid's approach, and has been developed in mathematical treatments of Euclid's geometry (Mueller 1981, 15–16; Hartshorne 1997, 18–22; Catton and Montelle 2012, 29; Meskens and Tytgat 2017, 27–36). Aside from the fact that every real instrument is bounded and real compasses carry length, Euclid's *problem-constructions* are built up from the postulates and from fully formed objects introduced directly from previously established, so that they always involve full circles, produced lines, and complete objects—they do not show how to produce intersecting arcs and points, as is usually done when working with real instruments.[57]

[56] In fact, in *Elem.* I.12 the given point that is introduced is on a certain side of a given line, so that there must be some constructive procedure for deciding this. Perhaps we could join the two given points and check whether or not there is an intersection with the given line.

[57] There are a few exceptions to the use of full circles, such as the use of a semicircle in *Elem.* VI.13, *Data* 43 and 90. These can be regarded as the production of a segment containing a right angle, *Elem.* I.11, III.33. I exclude cases where the text describes the production of a circle, but the manuscript figures show only an arc—these exhibit graphical choices that are related to the production of material objects and tell us little about the mathematical intention.

The *problem-constructions* in the *Elements* make no pretense to presenting optimal solutions to any but the earliest problems. This is because *problem-constructions* function by calling in fully formed objects whose constructions have been previously established—they are not constructed again from the postulates. Hence, obviously simpler constructions are overlooked in favor of constructions that call on previously established problems (Heath 1908, I.268, 270; Hartshorne 1997, 22; Joyce Online, Book 1, Proposition 10). One of the goals of the *problem-constructions* appears to have been producing at the same time and with the same constructions those objects which will complete the problem and those which will be used in the proof—that is, to reduce, so far as possible, the need to introduce further objects with *proof-constructions*. The result of this is that as the problems become more involved, the algorithms that they present become increasingly removed from an algorithm that would be devised by starting again from the postulates.[58] This, of course, is also relevant to my claim that *problem-constructions* are not meant to be instructions for the use of a compass and straightedge—as we will see in the example of *Elem.* II.11, below, the algorithms detailed by the *Elements* are almost prohibitively laborious to carry out with real instruments.

Euclid's *problem-constructions* do not provide us with instructions for producing a series of points as the intersections of lines or circles, created by either physical instruments or the postulates. This a common way to think of Euclid's constructions from a practical perspective (Catton and Montelle 2012, 35–36), and has been used as an approach to formalizing Euclidean constructions (Moler and Suppes, 1968; Seeland, 1978; Martin 1998, 1–28; Beeson 2010, 23–33), but it is not Euclid's approach. Euclid uses his postulates and previously established problems to bring in objects fully formed, and the points of intersection, which are the focus of most modern treatments, usually appear as secondary, almost accidental, elements in Euclid's *problem-constructions*. Of course, mathematically, there is no significant difference between these two approaches—nevertheless, they are conceptually quite distinct.

Finally, *problem-constructions* are not instructions yielding "the minimal graphic requirements" of the objects to be produced (Catton and Montelle 2012, 36). In order to follow the *problem-construction* of a later problem, we must treat it as an algorithm in which each previously established problem that is called on is a subroutine, provided with its own algorithm in its own *problem-construction*. The only constructive procedure licensed by the text, as it stands, is that produced by reading recursively back through all of the previously established problems. The constructions that we generally supply, as human geometers, in following the text are made up of construction shortcuts that we learned in school; they are not the constructions stipulated by the *Elements*.

With these remarks as preliminary, we may turn to a positive description of Euclid's *problem-constructions*: A *problem-construction* in *Elements* I–VI is an effective procedure that begins with some given objects and consists of a well-ordered, finite series of steps, each of which is either a postulate or a previously established problem. The

[58] For example, if we compare the algorithms developed in the *Elements* with those by Martin (1998, Chapter 1). In particular, compare the constructions for *Elem.* I.2 and II.11 with those in the *Elements* (Martin 1998, 8, 13).

result of each step of a *problem-construction* is the production of a given object—a point, a segment, a circle, or some more involved plane figure. This process may also involve some deductive inferences securing the possibility of producing certain objects—for example, in *Elem.* I.44, with the use of *Elem.* I.post.5, or often in *Elements* IV, in establishing the possibility of drawing certain circles.

Although, as noted above, some of the *problem-constructions* also include *proof-construction* steps and introduce objects not strictly necessary for completing the problem, taken together the constructions stipulated in both the *problem-construction* and the *proof-construction* result in the objects that we see in the diagram.[59] In fact, the diagram is simply a visualization of the construction—a symbolic representation of the objects that results from performing the operations stipulated in the construction. Deductive inferences that we tend to think of as originating in the diagram have their source not in the diagram itself, but in the construction that it points toward—usually through the definitions of the newly introduced objects, but occasionally through various aspects of their configuration.[60]

The objects that are introduced through *problem-constructions* can all be produced in a finite number of steps, each of which is ultimately reducible to the postulates, but we will see below that objects introduced into the discourse through *proof-constructions* are not subject to this constraint. Hence, the problems of *Elements* I–VI can be taken as proofs that there is an effective procedure for producing the objects at issue, given some assumed objects, arbitrary points, and perhaps lines, and the functions stipulated by the postulates—that is, they show that if the objects asserted in the enunciation are given, then the objects produced by the procedure can also be provided as given.

### 4.3 Routines and subroutines

We can regard a *problem-construction* as a routine made up of postulates and subroutines, each of which itself has the structure of a routine. For example, *Elem.* I.3, as read above, has the following structure:

> *Elem.* I.3:
> **Step 1**: *Elem.* I.2 (I.post.1, *Elem.* I.1 (I.post.3, I.post.3, I.post.1, I.post.1), I.post.2, I.post.2, I.post.3, I.post.3),
> **Step 2**: *Elem.* I.post.3.

In this case, when *Elem.* I.2 is called in **Step 1**, only the final object that it produces appears in the figure, and the constructive process that lead to its production, shown here in parentheses, is treated as a black box. In stating a routine, parentheses are used to describe a subroutine to make it clear that the auxiliary objects that are used in the subroutine are not introduced by the routine itself. Since all of *problem-constructions*

[59] Catton and Montelle (2012, 35–37) claim that this is not the case, but this is because the "actions" that they use to produce the construction do not agree with those given in the text.

[60] This emphasis on the construction as providing starting points for the deduction should be contrasted with accounts that place these starting points in the diagram itself, such as that provided by Netz (1999a, Chapter 1).

in *Elements* I–VI can be stated as routines, it is clear that a *problem-construction* acts as a series of operations that could in principle be mechanized. Of course, in order to make the procedure mechanical we would have to include various if-then rules in order to make certain decisions along the way. Euclid does not need to worry about this, because his instructions are written for human geometers who can make their own decisions and only need to be convinced that the operation in question is possible.

A characteristic of Euclid's practice in setting out *problem-constructions* is that a subroutine acts as a black box in the sense that the internal constructive operations that lead to its output are not directly recoverable. There is no logical or mathematical necessity that this should be so, since one could simply construct everything from scratch from the postulates, as is usually done in modern treatments of Euclidean constructions. Nevertheless, this results from the way that Euclid uses previously established problems as introduction rules to bring in new objects fully formed. An example that makes this practice clear can be taken from the use of *Elem.* I.10 in *Elem.* III.1—see step [3] in Sect. 2.1, above. In the process of bisecting line $AB$ at $D$, using *Elem.* I.10, the perpendicular to $D$ is produced as an auxiliary construction, as can be seen among the gray dotted lines in the figure for *Elem.* III.1 [3]. In the next step, [4], however, *Elem.* I.11 is used to produce a perpendicular anew, as though no perpendicular had been used in determining point $D$. The same situation—arising from a similar application of *Elem.* I.10 followed by *Elem.* I.11—is found in *Elem.* I.12, III.25, III.30, III.33 and IV.5. We see a similar situation in *Elem.* IV.12—to circumscribe a regular pentagon about a given circle. This *problem-construction* proceeds by taking the vertices of an inscribed regular pentagon, as established in *Elem.* IV.11, drawing the tangents to the circle though these points, *Elem.* III.16.corol., and then, among other things, finding the center of the circle. But, the unstated subroutine that produces these tangents involves finding the center of the circle, *Elem.* III.1, joining this with the vertices, *Elem.* I.post.1 (five times), and then erecting perpendiculars, *Elem.* I.11 (five times). Hence, although the center of the circle is already found in one of the subroutines, because it is required in the routine itself, it must be introduced separately. A particularly striking case is found *Data* 43. In this theorem, a given line is fitted into a circle, using *Elem.* IV.1, and then the circle that is used as an auxiliary object in *Elem.* IV.1 is introduced separately, using *Elem.* I.post.3.[61] In fact, there is no *problem-construction* in *Elements* I–VI or the *Data* in which an object that would have been involved in a subroutine is not later introduced by its own *problem-construction* step, if it is required. Hence, although this black-boxing of the subroutines is not mathematically required, it is consistently applied in Euclid's *problem-constructions* and can be used to explain the difference between Euclid's approach and that of modern accounts of Euclidean constructions, and to account for the diagrams found in the text.

The fact that the auxiliary objects used in a subroutine need to be reintroduced if they are to be used later in the routine, is related to our concept of the scope, or visibility, of an object used in a subroutine. In this sense, objects introduced in a subroutine have local scope and are not available after the subroutine is complete. That is, a problem acts as an introduction rule providing only the final object. An auxiliary object that

[61] Taisbak (2003, 128) refers to this as a "repeated construction." See also Sidoli (2018, Sect. 3.3).

may have been used in stating the construction rule is not also introduced when the rule is applied, unless it is an element of the object introduced by the rule. Hence, if such an auxiliary object is also needed in the routine, it must also be introduced by its own introduction rule. That is, a *problem-construction* uses previously established problems to introduce objects fully formed, having the properties demonstrated in the proofs of those problems—which has the advantage that we do not need to prove these properties, as would be necessary if we constructed the objects from the postulates themselves.

This discussion makes it clear how remote Euclid's procedures are from the actions of any human geometer—any person actually producing one of these diagrams would see that the objects produced in the subroutines are already there in the figure and do not need to be introduced again. In this way, although the *problem-constructions* of *Elements* I–VI emulate the *possible* actions of a human geometer, they quickly become rather far removed from any series of actions that a human geometer would actually perform.

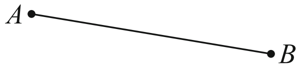
In the problem itself, taken as a whole, the demonstration serves a cognitive function beyond simply verifying that the produced object satisfies the requirements of the problem. It also identifies which of those objects produced in the *problem-construction* will be called in when the problem is later invoked. It does this by showing what object produced by the *problem-construction* satisfies the definition of the objects required by the problem in the enunciation. That is, it is the demonstration that determines which object meets the requirements of the enunciation and is returned when the problem is later invoked. For example, in *Elem.* I.2, which we can outline as follows,

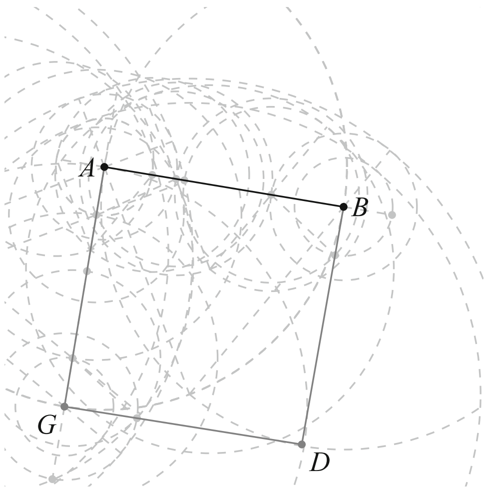
*Elem.* I.2:
**Step 1:** *Elem.* I.post.1,
**Step 2:** *Elem.* I.1 (I.post.3, I.post.3, I.post.1, I.post.1),
**Step 3:** *Elem.* I.post.2,
**Step 4:** *Elem.* I.post.2,
**Step 5:** *Elem.* I.post.3,
**Step 6:** *Elem.* I.post.3,

the previous problem, *Elem.* I.1, is invoked in **Step 2** as a subroutine, in such a way that of the four steps of the subroutine, only the final two, along with the line assumed as given, are involved in the production of the equilateral triangle that will appear when the proposition is invoked. We can regard the demonstration as the cognitive process that identifies the sought object among the various objects produced. The situation is more pronounced in a case like *Elem.* I.3, above. When we look at the constructive steps involved in the production of *Elem.* I.2, in parentheses, it is not at all clear, merely from reading the list of operations, which of the produced objects completes the problem and should appear in the figure. Only by following the demonstration is this made clear. Of course, we could also mechanize this process by writing an algorithm to identify the sought object and then drawing just these objects—Euclid, however, is not writing these procedures for machines, but for human geometers.

### 4.4 Example: *Elem.* II.11

We can summarize these various observations by going through the problem-construction of *Elem.* II.11—to cut a given line such that the rectangle contained by the whole and one of the sections is equal to the square on the other section.[62] This will serve as a concrete example that will make the syntax of Euclid's *problem-constructions* clear and illustrate the various points that have been raised above.




*Elem.* II.11: [1] Given objects [2] *Elem.* I.46

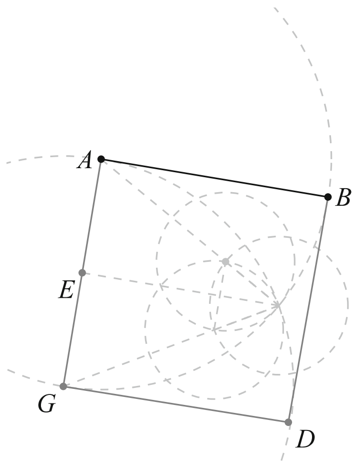
The enunciation of *Elem.* II.11, [1] sets out a given line—that is, given in position such that both of its endpoints are given, $ab_{m,p}$. The problem construction, then, begins with "For, let a square, $ABDG$, be erected on $AB$" (Heiberg 1883, I.108), which uses the same verb as was employed in *Elem.* I.46. This is expressed as a simple construction, but in order to see how a square would actually have been constructed according to the text we have to read *Elem.* I.46 and all of its subroutines. This gives,

*Elem.* I.46:
**Step 1:** *Elem.* I.11 (*Elem.* I.1 (I.post.3, I.post.3, I.post.1, I.post.1), *Elem.* I.9 (*Elem.* I.3 (*Elem.* I.2 (I.post.1, *Elem.* I.1 (I.post.3, I.post.3, I.post.1, I.post.1), I.post.2, I.post.2, I.post.3, I.post.3), I.post.3), I.post.1, *Elem.* I.1 (I.post.3, I.post.3, I.post.1, I.post.1), I.post.1)),
**Step 2:** *Elem.* I.3 (*Elem.* I.2 (I.post.1, I.1 (I.post.3, I.post.3, I.post.1, I.post.1), I.post.2, I.post.2, I.post.3, I.post.3), *Elem.* I.post.3),
**Step 3:** *Elem.* I.31 (I.post.1, *Elem.* I.23 (I.post.1, *Elem.* I.22 (*Elem.* I.3 (*Elem.* I.2 (I.post.1, *Elem.* I.1 (I.post.3, I.post.3, I.post.1, I.post.1), I.post.2, I.post.2, I.post.3, I.post.3), *Elem.* I.3 (*Elem.* I.2 (I.post.1, *Elem.* I.1 (I.post.3, I.post.3, I.post.1,

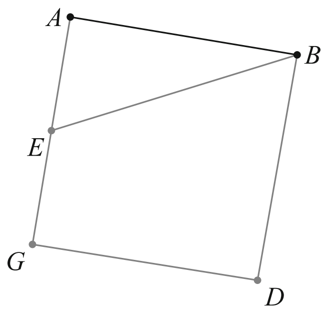
[62] As is often the case in *problem-constructions*, Heiberg (1883, I.153) did not attempt to provide a justification for every step, mentioning only *Elem.* I.46, once. In this he is followed by Heath (1908, 402). Vitrac (1990–2001, 353, 515) mentions no construction steps in his translation, although they are all listed in his table. Joyce (Online, Proposition II.11) and Fitzpatrick (2008, 63) give a fuller, although still incomplete, set of justifications in the text itself.

I.post.1), I.post.2, I.post.2, I.post.3, I.post.3), I.post.3, I.post.3, I.post.1, I.post1)), I.post.2),
**Step 4:** *Elem.* I.31 (I.post.1, *Elem.* I.23 (I.post.1, *Elem.* I.22 (*Elem.* I.3 (*Elem.* I.2 (I.post.1, *Elem.* I.1 (I.post.3, I.post.3, I.post.1, I.post.1), I.post.2, I.post.2, I.post.3, I.post.3), *Elem.* I.3 (*Elem.* I.2 (I.post.1, *Elem.* I.1 (I.post.3, I.post.3, I.post.1, I.post.1), I.post.2, I.post.2, I.post.3, I.post.3), I.post.3, I.post.3, I.post.1, I.post1)), I.post.2),

which appears in the diagram as the gray dotted lines of a subroutine, itself involving four subroutines, for a total of 80 uses of the postulates.[63] It goes without saying that no human geometer would ever actually construct a square in this way and that this is not a straightforward way to produce a square with a compass and straightedge.[64] The expression for invoking problem *Elem.* I.46, in fact, simply calls in a square as fully given, $\mathbf{S}(abgd)_{m,f,p}$. Of course, the square will appear on one side or the other of line $ab_{m,p}$, but the choice of which side is at the discretion of the geometer, since the subroutine itself can produce a square on either side.



*Elem.* II.11: [3] *Elem.* I.10



[4] *Elem.* I.post.1

The problem construction continues, "And [3] let $AG$ be bisected at point $E$" (Heiberg 1883, I.108)—using the same verbal phrase as was employed in *Elem.* I.10 to invoke that proposition. The subroutine that is called in with this expression is, in fact, the following:

*Elem.* I.10:
**Step 1:** Elem. I.1 (I.post.3, I.post.3, I.post.1, I.post.1)

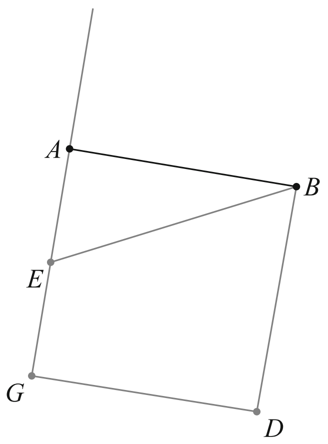
63 It should be noted that some of the steps of the subroutine, for example, in applications of *Elem.* I.3, will be degenerate, so that we will not actually produce all 80 circles and lines, but a somewhat smaller subset. (See the discussion of *Elem.* II.11 [6], below.)

64 Having worked through *Elements* I following the full constructions as stipulated in the text using a straightedge and compass, I found that proceeding through the full sequence of operations for *Elem.* I.46 has a tendency to produce a misshapen figure.

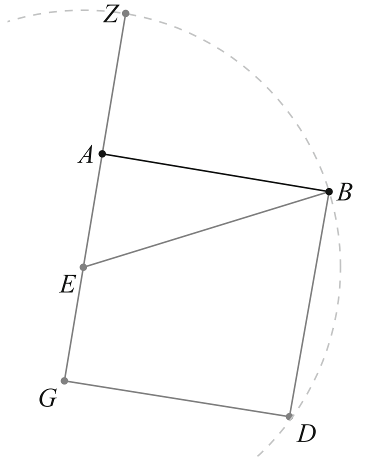
> **Step 2:** Elem. I.9 (Elem. I.3 (*Elem.* I.2 (I.post.1, *Elem.* I.1 (I.post.3, I.post.3, I.post.1, I.post.1), I.post.2, I.post.2, I.post.3, I.post.3), I.post.1, Elem. I.1 (I.post.3, I.post.3, I.post.1, I.post.1), I.post.1)

which appears as gray dotted lines in the diagram. Following these constructions back to the postulates would involve 19 operations.[65] As before, the only object from this subroutine that appears in the figure for *Elem.* II.11 is the point of bisection itself, $e_p$.

The *problem-construction* continues, "And [4] let $BE$ be joined" (Heiberg 1883, I.108), which is simply an application of *Elem.* I.post.1, as appears in the diagram.




*Elem.* II.11: [5] *Elem.* I.post.2




[6] *Elem.* I.3 [degenerate]

We then have the following two operations, "And [5] let $GA$ be extended to $Z$. [6] And let $BE$ be laid out equal to $EZ$" (Heiberg 1883, I.108). The first of these, [5], should be read as an application of *Elem.* I.post.2, in which we extend line $ga_{m,p}$ some arbitrary distance, so long as it is greater than $BE - AE$. At this point in the construction, $Z$ is not yet specified and simply names the extension of this line.

In the next step, [6], with the application of *Elem.* I.3, point $z_p$ is determined as a point on the extension of $AG$, which is given, by *Data* 25. This is an example, as often happens in *Elements* I–VI, of a degenerate application of this problem. In general, *Elem.* I.3 can be used to cut off a shorter line from a longer line, wherever the two lines lie in the plane. Here, as often however, the two lines share an endpoint, so that instead of all of the applications of the postulates that we see in the routine for *Elem.* I.3 above, we require only *Elem.* I.post.3.[66] Of course, there is no discussion in the text of degenerate and non-degenerate cases—probably because Euclid did not think of himself as explaining how to actually carry out these constructions, but only as demonstrating that an effective procedure exists, for which purpose, we need treat only the more involved case.

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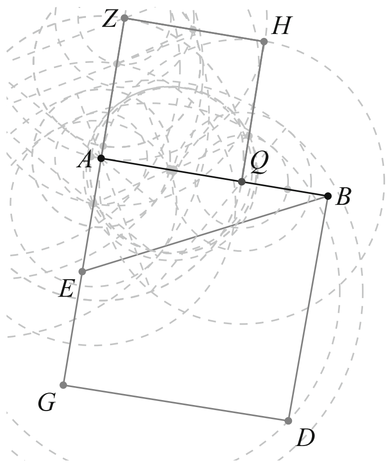
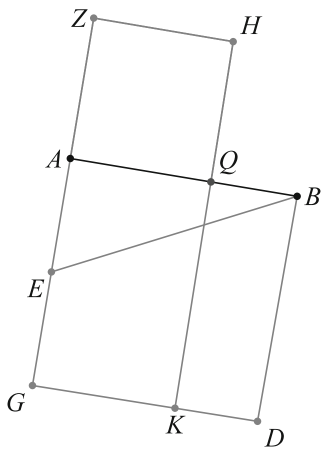
65 Again, in this case a number of the operations of Elem. I.3 are degenerate, and, hence, all of the applications of the postulates in the full algorithm will not appear as objects in the diagram. (See the discussion of *Elem.* II.11 [6], below.)

66 See notes 63 and 65, above.




*Elem.* II.11: [7] *Elem.* I.46 [8] *Elem.* I.post.2

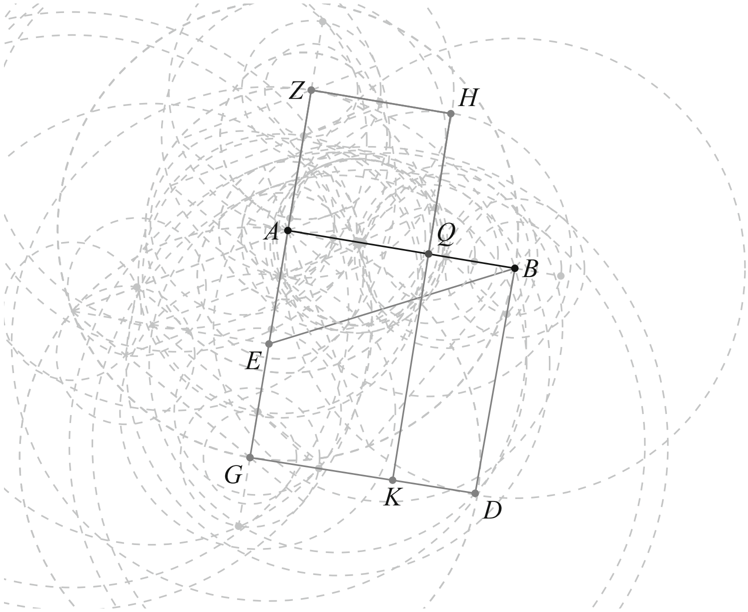
The final passages of the *problem-construction* read, "And [7] let a square, $ZQ$, be erected on $AZ$. And [8] let $HQ$ be produced through to $K$" (Heiberg 1883, I.108). This is another application of *Elem.* I.46, as discussed above, followed by an application of *Elem.* I.post.2. As usual, only the final square and line produced by these operations appear in the figure.

The production of line $QK$ brings up an interesting issue. The question arises whether or not this line serves a necessary function in the *problem-construction*, whether it is a *proof-construction*, or whether it serves a purely illustrative role. In fact, this depends to some extent on how we read the propositions of *Elements* II with regard to whether or not they are primarily geometrical or primarily demonstrate abstract relations that were meant be applied in other areas of mathematics as well. Certainly, if we are only interested in finding point $Q$, which divides $AB$ such that $\mathbf{R}(AB, BQ) = \mathbf{S}(AQ)$, there is no need for line $QK$. Furthermore, it is certainly possible to set up a more operational proof using the relations demonstrated in *Elements* II, as abstract relations—in the way that this material was handled by Apollonius and Archimedes and occasionally by Euclid—to show that the $\mathbf{R}(AB, BQ) = \mathbf{S}(AQ)$, without recourse to line $QK$. Indeed, as is well known, Heron rewrote much of *Elements* II is this more abstracted vein—making the treatment of the book stylistically compatible with its usage in the works Apollonius and Archimedes.[67] Hence, it would appear that line $QK$ plays an essentially illustrative role, like the lines constructed in the theorems *Elem.* II.1–8.[68] That is, this line is used to visually illustrate that point $Q$ divides $AB$ such that the square on one part, $\mathbf{S}(AQ)$, is equal to the rectangle formed

[67] Heron's reworking of *Elements* II is preserved only in al-Nayrīzī's commentary, in both Arabic and Latin (Besthorn and Heiberg 1897–1905, II.1.4–79; Tummers 1994, 73–89). For recent work in the long literature on Heron's rewriting of *Elements* II, see Corry (2013, 133–139) and Sialaros and Christianidis (2016, 652–654). Acerbi and Vitrac (2014, 31–39) give an overview of Heron's commentary and discuss Heron's more abstract proofs on p. 36. For a discussion of Euclid's use of this abstract approach, see note 69, below.

[68] For example, Acerbi (2007, 272) mentions that the constructions in *Elem.* II.1–8 serve such a purely illustrative function.

by the whole and the other part, $\mathbf{R}(BD, BQ)$. Just as Euclid's goals in *Elements* I–VI, in general, and *Elements* II, in particular, have to do with producing theorems about geometric objects, his problems are also situated in this geometric context.[69] This final line in the *problem-construction* of *Elem.* II.11 is a good illustration of the following general claim: for a Euclidean problem to be effected, it must, at the least, produce all of the objects explicitly stated in the enunciation—while other objects may also be necessary to either produce these, or for the proof.




All the steps of *Elem.* II.11

Having gone through the details of the *problem-construction* for *Elem.* II.11, it suffices to say that at each stage an object is called in directly through either a postulate, or the invocation of a previously established problem, in such a way that the construction produces only those objects that we see in the diagram. No object that serves in an auxiliary role in any of the subroutines appears in the final diagram. As with the previous examples, it would be possible to write out a full routine for this proposition, but since we have gone over the general principle for doing this, and as no new information would be presented, I will spare the reader this tedium. On the other hand, it is rather striking to see, in the final figure for *Elem.* II.11, what it would look like if the constructions were actually carried out according to the routine specified by the text. I will simply mention that the full routine would contain 183 applications

69 It should be noted that this was probably a stylistic choice on Euclid's part, not a conceptual limitation. As the arguments in *Elem.* III.35, III.36, IV.10, and *Data* 86 make clear, Euclid himself was capable of applying theorems of *Elements* II in the abstract, operational way that we find in the work of Archimedes and Apollonius, known as the application of areas, or geometrical algebra (Zeuthen 1917, 313–316(115–118); Taisbak 1996, 2003, 211–224), and there is no reason why he could not have written *Elements* II in the same abstracted vein as we find in Heron's commentary.

of the postulates.[70] I think it is likely that in all the centuries since Euclid composed his *Elements* no human geometer has ever carried out the full construction for this problem as specified by the text.

## 5 Constructions for theorems and demonstrations

In this section, I argue that the *proof-constructions* found in theorems and in the demonstrations of problems employ a broader range of introduction procedures or assumptions than we find in *problem-constructions*. Here, I present some examples of these practices in three categories that are neither mutually exclusive nor exhaustive.

In this section, the terminology of construction is used in a sense that is looser than usual. Here *construction* is employed in the sense of whatever is done in the construction section of a proposition as a means of introducing new objects into the discussion whose existence was not asserted in the enunciation. That is, in the sense that both *problem-constructions* and *proof-constructions* involve introduction procedures, while only *problem-constructions* are effective procedures, the introduction procedures in *proof-constructions* are sometimes purely hypothetical. As this section will establish, objects are introduced in *problem-constructions* through effective procedures, whereas they can be introduced in *proof-constructions* through purely hypothetical assumptions, whether or not any effective procedure for their production can be demonstrated. Nevertheless, following the terminology of the division of a proposition, I will call both of these introduction procedures *constructions*.

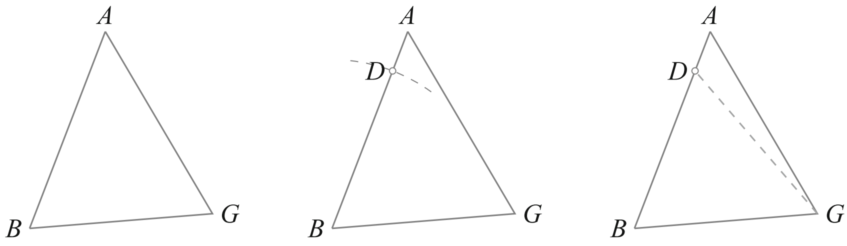
### 5.1 Counterfactual constructions

Probably the most straightforward type of *proof-construction* are those that seem to employ only postulates and previously established theorems, but produce a diagram that is then shown to be impossible—namely, for use in an indirect argument. Such *proof-constructions* may be called *counterfactual*.

As an example, we may look at the *proof-construction* for *Elem.* I.6, which is a theorem. The passages with which we are concerned, read as follows (Heiberg 1883, I.22):

> [*Proof-construction*:]
> [1] For, if $AB$ is unequal to $AG$, one or the other of them is greater. Let $AB$ be greater, and [2] let $DB$ equal to the lesser, $AG$, be removed (ἀφῃρήσθω) from the greater, $AB$. And, [3] let $AG$ be joined.

70 Of course, many of the applications of *Elem.* I.3 will be degenerate, as discussed above, but since this is not discussed in the text, and the only way to know which are degenerate and which are not is to go through the construction and count, I simply count the operations of the algorithm as presented by the text. The diagram shows only the lines and circles that would actually be produced—which is significantly less than 183, but still far more than a human geometer would require.




*Elem.* I.6: [1] Assumption [2] *Elem.* I.3 [*] [3] *Elem.* I.post.1

We may depict the procedure of this *proof-construction* with three diagrams. In [1] we simply have the assumption of the theorem, namely $\mathbf{T}(ABG)$, where $\angle ABG = \angle AGB$ but $AB > AG$. In [2], *Elem.* I.3 is applied using the same verb as was used in the enunciation of that problem. The issue, of course, is that the demonstration itself will show that this construction is, in fact, impossible. That is, we will demonstrate that there is no effective procedure that, starting with this configuration, will allow us to cut off $AD = AG$. Hence, we must take this construction step as purely hypothetical. Of course, we can say that once the assumption that one is greater than the other is made, it should be possible to carry out the application of *Elem.* I.3. But in fact, no effective procedure can be articulated that starts with an equiangular triangle and cuts off an equal to one of the opposite sides from the other.[71] Finally, in [3], once point $D$ has been assumed, we can use *Elem.* I.post.1 straightforwardly to join points $D$ and $G$.

This sort of construction appears to be an application of a postulate or a previously established proposition, but the construction cannot actually be carried through using the tools provided. Although we might construe such a *proof-construction* as a free procedure for producing a drawing, the postulates, and especially the problems, do not appear to be about making drawings, as I have emphasized. Indeed, there is nothing in Euclid's approach to indicate that the configurations depicted in the text diagrams were meant to be understood as anything but signs pointing toward the mathematical objects at issue—and, again, there is no way to produce the mathematical objects pointed to by the diagram for *Elem.* I.6 through an effective procedure. Hence, the *proof-construction* articulated in this theorem is actually a purely counterfactual assumption, introduced for the sake of the argument.

### 5.2 Semi- and non-constructive introductions

Another way of introducing objects, or properties of objects, into the argument is simply to assume that they exist, or are so, with no recourse to the language of construction. Since such assumptions are not clearly constructions we will call them *semi-* and *non-constructive introductions*.

[71] For example, although for the sake of an argument we might suppose that two numbers that are equal might be different and subtract one from the other, no effective procedure carried out on numbers would ever produce this difference as a natural number.

An interesting example of this style of reasoning comes from the first application of *Elem.* I.post.5 in *Elem.* I.29, which shows, among other things, that a straight line falling on two parallel lines makes the alternate angles equal. The indirect argument proceeds by the assumption that one of the angles is greater than the other, from which it can be shown that the two internal angles on the same side are less than two right angles. The text then reads, "But two [straight lines] extending indefinitely from less than two right [angles] meet" (Heiberg 1883, I.72). This is then shown to lead immediately to a contradiction.

In this case, the form of the verb—third person, present, indicative—simply indicates a fact, as was asserted in *Elem.* I.post.5. Nevertheless, as was argued above, we can read *Elem.* I.post.5 as a sort of introduction rule for the intersection of these two lines. Hence, the full indirect argument leads to an assumption of the production of a point that is also assumed, by *Elem.* I.def.23, not to be produced. We could also frame this as a property of the two lines—they both meet and do not meet. In either case, however, we can think of this as a semi-constructive assumption in that we are dealing with a new object, an intersection, or a property, a meeting, brought into the argument through an application of one of the postulates.

A second example of this type of non-constructive introduction can be taken from the *proof-construction* that we read above for *Elem.* III.1—to find the center of a given circle.[72] As we saw, the indirect proof for this problem begins with the assumption that some other point is the center of the circle. This is stated as a claim of existence—"let it be…"—asserted in a verb form that is used for both the exposition and the construction, but using the verb of being, which is more common for the exposition. In this case, we can simply think of this as taking any point and then assuming it has the property of being the center of the circle in question. The definition of a circle, *Elem.* I.def.15, immediately rules out any point outside the circle and the argument provided obtains for any point inside the circle. As with the previous example, we can regard this either as a semi-constructive assumption about the existence of an object—a center—or an assumption about a property—being a center—applying to an object that may be assumed unproblematically to exit—any point.

Both of these examples are in indirect arguments, so they are also counterfactual. Non-constructive assumptions, however, need not always be counterfactual, as we will see in the next section.

### 5.3 Superposition

We turn now to the most frequently discussed semi- or non-constructive assumption made in the *Elements*—namely the assumption of superposition made in *Elem.* I.4, I.8 and III.24. Although there is a long history of criticizing the practice in modern scholarship, it is a striking fact that no ancient or medieval commentator or mathematical scholar considered superposition to be illegitimate.[73] Hence, any attempt to explain

[72] This example was discussed in Sect. 2.1, above.

[73] There is an extensive literature on superposition in the *Elements*, which I will not pretend to survey. Axworthy (2018, 6–9) provides a recent overview. Vitrac (2005, 49–52) treats the ancient and medieval

how Euclid, and other ancient and medieval mathematical scholars, understood superposition must confront the fact that they did not consider it to be problematic. There have been various approaches taken to this by modern scholars.[74] Most explanations of Euclid's use of superposition can broadly be divided up into claims that the first three problems are meant to be used constructively to produce the superposition, claims that superposition depends on a transformation, or rigid motion, or some combination of both views.[75]

I am not convinced that Euclid intended superposition as an application of the previously established problems for the following reasons. (1) The construction of the elements of another triangle through *Elem.* I.1–3 will involve us in an argument about three triangles, $\mathbf{T}(ABG)$, $\mathbf{T}(A'B'G')$ and $\mathbf{T}(DEZ)$—but *Elem.* I.4 concerns only two triangles. (2) The use of *Elem.* I.1–3, which might explain *Elem.* I.4 and I.8, cannot serve for *Elem.* III.24. And, most importantly, (3) the grammatical form used for the invocation of superposition—the genitive absolute—which is used in each of *Elem.* I.4, I.8 and III.24, is not anywhere clearly used by Euclid for constructions, which are invoked with the imperative. Indeed, the use of the genitive absolute implies a hypothetical situation. Hence, in what follows I will argue that the argument by superposition involves a purely hypothetical assumption, and is explicable in terms of the concept of given.[76]

In order to formulate a new articulation of the argument by superposition, we will read through the opening passages of the proof *Elem.* I.4 after the exposition, followed by an exposition of how we should understand these passages. I will only discuss the details of this theorem, but the claims that I make about the argument it presents can be easily extended to *Elem.* I.8 and III.24—indeed, the grammatical similarity in the articulation of these three arguments indicates that Euclid intended that they be understood in the same way.

The exposition of *Elem.* I.4 sets out two triangles, $\mathbf{T}(ABG)$ and $\mathbf{T}(DEZ)$, such that $AB = DE$, $AG = DZ$, and $\angle BAG = \angle EDZ$. The text then reads as follows (Heiberg 1883, I.16–18):

Footnote 73 continued
discussions of superposition. Acerbi (2010) discusses the evidence for homeomeric lines in Greek mathematical authors, which makes clear the extent to which the technique of superposition was accepted and used by Greek mathematicians.

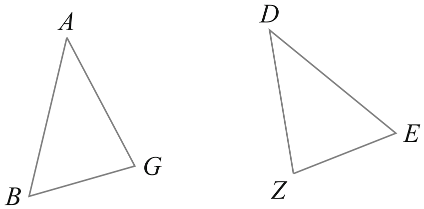
74 Here, I refer to scholarly work aiming to explain the ancient position, not mathematical work meant to criticize it and produce a new, more complete, formulation.

75 Without attempting to be exhaustive it is sufficient to point out that Levi (2003, 103–109), Wagner (1983), and Saito (2009, 807–809) argue for construction from the postulates, while Heath (1908, 225–228), Mueller (1981, 21), Vitrac (1990–2001, I.293–299), and Panza (2012, 92, n. 71) speak of a rigid displacement. Alvarez (2003) treats constructions by postulates as a sort of movement, or transformation. Finally, Dean and Mumma (2009, 725) argue that the assumption made in *Elem.* I.4 is essentially hypothetical.

76 My position is close to that put forward by Vitrac (2005, 49–52), except that I do not think we need to invoke the notion of *motion* in anything like the normal meaning of the word. Furthermore, I put greater emphasis on the notion of given to explicate the ancient and medieval understanding of superposition, and I agree with Dean and Mumma (2009, 725) that the supposition of superposition must be regarded as purely hypothetical. The case for the purely hypothetical status of superposition was clearly articulated by Commandinus (Axworthy 2018, 25–33).

[*Demonstration*:]
[1] For, when triangle $ABG$ is fit to triangle $DEZ$ and point $A$ placed on point $D$ and the straight line $AB$ upon $DE$, point $B$ will fit on point $D$ because $AB$ is equal to $DE$.[77] [2] And, when $AB$ fits on point $DE$, straight line $AG$ will also fit on $DZ$, because angle $BAG$ is equal to angle $EDZ$. [3] So that point $G$ will also fit on point $Z$, again because $AG$ is equal to $DZ$. [4] But, $B$ has also fit on $E$, so that base $BG$ will fit on base $EZ$.

The first thing to notice about this passage is that it begins, in [1], with a genitive absolute phrase in which the verbal participle is passive, but in the present tense. Hence, it describes a current state in which one triangle is being fit to another triangle—whatever this might mean. Although the passive construction leads us to read this as a kind of operation that has been performed *on* the triangle, we should remember that in the *Elements* constructive operations are expressed in the imperative, often in the perfect tense. Furthermore, the use of the genitive absolute, which acts as a conditional here, is not used to introduce constructions or operations in *Elements* I–VI. Hence, I propose that we read this opening clause as the hypothesis of a possible, current state which may obtain for $\mathbf{T}(ABG)$.




*Elem.* I.4[78]

The key to understanding this is to remember that in a theorem nothing has been specified as given, and, in particular, nothing is given in position. In *Data* 25–30, which are essentially corollaries of *Data* Def.4, the concept of *given in position* is developed by indirect arguments in which objects that are given in position are taken not to be so given, such that they are assumed "to undergo a change" (μεταπίπτειν). This is not a movement in the normal sense of the word—in which an object starts in one place and then after some time arrives at another place having passed through a series of

[77] I have translated the genitive absolute using *when*, to emphasize the difference between this type of hypothesis and that setting out the conditions of the theorem, which is introduced with ἐάν.
I have translated both the transitive ἐφαρμόζεσθαι and the intransitive ἐφαρμόζειν with "to fit," as opposed to the usual convention of using two different verbs, such as "to apply" and "to coincide" (Heath 1908, 224–225; Vitrac 1990–2001, I.181, n. 13). The reason for this is that we have a long tradition of reading "to apply" and "to coincide" as technical terms that we think we understand. For example, Heath (1908, 225) claims that the linguistic expression used to introduce superposition must be read to mean that one figure is "actually *moved* and *placed upon* the other." But, nothing in the text is so explicit. In fact, both the text and the argument are vague and subject to a range of interpretations. Hence, I use a literal translation, with an English verb that can be both transitive and intransitive.

[78] In most of the manuscripts used by Heiberg (1883), the two triangles are placed next to each other at the same rotation and reflection (Saito 2006, 100). In the Bodleian manuscript, Heiberg's **B**, there are two lines $EZ$, one curved below the other.

intermediate places—it is rather a more abstract concept of transformation.[79] Hence, objects that are not given in position, such as the elements of triangles $\mathbf{T}(ABG)$ and $\mathbf{T}(DEZ)$, can be imagined to undergo any sort of transformation relative to one another, so long as $AB = DE$, $AG = DZ$, and $\angle BAG = \angle EDZ$.

That is, so long as $AB = DE$, $AG = DZ$, and $\angle BAG = \angle EDZ$, the two triangles may be anywhere in the plane and have any orientation to one another. That is, we do not imagine that $\mathbf{T}(ABG)$ is first in one place and then *moved* to another place after the argument begins—for the conditions required by this theorem, it may as well be in any place, and with any orientation. Hence, it may just as well be in the same position and orientation as $\mathbf{T}(DEZ)$, somehow without any loss in the generality of the essential assumptions—namely that $AB = DE$, $AG = DZ$, and $\angle BAG = \angle EDZ$.

Continuing, in [1], we are then told that point $A$ is placed on point $D$, using a present passive participle. This simply orientates us to the details of the overall fit of the two triangles. The rest of the passage, [2]–[4], continues in the same vein, using genitive participles to state the conditions, and future or present verbs to state what follows. The unstated assumptions guiding this whole argument are that the components of $\mathbf{T}(ABG)$ are constant for any position and that lines that are equal can be fit to one another.[80] These were probably considered to be so obvious as to not require explicit formulation.

This is also the context in which we should understand the superposition argument in *Elem.* I.4, I.8, and III.24. Indeed, the implicit assumption of the argument by superposition is that the elements of a geometric object that are constrained in some way—as by definition, such as the equality of the radii of a circle in *Elem.* III.24, or through assumption, such as the equality of lines and angles in *Elem.* I.4 and 8—will not change no matter what the position or orientation of the object. In the case of *Elem.* I.4, the specific assumption is that the magnitude of $BG$ will not change, no matter what the position or orientation of $\mathbf{T}(ABG)$.[81] Whatever we may think of the soundness of arguments through superposition, the main point is that for Euclid they involved neither the constructive production of new elements, nor a movement of objects from one position to another. Rather, because the position of the objects is initially unspecified, we may assume, merely for the sake of the argument but without any loss of generality, that they may be superimposed on one another.

The majority of *proof-construction* steps in theorems can be carried out unproblematically with postulates and previously established problems. Moreover, in general, the overall structure of *Elements* I–VI is such that the problem showing that any particular object can be produced by an effective procedure from the postulates is introduced

[79] See the discussion of the verb μεταπίπτειν by Taisbak (2003, 93–94), who also argues that it denotes an abstract type of change. This point was also made by Russell (1938, 405–406), but note that he also claims that a "point of space is a position"—which is contrary to Euclid's view. An important goal of the *Data* and of the language of givens is to develop a meaningful way to differentiate between points which are positioned and points that have various degrees of freedom.

[80] The latter assumption is a sort of converse of *Elem.* I.c.n.7 (or 4). This is also all that is required for *Elem.* III.24, because the points and the lines are assumed to fit, from which the segment is shown to fit.

[81] One could argue that this assumption is mathematically equivalent to the assumption of Hilbert's sixth axiom of congruency, III-6.

before the said object needs to be used in a particular *proof-construction*—often immediately before, such as with *Elem.* I.11 and I.13, I.23 and I.24, I.31 and I.32, I.46 and I.47, and so on. As this short survey makes clear, however, in contrast to the situation with *problem-constructions*, this should be regarded, not as a strict necessity, but as a strong tendency. In fact, since calling on previously established problems allows us to bring in objects fully formed having the properties demonstrated in the problem, using problems as introduction rules assures us that the objects so introduced are compatible with the configuration at hand and have the required properties. Hence, the difference seems to be that whereas for theorems it is stylistically preferable that assumptions introduced for the sake of the argument, following the exposition, should be applications of postulates and previously established problems, for problems, this is an essential requirement.

## 6 Conclusion

We have seen that the concepts developed in the *Data* can be used to flesh out the conceptual background of the problems in *Elements* I–VI. This is not surprising because the *Data* itself makes extensive use of *Elements* I–VI and appears to have been written so as to mobilize *Elements* I–VI in analytical problem-solving (Taisbak 2003, 17–18). Indeed, the more fully developed treatment of given objects in the *Data* helps us to articulate what it means for objects to be given in the problems of the *Elements*, and to understand the sense in which objects are not given in the theorems of the same text.

In order to appreciate the role of problems in the text, we must clearly distinguish between *problem-constructions* and *proof-constructions*, despite the fact that these are not always structurally separated in the text. Indeed, the use of counterfactual and semi- and non-constructive introduction assumptions in theorems and in proofs, but not in *problem-constructions*, makes it clear that the author of the text intended such a conceptual distinction. The difference between these two is clearly exhibited in the full, potential structure of a problems, which is more involved that that of a theorem.

Moreover, by giving a constructive reading of all five postulates, on the basis of the notion of given, we see that they are all involved in problems, and four of them are employed as introduction rules in *problem-constructions*. This is a further indication that the theories of given developed in the *Data* provide a useful conceptual framework in which to understand the role of *problem-constructions* in the *Elements*.

A *problem-construction* is a well-ordered, finite routine made up of applications of postulates or previously established problems, which themselves act as well-ordered, finite subroutines. The objects involved in carrying out a subroutine are not directly recoverable when the object that the subroutine produces is invoked. The diagram in the text results from lines and circles produced by the postulates and from geometric objects produced directly from previously established problems. Insofar as the diagram has any role in the demonstration that follows, the possibility of this results from the construction, not from the diagram itself. The demonstration shows us what elements, of all those produced in the diagram, constitute the object the problem will return when it is later called upon—using the imperative and often the same verb. Taken as a whole, a problem is a proof that the object in question can be produced by an efficient procedure that is reducible to the postulates—and hence, can be directly called into the discourse, in either a problem or a theorem.

Theorems and problems both play a vital role in the development of the *Elements*—theorems are used in the proofs of all but the simplest problems and problems are used in the constructions of all but the simplest theorems. Indeed, we can read this interplay of problems and theorems in the *Elements* as Euclid's response to the debate reported by Proclus to have taken place between Menaechmus and Speusippus about the relative primacy of the two (Friedlein 1873, 78; Vitrac 2005, 40–42). Although there is a strong tendency to use postulates and problems in *proof-constructions*, as shown above, this is not absolutely necessary. *Problem-constructions*, however, depend essentially on postulates and previously established problems. Hence, the establishment of problems must be understood as a goal in its own right.

Since the *Elements* was meant to provide the fundamental tools necessary to do geometry—that is, to write new theorems, and to complete new problems—the primary purpose of problems must be the production of further problems, and the general development of problem-solving techniques. Hence, we should seek the motivation for the problems of the *Elements* in their use in ancient geometrical analysis. Now, of the seven works in geometry attributed to Euclid, four—*Data*, *Divisions [of Figures]*, *Porisms*, and *Loci on Surfaces*—are on topics entirely devoted to analysis. Furthermore, if Pappus' characterization of Apollonius' *Conics* is any indication, conic theory was regarded in antiquity as part of analysis (Jones 1986, 85)—or at least strongly related to analysis. Thus, Euclid's *Conics* must also have had as one of its goals the development of propositions and techniques of use in analysis. Finally, the *Elements* itself includes many problems and the close relationship between *Elements* I–VI and the *Data* shows that the former can also be read as a work of fundamental importance to analysis. In this way, we can see that the development of geometrical analysis, and the organization of problem-solving techniques, was one of Euclid's overall goals.

It is well known that ancient mathematical scholars divided problems up into (1) those that can be solved by straight lines and circles, (2) those that can be solved by conic sections, and (3) those that require more involved curves (Knorr 1986, 341–348; Vitrac 2005, 23–29). Indeed, it is possible that this distinction goes back to the division of topics in Euclid's own works devoted to geometrical analysis and problem-solving. Despite this, it is common to hear assertions that "Euclidean geometry" is restricted to the geometry of straight lines and circles—indeed, to straightedge and compass constructions. By this expression we mean, in fact, the geometry of the *Elements* and the *Data*—and, in particular, the way that this was reformulated, and eventually formalized, in the modern period. Given the original scope of Euclid's work, however, there is no reason to believe that he himself held that there was anything more than a practical distinction between these different arenas of problem-solving. Hence, the statement of the postulates, and their articulation in the problems of the *Elements*, can be taken as a foundational project, the goal of which was to provide a set of tools for demonstrating that certain objects can be produced as the result of an effective—that is a well-ordered and finite—procedure.

**Acknowledgements** The diagrams in this paper were created with Alain Matthes' tkz-euclide package. This package gives us tools for starting with a set of arbitrary points and producing new points as the intersections of objects produced with lines and circles, as is often done in modern treatments of Euclid's constructions. Of course, the underlying engine of this package assumes a Cartesian plane, and the location of each object is computed using modern formulas. Nevertheless, in this way, it is possible to emulate Euclid's actual constructions by drawing in only those lines and circles called for by each of his problems and building up a set of routines that can act as subroutines in subsequent constructions. The ideas in this paper were presented at the PhilMath InterSem, Paris, February 2016, at a Workshop in History of Greek Mathematics, Stanford University, October 2017, and the French-Japanese Workshop in Philosophy of Logic and Mathematics, Keio University, January 2018. I thank the organizers of these workshops for the chance to present my ideas. I received a number of useful comments and questions following my talks. I discussed my ideas about *Elem.* I.4 informally with Daryn Lehoux, and many times with Ken Saito and Marco Panza. Earlier drafts of this paper were read by Victor Pambuccian, Len Berggren, Marco Panza, Michael Fried, and Bernard Vitrac—all of whom made useful comments that helped me clarify my ideas. Bernard Vitrac will no doubt still disagree with my methodology, but his comments have nevertheless been helpful for me. I have greatly benefited from discussions with Marco Panza, who has helped me to sharpen my thinking on a number of fundamental points.

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