# Mathematical Methods in Ptolemy's *Analemma*

Nathan Sidoli

## 1. Introduction

This paper is an attempt to understand the mathematical methods found in Ptolemy's *Analemma* in the context of Greek mathematical practices. I first present an overview of the concepts we will need to read the text, followed by a close reading of select passages, which provide a clear overview of the structure of the argument and give examples of the mathematical methods involved.

In general terms, the *Analemma* provides a method for specifying the location of the sun in three pairs of locally orientated coordinate arcs as a function of three ostensibly empirical variables, namely the declination of the sun as a function of its longitude, $\delta(\lambda)$, the terrestrial latitude, $\varphi$, and the hour, $\eta$. The key to the approach is to represent the solid configuration in a plane diagram that Ptolemy calls the analemma. The mathematical argument begins with the presentation of a general argument that the analemma figure, or model, can be used to map arcs and lines of the solid configuration, through the example of a proof that this mapping is sound for one of the angles in question. It then proceeds to show that the analemma figure can be used to make computations of arc lengths of the solid sphere through two different methods: chord-table trigonometry, or what we can call analog, or nomographic, computation.[1] The final section of the received text details a series of physical manipulations through which we can compute the values of three pairs of coordinate arcs given the three variables $\delta(\lambda)$, $\varphi$, and $\eta$.

The *Analemma* has been the subject of a number of important studies, upon which I have drawn. The medieval Latin translation, accompanied by many useful notes that make sense of the mathematical methods, was first printed by F. Commandino in 1562, with no textual apparatus.[2] J. L. Heiberg made use

[1] The terminology of 'analog computation' is common in describing technical devices that use physical manipulation to produce a numerical result—such as a slide rule, or an astrolabe. The usage 'nomographic computation' follows that of Neugebauer, *A History*, pp. 839–856, in denoting a tradition of graphic procedures through which line segments or arcs can be physically measured.

[2] Commandino, *Ptolemaei Liber de analemmate*. This publication also contains Commandino's own work, *Liber de horologiorum descriptione*, which explains how to use the analemma methods set out in Ptolemy's *Analemma* to produce sundials.

*Ptolemy's Science of the Stars in the Middle Ages*, ed. by David Juste, Benno van Dalen, Dag Nikolaus Hasse and Charles Burnett, PALS 1 (Turnhout, 2020), pp. 35–77

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of this version in his critical edition of the Latin translation and the Greek fragments.[3] An excellent study of the mathematical conceptions underlying the text was made by P. Luckey.[4] All of this material was used by Neugebauer in his overview of ancient analemma methods.[5] D. R. Edwards provided a new critical edition of the Latin translation with an English translation, as well as a careful textual study of the work in his 1984 dissertation.[6] R. Sinisgalli and S. Vastola made an Italian translation, in 1992, accompanied by many useful notes and diagrams.[7] Finally, a full appreciation of the analemma methods must also involve some study of the substantial evidence of the medieval Arabic sources.[8]

In this paper, I make a close reading of key passages of the text, showing how each step of the argument can be understood as justified by other ancient mathematical sources, and how certain arguments are meant to be a justification, or summary of, mathematical practices that are not made explicit in the text. This results in an articulation and development of the approach of Luckey, Neugebauer and Edwards, which fleshes out many of the mathematical details in the context of ancient methods and which, I hope, helps us to understand Ptolemy's claim to be producing a more mathematical natural science.

## 2. Concepts and terminology

In this section, I introduce the concepts and terminology that we will need to read Ptolemy's text, without showing in detail how they can be derived from the sources.[9] This order of presentation—which may strike some readers as backwards—is motivated by the fact that the analemma approach is unknown to most modern readers, whereas it appears to have been well-known to ancient and medieval readers familiar with the mathematical sciences. Following this, I will make a close reading of key passages of the *Analemma*, arguing along the way that all of these concepts and techniques can be derived directly from the ancient and medieval concepts.

[3] Heiberg, 'Ptolemäus de Analemmate'. Heiberg somewhat revised this version in his *Opera astronomica minora*, pp. 189–223. I have mostly relied on the later version in this study.

[4] Luckey, 'Das Analemma'.

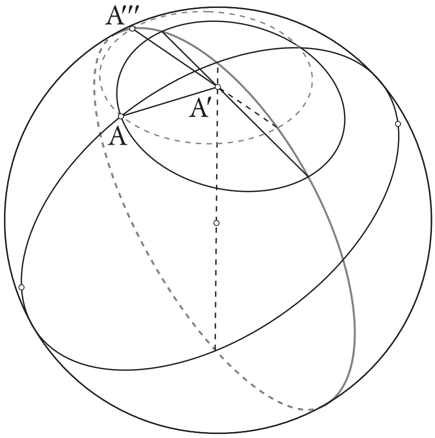
[5] Neugebauer, *A History*, pp. 839–856.

[6] Edwards, *Ptolemy's* Περὶ ἀναλήμματος.

[7] Sinisgalli and Vastola, *L'Analemma*.

[8] An incomplete selection of such material are the following: Schoy, 'Abhandlung'; Id, 'An Analemma Construction'; Kennedy and 'Id, 'A Letter of al-Bīrūnī'; Kennedy, 'Ibn al-Haytham's Determination'; Berggren, 'A Comparison'; Berggren, 'Ḥabash's Analemma'; Carandell, 'An Analemma', and Suzuki, 'A Solution'.

[9] This material takes its point of departure from the work of Luckey, 'Das Analemma'; Neugebauer, *A History*, pp. 839–856, and Edwards, *Ptolemy's* Περὶ ἀναλήμματος. Between when I wrote this paper and when it appeared in press, a useful summary of the text, including excellent diagrams, was made by Guerola Olivares, *El Collegio Romano*, pp. 67–132.



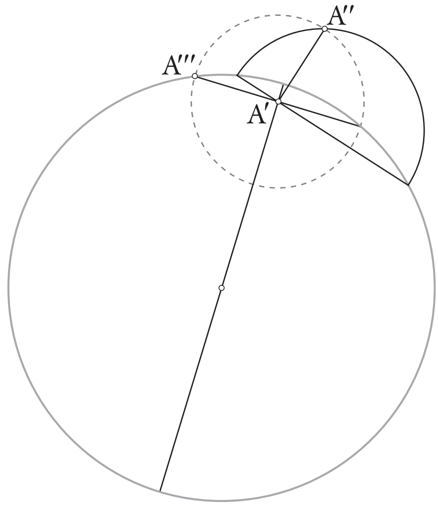




Figure 1: Analemma methods 1: (left) perspective diagram of a point $A$ on a sphere to be mapped to both a lesser circle orthogonal to the analemma and to the great circle joining it with the poles of the analemma circle in the receiving plane, in solid gray; (right) representation in the plane of the analemma of $A$, which appears in three different representations: (1) as $A \mapsto A'$ in its orthogonal projection into the analemma, (2) as $A \mapsto A''$ in its location on a lesser circle orthogonal to the analemma, and (3) as $A \mapsto A'''$ in its location on the great circle joining point $A$ with the pole of the analemma circle. The original point $A$ does not appear in the analemma representation, because, visually, it coincides with $A'$.

## 2.1. Analemma methods

The key to the use of the analemma as a problem-solving device lies in the application of four projective constructions, namely

**(M.1)** orthogonal projection of individual points into the receiving plane of the analemma,

**(M.2)** orthogonal projection of great and lesser circles into the lines of their diameters in the receiving plane of the analemma,

**(M.3)** orthogonal rotation of individual points into the receiving plane of the analemma, and

**(M.4)** orthogonal rotation of great and lesser semicircles into semicircles in the receiving plane of the analemma.

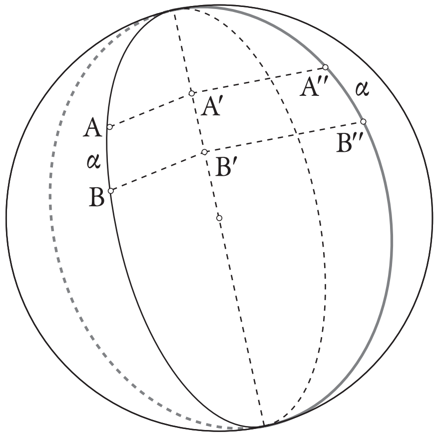
In all of these geometric transformations, the magnitudes of lines and arcs are preserved, and in the analemma figure we find the same object represented in multiple ways. Some examples will suffice to show the strategy.

A common way of mapping a point in two ways onto the analemma—which we will see Ptolemy perform three times in this account—is seen in Figure 1.

Here we see point $A$ on the sphere in Figure 1 (left), which we will represent both on a lesser circle perpendicular to the analemma, in gray, and on the great circle that joins point $A$ with the poles of the great circle of the analemma. In order to do this, in Figure 1 (right), we represent $A$ by its orthogonal projection in the plane of the analemma, $A'$ (M.1), and draw through $A'$ a line as the diameter of a lesser circle in the sphere that is perpendicular to the analemma, which can also be regarded as the orthogonal projection of the lesser circle into the analemma (M.2). We then rotate this lesser circle into the plane of the analemma by constructing a semicircle on this line and erecting $A'A''$ perpendicular to the diameter of the lesser circle (M.4 and M.3). The length of line $A'A''$ on the analemma will be constant no matter what lesser-circle diameter we draw through $A'$ and it is equal to $AA'$ on the sphere. Next, we effect the mapping of $A$ in its location on the great circle joining $A$ with the poles of the analemma onto the plane of the analemma by a two-stage process. First, (1) we join the orthogonal projection, $A'$ with that of the poles, the center of the sphere (M.1)—that is, by joining $A'$ with the center of the circle of the analemma. This gives us the diameter of this great circle as the orthogonal projection of the great circle into the plane of the analemma (M.2). Next, (2) we find the orthogonal rotation of $A$ on the sphere to $A'''$ on the analemma by rotating point $A$ along the circumference of a circle of radius $AA'$, shown in a gray dotted line, with the axis of rotation being the diameter of the great circle into which we project $A$ (M.3). We do this in the analemma by producing a circle around $A'$ with distance $A'A''$, since this length is equal to $AA'$ on the sphere. This circle, shown in a gray dotted line, however, is not drawn in the plane of the analemma—presumably because arc lengths are not preserved on this circle.[10] In this way, we can exhibit $A$ mapped to $A'$ by orthogonal projection, and to both $A''$ and $A'''$ by rotation into either a lesser circle or the great circle that joins $A$ with a pole of the analemma.

In Figure 2 (left), if points $A$ and $B$ in the sphere lay on a great circle passing through the poles of the gray analemma circle, they can be projected into the analemma plane orthogonal to the great circle between them by dropping perpendiculars to the plane of the analemma meeting the diameter of the great circle joining them at $A'$ and $B'$ (M.1). When this is represented in the analemma, Figure 2 (right), the line joining $A'$ and $B'$ must pass through the center of the circle, because $A$ and $B$ lie on a great circle that passes through the pole of the analemma (M.2). Arc $\alpha$ of the great circle between $A$ and $B$ is found in the plane of the analemma by constructing perpendiculars at $A'$ and $B'$, mapping them to $A''$ and $B''$ (M.1 and 2). In this way, lines $A'A''$

[10] That is, while $AA'''$ is always a quadrant on the sphere, $A''A'''$ is not generally a quadrant on the analemma.



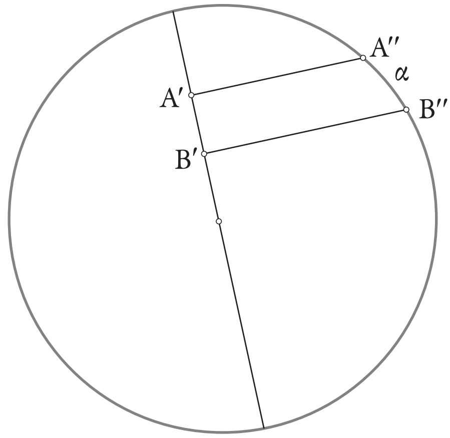




Figure 2: Analemma methods 1: (left) perspective diagram of two points $A$ and $B$ on a sphere, and the great-arc distance between them, $\alpha$, to be mapped to the receiving plane, in gray; (right) representation in the analemma of $A$ and $B$ as orthogonally projected onto the analemma such that $A \mapsto A'$ and $B \mapsto B'$, along with orthogonal rotation of the great circle into the analemma such that $A \mapsto A''$ and $B \mapsto B''$. The original points $A$ and $B$ do not appear in the analemma representation, because, visually, they coincide with $A'$ and $B'$.

and $B'B''$, in the analemma, are equal to the perpendiculars dropped from $A$ and $B$ into the receiving plane, in the sphere, and the length of arc $\alpha$, the great-arc distance between the two points, is preserved in the transformation.

In Figure 3 (left), if points $A$ and $B$ in the sphere lay on a lesser circle perpendicular to the gray analemma circle, they can be mapped into the plane of the analemma circle by rotating the lesser circle into the plane of the analemma—or rather, folding it into two semicircles that are rotated into the same position in the plane of the analemma. This is represented in the analemma, Figure 3 (right), by dropping perpendiculars into the receiving plane, such that $A'$ and $B'$ represent points $A$ and $B$ in the analemma (M.1), and the line joining them is a diameter of the lesser circle and its orthogonal projection into the analemma (M.2). The lesser circle is then folded and rotated into the analemma by erecting a semicircle on the diameter of the lesser circle (M.4). Arc $\alpha$ of the lesser circle is rotated into the plane of the analemma by constructing perpendiculars at $A'$ and $B'$ meeting the semicircle, such that $A$ maps to $A''$ and $B$ to $B''$ (M.3).[11] Once again, lines $A'A''$ and $B'B''$, in the analemma, are equal to the perpendiculars, $AA'$ and $BB'$, in the sphere,

[11] The practice, in dealing with a solid configuration, of rotating one plane into another by constructing the objects in the plane to be rotated directly in the receiving plane is common in Greek geometry. See, for example, the solid constructions by Diodorus or Eutocius; Hogendijk, 'The Geometrical Works', pp. 56, 70–71, and Sidoli, 'Review of *The Works of Archimedes*', pp. 160–61.

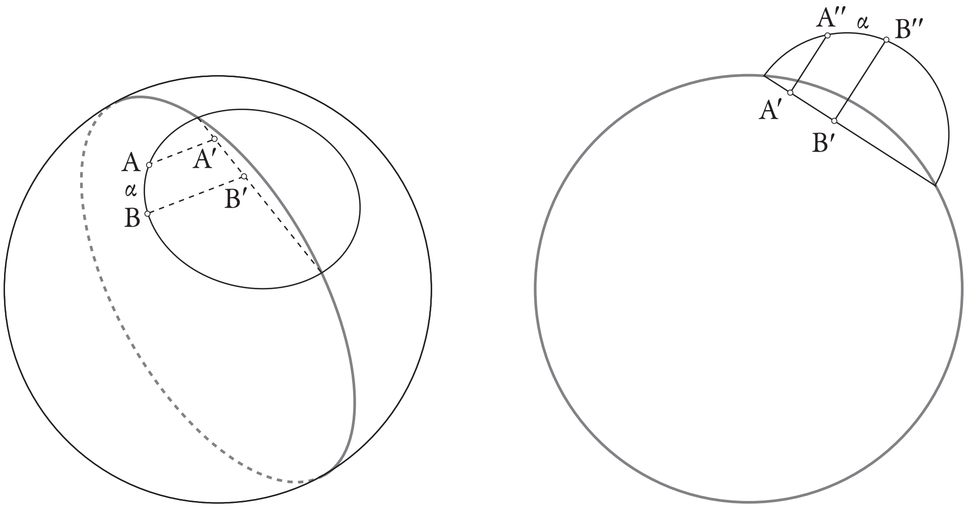



Figure 3: Analemma methods 2: (left) perspective diagram of two points $A$ and $B$ on a lesser circle of a sphere, and the lesser-arc distance between them, $\alpha$, to be mapped to the receiving plane, in gray, which must be perpendicular to the lesser circle and pass through its poles; (right) representation, in the analemma, of a semicircle of the lesser circle as rotated into the plane of the analemma, such that $A \mapsto A'$ and $B \mapsto B'$, by orthogonal projection, and to $A \mapsto A''$ and $B \mapsto B''$, by rotation.

dropped from $A$ and $B$ into the receiving plane, and arc $\alpha$ is equal to the lesser-arc distance between the two points.

In this section, I have used the term *analemma* as synonymous with the receiving plane of a projection, or mapping. This understanding of the term agrees with Ptolemy's usage in his *Analemma*, and, as Edwards has argued, best conforms with the various functions of the term in ancient sources.[12] Hence, the analemma is the receiving plane of a projection, which is performed by carrying out constructions directly in the plane.[13]

In Ptolemy's *Analemma*, we will see evidence for these various projective operations. Although he explicitly refers to rotating the semicircles of lesser circles, Ptolemy has no special terminology for orthogonal projection of points and circles, and the text speaks only of producing perpendiculars and of taking the diameters of circles in the analemma. Nevertheless, as we read through *Analemma* 6, below, we will see that the underlying solid configuration is essentially that described in this section.

[12] Edwards, *Ptolemy's* Περὶ ἀναλήμματος, pp. 1–10.

[13] This way of producing projective constructions is also found in Ptolemy's *Planisphere*; see Sidoli and Berggren, 'The Arabic Version'.

### 2.1.1. *Geometrical constructions and instrumental practice*

Although there are almost no synthetic theorems in analemma texts—and only one in Ptolemy's *Analemma*—there are constructions in all uses of the analemma in geometric problem-solving, in both ancient and medieval texts. The types of constructions employed, however, are clearly restricted. In fact, I am not aware of any constructive step in an ancient or medieval analemma problem that uses an operation that cannot be reduced to applications of *Elements* I.posts.1–3, I.11, and 12—that is, the first three postulates of Euclid's *Elements* and the two problems that produce perpendicular lines.[14] Even more, analemma constructions can all be regarded as abstractions of the use of a compass and a set square. Indeed, the three ancient texts that deal with the analemma make explicit mention of various types of instrumental practice, indicating that the mathematical methods of the analemma were closely associated with certain instruments.

As well as referring to operations performed on instruments such as specially prepared plates and hemispheres, analemma texts prescribe the instruments used to carry out geometric constructions. The analemma described by Vitruvius, *Architecture* IX.7, although not addressing a problem, is explicitly produced with a compass,[15] and Ptolemy, as we will see below, explicitly introduces both the compass and the set square. It seems clear that constructions in analemma texts were limited to abstractions of the operations that can be performed with these instruments—that is, a finite set square whose side is just a bit greater than the diameter of the great circle of the analemma, and a finite compass whose radius is just a bit greater than that of the great circle of the analemma, and which can be operated with a given radius.

In order to make this underlying instrumental practice explicit, in the following I will note how each construction on the analemma can be performed with either the compass or the set square.

## 2.2. Ptolemy's notion of 'model'

Ptolemy's uses of the term *hupothesis* (ὑπόθεσις), and the cognate verb (ὑποτίθεται), are far ranging. These terms often have the sense—implied by the basic meaning of the words—of what is set down in the beginning to be built upon, and, indeed, they are translated literally by Moerbeke, in his Latin translation of the *Analemma*, with *suppositio* and *supponere*. They may also, however, indicate the assumption of a fully elaborated depiction of the structure and function of the objects under investigation.

[14] As we will see below, the proof in *Analemma* 6 also requires *Elements* XI.12, but this is a theorem, not a problem.

[15] See Soubiran, *Vitruve. De l'architecture*, pp. 26–30.

Moreover, a Ptolemaic *hupothesis*, unlike a modern scientific hypothesis, is not subject to testing; in fact, it may sometimes be demonstrated, or even saved—as occasionally in the *Almagest* or in *Harmonics* I.2.[16] G. J. Toomer emphasized that Ptolemy's understanding of a *hupothesis* is often far removed from the modern sense of a hypothesis in scientific discourse, and pointed out that it was closer to our idea of a model.[17]

An *hupothesis* in Ptolemy's writings can be as simple as the assumption of the immobility and sphericity of the earth, or as complicated as the full geometric configurations for the moon or Mercury;[18] either purely arithmetical, as in harmonics, or fundamentally geometric, as in astronomy;[19] an idealized mathematization, as we will see in the *Analemma*, or closely connected with a physical representation, as in *Planetary Models* (commonly *Planetary Hypotheses*) I.1 and 2, or *Almagest* XIII.3.[20]

This broad notion of a conceptual tool for explanation and computation has more overlap with our concept of model than our concept of hypothesis. Hence, in my translation of passages of the *Analemma* I will translate ὑπόθεσις and its cognates with *model* and its cognates.[21] Some readers may find this excessively modern, but I hope the discussion here will help us avoid unwanted anachronism.

Ptolemy's use of modeling can be compared to that of Hellenistic authors working in the exact sciences such as Autolycus, Euclid, Aristarchus, Archimedes and Eratosthenes—all of whom used geometric modeling as the basis of their work in astronomy, mechanics, optics and harmonics.[22] A distinction can be drawn, however, between Ptolemy's use of *hupothesis* and that of Aristarchus in *On the Sizes and Distances of the Sun and the Moon* and that attributed to Eratosthenes by Cleomedes in *On Heavens* I.7.[23] Aristarchus and Eratosthenes used hypotheses both (a) to set out the overall geometrical configuration that

[16] Heiberg, *Syntaxis mathematica*, vol. II, pp. 26, 180, 461, and Düring, *Die Harmonielehre*, p. 5.

[17] Toomer, *Ptolemy's Almagest*, pp. 23–24.

[18] Heiberg, *Syntaxis mathematica*, vol. I, pp. 26, 350; vol. II, p. 255.

[19] Düring, *Die Harmonielehre*, p. 5.

[20] See Heiberg, *Opera astronomica minora*, pp. 70–74; Hamm, *Ptolemy's Planetary Theory*, pp. 72–76; Murschel, 'The Structure'; Heiberg, *Syntaxis mathematica*, vol. II, pp. 532–533. Jones, 'Ptolemy's Mathematical Models', gives an overview of the various ways mathematical modeling functions in Ptolemy's work.

[21] This is also done, for example, by E. Hamm, *Ptolemy's Planetary Theory*, in her translation of Ptolemy's *Planetary Models* I, Part A.

[22] There has been much debate over whether or not Euclid composed the *Division of the Canon*, but there seems to be no objective way to decide the issue; see Barbera, *The Euclidean Division*, pp. 3–29.

[23] Heath, *Aristarchus of Samos*, pp. 352–411, and Todd, *Cleomedis Caelestia*, pp. 35–37.

serves as the basis of the model, and also (b) to set out quantitative assumptions that are, at least in principle, empirically decidable and which serve as a basis for computation.[24] Ptolemy, however, does not use *hupothesis* in this second sense. For Ptolemy the *hupothesis* is only the general geometric configuration of the model, whereas the quantitative parameters to be determined through observation are not referred to by the term *hupothesis*.

Hence, by Ptolemy's time, and probably for a long while before, there was a fairly clear distinction between what we would think of as the model as a basis for computation and the given values that are used in the computation. As we will see in the *Analemma*, although there is no discussion of empirical practice and both the geometric model and the quantitative parameters are simply assumed in the course of the argument, there is a clear linguistic distinction between the two—the model itself is asserted as assumed and the parameters are asserted as fixed, or determined, although they may, of course, vary.

### 2.3. The two-sphere model

The *two-sphere model* is a name given by modern scholars to the model of the cosmos found in texts such as Autolycus' *Moving Sphere* and *Risings and Settings*, Euclid's *Phenomena*, and Theodosius' *Days and Nights* and *Habitations*. In this model, the sun is taken as located in varying positions on the ecliptic as a great circle in the sphere of the cosmos, which contains the fixed stars. The sphere of the cosmos, carrying the ecliptic, rotates about the celestial poles, $P_n$ and $P_s$, creating the celestial equator, to which the ecliptic is skew at the angle known as the obliquity of the ecliptic, $\varepsilon$. The sphericity of the earth is only accounted for by the fact that the horizon, which is also a great circle, is generally skew to the ecliptic and the celestial equator, and divides the cosmos into two hemispheres—above and below. In Figure 4, if the eastern point is taken to be in the direction of the viewer, since the horizon is immobile, the sphere of the cosmos is imagined to rotate clockwise. This configuration was used by ancient mathematicians to model the phenomena we associate with spherical astronomy—namely, the solar and stellar phenomena related to local coordinates, determined by the horizon and the local meridian.

Whatever the mathematicians of the Hellenistic period may have thought of this construction, by Ptolemy's time it must certainly have been thought of as a model in the sense discussed above—that is, as a simplified configuration that was known to not be a strict representation of reality, but which could be used mathematically without any significant loss of accuracy. Trivially, the earth is not actually a point, but a sphere—which is what allows us to speak

[24] Berggren and Sidoli, 'Aristarchus's *On the Sizes and Distances* ', pp. 231–234; Carman, 'Two Problems', pp. 55–58; Carman and Evans, 'The Two Earths'; Sidoli, 'Mathematical Discourse'.

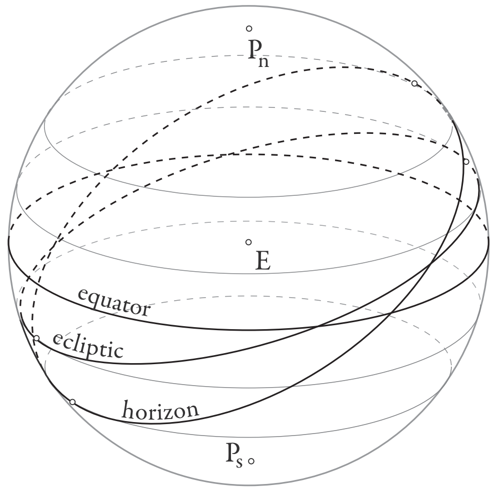



Figure 4: The 'two-sphere' model

of a region above and below the horizon. It is so small in comparison to the cosmos, however, that it can be regarded as a point. That this is a strictly false, but observationally adequate, simplifying assumption must also have been clear to the Hellenistic mathematicians. Secondly, at least by Ptolemy's time, the sun was not actually a point in the celestial sphere, but was rather closer to the earth, below the outer planets, and with a varying distance from the earth. Hence, the two-sphere model could not have been regarded as an accurate depiction of the sun in its relation to the earth, but simply as a mathematical model depicting the perceived location of the sun on the sphere of the cosmos from the perspective of the earth.

There are clear indications in *Analemma* 2 and 3 that Ptolemy thought of the overall model of the cosmos in just these sorts of perceptual terms. In *Analemma* 2, Ptolemy calls this simplified model of the sun in the cosmos, orientated to the local horizon, the 'world sphere', and he says that the great circles in this sphere that can be taken to determine the position of the sun 'move with the sun'.[25] Hence, they can be imagined to be great circles of the world sphere laying in planes that pass through the sun. Furthermore, when he describes the position of the sun in more detail, both in *Analemma* 2, in general terms, and in *Analemma* 3, in setting out the model with letter-names, he refers to the solar position as the 'solar ray'—which we can understand as the line along which we see the sun, drawn from the earth, through the sun, out to the sphere of the cosmos.

[25] These great circles are discussed in detail below.

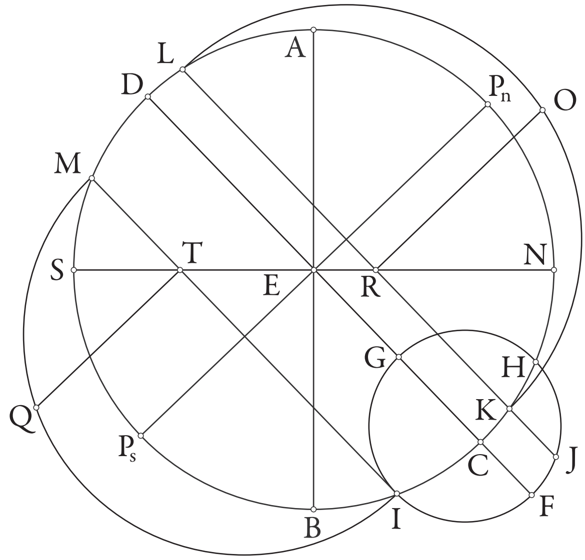



Figure 5: The analemma model

## 2.4. The analemma model

The analemma model can be found in three ancient sources—Vitruvius' *Architecture* IX.7, Heron's *Dioptra* 35, and Ptolemy's *Analemma*—and a few medieval Arabic sources. Although in medieval sources, analemma methods are used to solve a range of problems in spherical astronomy, in the extant ancient sources, they are almost always used on the analemma model described in this section.[26] The model itself is naturally suited to handle the seasonal hours of Greco-Roman daily life, $\eta_s$, and hence was closely associated with *gnomonics* (γνωμονική), the science of sundials. Here, I simply describe the model, with no attempt to derive this description from the sources.

The ancient analemma model is orientated towards a coordinate system of the local horizon and meridian, and can model the motion of the sun, on both its annual and daily paths—that is, the model can be used to specify the location of the sun, relative to local coordinates, given the terrestrial latitude, $\varphi$, the declination of the sun as a function of its longitude, $\delta(\lambda)$, and the time of the day, in hours, $\eta$.

In Figure 5, the local meridian is the great circle of the analemma *NBSA*, line *NS* is the orthogonal projection of the great circle of the horizon, line *CD* is that of the great circle of the equator, and line $P_nP_s$ is the line joining

[26] An exception is Ptolemy's *Planisphere* 18, which employs an analemma construction; see Sidoli and Berggren, 'The Arabic Version', pp. 132–133.

the celestial poles, perpendicular to $CD$. The orientation of lines $NS$ and $CD$ determine the terrestrial latitude, $\varphi$, because, in great circle $NBSA$, arc $NP_n$ is the elevation of the pole—that is, arc $NP_n = \varphi$.

Since the horizon is motionless, while the cosmos rotates about $P_nP_s$, the orthogonal projection of both the horizon and the equator into the meridian will remain fixed. The same is not true, however, of the ecliptic. Indeed, the ecliptic will only be orthogonal to the analemma twice daily, when the solstitial colure coincides with the local meridian—and, indeed, the ecliptic is not represented as such in Ptolemy's *Analemma*.

The proper position of the sun on the ecliptic, $\lambda$, can, however, be modeled on the analemma with the use of an auxiliary circle, $FHGI$, arranged such that its center lies on the diameter of the equator and it cuts the great circle of the analemma so that the arcs $CH$ and $CI$ are both equal to the obliquity of the ecliptic, $\varepsilon$. In this case, where $F$ represents the vernal equinox, $H$ the summer solstice, $G$ the autumnal equinox, and $I$ the winter solstice, if arc $FJ = \lambda$ is cut off equal to the arc of solar longitude from the vernal equinox at Ari 0°, then arc $CK$ of the analemma will be equal to the declination of the sun at this time, $\delta(\lambda)$.[27]

Then, since throughout the course of a day the sun will travel on a course roughly coinciding with the circle of its declination, which we can call its day-circle, the local position of the sun can be modeled on the day-circle folded and rotated into the plane of the analemma.[28] In Figure 5, when the sun is at $FJ = \lambda$, in its annual course, it can be imagined to travel uniformly on semicircle $KOL$ throughout the course of the day; or when it is at the winter solstice, $FJI = \lambda$, it will travel on semicircle $IQM$. For example, if a given day in the spring or summer begins at midnight, the sun will start at, say, $K$ and move along arc $KO$ until sunrise, passing over the horizon at point $O$, and then move up to midday at $L$ and return back along $LO$ in the afternoon to sunset at $O$, finally returning to $K$ at the following midnight. In fact, the second position of $K$ will be somewhat altered because of the daily longitudinal movement of the sun of about 1°, but analemma methods do not take this into account.

The final given magnitude, the hour $\eta$, is marked off along the day-circle. In the case of the seasonal hours of daily life, $\eta_s$, each of the arcs $LO$, $OK$, or $MQ$, $QI$ are divided into six equal parts for the six hours between the horizon and the meridian. Although this is not discussed in the ancient sources, it

[27] Neugebauer, *A History*, p. 845, shows this using methods consistent with Greek geometric practice.

[28] Heron, in his *Dioptra* 35, refers to this circle as the 'daily circle' (ἡμερήσιος κύκλος); see Schöne, *Herons von Alexandria Vermessungslehre*, pp. 302–306, or Acerbi and Vitrac, *Metrica*, pp. 103–106.

would also be possible to model the astronomer's equinoctial hours, $\eta_e$, by dividing the complete semicircle of the day-circle into twelve equals parts.

In fact, however, Ptolemy's *Analemma* works with further simplifying assumptions. In the first place, the circle *FHGI*, which is called the *menaeus* (from μηναῖος, meaning 'monthly') by Vitruvius,[29] is not found in Ptolemy's presentation. Instead, he simply takes $\delta$ as given at one of four solar declinations corresponding to the beginnings of the twelve zodiacal signs—namely,

$$\begin{array}{ll} \delta = 0° & \text{for } \lambda \approx 0°,\ 180°, \\ \delta = 11\,{}^2\!/_3°(= 11;40°) & \text{for } \lambda \approx 60°,\ 120°,\ 240°,\ 300° \\ \delta = 20\,{}^1\!/_2°(= 20;30°) & \text{for } \lambda \approx 30°,\ 150°,\ 210°,\ 330°, \text{ and} \\ \delta = 23\,{}^1\!/_2\,{}^1\!/_3°(= 23;50°) & \text{for } \lambda \approx 90°,\ 270°, \end{array}$$

and refers to the semicircle constructed at a given declination as the 'monthly circle' (μηνιαῖος κύκλος). Hence, in what follows, for the sake of explicating his text, I will use Ptolemy's terminology and refer to the day-circle of the sun as its month-circle.

These three declinations may have been determined, for example, through a table such as that in *Almagest* I.15, or they may have been values taken by Ptolemy from previous work in gnomonics, having been computed using chord-table trigonometry directly on the analemma model.[30] Whatever the case, although these declinations are those of evenly distributed longitudes of 30°, they are, as declinations themselves, rather unevenly distributed—since their differences are 11;40°, 8;50°, and 3;20° respectively. This is perhaps an indication that, in the *Analemma*, Ptolemy was more interested in the symmetry of his presentation, and the role of symmetry in his instrumental practice, than in the precision of any device that might be made with these methods.

As we have just seen, both Vitruvius and Ptolemy speak of a 'monthly' division of the annual solar cycle, presenting us with a kind of zodiacal month. Of course, there is no discussion of the duration of these months, and given the level of precision evident in Ptolemy's presentation this is probably not important, but these months are clearly a division of the sun's annual progress through the stars into twelve parts. The ancient tool that was used to track the course of the sun through the stars, often noting its passage into each of the twelve zodiacal signs, was the parapegma.[31] Hence, the analemma appears to have been directly related to the two most conspicuous devices used to

[29] Soubiran, *Vitruve. De l'architecture*, p. 29.

[30] It may be significant that Ptolemy states the declination using the proper parts (unit fractions) of standard Greek arithmetical practice, not the sexagesimal fractions of his mathematical astronomy. The values used in the *Analemma* are what we would get if we rounded the values in the *Almagest* to the nearest 0;05°—see *Almagest* I.15, Heiberg, *Syntaxis mathematica*, vol. I, p. 72. We do not know if Ptolemy derived these values in this way.

[31] See Lehoux, *Astronomy*, especially pp. 70–97.

regulate the cycles of daily life in the Greco-Roman world: the sundial and the parapegma.[32]

From the description given in this section, it is clear that, like the two-sphere model, the analemma model functioned as a simplified geometrical configuration that facilitated geometrical and computational problem-solving. We will see below, in reading passages of the *Analemma*, that Ptolemy's mathematical practice clearly distinguished between the assumption of the model itself, as an overall geometrical configuration, and the assumption of given values that may be assumed as parameters of the problem-solving activity.

## 3. Ptolemy's *Analemma*

The text of the *Analemma* is known to us from two sources—fragments of a 5th–7th century text palimpsested in *Ambrosianus graec.* L 99 sup., **Am**, and a 13th century Latin translation by William of Moerbeke contained in *Vaticanus Ottobonianus lat.* 1850, **O**—which is probably also incomplete. The Ambrosianus codex is an 8th century copy of Isidore's *Etymologiae* that contains eight bifolia that were repurposed from manuscripts that once contained mathematical material, of which twelve pages came from a copy of Ptolemy's *Analemma*.[33] The Ottobonianus codex is an autograph by Moerbeke of his translations of Greek mathematical works, focusing on Archimedes, of which the final three folia contain his translation of the *Analemma* (ff. 62–64)—ending somewhat abruptly with a single mathematical table and no colophon. The Latin text, with many corrections, was first published by Commandino,[34] the Greek fragments and the Latin text were critically edited by Heiberg,[35] and the Latin was reedited by Edwards, who also provided an English translation.[36]

[32] It is worth noting that Ptolemy's own parapegma text, the *Phases of the Fixed Stars*, does not work with zodiacal months, but rather divides a solar year into the twelve 30-day months of the Egyptian calendar, which was also used for astronomical purposes; see Lehoux, *Astronomy*, pp. 261–309. On the other hand, given the low level of precision evidenced in the *Analemma* itself, it is possible that Ptolemy regarded his 'monthly circles' as corresponding to these Egyptian months.

[33] The pages of the codex that contain the *Analemma*, in the order of the Ptolemaic text, are as follows: 119, 120, 139, 140, 137, 138, 143, 144, 129, 130, 117, 118.

[34] Commandino, *Claudii Ptolemaei Liber de analemmate*.

[35] Heiberg, 'Ptolemäus de Analemmate', and Id., *Opera astronomica minora*, pp. 189–223.

[36] Edwards, *Ptolemy's* Περὶ ἀναλήμματος. Since we do not have Greek for the whole text, I will often use the Latin text as the primary source. By relying on Moerbeke's translation of Ptolemy's *Tetrabiblos* and of the Archimedean corpus, I will not always translate the Latin literally, but will make some informed guesses about the original Greek terminology behind the Latin even where we do not have corresponding fragments in the palimpsest. See Clagett, *Archimedes*; and Vuillemin-Diem and Steel, *Ptolemy's* Tetrabiblos.

Although I have made my own translation of the text, I have often been guided by that of Edwards, and in the mathematical passages there is little significant difference.

Although the Greek manuscript numbers the text differently than Heiberg, and the Latin manuscript presents the treatise in continuous prose, I will follow Heiberg's numbering in presenting a short outline of the text:[37]

**1:** A short dedication to Syrus, explaining that we will take the approach 'of those men in lines' (*uirorum illorum in lineis*),[38] since there is need of a more mathematical conception of natural theory and a more natural-theoretic conception of mathematics.

**2, 3:** A discussion of the coordinate system from first principles, explaining that three dimensions are used to measure a volume, both in magnitude and in number, and that a point on a sphere can be determined by the motion of three circles of the 'world sphere' (*spera mundi*)—the *horizon*, the *meridian* and the *vertical*—about one of their own diameters as determined by their intersections, producing three pairs of angles—*hectemorius-meridian*, *horarius-vertical*, and *descensivus-horizon*.[39] A description of the analemma model using letter-names.

**4:** A description of the system of 'the ancients'—which did not use the *hectemorius circle*.

**5:** A few refinements to the conventions so that no coordinate arc need be taken as greater than a quadrant.

**6:** **(a)** An introduction of the mathematical goal of the treatise—an instrumental determination, using the analemma, of the six arcs set out in the introduction; **(b)** followed by a synthetic proof that a certain angle in the analemma diagram is equal to the *hectemorius angle* on the sphere.

**7:** Geometric construction of an analemma diagram containing all six angles, with no proof, for the situation in which the sun is near an equinox.

**8:** Geometric construction of an analemma diagram containing the same for any other longitudinal position of the sun.

**9:** **(a)** A short discussion of instrumental practice, in which Ptolemy points out that on the analemma any of the six principal arcs can be determined using 'linear demonstrations'—that is, computational, indeed, trigonometric methods (διὰ τῶν γραμμῶν; *per lineares demonstrationes*, *per numeros*);[40] **(b)** followed by a *metrical analysis* for the determination of all six arcs in the case in which the sun is at an equinox.

[37] Since not all of the numbers in the Greek fragments have been preserved, it is not possible to be certain where all of the divisions were placed in this version of the text.

[38] Presumably geometers.

[39] These angles are defined below; see page 51.

[40] See note 74 for a discussion of the meaning of the phrase διὰ τῶν γραμμῶν.

**10:** A *metrical analysis* for the determination of all six arcs for any other longitudinal position of the sun.

**11:** (**a**) General description of the drawing and the instruments (compass and set square) used to make analog, or nomographic, computations; (**b**) followed by a detailed description of the production of an analemma plate used for carrying out analog calculations.

**12:** Instructions for producing all six angles on the analemma plate while drawing no new lines, for the situation in which the sun is near an equinox.

**13:** Instructions for producing all six angles on the analemma plate while drawing no new lines, for any other longitudinal position of the sun.

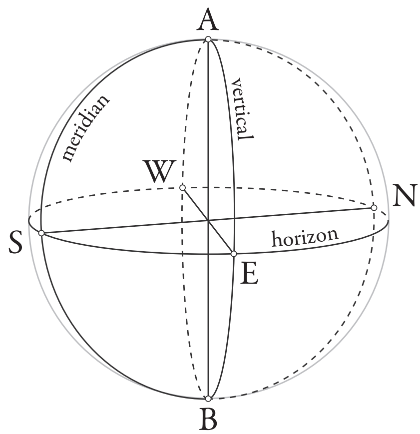
**14:** (**a**) Instructions for producing the angles of 'the ancients' on the analemma plate; (**b**) followed by a discussion of which pairs of angles, and taken in which direction, determine the position of the sun.

**15:** Table of all six angles for the latitude of Meroe, $\varphi = 16;25°$, when the sun is at Can 0°, $\delta(\lambda = 90°) = 23;50°$, for the position of the sun at the horizon, at the end of each of the five pairs of seasonal hours between the horizon and midday, and at the meridian.

It is likely that there were once more tables—perhaps 28 or 49, filling out the seven latitudes mentioned in the text and the four declinations of the beginnings of the twelve signs of the ecliptic.[41] It might be supposed that there was once more text following these tables, but nothing in the extant treatise compels us to this position.

In order to follow Ptolemy's mathematical approach, we will focus on explaining in detail only a few sections of the treatise—in each case, explicating only the *meridian-hectemorius* angle pair, since the three pairs are mathematically analogous and an understanding of one pair will suffice to grasp the overall approach. We will first look at the general exposition of the *world sphere* in *Analemma* 2 and 3. This will be followed by the synthetic proof in *Analemma* 6 that a certain angle of the analemma diagram is equal to the *hectemorius* angle. We will then read passages from the *metrical analysis* in *Analemma* 10, showing that if the terrestrial latitude, $\varphi$, solar declination, $\delta(\lambda)$, and seasonal hour, $\eta_s$, are given, then the *hectemorius* and *meridian* angles are also given.

[41] Edwards, *Ptolemy's* Περὶ ἀναλήμματος, p. 106, n. 506, states that the total number of tables should have been 49 tables—that is, the seven latitudes by the four declinations, three of which must be taken both to the north and to the south. Neugebauer, *A History*, pp. 854–855, states that there should have been 28 tables—presumably believing that Ptolemy would have made further use of the symmetries between the sets of seasonal hours to reduce the total number of tables.



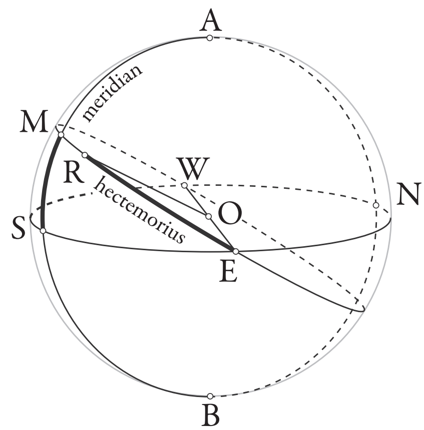




Figure 6: The *world sphere* as described in *Analemma* 2 (modern figures)

Finally, we will follow through the nomographic calculation of the *hectemorius* and *meridian* angles on the analemma plate in *Analemma* 13.

## 4. Ptolemy's *world sphere*

In *Analemma* 2, Ptolemy explains that three dimensions are sufficient to determine a volume (*moles*) in both magnitude and number, so that, in the *world sphere*, we need only three great circles and their diameters, set at right angles.[42] Because the discussion in *Analemma* 2 concerns the *world sphere* itself, it does not make reference to a lettered diagram—which makes it somewhat difficult to follow. In order to explicate this section, however, we will describe the objects that Ptolemy introduces using a modern diagram, Figure 6, which does not correspond to anything in the manuscript sources.

In Figure 6 (left), the three great circles of the *world sphere* will be taken as (a.1) the *horizon*, *NESW*, dividing the hemisphere above the earth from that below, (a.2) the *meridian*, *NBSA*, dividing the eastern and western hemispheres, and (a.3) the *vertical*, *EBW*A, dividing the northern and southern hemispheres; and the diameters will be (b.1) the *equatorial diameter*, *EW*, (b.2) the *meridional diameter*, *SN*, and (b.3) the *gnomon*, *AB*. Clearly, in this context, we are describing the cosmos using local coordinates, orientated to the position of the observer on the earth.

When these three circles are moved 'with the sun' (*cum sole*) about their diameters, (c.1) the horizon produces the *hectemorius*, rotating about the equatorial diameter, (c.2) the meridian produces the *horarius*, rotating

[42] Moerbeke often uses *moles* to translate ὄγκος, which in this context would mean 'volume'—see Clagett, *Archimedes*, p. 34.

about the meridional diameter, and (c.3) the vertical produces the *descensivus* (καταβατικός), rotating about the gnomon. When any one of these circles is raised 'above the earth with the solar ray' (*cum solari radio super terram*) it will produce two inclinations: (1) an angle contained by lines, between the solar ray and the diameter about which the circle rotates, and (2) an angle contained by planes, between the movable plane and its stationary counterpart—and 'as far as they are given, the position of the [solar] ray is also fixed' (*quibus datis et positio radii determinatur*). The final section of *Analemma* 2 is somewhat obscure because Ptolemy is setting out his own terminology at the same time as that of the ancients, but the main point is that each of the angle pairs *hectemorius-meridian*, *horarius-vertical*, and *descensivius-horizon* can be used to specify the location of the solar ray.

For example, in Figure 6 (right), when the hectemorius, *ERMW*, is inclined with the solar ray, *RO*, the equatorial diameter, *EW*, and *RO* create the rectilinear ∠*EOR*, which can be measured by arc *ER*, while the plane of the hectemorius creates an angle with its stationary counterpart, the horizon, *ESWN*, which can be measured by arc *MS* of the meridian. Hence, the position of the sun can be determined by arc *ER* of the hectemorius and arc *MS* of the meridian. Indeed, it is clear that if arc *ER* has the range 0°–180° and arc *MS* the range 0°–360°, any point on the sphere can be named in these coordinates. In fact, however, Greek geometers did not consider angles greater than 180°, and Ptolemy will introduce conventions in *Analemma* 5 that will insure that these six principal arcs will always have a range of 0°–90°.[43]

*Analemma* 2 is a discussion of how we can understand the position of the real sun in term of local coordinates—it speaks of the *world sphere* and of movable circles being carried with the sun with no reference to a lettered diagram. The angles that determine the position of the sun are described in terms relative to the position of the observer in the center of the cosmos. The use of the diagrams in Figure 6 helps a modern reader to follow Ptolemy's description, but it is not faithful to his approach, which is to describe the situation as taking place *around us*—with no appeal to the terminology of modeling or supposing.

This changes in *Analemma* 3, which Ptolemy introduces with the following words: 'In order that the sequence (*consequentia*) of the angles and what is modeled (*quod supponitur*) should fall more within our view, in fact, let there be a meridian circle, *ABGD*.'[44] He then proceeds to give a description using a

[43] There may have been practical reasons for this. For example, the graduated quadrants on his analemma plate only measure 0°–90°—which may have to do with the fact that a compass large enough that its radius would be equal to the diameter of his analemma plate would be rather cumbersome.

[44] In translating this and the following Latin passages, I have not attempted to render the

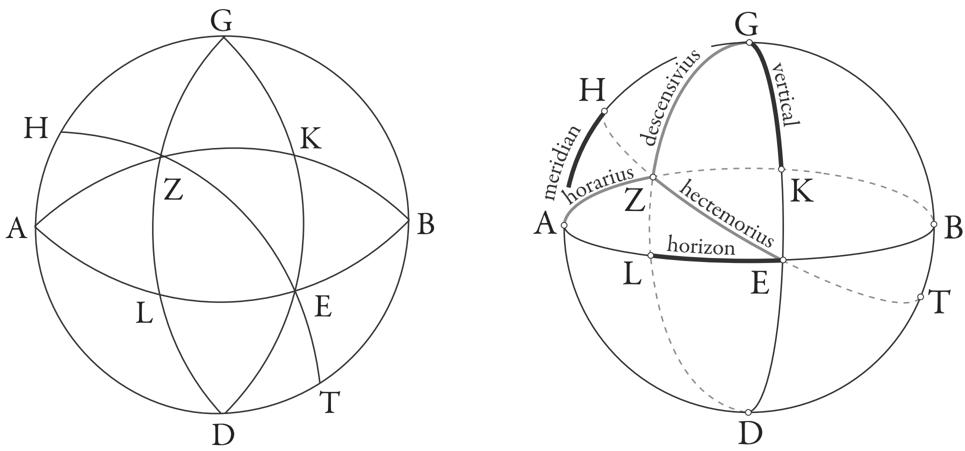



Figure 7: *Analemma* 3: (left) manuscript figure of the *world sphere*; (right) modern figure. All of the elements in the diagram are on the eastern hemisphere, which faces us.

lettered diagram, Figure 7, of the meridian circle and the circles to the east. Hence, Figure 7 depicts the eastern hemisphere of the model, such that all of the lines we see are on the outer surface of the sphere, facing us. Circle *ABGD* is the meridian, semicircle *AEB* is the horizon, and semicircle *GED* is the vertical. The semicircles *HZET*, *AZKB* and *GZLD* are the hectemorius, the horarius and the descensivus, respectively. Thus, the arcs which were mentioned as determining the solar ray are $(ZE, AH)$ as *hectemorius-meridian*, $(ZA, GK)$ as *horarius-vertical*, and $(ZG, EL)$ as *descensivius-horizon*.

*Analemma* 3 is devoted to the description of a certain geometric object and it does not deal with the *world sphere*, the real sun, or the actual solar ray.[45] Hence, the purpose of *Analemma* 3 is to describe a model—namely the geometric object introduced—that will henceforth stand in for objects in the real world. The rest of the *Analemma* deals only with this geometric model.

## 5. The mathematical approach

In order to understand the mathematical methods of the *Analemma*, we will follow through the determination of a solar position in local coordinates for one of the three angle pairs, namely the *hectemorius-meridian* pair.

Latin literally but have been guided by the fact that *scilicet* and various forms of *qui*, *ipse*, *idem*, and so on, are used by Moerbeke to translate various functions of the definite article, whose usage in Greek mathematical prose is well known; see Clagett, *Archimedes*, pp. 43–44; Vuillemin-Diem and Steel, *Ptolemy's* Tetrabiblos, p. 14, and Federspiel, 'Sur l'opposition'.

[45] The mention of the 'ray' that appears at the beginning of *Analemma* 3 is actually a reference back to the solar ray introduced in *Analemma* 2, stating that point $Z$ marks its position in the model.

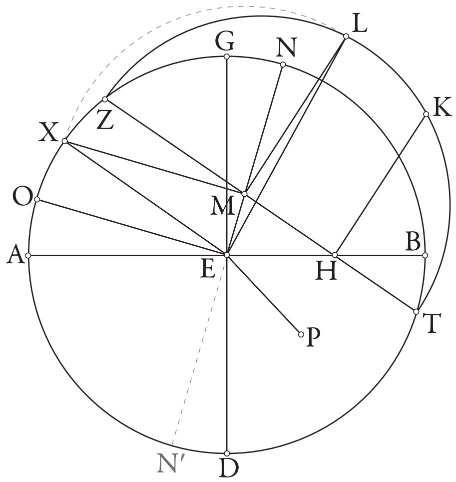



Figure 8: *Analemma* 6. Elements in gray do not appear in the manuscript diagram.

## 5.1. A synthetic proof

*Analemma* 6 begins with an introduction to the rest of the work. Ptolemy states that, with the foregoing as preliminaries, we will now set out the 'instrumental determinations' (ὀργανικαὶ λήψεις, *instrumentales acceptiones*)[46] of the coordinate angles. This appears to be a reference to the overall aim of the treatise of producing a physical analemma plate for making analog computations. Indeed, Ptolemy makes it clear that he will only supply a proof (ἀπόδειξις) for a single determination (λῆψις)—that of the new *hectemorius arc*, which he himself has introduced.[47] Hence, if we think of the remaining mathematical sections as establishing the methodological soundness of carrying out analog computations on the analemma plate, we can understand why Ptolemy would refer to this material generally as addressing 'instrumental determinations'.

Brushing off the case in which the sun is at one of the equinoxes as trivial, for the remaining solar positions Ptolemy gives a proof that a certain arc on the analemma is equal to the *hectemorius arc*, as follows (Figure 8 (left)):[48]

> Now, as for the remaining monthly [circles],[49] let there be a meridian circle, *ABGD*, in which a diameter of the horizon is *AB*, and at right angles to this along the

[46] The Greek is a conjecture by Edwards, *Ptolemy's* Περὶ ἀναλήμματος, p. 94, n. 454, based on the Greek of the following passage.

[47] Heiberg, *Opera astronomica minora*, pp. 194–195; Edwards, *Ptolemy's* Περὶ ἀναλήμματος, pp. 94–95, 136.

[48] The Greek for this passage is essentially complete, so I have translated Heiberg's text, but kept the letter-names of Moerbeke's Latin. In the palimpsest, this passage concludes Section 1.

[49] That is, besides the month-circles at the equinoxes.

gnomon is *GD*, and the center of the sphere of the sun is *E*,[50] while *ZHT* is a diameter of one of the monthly parallels north of the equator, upon which, in the same plane let an eastern semicircle, *ZKT*, be imagined (νοείσθω).[51]
And let *KH* be produced upright upon *ZT*,[52] such that section (τμῆμα) *ZK* of the parallel [circle] is made to be above the earth, and with arc *KL* being cut off,[53] let a perpendicular, *LM*, be produced from *L* to *ZT*.[54] And, with center *M* and distance *ML*, let a point, *X*, be determined on the meridian,[55] and let *EL*, *MN*, *EX*, and *MX* be joined,[56] and let *EO* be produced upright upon *EN*.[57] I say that angle *OEX* is equal to the sought angle.[58]

For, let semicircle *ZLT* be imagined (νοείσθω) as rotated (ἐπεστραμμένον) to its proper position, that is, the perpendicular to the plane of the meridian. And let a perpendicular, *EP*, be produced, as the equatorial diameter, to the same plane.[59] Then, *LM* being a perpendicular to the meridian, it is obvious that straight lines *EN*, *ML*, and *EP* are in a single plane perpendicular to *ABGD*.[60] Likewise, [it is obvious] that *EN* is the common section of the hectemorius circle and the meridian,[61] while *LE* is in line with the solar ray,[62] and the sought angle, which is contained by the ray and the equatorial diameter, is *LEP*. For, since *EL* is equal to *EX*,[63] and *ML* to *MX*,[64] and *EM* is common, therefore angle *MEL* is equal to angle *MEX*.[65] But angle *MEP* is right, and angle *MEO*,[66] therefore the remaining angle *LEP*[67]

[50] This is a clear indication that Ptolemy thinks of the analemma model as a simplifying assumption, since by his time it was well known that the sun does not orbit the earth in a simple sphere—although the model may have been developed at a time when this was still held to be so.

[51] The MS reads νοείσθαι, Heiberg corrects to νοείσθω.

[52] *Elements* I.11; set square.

[53] This is the arc of the seasonal hour, $\eta_s$.

[54] *Elements* I.12; set square.

[55] *Elements* I.post.3; compass. See the discussion of this construction below.

[56] *Elements* I.post.1; side of set square.

[57] *Elements* I.11; set square.

[58] Namely, the *hectemorius arc*.

[59] *Elements* XI.12. That is, the line about which the hectemorius rotates.

[60] *Elements* XI.6, 7 and 18.

[61] *Elements* XI.3.

[62] Note the clear language of modeling here. Ptolemy does not assert that *LE* is the line of the solar ray, but is in line with it—that is, models it.

[63] They are both radii of the sphere.

[64] By construction.

[65] That is, because $\triangle MEX \cong \triangle MEL$, by *Elements* I.8.

[66] Following this, both the Greek and the Latin include the phrase 'and since angle *EML*' (ἔπει καὶ ἡ ὑπὸ τῶν ΕΜΛ, *quoniam et qui sub* EML), which appears to be an interpolation, and has been noted as such by the modern editors.

[67] Here the manuscript includes the phrase 'to angle MEX, that is' (τῇ ὑπὸ ΜΕΞ τουτέστιν, *ei qui sub* MEX *hoc est*), which does not make sense and has been marked as an interpolation by the modern editors.

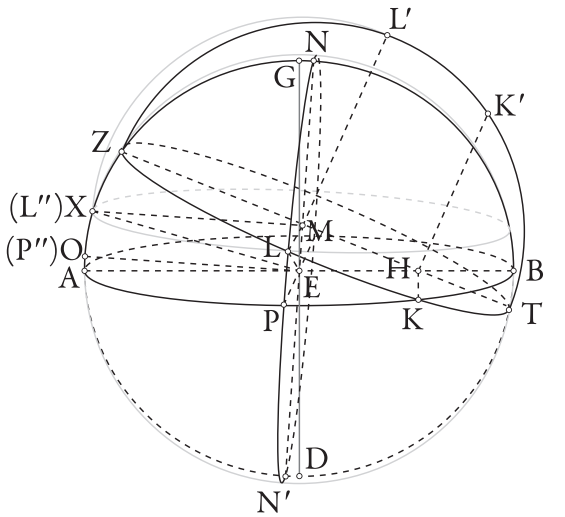



Figure 9: Perspective diagram for *Analemma* 6. Elements in gray do not correspond to any element in the manuscript diagram.

is equal to angle *XEO*. Which was to be shown.[68]

The key to understanding Ptolemy's argument is to consider the analemma figure as a representation of the solid sphere. There are a number of indications that this was Ptolemy's intention. The expression that he uses when he speaks of the monthly parallel, 'let it be imagined' (νοείσθω), is a standard expression in Greek geometric texts used to introduce solid constructions that cannot be fully, or accurately, represented by the plane figure.[69] When the semicircle of the parallel month-circle is introduced in the construction, it is *imagined* to be in the plane of the analemma, because it is, in fact, perpendicular to this plane. Ptolemy makes this clear in the proof when this circle is *imagined* rotated into its 'proper position'—namely, where it is found in the solid configuration. It seems likely that Ptolemy's readers could be expected to know how to view an analemma diagram as a solid configuration, or perhaps to have read the text while working with a solid sphere.

In order to illustrate this point, we will explain the argument with a perspective diagram. Considering Figure 9, let *BPA* be the local horizon and *BGA* the local meridian. Then the terrestrial latitude, $\varphi$, and the annual position

[68] Heiberg, *Opera astronomica minora*, pp. 195–198; Edwards, *Ptolemy's* Περὶ ἀναλήμματος, pp. 43–46.

[69] See, for examples, *Elements* XII.13, 16, *Conics* I.52, 54, 56, and Theo. *Spher.* I.19. The expression is also found many times in Archimedes' corpus. For a discussion of the various ways in which this verb is used to introduce objects, see Netz, 'Imagination'.

of the sun, $\delta(\lambda)$, are given by the inclination of the pole and the declination of the month-circle of the sun, *TKLZ*. Ptolemy makes no mention of this, simply assuming that they are fixed by the geometry of the figure. With *TKLZ* as the month-circle, section *KLZ*, being above the horizon, represents the motion of the sun from sunrise in the east to midday. Hence, the seasonal hour, $\eta_s$, must be taken as given along this arc, say as arc *KL*. This arc is taken as arbitrary in the construction. Because *Analemma* 6 is a theorem of synthetic geometry, however, there is no mention of any objects being given.[70]

Then, since *L* is the position of the sun and *P* is the east point of the local horizon, the hectemorius is the great circle *PLN* passing through *P* and *L*, and the *hectemorius arc* is *PL*. The construction then amounts to using analemma methods to project these points onto the analemma plane, while the proof amounts to showing that the arc that results from this projection is equal to arc *PL*.

The hectemorius circle is first projected orthogonally onto the analemma as line $NMEN'$, such that *P* maps to *E*, and *L* to *M* (M.1, M.2). Next, the location of *L* on the hectemorius is rotated into the plane of the analemma in two ways, by the method set out in the first example treating analemma methods above, Section 2.1. That is, *L* is mapped to the intersection of the circle about center *M* with distance *ML*, which is perpendicular to line *EN*, with the analemma circle, at point *X*. This construction is effected by taking *X* as the intersection of a circle drawn about center *M* with distance *ML*. In this way, *L* maps to *X*. The circle that produces *X* is not actually drawn in the plane of the analemma—probably because arc lengths are not preserved on it. Then, we project one of the endpoints of the diameter about which the hectemorius rotates into the analemma by erecting a perpendicular to the orthogonal projection of the hectemorius circle, $NEN'$, at *E*, so that *P* maps to *O* (M.3)—effectively, rotating the hectemorius circle into the plane of the analemma (M.4). Ptolemy does not talk about the hectemorius circle as rotated, he simply constructs the points *X* and *O* in the plane, first with the distance of a circle that is not itself drawn and then by constructing a perpendicular. The production of point *X* with distance *ML* is an interesting construction because it differs from any construction performed in the problems of *Elements* I–VI, insofar as the circle about center *M* is not actually drawn—only the point that is cut off by the circle is produced. In the *Elements*, such points are cut off on lines, as justified by *Elements* I.3, but not on circular arcs. We may regard this construction as an analemma construction, and we will see

[70] In general, Greek mathematicians only use the language of givens when treating problems, or in theorems written to facilitate certain problem-solving practices—for example, we do not read of objects being given in the synthetic theorems of the *Elements*; see Acerbi, 'The Language', and Sidoli, 'The Concept'.

that such constructions are often employed in the nomographic procedures on the analemma plate. Finally, with points $X$ and $O$ produced, it is a matter of elementary geometry to show that $\triangle MEL \cong \triangle MEX$, so that $90°-\angle MEL = 90°-\angle MEX$, that is arc $LP =$ arc $XO$.

The analemma diagram, Figure 8, represents three circles in three different planes superimposed upon one another in the plane of the figure—which is the analemma. The circle of the meridian, *ADBG*, lies in the plane of the figure. The circle of the hectemorius, $NPN'$, is perpendicular to the plane of the figure and intersects it in line $NEN'$. The month-circle, *TKLZ*, is perpendicular to the plane of the figure and intersects it in line *THMZ*. Although I have spoken of rotations and projections to help explain the solid configuration that motivates the construction, all but one of the constructive steps presented in the proof in *Analemma* 6 are carried out directly in the plane of the analemma—following what appears to have been a common practice among Greek geometers for handling solid configurations.[71] Moreover, because this is ostensibly a purely geometric argument, I have justified each constructive step with a problem, or postulate, from the *Elements* as well as by operations of a set square and compass as described in *Analemma* 11.

The next two sections of the *Analemma* set out the constructions for the remaining five angles with no proofs.[72] For our purposes, here, we will simply note that the arc of the meridian, which completes the angle pair with the hectemorius, is equal to arc *AO* in Figures 8 and 9.

This synthetic proof—which makes explicit reference to the solid sphere—provides the background to understanding analemma methods. Since we will not return to the solid configuration in this discussion, it may be helpful to summarize the analemma construction of the *hectemorius-meridian* angle pair. The *hectemorius arc* is found as follows:

**Hec.1:** The diameter of the hectemorius circle is found by taking the orthogonal projection of the solar position, $L$, onto the meridian plane, $M$ (M.1); and then joining this with the center, $E$, extended to produce $NMEN'$ (M.2).

**Hec.2:** The *hectemorius arc* is found by rotating the hectemorius circle into the plane of the meridian circle about its diameter, $NMEN'$ (M.4), such that $P$, or $E$, the east point, maps to $O$, and $L$, the solar position, maps to $X$. Using analemma methods, we find $X$ by taking the intersection of a

[71] See note 11, above.

[72] Luckey, 'Das Analemma', cols 25–26, gives a clear account of how the remaining analemma constructions are related to the solid sphere. Following these descriptions it would be a relatively simple matter to reconstruct proofs along the lines of *Analemma* 6. For a summary of the constructions of all the arcs in the analemma, see Guerola Olivares, *El Collegio Romano*, pp. 81–101.

circle of radius *ML* about center *M*, or a perpendicular erected to *NEN′* at *M*, with the analemma, and we find *O* by taking the intersection of a perpendicular to *NEN′* erected at *E* with the analemma (M.3).

Because the meridian circle is in the same plane as the figure, which is the analemma plane, the analemma construction of the *meridian arc* is somewhat simpler:

**Mer.1:** The diameter of the hectemorius circle is found, as before, by taking the orthogonal projection of the solar position, *L*, onto the meridian plane, *M* (M.1); and then joining this with the center, *E*, extended to produce *NMEN′* (M.2).

**Mer.2:** The *meridian arc* is found by rotating the hectemorius circle about *NMEN′*, such that, as before, *P*, or *E*, maps to *O*, and the *meridian arc* is cut off on the meridian circle between *O* and *A*—the south point (M.4). Again, using analemma methods, we simply erect the perpendicular from *E* to *NEN′* (M.3).

5.2. A metrical analysis

In *Analemma* 9, after introducing the methods of nomographic computation discussed above, Ptolemy explains that each of the six principal arcs can also be calculated using geometric, indeed trigonometric, means:[73]

> Such a determination, for those who prefer, would also exist precisely by means of lines (διὰ τῶν γραμμῶν),[74] although it would be easily brought about through

[73] Again, I have translated the Greek for this passage.

[74] Διὰ τῶν γραμμῶν is a technical expression in Ptolemy's writings; see Heiberg, *Syntaxis mathematica*, vol. I, pp. 32, 42, 251, 335, 380, 383, 416, 439; vol. II, pp. 193, 198, 201, 210, 321, 426, 427, 429; Heiberg, *Opera astronomica minora*, pp. 202, 203. It designates the geometric means through which a computation can be, or has been, carried out, either by elementary geometry, or by chord-table trigonometry. When it is used in the *Almagest*, it is in reference to either an actual calculation or to a metrical analysis, where the later is understood as showing that the former is, in principle, possible. As Edwards, *Ptolemy's* Περὶ ἀναλήμματος, p. 107, n. 512, suggests, the phrase is closely related to chord-table trigonometry, and in a number of places in the *Almagest* it clearly must indicate computation through chord-table trigonometry; see Heiberg, *Syntaxis mathematica*, vol. I, pp. 251, 335, 380, 439; vol. II, pp. 321, 426, 427. In a number of other places it refers to a computation, which, considering the context, was almost certainly carried out through chord-table trigonometry; see Heiberg, *Syntaxis mathematica*, vol. I, pp. 383, 416, 439; vol. II, p. 429. And, finally, in some cases it refers to a metrical analysis that is intended to justify a computation; see Heiberg, *Syntaxis mathematica*, vol. II, pp. 193, 198, 201, 426.

We find two uses of the phrase διὰ τῶν γραμμῶν in reference to the same metrical analysis, which, taken together, make it clear that we must understand this metrical analysis as justifying, or standing in for, trigonometric calculation; see Heiberg, *Syntaxis mathematica*, vol. II, pp. 426, 427, and Nathan Sidoli, 'Mathematical Tables', pp. 25–26.

> the analemma itself, even if it is not exactly the same as that through geometrical demonstrations (διὰ γραμμικῶν ἀποδείξεων), but rather near enough for theory in agreement with the senses, to which the practical goal of the proposed task leads.[75]

He then provides what I call a metrical analysis to show that given the declination of the sun, $\delta(\lambda)$, terrestrial latitude, $\varphi$, and seasonal hour, $\eta_s$, each of the six angles is also given—for $\lambda = 0°$ and $\lambda = 180°$ in *Analemma* 9, and for all other $\lambda$ in *Analemma* 10. A metrical analysis is a type of argument about what is *given* that is found in the writings of both Heron and Ptolemy, in which each step can generally be justified by reference to theorems of Euclid's *Data*, but which itself justifies, or establishes the possibility of, a computational procedure involving arithmetical operations—adding, subtracting, multiplying, dividing, and taking square roots—and, in the case of Ptolemy, entries into a chord table.[76] After reading one of Ptolemy's metrical analyses, we will discuss the significance of this type of argument.

In the foregoing passage, the distinction between producing the final determinations 'by means of lines', or 'through geometrical demonstrations', on the one hand, and those brought about 'through the analemma itself', on the other, addresses the fact that there will probably be slight differences in the values obtained through chord-table trigonometry, as justified by the metrical analyses of *Analemma* 9 and 10, and those obtained through the analog, or nomographic, methods, that will be provided in *Analemma* 12–14. Here, Ptolemy claims that the minor differences in these values will not undermine the overall validity of the analog calculations, which will produce values good enough for the practical goals of the treatise.

After giving the metrical analysis for the six angles in the case where the sun is at one of the equinoxes, in *Analemma* 9, the passages of metrical analysis in *Analemma* 10 that concern the *hectemorius-meridian* pair read as follows (Figure 10):[77]

[75] Heiberg, *Opera astronomica minora*, pp. 202–203; Edwards, *Ptolemy's* Περὶ ἀναλήμματος, p. 50.

[76] *Metrical analysis* is my terminology, but this type of argument is called an 'analysis' by Heron throughout his *Measurements*, and by Pappus in his commentary on Ptolemy's *Almagest* V; see Rome, *Commentaires*, p. 35. I have not found a passage where Ptolemy himself refers to this type of argument as an 'analysis'. I have discussed elsewhere the role of this type of argument with respect to mathematical tables in the *Almagest*; see Sidoli, 'Mathematical Tables', pp. 25–26. See also the discussion by Acerbi, 'I codici stilistici', pp. 201–208; note, however, that his attempt to rewrite Ptolemy's prose should be treated with caution—there is good reason why Ptolemy does not include the passages that Acerbi adds to the text (see n. 107, below).

[77] Only the beginning of this passage is preserved in Greek; I have translated first from the Greek and then from the Latin. In the palimpsest this passage begins Section 5. Here I translate only those passages necessary for the determination of the *hectemorius-meridian* angle pair.

As for the remaining monthly [circles],[78] let meridian *ABGD* be set out along with the diameters upright upon one another and axis *EZ*. And let a diameter, *HTK*, of any of the southern monthly parallels to the equator be produced through, upon which let semicircle *HLK* imagined (νοούμενον)[79] to the east be drawn.[80] And let axis *EZL* be extended,[81] obviously bisecting diameter *HTK* at *T* and semicircle *HK* at *L*.[82] And let line *MN* be produced through [as a perpendicular] to *HT*,[83] dividing the section, *HN*, of the semicircle above the earth from that below the earth. And with arc *NX* being determined as the given hours,[84] let a perpendicular, *XO* be produced from *X* to *HM*.[85] And through *O* let perpendiculars upon *AE*, *POR*, and upon *GE*, *SOC*, be produced through.[86]

Then, since arc *HTK* of the meridian is given,[87] and the double of line *ET* subtends its remainder in the semicircle, the ratio of *HTK* and *ET* to the diameter of the meridian will be given.[88] Likewise, since arc *AZ*, of the elevation [of the pole], is given,[89] angle *MET* of right-triangle *MET* will also be given. So, the ratio of *ET* to each of *EM* and *MT* will also be given,[90] and, moreover, that of diameter *HK* to each of them.[91] But, the double of line *MT* subtends the double of arc *LN*, so arc *LN* will

[78] That is, besides the month-circle at the equinoxes.

[79] The use of 'imagined' is a reminder that, in order to understand the analemma diagram, we must consider it as a representation of a solid configuration.

[80] The assumption of this configuration produces the first two given magnitudes: $\varphi$ and $\delta(\lambda)$.

[81] *Elements* I.post.2; side of the set square.

[82] *Elements* III.3, I.def.17.

[83] *Elements* I.11; set square. The location of *M* on the horizon is specified in the phrase that follows. The fact that this line must be perpendicular is made explicit in Moerbeke's Latin.

[84] This is the final given magnitude, $\eta_s$.

[85] *Elements* I.12; set square.

[86] *Elements* I.12; set square.

[87] In terms of the *Data*, this is so because both meridian *ADBG* and line *HK* are assumed to be given in position, so that, by *Data* 25 and 26, line *HK* is given in position and in magnitude, so that, by *Data* Def.8, arc *HTK* is given in position and magnitude. Edwards, *Ptolemy's* Περὶ ἀναλήμματος, p. 111, n. 537, points out that computationally this follows since arc $HZK = 180° - 2\delta(\lambda)$.

[88] That is $(HTK : \text{diameter}_m)$ and $(ET : \text{diameter}_m)$ are given. The first ratio is given by *Data* 1. The second ratio is given because, by *Data* 88, $\angle HEK$ is given, so that, by *Data* 2, half of it, $\angle HET$ is given. Then, by *Elements* I.32 and *Data* 4, the angles of $\triangle HET$ are given, so that, by *Data* 40, $\triangle HET$ is given in form. Then, by *Data* Def.3, $(ET : EH) = (ET : \text{radius}_m)$ is given, so that, by *Data* 8, $(ET : \text{diameter}_m)$ is given. In terms of computation, the use of *Data* 88 would involve entering into a chord-table, as would the computation of ratios based on angles implied by the use of *Data* 40.

[89] This is $\varphi$—assumed to be given by the geometric configuration, or by taking the meridional altitude of the pole, in degrees.

[90] That is, since by *Elements* I.32 and *Data* 4, the angles of $\triangle MET$ are given, by *Data* 40, $(ET : EM)$ and $(ET : MT)$ are given. The computation of these ratios would involve entering into a chord-table.

[91] That is, by *Data* 8, $(EM : \text{diameter}_m)$ and $(MT : \text{diameter}_m)$ are given.

also be given,[92] as well as the remainder *NXH* from a quadrant.[93] But *NX* is given,[94] therefore both *LX* and *XH* will be given.[95] But, the double of line *XO* subtends the double of arc *HX*, and the double of line *OT* [subtends] the double of arc *XL*,[96] so the ratio of *XO* and *OT* to diameter *HK* is given,[97] and because of this also to that of the meridian.[98] But, since the ratio of *TM* [to diameter$_m$] is also given,[99] the ratio of *MO* [to diameter$_m$] will be given.[100] And it is that as *EM* to *MO*, so is *TM* to *MP* and *ET* to *OP*,[101] for the triangles *EMT* and *OPM* are equiangular.[102] Therefore, the ratio of *MP* and *OP* to the diameter of the meridian is given.[103] On account of this, also the ratio of *ES* [to diameter$_m$], and of the whole of *EMP*, which is *OS* [to diameter$_m$, will be given].[104]

With these things demonstrated, with center *O* and distance *OX* let a point of the meridian, *Y*, be determined.[105] [...] And let *EY* [...] and *EO* [...] be joined.[106]

[92] That is, since by *Data* 2, 2*MT* is given, arc 2*LN* is given by *Data* 88. Computationally, we enter into a chord-table.

[93] *Data* 4.

[94] This is determined by $\eta_s$. Geometrically, it is simply assumed as given by taking the seasonal hour from sunrise to noon going from *N* through *X* to *H*, or from noon to sunset going back from *H* to *N*. Arc *NH* can be divided into six parts using one of the various trisections of an angle preserved in Greek sources; see Heath, *A History*, vol. I, pp. 235–244. Computationally, arc $NX = \eta_s \cdot$ arc $NH/6$; see Edwards, *Ptolemy's* Περὶ ἀναλήμματος, p. 112, n. 542.

[95] *Data* 3 and 4.

[96] That is, in semicircle *HLK*, by *Data* 87, 2*XO* and 2*OT* are given; so by *Data* 2, *XO* and *OT* are given.

[97] *Data* 1.

[98] That is, by *Data* 8, $(XO : \text{diameter}_m)$ and $(OT : \text{diameter}_m)$ are given. The Greek fragment ends here—we continue with Moerbeke's Latin. With regard to my translation choices, see note 44, above.

[99] That is, $(TM : \text{diameter}_m)$ is given, as shown above.

[100] That is, since $(TO : \text{diameter}_m)$ and $(TM : \text{diameter}_m)$ are given, by *Data* 8, $(TO : TM)$ is given. Hence, by *Data* 5, $(TO : (TO - TM))$ is given. Hence, again by *Data* 8, $((TO - MT) : \text{diameter}_m) = (MO : \text{diameter}_m)$ is given.

[101] *Elements* VI.4.

[102] *Elements* I.15 and 32.

[103] That is, $(EM : TM) = (MO : MP)$, and $(EM : ET) = (MO : OP)$, and each of $(EM : TM)$ and $(EM : ET)$ are given, so by *Data* 8, $(MP : \text{diameter}_m)$ and $(OP : \text{diameter}_m)$ are given.

[104] That is, since $(ME : \text{diameter}_m)$ and $(MP : \text{diameter}_m)$ are given, by *Data* 8, $(ME : MP)$ is given. Hence, by *Data* 6, $((ME + MP) : (ME))$ is given. Hence, again by *Data* 8, $((ME + MP) : \text{diameter}_m) = (OS : \text{diameter}_m)$ is given.

[105] *Elements* I.11; compass.
**O** reads *G* in place of *Y*—which error was noted by the modern editors.

[106] *Elements* I.post.1; side of the set square. Heiberg, *Opera astronomica minora*, pp. 206–209, Edwards, *Ptolemy's* Περὶ ἀναλήμματος, pp. 54–56.

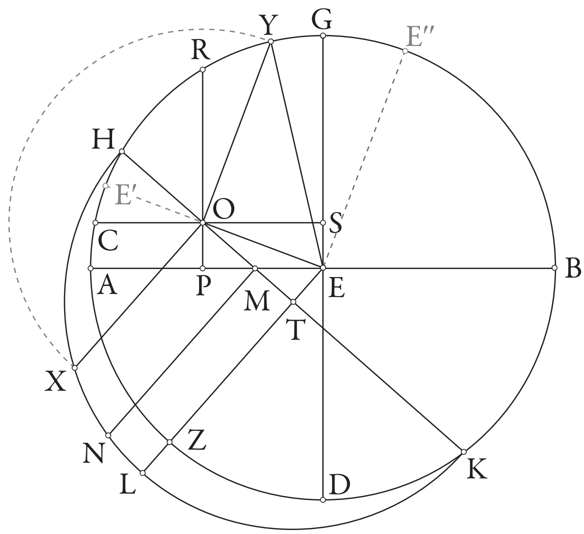



Figure 10: *Analemma* 10: Partial diagram of the analemma. Objects which do not concern the *hectemorius-meridian* angle pair have been omitted. Elements in grey do not appear in the manuscript diagram.

As before, in Figure 10, the solar position is projected into the analemma in two ways. Since $O$ is the orthogonal projection of the solar position, $X$, onto the plane of the meridian—produced by dropping a perpendicular from $X$ to the diameter of the solar month-circle, $HK$ (M.1)—the line $OE$, joining $O$ with the orthogonal projection of the east point, $E$, will be the diameter of the hectemorius circle (M.2). Hence, the *hectemorius arc*, $YE''$, will be produced by rotating the hectemorius circle into the meridian (M.4)—that is, by taking $Y$ on the analemma circle such that $OY = OX$, and erecting $EE''$ perpendicular to $OE$ (M.3). Since, by *Elements* I.29, $\angle YEE'' = \angle EYO$, Ptolemy will simply work with $\angle EYO$—probably to avoid having to produce $EE''$ in an already cluttered diagram. The *meridian arc* is found by extending the diameter of the hectemorius circle, $OE$, to meet the meridian at $E'$, and taking the arc between this intersection and the south point, arc $AE'$. Since arc $AE' = \angle PEO$, Ptolemy simply works with this angle—again to avoid producing any unnecessary lines.

Up to this point in the metrical analysis, Ptolemy has dealt with all lines in terms of ratios to other lines. This practice agrees with that found in the *Almagest* for plane trigonometry, and derives from the fact that when we enter into a chord table with an angle, the resulting chord is always given in terms of the radius of the circumscribing circle—that is, as a ratio.[107] In what

[107] This is the reason why Acerbi, 'I codici stilistici', pp. 204–208, has gone too far in attributing to Ptolemy claims about given lines in his rewriting of the ancient text.

follows, however, he will assert that the radius of the meridian is given, and he will then state the other lengths as also given—that is, given in terms of the diameter of the meridian.

The foregoing passage continues to argue that the six principal arcs are all given. We read only those passages pertaining to the *hectemorius-meridian* pair:[108]

> Since, then, in the preceding, the angle *EOY* was shown to be right,[109] while hypothenuse *EY*, being a radius of the meridian, is given,[110] as well as *OY*, being equal to *OX*,[111] angle *EYO*, containing that of the hectemorius circle, will be given.[112] [...] Then, since both *OP* and *EP*, of right-angled [triangle] *EOP* are given,[113] hypothenuse *EO* and angle *OEP*, which makes the *meridian arc*, will be given.[114] [...][115]

The foregoing metrical analysis constitutes a general argument that, where $\delta(\lambda)$, $\varphi$, and $\eta_s$ are assumed as given, the two arcs of the *hectemorius-meridian* angle pair are also given—that is fixed, or determined. The argument, as all extant metrical analyses, works on two levels: (1) as a purely geometrical proof, in which each step can be justified by theorems of the *Data*, and (2) as the articulation of an effective computational procedure, involving the basic arithmetic operations, taking square roots and entries into a chord-table.[116] That is, in the *Analemma*, for Ptolemy, *given* means both *geometrically given*—that is, producible using the constructive methods of *Elements* I–VI—and *numerically given*—that is, computable as some numerical value.[117]

As he stated at the beginning of *Analemma* 9, this argument constitutes a demonstration that these arcs can be determined 'by means of lines' (διὰ τῶν γραμμῶν)—that is, they are computable through geometric, or rather

[108] There is no Greek for this passage.

[109] This is a reference to the argument in *Analemma* 6, in which it was shown, in Figure 8, that $\triangle LME \cong \triangle XME$ and *LM* was imagined as constructed perpendicular to the plane of the meridian.

[110] That is, assumed as given—given by the geometry of the figure, or taken, as say $60^p$, following the practice of the *Almagest*.

[111] *Data* 2, since $(OX : \text{diameter}_m)$ was shown to be given.

[112] *Data* 43. Computationally, this involves entering a chord-table.

[113] *Data* 2, since $(OP : \text{diameter}_m)$ and $(OS : \text{diameter}_m)$ were shown to be given.

[114] *EO* is given by *Data* 52, 3 and 55; so that, by *Data* 39, $\angle OEP$ is given. Calculating this angle would involve Ptolemy's usual convention of taking *OP* when *OE* is assumed as given, and entering a chord-table.

[115] Heiberg, *Opera astronomica minora*, p. 209; Edwards, *Ptolemy's* Περὶ ἀναλήμματος, p. 56.

[116] Edwards, *Ptolemy's* Περὶ ἀναλήμματος, pp. 115–117, gives an example calculation following Ptolemy's methods that proceeds along the same lines as that established by Ptolemy's metrical analysis.

[117] Ptolemy does not express any concern with the fact that the numerical value used to express certain geometric objects will not be perfectly precise.

trigonometric, methods.[118] In the context of this treatise, this is contrasted with the analog computations that will be outlined below. The determinations, and likewise computations, 'by means of lines' are said to be more precise than those of the nomographic procedures to which we now turn.

### 5.3. An analog computation

In *Analemma* 11, Ptolemy describes the production of an analemma plate, on which 'with only the compass and the set square' (διὰ μόνου τοῦ τε καρκίνου καὶ τοῦ ὀρθογωνίου πλατύσματος)[119] constructions of the six principal arcs can be carried out.[120] The plate is a 'drum-shaped plane' (τυμπανοειδὲς ἐπίπεδον) on which certain, permanent lines are drawn. Ptolemy gives instructions for how this plate should be made using a lettered diagram that serves as a sort of mathematical recipe for a physical construction. The finished plate, however, would probably not have had labels on it, as in Figure 11.

The plate may be made of inscribed lines on bronze or stone, or colored lines drawn on wood, which is then covered with wax so that the horizon and gnomon can be drawn in the wax.[121] The wooden tablet—which we will treat here—is inscribed with red lines for the meridian and the diameter of the equator and black for three month-circles. Quadrants, graduated at 1° intervals, are produced on one or both sides of the equator, as well as in one of the quadrants of the outer circle. In each quadrant of the outer circle a set of seven marks is drawn for the elevation of the pole at seven well-known latitudes, $\varphi$, of Greco-Roman geography: 16 1/3 1/12°, 23 1/2 1/3 °, 30 1/3 °, 36°, 40 1/2 1/3 1/12°, 45°, and 48 1/2°.[122] Three month-circles are inscribed corresponding to the following solar declinations, $\delta(\lambda)$: 23 1/2 1/3°, 20 1/2°, and 11 2/3°.[123] The first month-circle is used when the sun is at the tropics, in the signs of Cancer or Capricorn ($\lambda \approx 90°, 270°$); the second is used when the sun is in Gemini, Leo, Sagittarius or Aquarius ($\lambda \approx 60°, 120°, 240°, 300°$); the third is used when the sun is in Taurus, Virgo, Scorpio or Pisces ($\lambda \approx 30°, 150°, 210°, 330°$); while the equator is used when the sun is in Aries or Libra ($\lambda \approx 0°, 180°$). The wooden plate is orientated by rotating it such that the elevation corresponding to the given terrestrial latitude, $\varphi$, is in the zenith and drawing the horizon and the gnomon in the wax. In the operations to be described below, arcs will be set

[118] See note 74 for a discussion of the meaning of the phrase 'by means of lines' (διὰ τῶν γραμμῶν).

[119] Literally, 'crab' and 'rectangular plate'.

[120] Heiberg, *Opera astronomica minora*, p. 212.

[121] Presumably working with bronze or stone involved having a number of different sets of plates for the different latitudes.

[122] These latitudes should be compared with those in *Almagest* II.6 and *Geography* I.23.

[123] These should be compared with the declinations in *Almagest* I.5.

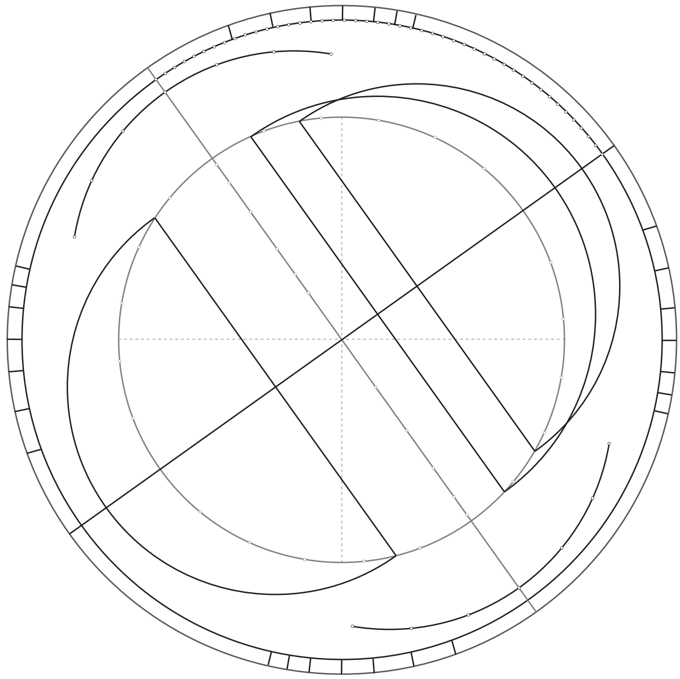


Figure 11: *Analemma* 11: The wooden analemma plate, rotated to carry out computations at the latitude of Rhodes, $\varphi = 36°$. Red lines are shown in gray, dotted lines are to be drawn in the wax.

out on the meridian. These arcs can then be measured by carrying them, with the compass, to one of the graduated quadrants at the side, which have the same diameter as the meridian.

In order to follow the method of the analog computation, we will read *Analemma* 13 closely, with a new diagram for each step of the procedure.[124] *Analemma* 13, in which Ptolemy describes the analemma-plate computation for the six principal angles, begins as follows:[125]

> Again, [1] let a diameter of any one of the monthly circles be modeled, and let it be *ZHTK*, upon which is the eastern semicircle *ZLK*.[126] [2a] And with center *T* and

[124] Luckey, 'Das Analemma', cols 32–39, gives a complete account of the nomographic computation for all of the angles. See also the account by Guerola Olivares, *El Collegio Romano*, pp. 122–131.

[125] There is no Greek text for this part of the treatise; I translate Moerbeke's Latin—omitting those passages unnecessary to the computation of the *hectemorius-meridian* pair.

[126] *L* has not actually been produced yet, so at this point it serves as an unspecified name for the semicircle. The positioning of *L* and *M* in the diagram provided by Heiberg, *Opera astronomica minora*, p. 219, which accurately reflects that on **O** f. 64v, is incorrect.

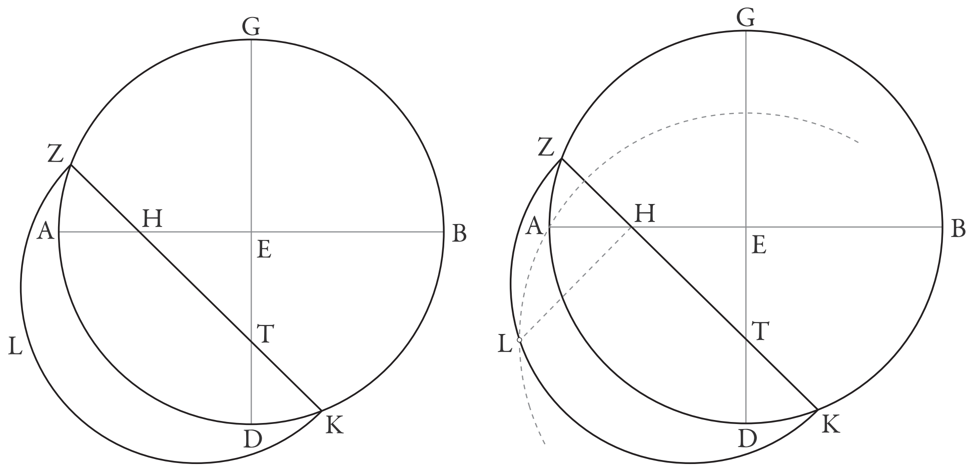



Figure 12: *Analemma* 13: Computing on the analemma plate, steps [1] and [2]: (left) initial set up, determination of $\delta(\lambda)$ and $\varphi$; (right) determination of $L$, the position of the horizon on the month-circle.

> distance $TA$, let a point of the meridian, $L$, be determined,[127] by which $ZL$—the semicircle above the earth—and $LK$—below the earth—are separated. [2b] But point $L$ is determined with the set square (*per platinam rectangulam*) if the angle will have been brought to $H$ such that the other side is fitted to $ZK$—for what the remaining side cuts on the semicircle will be the point [$L$ ], because the perpendicular produced from $H$ of $HK$ will be the [common] section of the planes of the horizon and the monthly circle.[128]

In the first step, [1], we orientate the plate to the given latitude—say $\varphi = 36°$, the latitude assumed for Rhodes—and draw the horizon, $AB$, and the gnomon, $GD$, into the wax on the plate. In the actual procedure, we would not need to label these lines, but we label them in Figure 12 for the sake of clarity. We then chose one of the month-circles, which will determine $H$ and $T$—the plate can be rotated 180° so that any month circle can be taken in the northern or southern direction.

The second step, [2], can be carried out in two ways. We determine the point on the month-circle that divides between day- and night-time, $L$, by [2a] either using the compass to take the intersection of a length $TL = TA$ on the month-circle, [2b] or using the set square to take the perpendicular from $H$, the intersection of the diameter of the month-circle and the horizon. That we must set $TL = TA$ is clear from considering the solid configuration—as hinted

[127] The fact that $TA = TL$ can be shown by considering the solid configuration—see below.

[128] Heiberg, *Opera astronomica minora*, p. 219; Edwards, *Ptolemy's* Περὶ ἀναλήμματος, pp. 68–69.

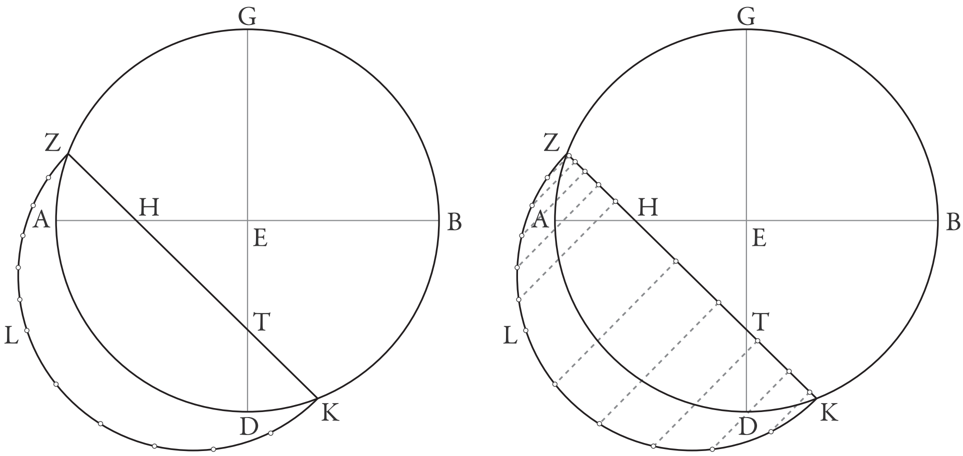



Figure 13: *Analemma* 13: Computing on the analemma plate, steps [3] and [4]: (left) determination of the day- and night-time hours; (right) projection of the hours onto the diameter of the month-circle.

at in the text. If *ZLK* is rotated into its proper position, *A* and *L* will both lay on the great circle of the horizon and *T* is some point on the gnomon. Since the gnomon is perpendicular to the horizon and passes through its center, by an argument similar to that in Theodosius' *Spherics* I.1, the distances from *T* to every point on the great circle of the horizon are equal.

*Analemma* 13 continues, as follows:

> Then, [3] let each section [*ZL* and *LK*] be divided equally in 6, and with these points, [4] by an application of the set square let points on *ZK* made by perpendiculars to it from the divisions determined on the semicircle be determined.[129]

It is not stated, in step [3], how to perform the division of the daytime arc into six parts. Various possibilities come to mind. We could use one of the purely geometrical solutions to this problem that are extant in the ancient sources—for example, one from among those treated by Pappus in *Collection* IV.[130] Indeed, Pappus tells us that he showed how to trisect an angle in his lost commentary to the lost *Analemma* of Diodorus.[131] Alternatively, the plate itself could be used to perform this division as follows:

- We draw an auxiliary circle with the same radius as the month-circle, concentric with the meridian,

[129] Heiberg, *Opera astronomica minora*, p. 219, Edwards, *Ptolemy's* Περὶ ἀναλήμματος, p. 69.

[130] See Hultsch, *Pappi Collectionis*, pp. 272–288, and Sefrin-Weis, *Pappus. Book 4*, pp. 146–155. Heath, *A History*, vol. I, pp. 235–244, gives an overview of the ancient solutions to trisecting an angle.

[131] See Hultsch, *Pappi Collectionis*, pp. 244–246.

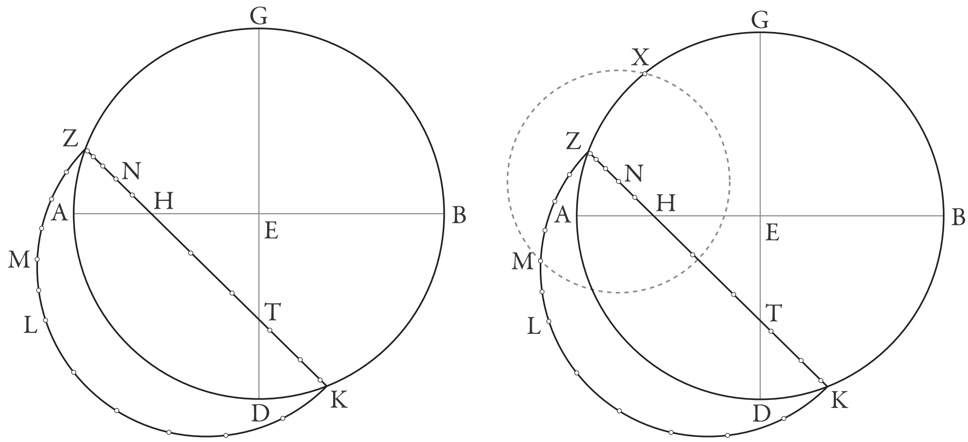



Figure 14: *Analemma* 13: Computing on the analemma plate, steps [5] and [6]: (left) determination of the solar position, $M$, and its projection onto the diameter of the month-circle; (right) determination of the solar position on the hectemorius circle in the plane of the meridian, $X$.

- we transfer the daytime arc to this auxiliary circle with the compass, such that one endpoint falls on the axis that bounds the outer graduated quadrant,
- we project the other endpoint onto the outer graduated quadrant with the set square,
- we read off the angle measure on the graduated quadrant and divide this value by six,
- we mark this value off on the graduated quadrant and project this arc back onto the auxiliary circle, and
- we transfer this arc back to the month-circle with the compass and mark it off six times.

Since both the geometrical and analemma plate methods of producing the hours are non-trivial, it seems likely that Ptolemy took his readers to have some familiarity with these sorts of constructive procedures.

Step four, [4], is carried out by lining up one side of the set square on the diameter of the month-circle such that the other side passes through the hour points—as is made explicit in the text. The points on the diameter of the month-circle are then marked at the angle of the set square.

The text continues, as follows:

But, [5] let one of them that is above the earth be that at $M$,[132] and the ordinate

[132] This is the given hour, $\eta_s$.

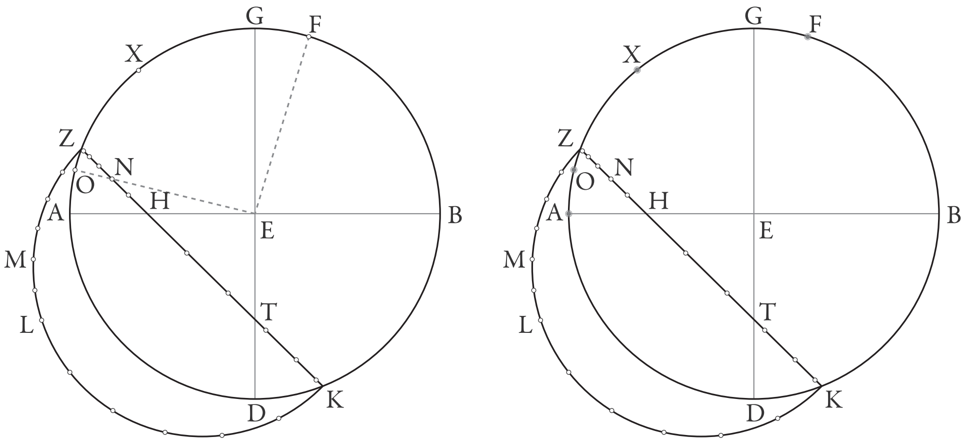



Figure 15: *Analemma* 13: Computing on the analemma plate, steps [7] and [8]: (left) determination of the diameter of the hectemorius circle and of the east point of the hectemorius circle in the plane of the meridian; (right) determination of the *hectemorius-meridian* pair, *XF* and *OA*.

> with it, *N*, of those on *ZH*. Then, [6] with center *N* and distance *NM*, let point *X* be determined on the meridian.[133]

Step five, [5], simply consists in choosing a pair of corresponding points along arc *LZ* and line *HZ* for the solar position of the sun at the given hour, *M*, and its projection onto the diameter of the month-circle, *N*.

In step six, [6], we find the projection of the solar position onto the rotation of the hectemorius into the plane of the meridian—that is, the plane of the analemma. Following the first example of the analemma methods in Section 2.1 and the construction provided in *Analemma* 6, this is found by setting the stationary end of the compass on *N*, the mobile end on *M*, and then marking the intersection of the mobile end with the meridian at *X*.

The material from *Analemma* 13 that concerns the *hectemorius-meridian* pair concludes as follows:

> And, [7] with the side of the set square brought to points *E* and *N* such that it cuts the meridian at *O*, [8] arc *XO* will make the complement of the hectemorius,[134] and that from *X* to the other intersection of the set square and the meridian, [*F*,] is the hectemorius [...] Again, arc *AO* will make that of the meridian [...][135]

[133] Heiberg, *Opera astronomica minora*, p. 219; Edwards, *Ptolemy's* Περὶ ἀναλήμματος, p. 69.
[134] **O** reads *ZO* in place of *XO*—which error was noted by the editors.
[135] Heiberg, *Opera astronomica minora*, pp. 219–220; Edwards, *Ptolemy's* Περὶ ἀναλήμματος, p. 69.

In step seven, [7], we determine the diameter of the hectemorius circle by placing the angle of the set square at the center of the figure and passing one side over point *O* so that the other side falls above the earth, at point *F*.

In the final step, [8], we note that *XF* is the *hectemorius arc* and *AO* is the *meridian arc*. These can be measured by placing the compass points at their endpoints and then transferring them to the graduated quadrant at the equator.

By following a series of physical manipulations of this sort, each of the six principal arcs can be computed nomographically.

## 6. Conclusion

Following the details of Ptolemy's presentation of the *Analemma*, as we have done in this paper, has made it clear that the analemma methods, as a loose collection of problem-solving methods in the science of *gnomonics*, were closely associated with various instrumental practices. We have seen both a restriction to operations that can be performed by abstractions of realizable instruments and explicit instructions for the production and use of an analemma plate as a tool for analog computations. This basis in instrumental practice, and its justification through metrical analysis, appears to have been a significant part of Ptolemy's mathematical bequest to scholars of the mathematical sciences in the late ancient and medieval periods. This explicitly instrumental approach, which is not found in the extant writings of authors like Euclid or Diophantus, was, nevertheless, an important aspect of the Greek mathematical sciences.[136]

Although there is one theorem in the *Analemma*, the analemma methods, as they are preserved in ancient and medieval sources, were clearly focused around problem-solving—based on operations that can be performed with a real compass and set square. This provides us with an interesting example of a mathematical practice that is clearly the articulation and abstraction of an actual instrumental practice. In fact, the contrast between the constructive methods of gnomonics and those of Euclid's *Elements*, allows us to cast the Euclidean problems in a new light. It is often claimed that Euclid's postulates derive from the operations of a compass and a straightedge,[137] but in fact they are more abstract than this. For example, *Elements* I.post.1 can be used to join points that are any distance apart, such as in *Elements* I.2, which a straightedge cannot do. Of course, one could argue that the postulates in the *Elements* suppose an indefinitely long straightedge—but there is no such thing. Again, *Elements* I.16 requires that *Elements* I.post.2 be used to extend a line to any assumed length, which a straightedge cannot do, since every actual straightedge is finite.

[136] I have used this interpretation of Ptolemy's *Analemma* as both computational and instrumental to give an interpretation of the mathematical methods underlying Heron's *Dioptra* 35 as an application of analemma methods; see Sidoli, 'Heron's *Dioptra* 35'.

[137] See, for one of many examples, Mueller, *Philosophy*, pp. 15–16.

*Elements* I.post.3 is used to produce a circle about a given point as center and passing through another given point, which can be at any distance from the center. Again, this is not possible with a real compass—since every compass has a fixed finite radius. Moreover, as its application in *Elements* I.2 shows, *Elements* I.post.3 cannot be used to produce a circle about a given point with a given radius—but any actual compass can perform this operation. Hence, it has sometimes been argued that *Elements* I.post.3 concerns the operation of a compass that closes when it is no longer in contact with the plane—but again, there is no such compass.[138] In this way, we can contrast the level of abstraction allowed in the *Elements* with that allowed in the *Analemma*, in which every problem can be carried out with constructive operations that are direct abstractions of the physical manipulations of a compass that can operate with given radii and a set square—both of some preassigned, definite size.

Another interesting feature of the *Analemma* is its concern with providing multiple methods for computing the same value. It is clear from the way in which Ptolemy presents his procedures that a primary goal of the text is to provide nomographic techniques, but this is proceeded by a full argument that the values in question are both geometrically determined and computable through chord-table trigonometry. This presentation constitutes a multilevel argument that the procedure—geometrical, computational, and nomographical—is complete. At the most basic level, the theorems of the *Data* implied by the steps of the metrical analysis insure that the geometric magnitudes are fixed; at the next level, the metrical analysis itself provides confirmation that there is an effective procedure for computing the value; and at the final level, the articulation of a geometric and computational procedure assures us that the physical manipulations of the analog computation will produce results that, although perhaps not terribly precise, are, in principle, sound.

Although there is no evidence that Ptolemy's *Analemma* was translated into Arabic, the gnomonics of his predecessor Diodorus certainly was, and there is clear evidence that mathematicians working in Arabic were familiar with analemma methods already at the time of Ḥabash al-Ḥāsib in the early 9th century.[139] These methods must have come directly or indirectly from Greek sources—since there is no evidence of analemma methods in other ancient cultures.[140] Hence, since Ptolemy's *Analemma* presents the most complete

[138] Heath, *A History*, vol. I, p. 246, recounts this interpretation of Euclid's postulate by Augustus De Morgan.

[139] Edwards, *Ptolemy's* Περὶ ἀναλήμματος, pp. 152–182, gives a full study of what is known of Diodorus' life and work. For evidence of the translation of, at least parts of, Diodorus' work into Arabic, see Kennedy, *The Exhaustive Treatise*, pp. 157–166, and Hogendijk, 'Geometrical Works'.

[140] It used to be argued that analemma methods provide the best explanation for certain

explanation we have of the mathematical conceptions underlying analemma methods, and since these methods were so fruitfully applied in the classical Islamic period, if we want to fully understand the medieval development of analemma methods, we should begin with a firm basis in Ptolemy's text. The key features of the *Analemma* that should inform our reading of the medieval sources are (1) its essentially projective approach and (2) its interest in the mathematical justification of the methods of analog computation.

Finally, the *Analemma* provides a well-contained example of the approach to mathematical astronomy developed in the Hellenistic period and still practiced by Ptolemy and others in the Roman Imperial period.

- A geometric model is posited, with relatively little attempt to argue that it is a sound representation of the physical world.
- The model itself becomes the object of geometrical investigation and geometrical claims that can be made about the model are assumed, without comment, to apply also to the world.
- Numerical values, which are ostensibly empirical, enter into the model as given parameters for computation.
- The mathematical methods of computation (λογιστική) are mixed with the constructive methods of geometry, with no evidence for the division of these two areas of mathematics that we find, for example, in the *Elements* and certain philosophical authors.
- The geometrical methods of the *Data* are used as a theoretical basis for a computational practice that is understood as producing measurements of various aspects of the underlying geometric model.

Mathematical scholars of the late ancient and medieval periods, who read Ptolemy as a mathematician, found in these aspects of his approach various methods to articulate, critique and revise in their effort to further develop the mathematical sciences.

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Indian sources, but there is no Indian source that clearly contains an analemma construction and the spherical methods in Indian sources can also be explained through other approaches to spherical geometry; see Mimura, *The Tradition*, pp. 36–55.

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