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Naṣīr al-Dīn al-Ṭūsī's Comments on Euclid's *Data*

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Abstract

This paper is a study of Naṣīr al-Dīn al-Ṭūsī's comments on Euclid's *Data*. We produce a critical edition, translation and commentary of comments and marginal notes made by al-Ṭūsī in his *Revision* of the Euclidian text (*Taḥrīr kitāb al-Muʿṭayāt li-Uqlīdis*). The study results in some insight into what Ṭūsī thought was worth explaining from a mathematical perspective, some information about his manuscript sources, and, perhaps most importantly, some of his scholarly practices in producing his edition of this canonical mathematical text. Another result is that of two versions of this text that can be read in the manuscripts, one can be identified as the more polished draft.

概要

この論考の分析対象は、ナスィール・アッディーン・トゥースィー (Naṣīr al-Dīn al-Ṭūsī) による、エウクレイデス『デドメナ』に対する注釈である。我々はトゥースィーがその著書『エウクレイデス・デドメナ再述』 (*Taḥrīr Kitāb al-Muʿṭayāt li-Uqlīdis*) のなかで為した注釈および欄外注の校訂英訳注を提示する。分析の結果、トゥースィーが数学的観点から何を重要だと見なしていたのかについて、また現存諸写本の関係についていくつかの新知見が明らかになる。そのなかでおそらく最も重要だと思われるのは、彼がこの数学の重要書に対して行った"書き直し"の具体相である。他にも、現存写本群には2つ系統があること、その一方がより推敲を経たヴァージョンであることが明らかになる。

MSC: 01A20; 01A30; 01A35

Keywords: Naṣīr al-Dīn al-Ṭūsī; Tusi; Euclid; *Data*; Commentary

1. Introduction

This paper is a study of the comments and notes that Naṣīr al-Dīn al-Ṭūsī introduced into his *Revision of Euclid's Data* (*Taḥrīr kitāb al-Muʿṭayāt li-Uqlīdis*). Ṭūsī's version of Euclid's text was clearly based on

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Thābit ibn Qurra's *Restoration (iṣlāḥ) of Euclid's Data*, since it agrees with this text in all places where they both differ from the edited Greek and Latin texts (Menge, 1896; Ito, 1980).[1] While there are few global differences between these two texts, there are local differences having to do with the style of the Arabic expression, the details of the arguments and the drawing of the diagrams.[2] Some of these differences, and al-Ṭūsī's interventions in the text are the subject of this paper.

Al-Ṭūsī's *Revision* has been studied by Thaer (1942), who compared the whole text with the Greek text edited by Menge (1896), before the manuscripts of Thābit's *Restoration* were known, and *Data* 62 (Ṭūsī Prop. 64) with an older version of the same theorem contained at the end of one of the Ṭūsī manuscripts (**W**, f. 268).[3] For the purposes of this study, we have produced a critical edition and translation of the comments that Ṭūsī introduces into the text with the words "I say" (أقول), along with those marginal notes found in the manuscripts we studied that we are fairly certain were written by Ṭūsī himself. Our text can be compared with the Hyderabad (1939/40) edition, which was made on the basis of at least three manuscripts, but without a critical apparatus.

1.1. Notation, naming and editorial conventions

Euclid's *Data*, which provides a theory, or loose grouping of theories, of the ways in which geometric objects are given, is a fairly peculiar text—unlike anything we encounter in our own education.[4] Although this is not the place to provide a full treatment of the *Data*, in order to explicate al-Ṭūsī's comments, we are obliged to look at the details of the argument is some places.[5] In the Greek text, the notion of given is handled with the verb "to give" (διδόναι) and its participles. In the Arabic versions of this treatise, however, the same concept is handled exclusively with the past participle "known" (معلوم).[6] Since, however, the one translates the other, and since in this treatise we can discern no mathematical distinction between the two expressions, for the purpose of discussing the Arabic versions of the *Data*, we use *known* and *given* interchangeably.

In order to explicitly indicate that certain objects are given, we use the following notational conventions. We denote geometric objects such as points, and occasionally lines, with the same letter-name as used in the text under discussion, in italic type, such that A denotes a general point, while a denotes the same point when it is *known*—that is, *known in position*, a_p.[7] Hence, we can denote a general line as AB, and the same line as AB_p, AB_m or $ab_{p,m}$ when it is *given*, since a line can be *known in position*, *in magnitude*, or both.[8] Rectilinear figures are denoted with bold type, such that a general, rectilinear figure, constructed from points A, B, C, ... is denoted as $\mathbf{F}(ABC\ldots)$, a triangle as $\mathbf{T}(ABC)$, a square as $\mathbf{S}(ABCD)$ or $\mathbf{S}(AB)$, a rectangle as $\mathbf{R}(ABCD)$ or $\mathbf{R}(AB, BC)$, and so on. A figure can be *known in magnitude*, $\mathbf{F}(ABC\ldots)_m$, *in*

[1] We have completed a text and translation of Thābit's *Restoration*, which includes, in the commentary, comparisons between this text and those of the edited Greek version and Ṭūsī's *Revision* (Sidoli and Isahaya, 2018).

[2] We have made a comparison of the substantial differences in the mathematical argumentation between the two text in our commentary to Thābit's *Restoration* (Sidoli and Isahaya, 2018, 218–312).

[3] The two known manuscripts of Thābit's *Restoration* are referred to as **A** and **K**; see References.

[4] See Taisbak (2003), Acerbi (2011), Sidoli (2018), and Sidoli and Isahaya (2018) for recent studies of the *Data* and the concept of given in Greek mathematics.

[5] In order to read the text, we have used the Hyderabad (1939/40) edition, simply checking it against a few of the oldest manuscripts.

[6] Other Arabic words are used to translate the various forms of διδόναι in other places—such as "assumed" (مفروض) in other treatises (Rashed and Bellosta, 2010, 467–469), and "givens" or "data" (معطيات), in the title of the *Data* itself.

[7] A point can only be *known in position*.

[8] In geometrical analysis, a line may have one point *known*, aB, or no points *known* but it is *known in position*, AB_p, or no points *known* but it is *known in magnitude*, AB_m, or both points known and it is *known in position* and *in magnitude*, $ab_{p,m}$, and so on. The *Data* and related texts handle and differentiate between all of these situations, so our notation must as well.

Table 1
Concordance of proposition numbers in the *Data*: **G** is the edition of the Greek text by Menge (1896), **Th** is the our edition of Thābit's *Restoration* (Sidoli and Isahaya, 2018), and **Tu** is Ṭūsī's *Revision* in the Hyderabad (1939/40) edition. The designations $n.1$ and $n.2$ refer to parts of a proposition, while na, nb, and so on, refer to alternate proofs.

Th	Tu	Gr	Th	Tu	Gr	Th	Tu	Gr	Th	Tu	Gr
1	1	1	27	28	28	48	51	50	69	73	72
2	2	2	28	29	29	49	52	51	70	74	73
3	3	3	29	30	30a	50	53	52	71	75	74
4	4	4	–	–	30b	51	54	53	72	76	75
5	5	5	–	–	30c	52	55	54a	73	77	76
6	6	6	–	–	30d	–	–	54b	74	78	77
7	7	7	30	31	31	53	56	55a	75	79	78
8	8	8	31	32	32	–	–	55b	–	–	79
9	9	9	32	33	33a	54	57	56	–	–	80a
10	10	12	–	–	33b	55	58	57	76	80	80b
11	11	10	33	34	34.1	56	59	58	77	81	81.1
12	12	11.1	33	34	34.2	57	60	59	–	–	81.2
–	–	11.2	34	35	35	58	61	60	78	82	82
13	13	13	–	–	36	59	62	61	79	83	83
14	14	14	35	36	37	60a	63	62	80	84	84
15	15	15	36	37	38	60b	64b	–	81	85	85
16	16	16	37a	38	39	–	64a	–	82	86	v.87a
17	17	17	37b	39	–	–	–	63	–	–	v.87b
18	18	18	38	40	40	61	65	65	83	87	86
19a	19	19a	39	41	41	62	66	64	84	88	87
19b	20	19b	40	42	42	63	67	66	85	89	88
20	21	20	41	43	43	64	68	67a	86	90	89
–	–	21	42	44	44.1	–	–	67b	87	91	90
21	22	22	–	–	44.2	–	–	67c	88	92	91a
22	23	23	43a	45	45b	–	–	67d	–	–	91b
–	–	24a	43b	46	45a	65	69	68a	89	93	92
23	24	24b	–	–	46a	–	–	68b	90	94	93a
24	25	25	44	47	46b	66	70	69	–	–	93b
25	26	26	45	48	47	67	71	70	–	–	93c
26	27	27a	46	49	48	68	72	71	91	95	94
–	–	27b	47	50	49						

form, $\mathbf{F}(ABC\ldots)_f$, and so on. Another convention that we use is to put the object that was originally known, or assumed, to be *known* on the right-hand side of an equation and the object that is shown to be *known* on this basis on the left-hand side. In this way, $(A : X)_r = (D : E)_r$ means that ratio $(A : X)_r$ is *known* because it is set as equal to $(D : E)_r$, which was previously taken, or shown, to be *known*.[9]

The numbering of the propositions of the *Data* is slightly different between the Greek text edited by Menge (1896), Thābit's *Restoration*, and al-Ṭūsī's *Revision*, see Table 1.[10] In order to refer to the propositions by number, we will use the numbers of the edited Greek version, with the number of the same proposition in the Hyderabad (1939/40) version of Ṭūsī's *Revision* in parentheses, if this is different from the Greek version—that is, *Data* 67 (Ṭūsī Prop. 68) indicates the 67th proposition of the Menge edition, which is the same proposition as the 68th proposition of the Hyderabad edition of al-Ṭūsī's *Revision*.

9 For the introduction of this notation, a discussion of the meaning of these modes of being given, and their use in Greek mathematical works, see Sidoli (2018).

10 In fact, the numbering is also often slightly different between different manuscripts of the same version of the text. For the numbering of al-Ṭūsī's *Revision*, we have followed that of the Hyderabad (1939/40) edition.

We have not noted orthographic variations in our apparatus, unless some meaningful difference is at stake. Nor have we noted differences in dotting—often silently correcting or supplying the gender of verbs. In the critical apparatus, we note exactly what we see in the manuscript, with no attempt to point, or to correct, for grammar or sense. In the critical notes, we use the following abbreviations:

hand	in a different hand, or ink;
line	a gloss, addition or correction found between lines of text;
marg.	a gloss, addition or correction found in the margin;
(−)	omitted.

2. Manuscript sources

There are many known manuscripts of the *Revision of Euclid's Data*. Sezgin (1974, 116) lists more than 20 copies, and more are now known through the library catalogs and online databases that have become available since the 1970s. We have used seven manuscripts in order to establish our text of al-Ṭūsī's comments to his *Revision of the Data*, which are noted in our critical apparatus by the following sigla:

Th: Tehran, Sipahsalar (now, Kitābkhāna-yi Madrasa-yi ʿĀlī Shahīd Muṭahharī) 4727, 671 AH (1272 CE). pp. 99–110.[11]
Ia: Istanbul, Topkapı Sarayı Library, Ahmet III 3453, 671 AH (1272 CE). ff. 65a–72b.[12]
Ih: Istanbul, Haci Selim Ağa Library 743, 671 AH (1272 CE). ff. 244b–256a.
Tb: Tabriz, National Library, 3484. Late 7th–early 8th c. AH (late 13th–early 14th c. CE). ff. 0a–11a, pp. 0–22.[13]
Is: Istanbul, Süleymaniye Library, Aya Sofya 2758. Early 8th c. AH (early 14th c. CE). ff. 93b–100a.
Ts: Tehran, Sipahsalar (now, Kitābkhāna-yi Madrasa-yi ʿĀlī Shahīd Muṭahharī) 597, 781 AH (1380 CE). ff. 4b–14a.[14]
W: Kraków[15] (formally Berlin), Jagiellonska Library, Ms. or. fol. 258. 12th c. AH (17th c. CE). ff. 250b–268b.

In order to make clear the differences between our text and that in the Hyderabad edition, we refer to that edition with the following siglum:

ℍ: Hyderabad, 1939/40 (1358 AH), Naṣīr al-Dīn al-Ṭūsī, *Taḥrīr kitāb al-Muʿṭayāt li-Uqlīdis*.

The selection of these manuscripts was made by, first, taking the oldest three manuscripts known to us, **ThIaIh**, and, next, by choosing three manuscripts each from two different families that we have identified—which we call the **Th**-family and the **Ih**-family. **W** was included as well because it was studied

11 Note that Ragep (1993, 81), presumably following Mudarrisī (1956, 114), incorrectly dated this manuscript to 1360 CE. This manuscript was printed in facsimile by Qāsimlū (2010), but since the images of the facsimile appear to have been digitally altered, we have consulted photographs of the original manuscript. We also follow the page numbering of the original manuscript, not that of the facsimile.

12 Sezgin (1974, 116) dates this as 677 AH, but the manuscript reads 671 (**Ia**, f. 72b).

13 This manuscript was printed in facsimile by Aghayanī-Chavoshī (2005). In this codex, both the folia and the pages are counted; however neither the first page, nor the first folio are included in the count. Furthermore, in the *Data*, some folia are missing and others are bound out of order.

14 Most of the folia of this manuscript are numbered according to two different numerations. The larger numbers, which we follow, numerate all of the folia of the current codex.

15 Note that we elsewhere incorrectly stated the location of this manuscript as Warsaw (Sidoli and Isahaya, 2018, 343).

by Thaer (1942), includes readings from both families, and contains a fragment of an interesting version of the text that appears to predate Thābit's *Restoration*.[16]

Although we have not studied these manuscripts in their entirety, it is clear that with respect to Ṭūsī's comments, the oldest manuscripts contain two different versions of the text—as we will show below. **ThIa**, which were copied a couple of years before al-Ṭūsī died, share the same text and often have the same marginal notes—see for example the marginal notes to *Data* 44, 62 and 78, below. Moreover, since the colophon to Ṭūsī's *Memoir on Cosmography* (*al-Tadkira fī ʿilm al-hayʾa*) in **Th** tells us that this manuscript is a copy made from a copy that was "read to" al-Ṭūsī,[17] including marginal notes in the master's own hand (**Th**, p. 424), we have assumed that most of the other marginal notes in this manuscript are also due to Ṭūsī himself.[18] On the other hand, **Ih**, which was copied in the same last year, contains a number of clear improvements over this text—see, in particular, the material to *Data* 62 and 80, below. Hence, we put forward the working hypothesis that the **Th**-family represents an earlier version, and the **Ih**-family a later version, of Ṭūsī's scholarship on this treatise. If this picture is accurate, it would mean that al-Ṭūsī continued to edit his scholarship on the *Middle Books*, based on his experience of reading these texts with students—in a similar vein to his practice with the *Memoir* (Ragep, 1993, 70–75). Of the sources we have used, **ThIaTs** form the **Th**-family, and **IhTbIs** form the **Ih**-family. **W**, which is much later, is not straightforwardly classifiable in terms of these two families.

3. Ṭūsī's comments

In the following sections, we provide a text, translation, and remarks, for each of al-Ṭūsī's comments as incorporated into the text, as well as for the marginal notes to this material in the **Th**-family of manuscripts, which probably go back to Ṭūsī's own scholarly work on the text.

Al-Ṭūsī's marginal notes and comments can be summarized as follows:[19]

***Data* 14, 15:** Comments dealing with different mathematical cases.
***Data* 25:** A comment clarifying the conditions of the theorem.
***Data* 28–30:** Comments relating terms introduced in *Data* Defs. 13–15 to geometric objects in these theorems.
***Data* 44:** A comment dealing with different geometric configurations that do not amount to mathematical cases. A marginal note in some manuscripts (**ThIaTs**) explaining that the argument in the text covers all possible cases. (*)
***Data* 62 (Ṭūsī Prop. 64):** Marginal notes and comments dealing with some textual variation in the manuscript sources and making a critique of mathematical difficulties.
***Data* 67 (Ṭūsī Prop. 68):** Comments providing two lemmas necessary to the proof.

16 When Thaer (1942, 203–205) studied this manuscript, which was then in Berlin, he was unaware of the existence of Thābit's *Restoration* in **AK**—the only known manuscripts of Thābit's version of Euclid's *Data*. Hence, he took the fragment at the end of **W** (f. 268a,b) to be a passage of al-Ṭūsī's source. We can now be virtually certain, however, that Ṭūsī worked with sources similar to **AK**.

17 The expression "read to" (قرأ على), here and in the following, probably refers to the educational practice of the student reciting, or vocalizing, a text to the master as a way to demonstrate control of the material, and in the hope of receiving a *ijāzat al-qirāʾah*. In the case of the mathematical sciences, it seems that students often made a copy from the master's model and then read this copy to the master. For the general educational context see discussions by Makdisi (1981, 147–152), Berkey (1992, 21–43), Chamberlain (1994, 87–90), and Brentjes (2018, 161–168). For the various cognates, see Gacek (2001, 113).

18 See Ragep (1993, 74), for a translation of this colophon to the *Memoir*, in which the scribe mentions copying Ṭūsī's notes.

19 A (*) indicates a comment which is technically problematic or uncertain. In the case of the comments to *Data* 74 and 80, these problems were resolved in what we will argue was a more polished draft of his *Revision*.

***Data* 74 (Ṭūsī Prop. 75):** A marginal note in some manuscripts (**ThIaTs**) explaining a specification that can be made to the enunciation for one of the cases of the proposition. (*)

***Data* 80:** A comment in some of the manuscripts (**ThIaTs**) that incorrectly claims that the proposition deals with only one case. (*)

***Data* 82, 83:** Comments making explicit the contrivances introduced in the course of the proof.

3.1. Data *14 and 15*

Data 14 and 15 treat the relation *greater-by-a-known-than-in-ratio*—that is, the ratio $((A - C_m) : B)_r$ is known, where magnitude C_m is known.[20] These two propositions show that if two known magnitudes are added to, or subtracted from, the terms of a known ratio, then either the ratio of the sums and differences are known, or the first sum, or difference is greater by a known magnitude than in ratio to the second sum or difference—that is, in Figure 1, where the known magnitudes AE_m and GZ_m are added or subtracted from the terms of the ratio $(AB : DG)_r$, then either, in Case 1, $(BE : DZ)_r$ is known, where $(AE_m : GZ_m) = (AB : DG)_r$, or, in Case 2, $((BE - HE_m) : DZ)_r$ is known, where $(AH_m : GZ_m) = (AB : DG)_r$ and $HE_m = AE_m \pm AH_m$. Al-Ṭūsī's comments to these propositions deal with an alternate version of Case 2 that is not treated in the main proof, say Case 2b. In Case 2 of both propositions, since AE_m and GZ_m are known magnitudes of any size, while the magnitude AH_m is determined by the ratio $(AH_m : GZ_m)_r = (AB : DG)_r$, it may happen that $AE_m < AH_m$, in *Data* 14, or $AE_m > AH_m$, in *Data* 15. If this is happens, then the argument that is presented in the main text will not work exactly in the terms in which it is stated.

Ṭūsī addresses this issue for *Data* 14 as follows:

أقول: إن كان اح أعظم من اه، كانت نسبة ما هو أصغر من جز إلى اه كنسبة جد إلى اب. فيكون زد
كله أعظم بقدر معلوم على قدر نسبته إلى هب كله معلومة.

I say: If *AH* is greater than *AE*, the ratio of something less than *GZ* to *AE* is as the ratio of *GD* to *AB*. So, the whole of *ZD* is greater by a known magnitude than a magnitude whose ratio to the whole of *EB* is known.

The argument can be fleshed out a little—introducing *GX* for "something less than *GZ*". That is, in Figure 1 (left), if $AE'_m < AH_m$, since $(GZ_m : AH_m) = (GD : AB)_r$, then we can set $(GX : AE'_m) = (GD : AB)_r$, where $GX < GZ_m$. Hence, GX_m is known, by *Data* 2, so that $ZX_m = GZ_m - GX_m$ is known by *Data* 4. Therefore, $((DZ - ZX_m) : E'B)_r$ is known, by *Elem.* V.12 and *Data* Def.2. That is, the claim made in the enunciation still holds, but for opposite terms.

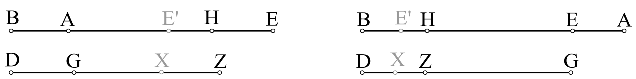

Figure 1. Diagrams for *Data* 14 (left) and 15 (right). Elements shown in gray do not appear in the manuscript diagrams.

Ṭūsī's treatment of the equivalent case for *Data* 15 is as follows:

[20] See Taisbak (2003, 57–61), Acerbi (2011, 124), and Sidoli and Isahaya (2018, 235–246) for discussions of this relation.

1 كان اح] كان لع، **Ih**، كان ~~سبه احد هما~~ اح **Ts**   كانت] كان **IhTb**   أصغر من] أصغر من ~~قدر~~ **Tb**

أقول: إن كان اح أصغر من اه، كانت نسبة ما هو أعظم من جز إلى اه كنسبة جد إلى اب. وتمّم البيان كما مرّ.

I say: If *AH* is less than *AE*, the ratio of something greater than *GZ* to *AE* is as the ratio of *GD* to *AB*. And the explanation is as discussed [above].

This remark, which is even more elliptical, can be fleshed out along the same lines—again, introducing *GX* for "something greater than *GZ*." That is, in Figure 1 (right), if $AE'_m > AH_m$, since $(GZ_m : AH_m) = (GD : AB)_r$, then we can set $(GX : AE'_m) = (GD : AB)_r$, where $GX > GZ_m$. Hence, GX_m is known, by *Data* 2, so that $ZX_m = GZ_m + GX_m$ is known by *Data* 3. Therefore, $((DZ - ZX_m) : E'B)_r$ is known, by *Elem.* V.12 and *Data* Def.2, so that, once again, the claim made in the enunciation holds for the opposite terms.

This coverage of the alternative Case 2b is not found in either of the known manuscripts of Thābit's *Recention* (**A**, 2b–3a; **K**, 3b), nor in the Greek scholia that Menge (1896, 277) edited. It is possible that al-Ṭūsī produced it himself.

3.2. *Data 25*

Data 25 shows that if two lines are given in position, their intersection is given in position. Ṭūsī's comment to this proposition is a simple clarification of the fact that the lines need not be straight. The remark reads:

أقول: ليس من شرط الخطين أن يكونا مستقيمين.

I say: There is no condition on the two lines that they be straight.

Indeed, in the *Data* this proposition is often used for circles and circular arcs while in the *Conics* it is also used for conic sections. Although, the Arabic expression in the enunciations of both Thābit's *Restoration* and Ṭūsī's *Revision*, خطين, could, indeed, be read as meaning two straight lines, in the context of a Greek mathematical text the expression in the Greek versions, δύο γραμμαί, would naturally be understood to mean "two [straight or curved] lines" (Menge, 1896, 46)—for example, when γραμμή must mean a straight line in the *Elements*, it is qualified by εὐθεῖα.[21] Hence, Ṭūsī's comment is useful for the reader of the Arabic text.

3.3. *Data 28, 29 and 30*

Data 28, 29 and 30 treat lines given in position. Al-Ṭūsī's comments to these propositions connect one of the lines introduced in each one of the propositions to the terminology of *Data* Defs. 15, 14 and 13, in that order. These three definitions are not required anywhere in the course of argument itself, and they are asserted in a scholium to have been introduced by Apollonius (Menge, 1896, 264).

Data 28 shows that if a line passes through a point known in position, parallel to a line known in position, it is itself known in position. In order to carry out the proof, a transformation of the line in question is introduced as also parallel and shown to be impossible. Ṭūsī's comment to *Data* 28 reads:

21 This distinction is made clear in the exposition of *Elem.* XI.3 (Heiberg, 1883–1885, IV.12).

1 كانت] كان **ThIaIhTbIsTs**   البيان] اَلسَاَىَ **Ih**، البرهان ℍ   3 يكونا] يكونان **Is**، يكون **W**

1 أقول: وهذا الخط هو الذي يسمّى بالقارن للخط الموضوع، أعني الأول بأحد المعنيين.

> I say: This line is that called the *associated* with the positioned line—that is, the first with one of the two meanings.

Al-Ṭūsı's wording of *Data* Def. 15, reads: "The *line associated with the positioned line* is that [1] which is produced from a known point parallel to the positioned line, or [2] passes through a known point, and joins a positioned line, and creates with it a known angle" (Hyderabad, 1939/40, 3). Hence, his remark appears to say that the first line introduced in the argument of *Data* 28 can be understood to be the *associated* line, in the first of the two senses defined in *Data* Def. 15—namely, that stated as [1].

Data 29 shows that if a line is erected from a known point on a line known in position at a known angle, it is itself known in position. Ṭūsī's comment to *Data* 29 reads:

2 أقول: وهذا الخط هو الذي يسمّى بالصاعد عن الخط الأول.

> I say: This line is that called the *ascendant* from the first line.

Al-Ṭūsī's version of *Data* Def. 14 reads, "The *ascendant [line]* is that which raises from a known point that is on a positioned line, and creates with it a known angle" (Hyderabad, 1939/40, 3). Hence, in the comment he is simply pointing out that the line introduced in *Data* 29 satisfies the terms of *Data* Def. 14.

Data 30 shows that if a line passes through a point known in position and makes a known angle with a line known in position, it is itself known in position. Ṭūsī's comment to *Data* 30 reads:

3 أقول: وهذا الخط هو الذي يسمّى بالمنحدر إلى الخط الموضوع الأول.

> I say: This line is that called the *descendant* to the first positioned line.

Ṭūsı's wording of *Data* Def. 13, reads: "The *descendant line* is the straight line that descends from a known point to a positioned straight line, and creates with it a known angle" (Hyderabad, 1939/40, 3). Again, it is clear, as al-Ṭūsī remarks, that the line introduced in *Data* 30 satisfies *Data* Def. 13.

Although *Data* Defs. 13–15 are purely descriptive and are not required for any argument in the text, al-Ṭūsī is correct to point out that *Data* 28–30 introduce lines that satisfy these definitions. Hence, Ṭūsī's remarks are helpful to the reader in understanding this final set of definitions. Since there are no comments related to the other definitions, which are actually required in the text,[22] Ṭūsī probably felt that the mathematical purpose of Defs. 13–15 in the overall development of the treatise called for some explanation.

It should be pointed out that in Defs. 13 and 14 in Thābit's *Restoration*, the orientation of the two lines is reversed. That is, the *descendant* is defined as descending "from" (على) a known point on a line known in position, whereas the *ascendant* is defined as ascending "to" (إلى) a line known in position (Sidoli and Isahaya, 2018, 39). Al-Ṭūsī appears to have realized that the orientation of these lines must be reversed by thinking through how *Data* Defs. 13 and 14 are related to the lines in *Data* 29 and 30. The orientation of the

[22] Although of the four groups of definitions, *Data* Defs. 1–4, 5–8, 9–12, 13–15, not all of the definitions in the second and the third group, Defs. 5–8, 9–12, are used, the final group, Defs. 13–15, is the only group from which no definitions are used.

1 الذي يسمّى] المسمى **Th**   المعنيين] المعنن **Th**   2 بالصاعد] الصاعد ℍ   عن الخط] عن الخط الموضوع ℍ   3 بالمنحدر] المنحدر **ThIaIh** ℍ

lines in Ṭūsī's definitions agrees with that in the Greek—against the orientation in the extant manuscript of the *Restoration*.[23]

3.4. Data *44*

Data 44 shows that if, in a triangle, an angle and the ratio of the sides about one of the other angles are known, then the triangle is known in form—that is, in Figure 2, $\mathbf{T}(ABG)$, $\angle BAG_m$, and $(AB : BG)_r \Rightarrow \mathbf{T}(ABG)_f$. The proof proceeds by dropping $BD \perp AG$, which will result in three possibilities for the arrangement of points A, G and D along line AG extended. Al-Ṭūsī is clearly discussing this situation, but as we will argue below, it is not clear where he is going with it.

Ṭūsī's comment to *Data* 44 reads as follows:

أقول: إن كانت زاويتا آ جـ حادتين أو كانت زاوية آ المعلومة منفرجة، فالحكم كما ذكره. أمّا إن كانت زاوية آ
حادة فينبغي أن يعلم إن زاوية جـ هي حادة أم ليست بحادة. وذلك لأنّها إن كانت حادة، وقع عمود ب د
داخل المثلث، وإن كانت منفرجة، وقع خارجه. وكان للمثلث مع كون زاوية آ بحالها ونسبة آ ب إلى ب جـ
بحالها صورتان، لأنّه يكون تارةً جزءًا من المثلث القائم الزاوية وتارةً يكون المثلث القائم الزاوية جزءًا
منه.

I say: [1] If angles *A* and *G* are acute, or [2] if known angle *A* is obtuse, then the theorem is as stated. When, however, angle *A* is acute it is necessary to know if angle *G* is acute or not acute. And that is because [1] if it is acute, the upright *BD* falls inside the triangle; but [3] if it is obtuse, it falls outside. And, regarding the triangle, with angle *A* in its situation and ratio *AB* to *BG* in its situation, there are two forms. Because it is sometimes a part of the right triangle and sometimes the right triangle is a part of it.

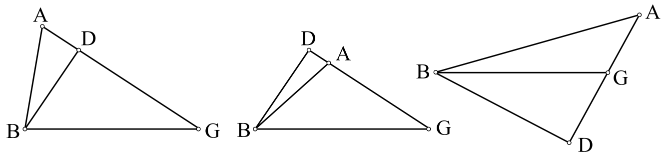

Figure 2. Diagrams for *Data* 44. Only the leftmost diagram appears in the manuscripts.

Al-Ṭūsī sets out various possibilities for the configuration of the two triangles $\mathbf{T}(ABG)$ and $\mathbf{rightT}(BDG)$ —the numbers that we have included in the text correspond to Figure 2 such that [1] is the left, [2] is the middle, and [3] is the right diagram. What is less certain is why Ṭūsī claims that it is "necessary to know" whether or not angle *G* is acute. That is, it is not evident that anything in the argument depends on this distinction.

In fact, the argument in the main text of the *Revision* will work irregardless of the placement of line *BD*. The following is a summary of the argument in al-Ṭūsī's text. In Figure 2, where $\angle GAB_m$ and ratio $(AB : BG)_r$ are known, we set $BD \perp AG$. Then, since the angles $\angle A_m$ and $\angle D_m$ are known, by *Data* 40,

[23] At this place, we only have one manuscript for the *Restoration*, because the first folio of the text is missing in **A**.

1 كانت] كان **Ts**   زاويتا آ جـ ... المعلومة منفرجة] زاوية آ المعلومة منفرجة **ThIaTs** ℍ   أمّا] واما ℍ   2–1 زاوية آ حادة] حادة **ThIaTs**، ℍ
2 جـ هي] جـا هى **ThIaIhTbIsW**، جـ اعى **Ts**، جـا هي ℍ   لأنّها] انها ℍ   إن] لو **TbIs**   3 وإن] ولو **Ih**   وقع] وقعت **Tb**   4 يكون
تارةً] تارة يكون **ThIaTsW** ℍ   القائم الزاوية] القآئم العامه *line* الزاوية **Is**

$\mathbf{T}(ABD)_f$ is known in form. So, ratio $(AB : BD)_r$ is known, by *Data* Def.3. But, by assumption, ratio $(AB : BG)_r$ is known, so by *Data* 8, $(BD : BG)_r$ is known. So, by *Data* 43, $\mathbf{rightT}(BDG)_f$ is known in form. So, by *Data* Def.3, $\angle G_m$ is known. Therefore, by *Data* 4 and 40, $\mathbf{T}(ABG)_f$ is known in form. QED.

Reading through this argument using the middle and right diagrams, in which either $\angle A$ or $\angle G$ are inside $\mathbf{rightT}(BDG)_f$, since both of the supplementary angles at these points will be known, by *Data* 4, it is clear that this argument is valid for all three cases that al-Ṭūsī discusses.

A note in the margin of **ThIaTs**, and included as a footnote in $\mathbb{H}$, which was probably written by al-Ṭūsī himself indicates that he was aware of this (**Th**, p. 103; **Ia**, f. 68a; **Ts**, f. 8b; Hyderabad, 1939/40, 19, n. 1).[24] The note reads as follows:

وإن كان يقع عمود ب د خارج ا أو تكون زاوية ا في مثلث ب د ا من الجهة الأخرى معلومة لكونها
مع ا المعلومة كما تمتين، وباقي البرهان بحاله.

If the upright *BD* falls beyond *A*, or angle *A* in triangle *BDA* in the other direction is known, due to the fact that it is the known with *A*, as is a tent-rope,[25] and the rest of the proof is with its situation.

Since this is not the only theorem that could have different configurations, and since these configurations do not amount to true geometrical cases requiring different arguments, it is not clear why Ṭūsī felt the need to discuss the different configurations in such detail.

3.5. *Alternative proof for* Data *62 (Ṭūsī Prop. 64)*

Data 62 shows that if there are two lines that have a known ratio, and if a figure known in form is erected on one of them and a parallelogram with a known angle is erected on the other, and if the ratio of the two figures is known, then the parallelogram is known in form—that is, in Figure 3, $(\mathbf{F}(ABE\ldots)_f : \mathbf{P}(AB, AG))_r$, $\angle BAG_m$, and $(AB : GD)_r \Rightarrow \mathbf{P}(AB, GD)_f$. For reasons that are not clear to us, there are more different versions of this theorem in the Arabic tradition than any other proposition—there are three versions that present trivially different mathematical arguments, and one version that contains a false proof. This diversity may be partly due to the presence of this false proof, which is found both as an alternative proof in Thābit's *Restoration* and is also recorded as a marginal note in some of the manuscripts of al-Ṭūsī's *Revision* and as a comment in other manuscripts of Ṭūsī's text.

The treatment of this false proof in the *Revision* provides us with one of the most historically interesting of Ṭūsī's comments to this treatise. In **ThIaTs** there is no comment to this proposition in the main text, but we find the following marginal note (**Th**, p. 105; **Ia**, f. 69b; **Ts**, f. 10b).[26]

وجدت في نسخة قرأها أبو نصر أحمد ابن إبراهيم بن محمد السجزي على الأستاد المختصّ علي بن
أحمد النسوي شكل سد على هذا الوجه. ونعمل هذا الشكل على جهة أخرى أيضًا. ونجعل نسبة ا ب
إلى جـ د معلومة، ونقيم على ا ب شكل ا ه ب معلوم الصورة، وعلى جـ د متوازي الأضلاع ا جـ د ب. وأقول:

[24] The note in **Ts** is slightly different in the latter part, but as it is difficult to read, we have not included its readings.
[25] This is our best guess for a word that is illegible in **Ts**, barely legible in **Ia** and completely undotted in **Th**.
[26] It is worth pointing out that this note is not mentioned by $\mathbb{H}$, which appears to be otherwise based on manuscripts of the **Th**-family.

2 كما تمتين] كقا تمتين $\mathbb{H}$ 4 أيضًا] (–) **Ia** 5 الأضلاع] اضلاع **ThIa**

إنّه معلوم. لأنّه قد أقيم على ا ب شكلان كيف ما اتّفقا ا ه ب ا ج د ب، فنسبة ا ه ب إلى ا ج د ب
معلومة. وا ه ب معلوم الصورة، فا ج د ب معلوم الصورة.

I found, in a manuscript that Abū Naṣr Aḥmad ibn Ibrāhīm ibn Muḥammad al-Sizjī read to the distinguished master ʿAlī ibn Aḥmad al-Nasawī, Proposition 64 in this way: We also do this proposition by another approach. We set ratio *AB* to *GD* as known,[27] and we erect figure *AEB*, known in form, on *AB*,[28] and on *GB* parallelogram *AGDB*.[29] I say it is known. Because two arbitrary figures, *AEB* and *AGDB*, have been erected on *AB*, so the ratio *AEB* to *AGDB* is known.[30] And *AEB* is known in form, so *AGDB* is known in form.[31]

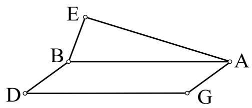

Figure 3. Diagrams for Ṭūsī's comments to *Data* 62 (Ṭūsī Prop. 64).

The argument provided here is fine as far as it goes but it cannot serve as a proof of the proposition, because of the way that $\mathbf{P}(AGDB)$ has been constructed, involving the unnecessary supposition that $AB = GD$, which is implicit in the construction and is more restricted than the simple claim that $(AB : DG)_r$ is known, as required by the proposition.[32]

The claim in the marginal note edited above is that this alternative, false proof was found in a manuscript that was read to al-Nasawī, the well-known mathematical scholar of the 11th century, and which manuscript was apparently not used in producing the initial draft of al-Ṭūsī's *Revision*. That is, whether or not this marginal note is by Tūsī himself, the original draft of the *Revision* in these manuscripts shows no sign of familiarity with this false proof. In what we believe was a later draft of the treatise, however, Ṭūsī moved this material into the text in the form of a comment, acknowledged that the false argument represents what is in the manuscripts, gave a summary of the argument, and then pointed out the problem with this material—as we will see below.

We find the following commentary in the main text of **IhIsW**,[33] as well as printed in $\mathbb{H}$, which mentions that it only comes from one of the manuscripts consulted (Hyderabad, 1939/40, 28, n. 1).[34]

أقول: الموجود في النسخ هكذا. ونعمل هذا الشكل على جهة أخرى أيضًا. ونجعل نسبة ا ب إلى ج د
معلومة، ونقيم على خط ا ب شكلًا معلوم الصورة، وهو ا ه ب, ونقيم على خط ج د سطحًا متوازي

[27] This is the assumption of the original proposition, but we will see below that the construction requires that $AB = GD$.
[28] That is, the figure is constructed arbitrarily as *known*.
[29] *Elem.* I.31, twice. This construction implies that $AB = GD$, which should not necessarily be the case. We also assume the angles of the parallelogram as known and we assume the ratio $(AB : BD)_r$ is known through the construction.
[30] *Data* 49.
[31] *Data* 61.
[32] In fact, the argument for this alternative proof could be saved if we set the given ratio between lines as $(AB : BD)_r$, and we erect the parallelogram $\mathbf{P}(ABDG)_f$ on line *AB* with arbitrarily given angle $\angle ABD_m$ (Sidoli and Isahaya, 2018, 282–283). Since, however, al-Ṭūsī also understood this argument as problematic, we have decided against trying to correct the text in this way.
[33] Notice in the critical apparatus to the final part of the mathematical argument that the text of the argument is more complete in **W**. It is possible that the source for **W** had a fuller text at this point, but it is also possible that the copyist of **W** fleshed out this argument based on the second of the two alternative versions of this proposition found following the text of the treatise (**W**, f. 268).
[34] The folio containing this proposition is missing from **Tb**.

الأضلاع، وهو ا جـ د ب. وأقول إنّه معلوم الصورة. وعملوا له شكلًا هكذا. وإذا كان الشكل والسطح على خط ا ب، كان خط جـ د مساويًا لـ ا ب، ولا يحتاج إلى أن يقال ونجعل نسبة ا ب إلى جـ د معلومة. وأيضًا يصير العلم ما ذكر في الشكل المتقدّم بعينه، فلينظر فيه فأنّ هذا الشكل مخبط.

I say: The [proposition] found in the manuscripts is thus. We also do this proposition according to another approach. We set ratio *AB* to *GD* as known,[35] and we erect on line *AB* a figure known in form, which is *AEB*,[36] and we erect on line *GB* a parallelogramic surface, which is *AGDB*.[37] I say it is known in form. They constructed for it a figure thus. But when the figure and the surface are on line *AB*, line *GD* is equal to *AB*, and it is not required for it to be said that we make ratio *AB* to *GD* as known. And, again, the theorem will be the same as stated in the previous proposition. So, let it be observed in it, that this proposition is a mess.

Although it is impossible now to know precisely the sequence of events that led al-Ṭūsī to revise his text in the way that he did, the evidence of this proposition is one of our clearest indications that we now have two different versions of Ṭūsī's *Revision*. It seems that when he initially made his draft of the *Data*, he worked with a copy of Thābit's *Restoration* that was basically similar to that contained in **AK**, with the exception of the alternative proof of *Data* 62 (*Restoration* Prop. 60b, Ṭūsī Prop. 64). The idea that there were such alternative versions of this proposition in the tradition of the *Restoration* is supported by yet another alternative version of this theorem on the final folio of **W** that is different from the alternative versions in both the *Restoration* and the main text of Ṭūsī's *Revision* (**W**, f. 268b). Then, at some later point, Ṭūsī apparently found a copy of the *Restoration* that had been read to al-Nasawī and made a note about this in the margin of his working text. This copy that had been read to al-Nasawī was probably the same text as now found in **AK**, and was certainly the same at *Data* 62 (*Restoration* Prop. 60b). Finally, by the time that Ṭūsī came to produce his final version of the *Revision of the Data*, he had realized that this false proof in the al-Nasawī manuscript was that of the majority of the manuscripts, and he noted it as such in his commentary—along with a short discussion of the mathematical issues.

3.6. *Data 67 (Ṭūsī Prop. 68)*

Data 67 shows that if an angle of a triangle is known, then the ratio of the difference between the square of the sum of the sides containing the known angle less the square of the opposite side to the triangle itself is known—that is, in Figure 4, $\mathbf{T}(ABG)$, $\angle BAG_m \Rightarrow (\mathbf{S}(BA+AG) - \mathbf{S}(BG) : \mathbf{T}(ABG))_r$. In all known versions of the text, there are two steps in the argument that do not immediately follow from propositions of the *Elements*, and for one of which the justification requires the introduction of an auxiliary line. Namely, the claims that

$$\mathbf{R}(DG, GE) + \mathbf{S}(BG) = \mathbf{S}(BD), \tag{1}$$

[35] See Note 27, above.
[36] See Note 28, above.
[37] See Note 29, above.

1 ا جـ د ب] ا جـ د ب W ℍ   وأقول] فمول Ih   3–1 وعملوا ... مخبط] وهو ا ه ب لنه قد اقيم على ا ب شكلان كيف اتفق وهما ا ه ب ا جـ د ب فنسبة ا ه ب الى ا جـ د ب معلومة وا ه ب معلوم الصورة فا جـ,د ب معلوم الصورة W   3 العلم] الحكم Is ℍ   فلينظر ... مخبط] Ih *marg.*، هذا] في هذا ℍ, مخبط] خبطا ℍ

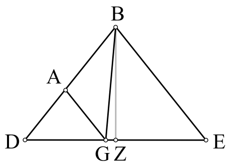

Figure 4. Diagram for *Data* 67 (Ṭūsī Prop. 68). In the diagrams of the *Revision*, the line BZ, in gray, is generally drawn in black whereas the rest of the diagram is generally drawn in red. It does not appear in the manuscripts of the other versions of the *Data*.

and

$$(\mathbf{S}(DG) : \mathbf{S}(DA)) = (\mathbf{R}(DG, GE) : \mathbf{R}(DA, AB)). \quad (2)$$

Ṭūsī provides the following short lemmas to verify these claims:

أقول: إنّما كان سطح د ج في ج ه مع مربع ب ج مساويًا لمربع ب د. لأنّا إذا أخرجنا من ب عمود ب ز على د ه، كان خط د ه قد نصف على ز، وقسم على ج، فسطح د ج في ج ه مع مربع ز ج يساوي مربع ز ه. ونجعل مربع ب ز مشتركًا، فيصير سطح د ج في ج ه مع مربعي ز ج ز ب، أعني مربع ب ج مساويًا لمربعي ز ه ز ب، أعني مربع ب ه بل مربع ب د. وإنّما كانت نسبة مربع د ج إلى مربع د ا كنسبة سطح د ج في ج ه إلى سطح د ا في ا ب. لأن نسبة د ج إلى ج ه كانت كنسبة د ا إلى ا ب من جهة موازاة ا ج لب ه، فنسبة مربع د ج إلى سطح د ج في ج ه كنسبة مربع د ا إلى سطح د ا في ا ب. وإذا أبدّلنا، كان كما ذكرنا.

I say: [(1)] But, the surface DG by GE with the square of BG is equal to the square of BD. Because, when we produce the upright BZ from B onto DE, line DE has been bisected at Z,[38] and it is sectioned at G, so the surface DG by GE with the square of ZG is equal to the square of ZE.[39] And we make the square of BZ a common, so the surface DG by GE with the two squares of ZG and ZB, that is the square of BG, is equal to the two squares of ZE and ZB,[40] that is the square of BE, or rather the square of BD.[41] [(2)] But, the ratio of the square of DG to the square of DA is as the ratio of the surface DG by GE to the surface DA by AB. Because, of the ratio DG to GE, it is as the ratio DA to AB, from the fact of the parallelism of AG to BE,[42] so the ratio of the square of DG to the surface DG by GE is as the ratio of the square of DA to the surface AD by AB.[43] And when we alternate, it is as what was stated.[44]

38 *Elem.* I.4.
39 *Elem.* II.5. That is, $\mathbf{R}(DG, GE) + \mathbf{S}(ZG) = \mathbf{S}(ZE)$.
40 That is, $\mathbf{R}(DG, GE) + \mathbf{S}(ZG) + \mathbf{S}(ZB) = \mathbf{S}(ZE) + \mathbf{S}(ZB)$.
41 *Elem.* I.47. That is, $\mathbf{R}(DG, GE) + \mathbf{S}(BG) = \mathbf{S}(BE) = \mathbf{S}(BD)$.
42 *Elem.* VI.2.
43 *Elem.* VI.1.
44 *Elem.* V.16. That is, $(\mathbf{S}(DG) : \mathbf{S}(DA)) = (\mathbf{R}(DG, GE) : \mathbf{R}(DA, AB))$.

1 من ب] من ت د **Ia**   عمود ب ز] عمود ت د **Ts**   2 ج] ة ℍ   4–2 ز ج يساوي مربع ... أعني مربع] **Tb** *marg.*   3 مربع ب ز] مربع ت د **Ts**   ز ج ز ب] ز ج ت **W**   4–3 أعني مربع ب ج مساويًا لمربعي ز ه ز ب] (–) **W**   4 ز ه ز ب] ز ب ز ه **Ia**   كانت] كان **ThIaIhTbIsTsW**   5 موازاة] موازة ℍ   6 إلى سطح] (–) **Ts**   د ج في ج ه] د ح في ج ه ℍ   مربع د ا] مربع د **Is**

Each lemma is introduced with the conjunction إنّما, “but”, which restates the claim made in the course of the main argument, followed by لأن, “because”, which introduces the argument. This usage is a clear linguistic signal that Ṭūsī is providing lemmas to the main proof. Indeed, his justification of the first claim, (1), is the same as that found as a scholium in certain Greek manuscripts, and edited as Scholium 133 by Menge (1896, 296–298). This lemma is also shown with a full argument including enunciation and separate figure along with a geometrically related proposition following the main text of Thābit’s *Restoration* in one of the manuscripts of that version of the text (**A**, f. 19a,b; Sidoli and Isahaya, 2018, 28).

The claim of (1) in the argument of *Data* 67—because it involves an auxiliary line—does seem to call for some sort of lemma. The claim of (2), however, is trivial—indeed, it is not uncommon in geometrical texts of the Hellenistic period to find steps requiring three propositions of the *Elements* for their justification. Al-Ṭūsī may have taken his argument for claim (1) from his sources, but he may also have come up with it himself. The argument for claim (2) is simple enough that any reader familiar with ancient and medieval geometry could have supplied it.

3.7. Data *74 (Ṭūsī Prop. 75)*

Data 74 shows that if the ratio of two parallelograms is known and their angles are either mutually equal or unequal and known, then the ratio of a side of the first parallelogram to its correlate in the second is as the ratio of the other side of the second parallelogram to a line whose ratio to the remaining side of the first parallelogram is known—that is, in Figure 5, $(\mathbf{P}(AE, EB) : \mathbf{P}(GZ, ZD))_r$, and $\angle AEB = \angle GZD$ or $\angle AEB_m \Rightarrow (EB : ZD) = (GZ : EH)$ where $(AE : EH)_r$.

The argument begins by setting out line EH such that $(EB : ZD) = (GZ : EH)$, which, by *Elem.* VI.14, amounts to setting out $\mathbf{P}(BH) = \mathbf{P}(GD)$ such that $\mathbf{P}(BH)$ is under the same height as $\mathbf{P}(AB)$. Although there is no comment following this proposition in the main text of any of the manuscripts that we have consulted, **ThIaTs** contain a marginal note to this proposition, which is probably due to al-Ṭūsī himself and which addresses this situation (**Th**, p. 107; **Ia**, f. 70b; **Ts**, f. 11b). The following is our best guess for its text:

أقول: الأولى أن يقال فنسبة ضلع من أحدهما إلى ضلع من الآخر كنسبة الضلع الباقي من الآخر إلى خط نسبته إلى الضلع الباقي من الأولى كنسبة السطح الثاني إلى السطح الأول، يعدم انحصار الخط الذي نسبته إلى اه معلومة في ه ح. فلا يكون ه ب إلى ز د كـ ج ز إلى خط كيف ما كان نسبته إلى اه معلومة، فلا يطابق البرهان إلّا إن نخصص الدعوى.

I say: Of the first [case], if it is said that the ratio of a side of one of them to a side of the other is as the ratio of the remaining side of the other to a line whose ratio to the remaining side of the first is as the ratio of the second surface to the first surface, the restriction of the line whose ratio to AE is known to EH is missing. So, EB to ZD is not as GZ to an arbitrary line whose ratio to AE is known. So, the proof does not fit except if we qualify the enunciation.[45]

That is, al-Ṭūsī appears to be claiming that in the first case, the enunciation can be stated in such a way that the known ratio between lines is in fact the known ratio between the two parallelograms, so that the

[45] Literally, دعوى means *claim*, but Ṭūsī appears to use it to mean the part of the proposition known as the enunciation. See Sidoli and Isahaya (2018, 212–213) for a discussion of various Arabic terms used for the parts of a Greek proposition—which, however, does not include دعوى.

1 فنسبة] ونسبة **Ia** من أحدهما] من من أحدهما **Th**

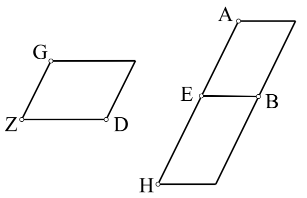

Figure 5. Diagram for *Data* 74 (Ṭūsī Prop. 75).

known ratio between lines is not that of an arbitrary line to one of the sides of the first parallelogram. While this is true, it is only relevant for the first case, so that the enunciation as stated in the text is still the most general expression for the entire propositions. Hence, it is clear why Ṭūsī decided that this overly pedantic note did not warrant being included in the final version of his *Revision*.

3.8. Data *80*

Data 80 shows that if a triangle has a given angle and if the ratio of the rectangle of the sides containing it to the square on the opposite side is known, then the triangle is known in form—that is, in Figure 6, $\mathbf{T}(ABG)$, $\angle BAG_m$, and $(\mathbf{R}(BA, AG) : \mathbf{S}(BG))_r \Rightarrow \mathbf{T}(ABG)_f$.[46] There is a comment by Ṭūsī to this proposition in some of the manuscripts, but not in others. Among the manuscripts that we have consulted, this comment is found in **ThIaTsW**, but not in the main text of **IhTbIs**.[47] It is also contained in the Hyderabad (1939/40, 36, n. 1) edition, which notes that it is missing from two of the manuscripts that were consulted. The comment reads:

أقول: هذا البيان خاصّ بالصورة التي تكون زاوية آ منها حادة، والدعوى عامّة، فينبغي أن نورد مع التركيب التفصيل، ونجعل البيان عامًّا ليشتمل المنفرجة أيضًا.

I say: This demonstration is specific to the case in which angle A is acute, but the enunciation is general. So, it is necessary that we come forth with a detailed assemblage[48] and make the demonstration general, so as to include the obtuse [angle] as well.

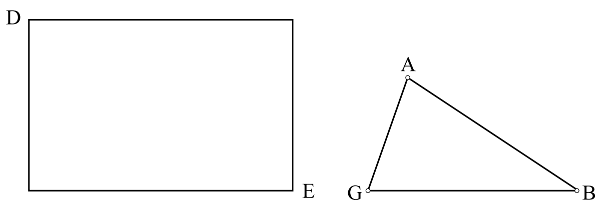

Figure 6. Diagram for *Data* 80b.

[46] The proof found in the Thābit's *Restoration* and Tūsī's *Revision* is that found as the alternative proof in the Greek text (Menge, 1896, 218–220; Sidoli and Isahaya, 2018, 169–171, 295–296).

[47] It is included in the margin of **Ih**, but by a later hand writing in the *ruqʿa* script.

[48] That is, a construction (تركيب). This is a standard term, but we differentiate it from عمل, which we translate as construction (Sidoli and Isahaya, 2018, 212–213).

2–1 أقول ... أيضًا] **Ih** *marg. hand*، **TbIs** (–)

In fact, however, the proof of *Data* 80 relies on *Data* 67 and 66, which both hold for triangles of any angle. Hence, this comment is, strictly speaking, not correct. The most likely scenario is that al-Ṭūsī himself noticed his own error in an earlier draft of the *Revision* and corrected himself in a later version by omitting the comment. If this is, indeed, what happened, it would serve to strengthen our hypothesis that **Th** represents an early draft, whereas **Ih** is an early copy of Ṭūsī's polished *Revision*.

3.9. Data *82 and 83*

Data 82 and 83 come from a short group of propositions that treat proportions of lines. They both use an argumentative strategy that we can call the introduction of a *contrivance*—namely, the semi-constructive introduction of a fourth proportional, which is not stated in the enunciation, but without which the proposition would not hold. *Data* 82 introduces the contrivance directly at the beginning of the proof, where we would normally expect the construction, while *Data* 83 introduces it implicitly by applying *Data* 82. The contrivance is semi-constructive in the sense that in certain cases—as here for lines—the fourth proportional can be introduced by a problem from the *Elements*—such as *Elem.* VI.12—but it is of a different logical status from a usual construction introduced in the course of a proof, because the proposition will not hold without the assumption of the contrivance, whereas a proposition can often be shown through a number of different constructions. In Greek sources, the contrivance is often introduced with the verb "to make" (ποιεῖν). The most famous use of the contrivance in Hellenistic geometry is in *Conics* 11–13 with the introduction of the *upright side* of the conic sections as a fourth proportional determined as given by its relation to certain fixed elements of the generating cone. Apollonius is quite clear about this by introducing his contrivances in the enunciation, where they belong, but in *Data* 82 the contrivance is introduced where we would normally expect the construction, which may make its logical status confusing. Al-Ṭūsī's comments to these two propositions directly address this issue, and make it clear that the contrivance is a fundamental assumption, without which the theorem would not hold.

Data 82 shows that if there are four proportional lines, then the ratio of the first of them to a line whose ratio to the second is known is as the ratio of the third to a line whose ratio to the fourth is known—that is, $(A : B) = (G : D) \Rightarrow (A : E) = (G : Z)$, where $(B : E)_r$ and $(D : Z)_r$ are known, under the assumption that $(D : Z)_r = (B : E)_r$.[49] The assumption is essential to the theorem, as can be seen from the argument, which we can summarize as follows. We set E as the line whose ratio to B is known, by *Data* Def.2, and then set $(D : Z) = (B : E)_r$, by *Elem.* VI.12, so that $(D : Z)_r$ is known, by *Data* Def.2. Then, since $(A : B) = (G : D)$ and $(B : E)_r = (D : Z)_r$, by *Elem.* V.22, $(A : E) = (G : Z)$. QED.

Al-Ṭūsī's comment to *Data* 82 addresses this situation as follows:

أقول: الأوضح إن يقال في الدعوى فنسبة الأول إلى خط نسبته إلى الثاني معلومة كنسبة الثالث إلى
خط نسبته إلى الرابع تلك النسبة حتى يطابقه البرهان.

I say: The clearest is if it is said in the enunciation that the ratio of the first to a line whose ratio to the second is known is as the ratio of the third to a line whose ratio to the fourth is that ratio[50] in order that the proof fits with it.

[49] For these propositions, we will not include the diagrams, since they are simply a series of lines with the associated letter-names.
[50] Namely, the known ratio—$(D : Z) = (B : E)_r$.

[1] يقال] فاو **W**   نسبته] نسبة ℍ   [2] نسبته] نسبة **W** ℍ   يطابقه] تطابقه ℍ

Ṭūsī's comment makes it clear that he considers the full statement of the conditions of the theorem to be $(A : E) = (B : Z)$ where $(D : Z) = (B : E)_r$. That is, he is pointing out that the contrivance, $(D : Z)_r = (B : E)_r$, is a fundamental assumption of the proposition, and in his opinion should be stated as such in the enunciation.

Data 83 shows that if there are four lines, and from them three are taken and with the three a fourth line is taken whose ratio to the remaining line of the first four is known, and the other four are proportional, then the ratio of the remaining line of the first four to the third line of them is as the ratio of the second line to a line whose ratio to the first line is known—that is, $(A : B) = (G : E)$, and $(D : E)_r \Rightarrow (D : G) = (B : X)$ and $(A : X)_r$, under the assumption that $(A : X)_r = (D : E)_r$.[51]

Al-Ṭūsī's comment to *Data* 83 reads as follows:

أقول: ينبغي في الدعوى أن يقال فنسبة الخط الباقي من الأربعة الأولى إلى الثالث منها كنسبة الثاني
إلى خط نسبته إلى الأول هي النسبة المعلومة المذكورة، أعني نسبة الرابع المأخوذ إلى الباقي من
الأربعة الأولى فإنّ نسبة د̅ إلى ج̅ كنسبة ب̅ إلى خط نسبته إلى ا̅ كنسبة ه̅ إلى د̅.

I say: It is necessary that it be said in the enunciation that the ratio of the remaining line from the first four to the third of them is as the ratio of the second to a line whose ratio to the first is the stated known ratio—namely, the ratio of the taken fourth to the remaining from the first four. For, indeed, the ratio D to G is as the ratio of B to a line whose ratio to A is as the ratio E to D.

Once again, al-Ṭūsī is asserting that a full statement of the conditions of the theorem would include the claim that $(X : A) = (E : D)_r$. That is, he argues that this is a fundamental assumption of the proposition, which should, in his opinion, be included in the enunciation.

Al-Ṭūsī's comments to these propositions are indeed helpful for elucidating the logical structure of the propositions and making clear the role of contrivances. In both the Greek and the Arabic traditions, the grammatical idiom used to state these claims—"let it have been made" (πεποιήσθω) and "we make" (نجعل) —and their placement in the structure of the overall argument—in the beginning of the argument, where we would normally have the construction—could lead to confusion about their logical role in the argument (Menge, 1896, 162; Hyderabad, 1939/40, 37). Ṭūsī points out that these are not constructions in the normal sense, but are, in fact, fundamental assumptions of the theorems, without which they would not hold.

4. Conclusion

It may be worth pointing out that al-Ṭūsī provides no overall discussion of what he thinks is important about the *Data*, why we should read it, its relation to geometrical analysis, or why it is found in the *Middle Books*—a collection of classical texts in the mathematical sciences by Greek and Arabic authors that was organized so as to be read between Euclid's *Elements* and Ptolemy's *Almagest*. We may presume that the treatise had originally been included in this collection because of its use in reading through arguments of metrical analysis in the *Almagest* that proceed by making claims about what is given, or known.[52]

[51] Line X is unnamed in the text. It is only introduced implicitly through an application of *Data* 82.

[52] *Metrical analysis* is the terminology introduced by Sidoli for the argumentative style of Heron and Ptolemy that involves claims about what is given that can be justified by theorems of the *Data* and which can be used to justify, or summarize, a computation involving arithmetical operations and, in Ptolemy's case, entries into a chord table (see Sidoli, 2004, 78; Sidoli, 2005, 253, and

1 في الدعوى] *line* **Ia**   3 الأولى] الاول ℍ   ب̅] خط ℍ

Nevertheless, Ṭūsī does not address any of this and he probably simply took the placement of the *Data* in the *Middle Books* for granted, and as not requiring any explanation on his part.

From a mathematical perspective, the most interesting comments are those to *Data* 14, 15, 67, 82 and 83—but even these are at a fairly elementary level. The first three of these deal with cases and lemmas, and are the sort of remarks that we often find in ancient and medieval commentaries and scholia to mathematical works. Perhaps the two most mathematically significant comments are those that treat the issue of contrivances. These deal with the use of contrivances in the proof and directly address the question of where, in the structure of a proof, the contrivance should be introduced. Since, in a Greek mathematical proposition, certain aspects of the deductive force are implicitly dealt with through the structure of the argument, these comments can be regarded as treating the foundations of mathematics—insofar as they give al-Ṭūsī's opinion of how the argument *ought* to be structured. On the other hand, contrivances are also used in the group of theorems dealing with the *greater-by-a-known-than-in-ratio* relation, *Data* 10–23 (Sidoli and Isahaya, 2018, 235–247), but these are not discussed by Tūsī, so it seems that his treatment of contrivances was not systematic.

From the perspective of the text history, the most interesting comments are those to *Data* 62 and 80. From the first, we get a clear indication of the fact that al-Ṭūsī had access to at least two or three manuscripts of the treatise, one of which goes back to the scholarship of al-Nasawī. From both we form the impression that Ṭūsī himself produced at least two different drafts of his *Revision*, the second of which shows certain clear improvements over the first.

From this study we can develop a more detailed understanding of the way that al-Ṭūsī worked with and studied Euclid's *Data* that complements the picture of Ṭūsī's scholarship with regard to his original works as developed by Ragep (1993, 65–75) in his study of Ṭūsī's *Memoir*. In particular, Ṭūsī appears to have started his edition on the basis of a manuscript of Thābit's *Restoration of the Data* that was similar to that contained in **AK**, except at *Data* 62 (*Restoration* Prop. 60b, Ṭūsī Prop. 64). He then wrote marginal notes in this text, and worked some of these into the text itself in the form of comments. Over time, he accumulated more manuscripts of the *Restoration*, in particular one that had been read to the famous mathematical scholar Abū al-Ḥasan ʿAlī ibn Aḥmad al-Nasawī. As he collected these manuscripts he continued to read the text—probably with students who copied his text from him and read it back to him as a way of collating their copies and demonstrating their ability to pass on the master's teachings. Finally, he produced a more polished version that was based on a somewhat wider purview of the manuscript evidence and which excised some comments and notes that were either incorrect or trivial.

A final result of our investigation of the manuscript sources of al-Ṭūsī's *Revision of Euclid's Data* is the realization that the version of this text contained in what we have called the **Ih**-family is more polished, and hence should probably be regarded as Ṭūsī's more considered draft of his work. This may be of significance in studies of other treatises of Ṭūsī's work contained in these manuscripts.

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