Naṣīr al-Dīn al-Ṭūsī's revision of Theodosius's *Spherics*

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Abstract

We examine the Arabic edition of Theodosius's *Spherics* composed by Naṣīr al-Dīn al-Ṭūsī. Through a comparison of this text with earlier Arabic and Greek versions and a study of his editorial remarks, we develop a better understanding of al-Ṭūsī editorial project. We show that al-Ṭūsī's goal was to revitalize the text of Theodosius's *Spherics* by considering it firstly as a product of the mathematical sciences and secondarily as a historically contingent work. His editorial practices involved adding a number of additional hypotheses and auxiliary lemmas to demonstrate theorems used in the *Spherics*, reworking some propositions to clarify the underlying mathematical argument and reorganizing the proof structure in a few propositions. For al-Ṭūsı, the detailed preservation of the words and drawings was less important than a mathematically coherent presentation of the arguments and diagrams.

Introduction

If the number of the manuscripts may be taken as any indication, during the medieval period, some of the most popular versions of ancient Greek mathematical texts were those made in the middle of the 13$^{\text{th}}$ century by Abū Jaʿfar

Muḥammad ibn Muḥammad ibn al-Ḥasan Naṣīr al-Dīn al-Ṭūsī. During the course of a long scholarly career, al-Ṭūsī made new Arabic editions of the texts that were then the classics of the mathematical sciences. Along with Euclid's *Elements* and Ptolemy's *Almagest*, Ṭūsī edited some fifteen works by Greek mathematicians as well as a few original treatises by Arabic mathematicians that circulated with them under the name of the *Middle Books*.[1] This paper is a study of Ṭūsī's editorial practices in the production of his version of Theodosius's *Spherics*.

Such a study, however, presents an immediate problem. The medieval manuscripts preserving Arabic texts of Theodosius's *Spherics* reveal that there were at least four versions from which al-Ṭūsī could have made his revision and it is not clear which of these he used.[2] Although Ṭūsī mentions that he worked with a number of different copies of the text, it is clear from his comments that he regarded these copies as essentially representing a single version of the text, the correction of an Arabic translation made by Thābit ibn Qurra. Nevertheless, the changes Ṭūsī made to the text are so pervasive, and the differences in the older versions in comparison so relatively minor, that it has not yet been possible to determine precisely which of these versions he used. Hence, in this paper, we work around this problem by focusing on some of the unique features of Ṭūsī's revision. In a number of passages, he makes it clear that he is adding material, which, indeed, is not found in the other versions. In some propositions, Ṭūsī's text has distinctive characteristics that distinguish it from both the older Arabic versions and the Greek text. By examining these types of passages in his edition of Theodosius's *Spherics*, we hope to shed some light both on Ṭūsī's editorial procedures and on the interests that may have guided his entire editorial project.

In the first section of the paper, we examine Ṭūsī's additions, including remarks on the text tradition and lemmas apparently carried over from the Greek transmission and Ṭūsī's predecessors in the Arabic transmission. In the second section, we compare al-Ṭūsī's version of *Spher.* II 15 & 16 with that in the Greek text and in earlier Arabic editions.

For al-Ṭūsī's recension, we have used the text printed in Hyderabad, often correcting it against manuscript 3484 of the Tehran National Library.[3]

[1] A list of Ṭūsī's editions is given by Krause (1936a, 499–504).

[2] See Lorch (1996, 164–165) for a stemma of the early Arabic versions.

[3] See al-Ṭūsī (1940a); Aghayanī-Chavoshī (2005). For Ṭūsī's revision of Theodosius's *Spherics*, the text in *T.* 3484 occupies pp. 23–56 but the beginning of the treatise is in disarray. The opening pages should be read in the following order: 27, 28, 25, 26, 23, 24. Then, four

Although we cannot be certain what version of the text Ṭūsī used as his source, that found in the Kraus MS and *Leiden or.* 1031 is often close to the Greek text, whereas Ṭūsī's edition differs from these according to fairly consistent patterns.[4] Despite the fact that the text in these manuscripts is different in many places in ways that would present real issues for the editor of a critical edition, for the passages of interest to us and for the purposes of broad structural comparison, we find that the they agree closely enough that they can be used to correct against each other.[5] For the purposes of comparison with the Arabic texts, we have preferred the Greek edition made by Czinczenheim (2000) to that of Heiberg (1927).[6]

Al-Ṭūsī's editorial interjections

Including his opening remarks about the history of the text, al-Ṭūsī's edition of the *Spherics* contains nine passages that were certainly included by him and are not found in other Arabic versions of the treatise.[7] These are clearly identified by him as additions because they either open with the statement "I say" (أقول) or they make explicit reference to the manuscript tradition with the words "in some copies" (في بعض النسخ).[8]

pages have gone missing (midway through *Spher.* I 19 to midway through *Spher.* II 8). The text picks up again on p. 29 and is thereafter continuous and complete. After the pages were jumbled in this manner, they were bound together and numbered continuously in Eastern Arabic numerals. (The missing pages of Theodosius's *Spherics* do not appear to be bound in elsewhere in the manuscript.)

[4]The Kraus MS is so called because it once belonged to the book dealer H. P. Kraus. It has been described by Kraus (1974, 45, n. 18), Kheirandish (1999, xxvii) and Lorch (2001, 28). We are grateful to the owner of the manuscript for making reproductions available to scholars.

[5]Although perhaps an artificial construct, for the sake of brevity, we will call the text agreed upon by these MSS the **KL** version. In this paper, we present some passages of this so-called **KL** version, but this is done only for the purposes of comparison with al-Ṭūsī's revision and the Geek. Hence, we do not note all of the variant readings and overlook a number of the real difficulties involved in assuming that these two manuscripts represent a single version of the text.

[6]In fact, the agreement between Czinczenheim's Greek text and the early Arabic version represented by the Kraus MS and *L. or.* 1031 often lends further support to her editorial choices.

[7]Because some of these are of a rather trivial nature and will not be discussed below, it may be useful to provide a full list of the page numbers in the Hyderabad version: 2, 3, 13, 19, 22, 33, 46, 48-50, 50 (al-Ṭūsī 1940a).

[8]There are also a few short passages found woven into the propositions that can also almost certainly be attributed to Ṭūsī, because they make reference to his editorial remarks and are not included in other Arabic versions.

These remarks are primarily concerned with the logical structure and mathematical consistency of the text. This situation may be compared with some of al-Ṭūsī's revisions of other Greek mathematical texts in the *Middle Books*. Indeed, the style of Ṭūsī's remarks in his Theodosius's *Spherics* is a sort of intermediary between his revision of shorter works in the *Middle Books* and his much more extensive work on Menelaus's *Spherics*, the mathematical contents of which was still of great interest to working mathematicians in his time. Ṭūsī's revision of Aristarchus's *On the Sizes and Distances of the Sun and the Moon* contains little in the way of editorial comments, although there is minor reworking of the logical argumentation in some theorems.[9] His revision of Menelaus's *Spherics*, on the other hand, is bristling with editorial remarks following the majority of the propositions and treating many different aspects of the text, such as the logical structure of the individual propositions and of the work as a whole, the complicated textual history of the work in Arabic (including alternative proofs found in some manuscripts and the editorial remarks of some of his predecessors), a few comments on how the theorems relate to other medieval work in the exact sciences and explanations of how the geometrical propositions of the text can be interpreted in terms of spherical astronomy.[10] Since, many of the theorems in books II and III of Theodosius's *Spherics*, like a number of those in Menelaus's, clearly have an astronomical purpose, this difference is noteworthy.[11] Hence, Ṭūsī's interest in explaining and supplementing Theodosius's *Spherics* appears to have been primarily structural and mathematical, as opposed to astronomical.

Comments on the history of the text

The supplementary comments fall into two basic types; those based on the text tradition, and those based on Ṭūsī's own assessment of the mathematical requirements of the text. There are three comments in which Ṭūsī makes mention of his manuscript sources, in each of which he refers to a variant by stating that it occurs "in some copies" (في بعض النسخ).[12] Since this phrase

[9]al-Ṭūsī (1940c); Berggren and Sidoli (2007).

[10]See al-Ṭūsī (1940b); Nadal, Taha and Pinel (2004). The astronomical remarks in Ṭūsī edition of the Menelaus's *Spherics* are largely based on the comments included by Abū Naṣr Manṣūr ibn ʿAlī ibn ʿIrāq, in his edition of the same text (Krause 1936b).

[11]Discussions of the astronomical content of Theodosius's *Spherics* are given, for examples, by Schimdt (1943) and Berggren (1991).

[12]al-Ṭūsī (1940a, 1, 19 & 48).

implies that the variant occurred in multiple copies of the text, from this we learn something of Ṭūsī's working habits and assumptions in revising Theodosius's *Spherics*. In the process of carrying out this work, he appears to have used at least three different manuscripts. While making his revisions, he collated these against each other and believed that it would be worthwhile to inform his readers of some of the differences.

After the title and before the definitions, al-Ṭūsī briefly described what he understood to be the history of the text in the Arabic language. Concerning the *Spherics* he says,

> It is three books and fifty nine propositions, with a proposition omitted in number in some copies. Abū al-ʿAbbās Aḥmad ibn Muʿtaṣim bi-llāh commissioned its translation from Greek into Arabic and Qusṭā ibn Lūqā al-Baʿlabakkī carried out its translation as far as the fifth proposition of the third book. Then someone else carried out a translation of the rest of it, and Thābit ibn Qurra corrected it.[13]

According to Ṭūsī, the translation of Theodosius's *Spherics* was carried out in the midst of the Baghdad translation activity in a manner that was typical of the time. It was commissioned by one of the period's most supportive patrons, begun by one of its most active translators and corrected by one of its most able mathematicians.[14] From the way he describes this, it seems Ṭūsī believed this activity produced a single text. Moreover, it seems that Ṭūsī believed he was working with Thābit's correction. Indeed, although he mentions some textual variations in his comments, it will become clear as we proceed that these were not great enough to support an argument for significantly different text traditions and Ṭūsī probably believed these variants occurred in the Arabic tradition of Thābit's correction.

Ṭūsī's next mention of the manuscript variants provides us with an explanation for his use of the otherwise odd phrase "a proposition omitted in number" in his introduction. Following *Spher.* II 12, al-Ṭūsī states that in some MSS *Spher.* II 11 & 12 are counted together as a single proposition.[15]

[13] al-Ṭūsī (1940a, 2).

[14] See Gutas (1998, esp. 125–126) for a discussion of the Baghdad translation activity and especially for Ibn Muʿtaṣim's role in it.

[15] This is the case, for example, with *L. or.* 1031. In this MS, *Spher.* II 12–23 are numbered as 11–22.

Hence, while the proposition is not counted individually, its substance is still included in the text. Referring to *Spher.* II 12, he says,

> In some copies, this is not enumerated as a separate proposition but is, rather, enumerated under the reckoning of the previous proposition. For in the first, the equality of the two opposite arcs that are the sides of the perpendicular is established from the equality of the two arcs that are the other sides and from the equality of their chords, while in the second the equality of the chords is established from the equality of the arcs of the sides, each due to its correlate.[16]

In his note, Ṭūsī points out that *Spher.* II 11 & 12 were probably numbered together, not merely because each is the converse of the other, but because the arguments themselves are closely related and simply start with a different set of equal objects. Nevertheless, despite the fact that he understands the reason for numbering these propositions together, Ṭūsī gives them individual numbers in his edition, following the Greek practice of individuating converse theorems.

Al-Ṭūsī's final reference to his manuscript sources is at the beginning of a series of lemmas he provides following *Spher.* III 11. Before setting out the lemmas, Ṭūsī mentions that "a proposition for a proof" (شكل لبيان) of this lemma, "by Thābit" (لثابت), is set out "in some copies."[17] Because these lemmas are of interest in their own right, we will discuss them in some detail below.

From these three passages, we see that although Ṭūsī had access to a number of manuscripts as he worked on his edition, the differences that he found, and saw fit to mention, were rather slight. These were a slight difference in enumeration and a lemma, probably found in the marginal notes of some of his sources. Hence, it is clear that based on the evidence available to him, Ṭūsī believed that there was in circulation only a single version of the Arabic translation of Theodosius's text, the correction made in Baghdad by Thābit ibn Qurra.[18] His references to the manuscript tradition, and his care to always inform the reader when he is interjecting his own comments, make it clear that Ṭūsī was attempting to produce what he regarded as a faithful

[16] al-Ṭūsī (1940a, 19).
[17] al-Ṭūsī (1940a, 48).
[18] This may be contrasted to the situation evident from a recent assessment of the medieval

preservation of his sources. As we will see below, in the few places where he engages in a revision of the text itself, it can be shown that this stems from a belief that the mathematical argument in his sources is insufficient.[19]

Comments on mathematical structure and lemmas

Al-Ṭūsī's overall interest in the logical integrity of the treatise is established at the outset with a new set of first principles, which he says will be used in the problems, that is in the seven propositions that demonstrate the validity of certain geometric constructions (*Spher.* I 2, 18–21, II $13_a(14_g)$, 15).[20] In fact, the presentation of all of the first principles is somewhat different from what we find in the Greek texts, but since the early definitions are all mathematically related to those in the Greek MSS and agree closely with those found in other Arabic MSS, we may assume that the arrangement and content of these are not due to Ṭūsī. Indeed, as usual, al-Ṭūsī makes his intervention clear.

> I say: It is necessary that we establish that it is the case [1] that we make any point that happens to be on the surface of the sphere a pole and we draw about it with any distance, less than the diameter of the sphere, a circle on that surface; and [2] that we produce any arc that there is until it completes its circle; and [3] that we cut off what is equal to a known[21] arc from an arc greater

manuscripts. See note 2, above.

[19]This faithfulness to the overall structure of his sources has also been shown in studies of his revisions of the Banū Mūsā's *Treatise on Measuring Plane and Spherical Figures* and Aristarchus's *On the Sizes and Distances of the Sun and the Moon* (Rashed 1996, 9, 12–27; Berggren and Sidoli 2007, 238–247). Dold-Samplonius (1995), on the other hand, argues that he made rather extensive changes in the structure of Thābit ibn Qurra's *Treatise on Assumptions* when he revised this treatise to be included in his edition of the *Middle Books*. Since such structural changes are not common in his revisions, however, we should also consider the possibility that Ṭūsī worked with a different version of Thābit's text than that preserved in Aya Sofya 4832, today the only known copy of Thābit's treatise made prior to Ṭūsī's revision.

[20]The ordering of propositions II 13 & 14 is switched between the Arabic and Greek versions of the text, so that the problem which is II 14 in the Greek, is II 13 in the Araibic versions.

[21]Ṭūsī, following standard practice, uses perfect participles from two different roots, to render Theodosius's idea of *given*: معلوم and مفروض. Greek mathematicians, however, used two participle forms of a verb meaning "to give" (δίδωμι), δοθείς and δεδομένος (aorist and perfect participles, respectively), with a range of meanings including assumed at the mathematician's discretion, fixed by the mathematical constraints of the problem and determined on the basis of these. Since it is possible to distinguish between these different usages, and since al-

> than it, when they belong to equal circles; and [4] that it is not the case that a single circle has more than two poles; and [5] that the arcs similar to an arc are similar to one another and so on; which treats this in the same way with respect to what results in the course of the problems.[22]

Statements [1]–[3] are construction postulates, whereas [4] & [5] are common notions asserting the transitivity of similarity and the uniqueness of the pair of poles of any circle in a sphere. The final, obscure phrase seems to imply that the new postulates are specifically related to the problems in the text. That is, that the new material deals with suppositions that are necessary for what occurs in the problems in the same way as the original definitions treat suppositions that are necessary for the theorems. Most of these five new first principles are, indeed, used fairly frequently throughout the treatise, without any explicit reference.[23] The one exception is [4], which appears to be used only once and with an explicit reference.

Spher. II 7 shows that if, in a sphere, there are two equal parallel circles and a great circle is tangent to one of them, it will also be tangent to the other. The argument is indirect. Since *Spher.* II 6 has just shown that if a great circle is tangent to a lesser circle it will also be tangent to an equal parallel, in *Spher.* II 7, Theodosius argues that if one of the original equal parallel circles is not tangent to the great circle, then there will be another equal parallel circle that is. There will then be three equal parallel circles in the sphere, which is asserted as impossible. In Ṭūsī's edition the statement of impossibility is followed by an explanation that is not found in any of the other versions and is almost certainly due to Ṭūsī himself. Concerning the claim that it is impossible to have three equal parallel circles in a sphere, the text reads, "That would require either that a single circle have three poles or the equality of the whole and its part."[24] Although he does not give the details, he means that the argument can be brought to this dichotomy, both

Ṭūsī may have been attempting to make such a distinction, we have translated the former as *known* and the latter as *determined*. The claim by Fournarakis and Christianidis (2006) that the two Greek particles have different meanings should be compared with the Arabic terminology.

[22] al-Ṭūsī (1940a, 3). Here, and throughout, we include some material not found in the text for convenience, in square brackets, [].

[23] For a discussion of the uses of the three construction postulates, see Sidoli and Saito (Forthcoming). The transitivity of similarity is used frequently.

[24] al-Ṭūsī (1940a, 15).

sides of which are false. While the falsity of the second claim is secured by the Euclidean common notion (*Elem.* I c.n. 8), Ṭūsī appears to have written his postulate [4] specifically to refute the first claim.

The first three postulates are meant to supply constructions that are used in the text, especially in the seven problems. Ṭūsī's interest in asserting these postulates appears to be purely logical. The constructions they provide are in fact used in the treatise and hence Ṭūsī considers that the argument will be more sound if the treatise contains some statements asserting their validity. From this perspective, it is interesting to consider why Theodosius may not have included such postulates in his work. The fundamental difference is probably due to the contexts in which Theodosius and Ṭūsī believed the text would be read. While Theodosius wrote the *Spherics* to be read as logically founded on the *Elements* and probably some other works in elementary geometry,[25] Ṭūsī sought to give the work a more independent status.

The next three examples we discuss reveal Ṭūsī's interest in guiding the reader through the sometimes intricate thicket of the argument. They are also closely related to scholia found in the Greek MS tradition and include a scholium apparently written by Thābit ibn Qurra. In the first, Ṭūsī gives a somewhat different argument to make the same claim as one of the Greek lemmas, in the second he appears to have tacitly adopted the lemma and in the third he mentions that the lemma was found in some of his sources.

Lemma to Spher. *III 12*

In the course of the demonstration of *Spher.* III 12, Theodosius claims, in Figure 2, that where ETK and FTO are great circles tangent to the same circle, EF, parallel to circle OK, and LTQ is a great circle through their intersection, T, and the pole of the parallels, L, then arc OQ and arc QK are equal.

An argument for this can be based on *Spher.* II 11, which shows, in Figure 1, that where AHG and DTZ are equal segments perpendicular on the diameters of equal circles and arc TD equals arc HA and line TE equals line HB, then arc AB is equal to arc ED.[27] Following *Spher.* III 12, Ṭūsī

[25] See Sidoli and Saito (Forthcoming) for a discussion of the relationship between Theodosius's *Spherics* and Euclid's *Elements*.

[26] For the MS diagrams we follow those in *Tehran* 3484 (Aghayanī-Chavoshī 2005).

[27] Al-Ṭūsī's proof as given below should be compared with that in the scholia to the Greek

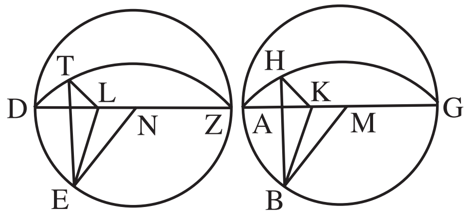

Figure 1: ***Spher.* II 11**, MS Diagram [*T.* 3484, 30][26]

gives a lemma using *Spher.* II 11. He anticipates this argument, however, with a short lemma following *Spher.* II $14_a(13_g)$, which states,[28]

> I say: It is obvious from this proof that each of the two arcs FO and EK, and of the two arcs TF and ET, and of the remaining two arcs TO and TK, are mutually equal; and this will become requisite in what will be done below.[29]

On this basis, Ṭūsī provides a lemma showing that, in Figure 2, arc OQ equals arc QK. As usual, the lemma comes after the theorem in which is it used. In what will be his final commentary, following *Spher.* III 12, al-Ṭūsī writes,

> I say, in proof that circle LTQ bisects arc KO: The equality of arcs TK and TO has been shown from what took place following the fourteenth proposition of the second book. Circle LTQ, passing through the pole of circle KO, bisects it orthogonally [*Spher.* I 15], so, the segment TQ, which is united with it, constructed on the diameter of circle OK passing through the point Q, is upright on the plane of circle OK. And the chords of the arcs TK and TO, extending from point T to the circumference OK, are mutually equal. So, arcs QK and QO are mutually equal, just as took place in the eleventh proposition of the second book. The distinction is that there the proof was with respect

MSS (Czinczenheim 2000, 436–437). See also Ver Eecke (1959, 116, n. 4).

[28]Since the objects mentioned in the lemma to *Spher.* II $14_a(13_g)$ can all be located in Figure 2, in the following translation we have changed the letter names of the geometric objects to agree with the names of the same objects in *Spher.* III 12.

[29]al-Ṭūsī (1940a, 22).

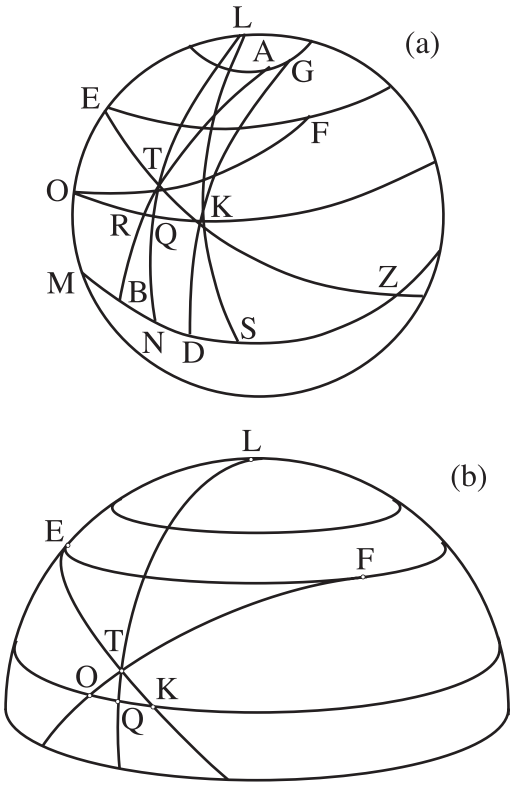

Figure 2: ***Spher.* III 12**, (a) MS Diagram [*T.* 3484, 54], (b) Simplified Reconstruction

to two circles and here with respect to a single circle.[30]

Both the set of five new first principles and the anticipation of the requirements of the lemma to *Spher.* III 12 by the lemma to *Spher.* II 14 show that al-Ṭūsī composed his revision of the Arabic text after he was already thoroughly familiar with its mathematical content. Although he may have been guided by the scholia in his sources, his reworking of the lemma to *Spher.* III 12 shows that he sometimes went beyond his sources in anticipating the need of the argument and reworking the text. While the gap filled by his lemmas to *Spher.* III 12 and *Spher.* II 14 flesh out the steps of the argument based on material that is already in the *Spherics*, in the final two lemmas we will

[30]al-Ṭūsī (1940a, 50–51).

examine, al-Ṭūsī brings material found as scholia in his sources into the text itself.

Lemma to **Spher.** *III 10*

The first substantial lemma introduced into the text shows that, given two unequal magnitudes, an intermediate magnitude can be constructed that is commensurable with any given magnitude. This lemma is used in a number of places in the Greek mathematical corpus and it may, in fact, derive from a pre-Euclidean theory of proportion.[31]

In what was apparently a common strategy for demonstrating the assertion of a proportion, or ratio inequality, a Greek mathematician might first establish the theorem for the case where the objects in ratio are commensurable and then, on the basis of this, prove the case where they are incommensurable. This method was used by Archimedes in *Equilibrium of Planes* I 6 & 7,[32] by Theodosius, here, in *Spher.* II 9 & 10, and later by Pappus in his *Commentary to Almagest V* (and *Collection* V 12),[33] and in his comments to the Theodosian propositions in *Collection* VI 7 & 8.[34] In each of these pairs, the proof of the second, incommensurable case relies on the assumption that an intermediate commensurable can be constructed.

A scholium establishing this lemma was transmitted in the Greek manuscript tradition of the *Spherics*.[35] Although Ṭūsī makes no mention of any textual basis for the lemma in his edition, its similarity with the Greek version indicates that it was probably based on some transmission of this proof into the Arabic tradition. On the other hand, there are some differences of conception and approach between the Greek scholium and Ṭūsī's insertion that may indicate that he, or someone before him in the Arabic transmission,

[31] Knorr (1978); Mendell (2007).

[32] Heiberg (1910–1915, 132–138).

[33] Rome (1931, 256–257); Hultsch (1876, 336–340)

[34] Hultsch (1876, 482–486). See Mendell (2007, 5, n. 5) for a discussion of all the known examples of this two-stage method. Berggren (1976, 96–99) regards *Equilibrium of Planes* I 6 & 7 as later additions to the text, which should not be attributed to Archimedes. It should be noted, however, that these propositions, like the rest of the work, are written in Archimedes' native Doric. Hence, if they are not due to Archimedes himself, they must have been added by someone who was interested in preserving the original dialect and historicity of the text and considered the two-stage method a viable means of demonstrating the law of levers. Thus, whether or not we read these theorems as due to Archimedes, they support our basic claim that the two-stage method was a common strategy for asserting a proportion.

[35] Heiberg (1927, 193–194); Czinczenheim (2000, 431); Mendell (2007, 6).

simply read and understood the proof in the manuscripts and then rewrote it based on an understanding of the mathematics involved. As is usually the case with scholia, the proof is fairly simple, and it would almost certainly have been easier for a mathematically inclined reader to simply rewrite it than to carefully transcribe and collate it against a manuscript source. Following *Spher.* III 10, al-Ṭūsī inserts the following passage.

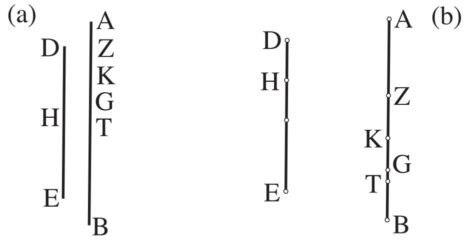

Figure 3: **Lemma to *Spher.* III 10**, (a) MS Diagram [*T.* 3484, 51], (b) Reconstruction

> I say: For a proof of a lemma, used in this proposition and the proposition before it, let AB and BG be two unequal magnitudes and DE a third of their kind. The requirement is finding a magnitude less than AB and greater than GB, being commensurable[36] with DE. So, let us bisect AG at Z, and we bisect DE over and over again until it becomes less than GZ. Let DH be the part of it that is less than GZ. We measure BG with DH by diminishing it by it over and over again until it vanishes or what remains of it, TG, is less than DH. So, BT is measured by DH.[37] If we add DH to BT, as BK, it will be greater than BG.[38] So, BK is a magnitude less than AB and greater than GB, and it is commensurable with DE, because DH measures them both, which is the requirement.[39]

If we compare Ṭūsī's proof with that in the Greek scholia, we find that

[36]Here and elsewhere, al-Ṭūsī uses the term مشارك to express the idea of commensurability denoted by σύμμετρος in the Greek.

[37]In the case that DH "vanishes," T and G will coincide. That is, $BG = BT$ is measured by DH.

[38]In the case that DH "vanishes," we add DH to BG resulting, again, in BK.

[39]al-Ṭūsī (1940a, 46).

there is one significant conceptual difference that results in a somewhat different proof.[40] In the Greek scholium, once DH has been constructed, the text asserts that it "either measures" (ἤτοι μετρεῖ) BG "or it does not" (ἢ οὐ).[41] The proof then breaks into two cases. In Ṭūsī's version, however, "we measure (نقدّر) BG with DH by diminishing it by it over and over again until it vanishes," or the remainder is less than the measuring magnitude. Hence, whereas in the Greek proof, the property of measuring, or not measuring, inherently belongs to two given magnitudes, for Ṭūsī measuring a magnitude is something a mathematician does through a certain process leading to some result. Because Ṭūsī's proof is based on the result of this process, there is really only one case.

Although there is no textual evidence that Ṭūsī was aware of the other uses of this lemma in the Greek mathematical corpus, it is noteworthy that he selected it for rewriting and inclusion in his edition. In some way or another, Ṭūsī probably was aware of the historical importance of the lemma. As we will see, the next lemma that he included was also of historical significance, and, in this later case, we are certain that Ṭūsī did know its significance.

Lemmas to **Spher.** *III 11*

Ṭūsī's most substantial addition to the text of Theodosius's *Spherics* is a lemma of some importance for the history of exact sciences in antiquity. From the early Hellenistic period, mathematicians working in the exact sciences, such as Euclid, Aristarchus and Archimedes, were assuming and demonstrating a group of lemmas that establish ratio inequalities for ratios of sides and angles of right triangles under the same height.[42] In the Roman Imperial period, Theodosius assumes such a lemma, here, in *Spher.* III 11,[43] and Ptolemy, in his *Almagest*, proves two related lemmas, one of which he says comes from Apollonius.[44] In late antiquity, a number of proofs of these lemmas were circulating in the commentaries on, and scholia to, the above

[40] In fact, there are also some other minor differences that indicate that Ṭūsī, or his source, did not simply copy the Greek proof.

[41] Czinczenheim (2000, 431). Note that Heiberg's text is slightly different in this passage (Heiberg 1927, 193).

[42] Knorr (1985); Berggren and Sidoli (2007, 223–225).

[43] Heiberg (1927, 158); Czinczenheim (2000, 169–170).

[44] Heiberg (1898–1903, p. 1, 43–45 & p. 2, 456–458). Whereas most of these lemmas concern right triangles, that shown by Apollonius concerns any triangle. Nevertheless, both the subject matter and proof are closely related to the lemmas concerning right triangles.

mentioned uses of them.[45] A number of the uses and proofs of these lemmas were known to al-Ṭūsī and included in his editions of Greek mathematical texts.

The two basic versions of these trigonometric lemmas, to which all others can be reduced, were asserted by Archimedes in his *Sand Reckoner*.[46] Considering Figure 4, Archimedes claims that

$$\beta : \alpha < BA : BD, \qquad \text{(T.L. 1)}$$

and

$$\beta : \alpha > GA : GD. \qquad \text{(T.L. 2)}$$

Although the *Sand Reckoner* does not appear to have been known to the Arabic mathematicians, both of these versions of the lemmas were also asserted by Aristarchus in his *Sizes and Distances of the Sun and the Moon*, a text which al-Ṭūsī knew and edited.[47] Another text that Ṭūsī edited, however, contains a version of T.L. 2 that is closely related to what we find in his edition of the *Spherics*. Ptolemy, in *Alm*. I 10 in the course of deriving his chord table, proves that $arc\,GA : arc\,GD > GA : GD$ (see Figure 4 (b)), which is an immediate consequence of T.L. 2.[48] Not only does Ptolemy's version of the lemma assert a property of arcs of the circle drawn through points A, G, and D, but his proof is based on the geometry of this circle. As we will see below, one of Ṭūsī's lemmas, which he attributes to Thābit, involves the same circle reflected about line GA, such that it is drawn through A, D' and G.

[45]Knorr (1985) provides a useful discussion of almost all of the ancient versions of these lemmas. Knorr believes that there was a single source for these lemmas in some 3rd or 4th century treatise of mathematical astronomy. This may, indeed, have been the case; however, the historical precision suggested by his detailed textual analysis is largely a chimera. Notice, for example, that the version of the scholium to Theodosius's *Spherics* III 11 that he presents contains not only excerpts imported from other texts but a number of manuscript readings from a collection of scholia made by Andreas Darmarius in the end of the 16th century, which are of no value for the purposes of establishing an ancient or medieval text, and were wisely omitted by both Heiberg and Czinczenheim (Knorr 1985, 264–265). Furthermore, the selection of Greek passages cited to facilitate verbatim comparison of the different versions are almost all idiomatic mathematical expressions (Knorr 1985, 387–388). Given the rigidly formulaic nature of Greek mathematical prose, and the fact that these passages are found in proofs of a closely related set of theorems, it would be rather surprising if there were less verbatim agreement (Netz 1999, 127–167).

[46]Heiberg (1910–1915, vol. 2, 232).

[47]al-Ṭūsī (1940c); Berggren and Sidoli (2007, 225).

[48]Heiberg (1898–1903, 43–45).

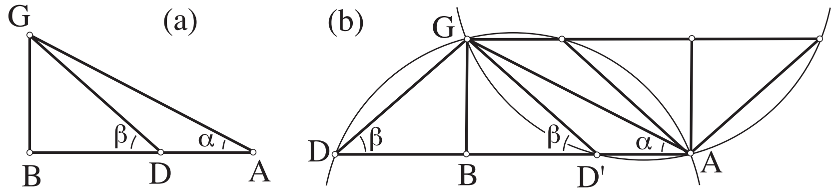

Figure 4: Diagrams for the trigonometric lemmas

Following *Spher.* III 11, al-Ṭūsī includes two different proofs of T.L. 1. Although his expression is somewhat vague, he seems to be attributing the first of these to Thābit ibn Qurra and the second to an unnamed source. Ṭūsī's addition is as follows.

> I say: In some copies is found a proposition[49] by Thābit for a demonstration of the lemma used here. His arrangement is thus.
>
> In triangle ABG, let angle B be right, and we produce in it GD at random.[50] I say that the ratio of AB to BD is greater than the ratio of angle BDG to angle BAG.
>
> [Fig. 5 (a)] Its proof: We draw circle $ADGE$ about triangle ADG, and we produce line DZE from point D as a parallel to BG. We join AE and GE. So, because angle ADE, equal to right angle ABG, is right, line AE is a diameter of the circle.[51] So, it is longer than chord EG. Because angle AGE, occurring in the semicircle, is right, and angle EZG is acute, then EZ is longer than EG. So, if we draw the section (قطعة) of circle HZT about center E with distance EZ, and we extend EG to H, sector (قطاع) TZE is less than triangle AZE, and sector ZHE is greater than triangle ZGE, and the ratio of triangle AZE to triangle ZGE, that is the ratio of AZ to ZG, or rather

[49]The word we have translated as "proposition," شكل, can mean either a figure or a proposition. Since, however, another word for "diagram," صورة, is used elsewhere in the text to specifically refer to the figure itself, we have assumed that the شكل mentioned here was accompanied with some argument and translated as such (see, for example, page 36).

[50]That is, point D may be chosen arbitrarily on line AB, and GD is joined.

[51]Where the Hyderabad version prints لكون خط اه, we follow *Tehran* 3484 and read يكون خط اه (al-Ṭūsī 1940a, 48; Aghayanī-Chavoshī 2005, 52).

the ratio of AD to DB, is greater than the ratio of sector TZE[52] to sector ZHE, that is the ratio of arc TZ to arc ZH, or rather the ratio of angle AED to angle ZEH, which is the ratio of angle DGA to angle DAG. If we compound,[53] the ratio of AB to BD is greater than the ratio of the sum of angles DGA, DAG to angle BAG. QED.

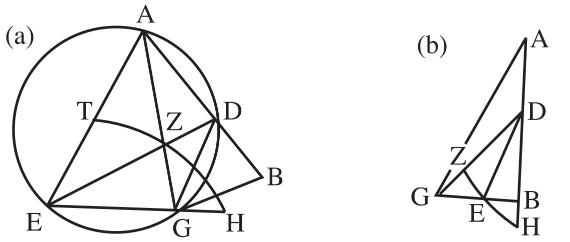

Figure 5: **Lemma to *Spher.* III 11**, (a) First Proof, (b) Second Proof [*T.* 3484, 53][54]

[Fig. 5 (b)] In another way: We repeat triangle ABG and line GD, and the presupposition of their situation, and we produce DE as a parallel to AG. We draw the section of a circle, ZEH, about center D and with distance DE. So, because angle DBE[55] is right and angle DEB is acute, DE is longer than DB. Likewise, because angle DEG is obtuse and angle DGE acute, DG is longer than DE. So therefore, the arc of the section intersects line DG at Z, and it passes beyond DB.[56] So we produce AB until it intersects it at H. Triangle DGE is greater

[52]Where the Hyderabad version prints زه, we follow *Tehran* 3484 and read ط ز ه (al-Ṭūsī 1940a, 48; Aghayanī-Chavoshī 2005, 53).

[53]Where the Hyderabad version prints ركبتا, we follow *Tehran* 3484 and read ركبنا (al-Ṭūsī 1940a, 48; Aghayanī-Chavoshī 2005, 53).

[54]In (a), *Tehran* 3438 omits line DG (Aghayanī-Chavoshī 2005, 53). Both *Tehran* 3438 and the Hyderabad version include a third diagram which seems to be related to this material. Since, however, it is not possible to make sense of this diagram in conjunction with the text as it stands, we have omitted it.

[55]Where the Hyderabad version simply prints ا ب ه, we follow *Tehran* 3484 and read زاوية د ب ه (al-Ṭūsī 1940a, 49; Aghayanī-Chavoshī 2005, 53).

[56]A number of words have been omitted in the Hyderabad version. Following *Tehran* 3484, we read this sentence as فلذلك يقطع قوس القطعة خط دج على ز وتمر خارجة من دب (Aghayanī-Chavoshī 2005, 53).

> than sector DZE and triangle DEB is less than sector DEH. The ratio of triangle DGE to triangle DEB, that is the ratio of GE to EB, or rather the ratio of AD to DB, is greater than the ratio of sector DZE[57] to sector DEH,[58] that is the ratio of angle GDE to angle EDH. Angle GDE, however,[59] is equal to its alternate (لمبادلتها),[60] which is angle DGA, and the external angle EDH is equal to the internal angle BAG, then the ratio AD to DB is greater than the ratio of angle AGD to angle BAG. By composition, the ratio of AB to BD is greater than the ratio of the sum of angles AGD, GAD, that is angle BDG, to angle BAG. QED.[61]

Both of the versions of T.L. 1 included by al-Ṭūsī are closely related to the treatment of similar material in the Greek tradition. As mentioned above, Thābit's proof is closely related to that given by Ptolemy in *Alm.* I 10. Whereas, in Figure 4 (b), Ptolemy uses the properties of the circle drawn through points A, B and G to demonstrate a version of T.L. 2, Thābit uses the properties of the same circle, reflected about AG, to prove T.L. 1. Although Thābit's proof concerns triangle ABG, which partially falls outside the circle used in the proof, $AD'G$, considerations of symmetry make it clear that the triangle can also be constructed inside the circle. In this way, Thābit shows that $arc\,GA : arc\,GD < BA : BD$, which is a version of T.L. 1 that is fully analogous to that shown by Ptolemy for T.L. 2.[62] This shows that

[57] Where the Hyderabad version prints دحه, we follow *Tehran* 3484 and read دزه (al-Ṭūsī 1940a, 49; Aghayanī-Chavoshī 2005, 53).

[58] The Hyderabad text contains a dittograph of the entire passage, اعني نسبة جه الى هب بل نسبة من اد الى دب اعظم من نسبة قطاع دزه الى قطاع دهح (al-Ṭūsī 1940a, 49). We have omitted this, following *Tehran* 3484.

[59] Where the Hyderabad version prints the beginning of this sentence as لتكن, we follow *Tehran* 3484 and read ولكن (al-Ṭūsī 1940a, 49; Aghayanī-Chavoshī 2005, 53).

[60] The term مبادلة is based on the third form of بدل, the same root used to express the idea of the alteration of a proportion ($a : b = c : d \implies a : c =: b : d$, *permutando*, ἐναλλάξ). Here, however, it is used to refer to the equality of the alternate internal angles, formed when a transversal falls on a pair of parallel lines.

[61] al-Ṭūsī (1940a, 48–49).

[62] Furthermore, this configuration of T.L. 1 also makes it clear that it is a stronger version of the lemma demonstrated by Apollonius, somehow in connection with a discussion of the retrogradations of the planets, and preserved by Ptolemy in *Alm.* XII 1 (Heiberg 1898–1903, p. 2, 456–458). In Figure 4 (b), where $x \geqq AG$, Apollonius's lemma shows that $\beta : \alpha < x : (AD - x)$. Since Apollonius was presumably aware of the use of these lemmas by

Thābit was well aware of the historical significance of this group of lemmas in the Greek mathematical sources, and wanted to write a version of T.L. 1 that would reveal more explicitly its geometric analogy with the other proofs in the Greek mathematical corpus. Ṭūsī, in turn, saw the importance of all this material to anyone reading the classics of the mathematical sciences and made the decision to insert it into the text itself.

The second version of T.L. 1 that Ṭūsī includes in the text is fairly close to the proof preserved in the oldest Greek MSS of the lemmas to the *Spherics*.[63] The version of the proof provided by Ṭūsī is somewhat longer than that in the Greek *Spherics* but it proceeds by the same constructions and the same chain of argument. Indeed, the proof in the scholia, having been written in the margin of the text, is in many ways only a proof sketch. Since al-Ṭūsī's version was intended to be read as part of the text, it is only natural that the details of both the constructions and the proof should be fleshed out.

Rewriting the text: *Spher.* II 15 & 16

Having discussed al-Ṭūsī's explicit additions to the text, we should make it clear that his general tendency was not to expand, but rather to condense. For the most part, Ṭūsī's edition stays close to the older versions in terms of the overall content, the order of the propositions and the internal structure of the arguments. As noted in studies of some of Ṭūsī's other editions, he rewrites the text itself in less verbose Arabic prose by reorganizing the syntax, eliminating repetition, streamlining the proof structure and generally ignoring features of the older versions that were an attempt to reproduce the style of the Greek source.[64] In his edition of Theodosius's *Spherics*, it appears that the only times when he rewrote entire passages or propositions was when he believed there was some sort of mathematical or logical problem with his source material. Even in these cases, however, many passages were simply rewritten in his usual concise style with no substantial alteration. In order to develop some examples of Ṭūsī's editorial practices, we will examine a number of passages from *Spher.* II 15 & 16, which are useful examples because these two propositions show considerable differences between Ṭūsī's

Aristarchus and Archimedes in their work employing pre-chord table trigonometric methods, he must have put forward his weaker version of the lemmas in a rather different context.

[63] Heiberg (1927, 195–196); Czinczenheim (2000, 435). Since the version of this lemma translated by Knorr (1985, 264–265) is not attested by the oldest Greek MSS, we have translated the older version to facilitate comparison with Ṭūsī's version. See the Appendix.

[64] Rashed (1996, 12–27); Berggren and Sidoli (2007, 238–247).

version and all other Greek and Arabic versions.

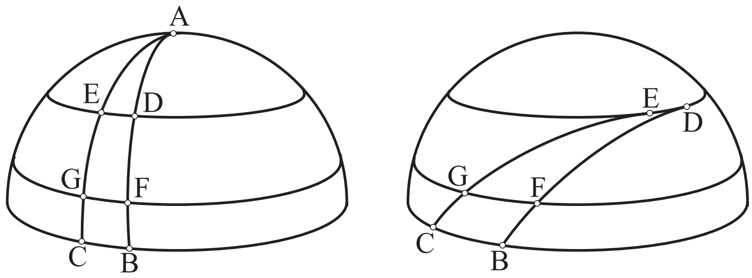

Figure 6: Diagram for *Spher.* II 10–16

The group of propositions *Spher.* II 10–16 constitutes a theory of the conditions under which great circles cut off similar arcs of lesser circles. Taken together, the series shows that these arcs are similar if and only if the great circles either (a) both pass through the poles of the parallel circles, or (b) are both tangent to the same parallel circle. In fact, only three theorems are directly used to prove these statements. In Figure 6, *Spher.* II 10 shows that if great circles DB and EC are through pole A, they will cut parallel circles ED and GF such that arc ED is similar to arc GF, *Spher.* II $14_a(13_g)$ shows that if they are both tangent to parallel circle ED they will again make arc ED similar to arc GF, and *Spher.* II 16 uses indirect arguments to show that if the arcs of the parallel circles are similar there are no other possible configurations. The other four propositions in this series are all auxiliaries. *Spher.* II 11 & 12 are lemmas required in the proofs of *Spher.* II $14_a(13_g)$ and *Spher.* II 16, while *Spher.* II $13_a(14_g)$ & 15 are problems required in the construction of *Spher.* II 16. With this summary providing the mathematical context, we examine a number of passages from *Spher.* II 15 & 16.

Rewriting sentences

Al-Ṭūsī's most persistent tendency as an editor is to rewrite individual sentences so that they are more concise and presumably more consistent with his idea of contemporary Arabic usage. As a representative example, we examine the same sentence, taken from *Spher.* II 15, in the Greek, the early Arabic version **KL** and Ṭūsī's edition.

Καὶ τῇ ὑφ' ἣν ὑποτείνει ἡ τοῦ τετραγώνου πλευρὰ τοῦ εἰς τὸν μέγιστον κύκλον ἐγγραφομένου περιφερείᾳ ἴση ἀπειλήφθω ἡ ΒΘ.[65]

Let BT be cut off equal to an arc subtending the side of the square inscribed in the great circle.

ونفصل منها قوساً مساوية للقوس التي يوترها ضلع المربع المرسوم في الدائرة العظيمة، ونفرضها قوس $\overline{\text{ب ط}}$.[66]

We cut off from it an arc equal to the arc subtending the side of the square drawn in the great circle, and we assume it as arc BT.

These two passages agree both in terms of expression and syntax, including the terminal position of the topic of discussion, the letter name BT. In the Greek, BT is the grammatical subject of a single sentence and it immediately follows the verb. Since this delayed position of the verb would not work in Arabic, the author has preserved the position of BT by embedding it in a second, subsidiary sentence. Ṭūsī, however, having no interest in attempting to preserve such specific features of the prose, which were presumably motivated by familiarity with Greek syntax, rewrote as follows.

ونفصل منها $\overline{\text{ب ط}}$ بقدر ما يوتره ضلع المربع الواقع الدائرة العظيمة.[67]

We cut BT off from it at the size of what subtends the side of the square occurring in the great circle.

Al-Ṭūsī's expression, although idiomatically briefer, expresses the same mathematical content as the earlier version. In this way, by consistently reworking the individual passages of the text, Ṭūsī strove for more concision and greater mathematical clarity. Although this kind of local rewriting pervades the whole text, Ṭūsī's tendency toward concision is most pronounced when he rewrites an entire proposition to clarify the mathematical argument.

Rewriting arguments

For reasons that we will examine below, al-Ṭūsī apparently found the entire exposition of *Spher.* II 15 & 16 unsatisfactory and decided to rewrite them. *Spher.* II 15 is a problem showing how to draw a great circle tangent to a

[65]Czinczenheim (2000, 102).
[66]Kraus MS, 46v; *L. or.* 1031, 43v.
[67]al-Ṭūsī (1940a, 22).

given lesser circle and passing through a given point. In the older Arabic versions of *Spher.* II 15, the problem is demonstrated in two cases, although the second case is an obvious interpolation.[68]

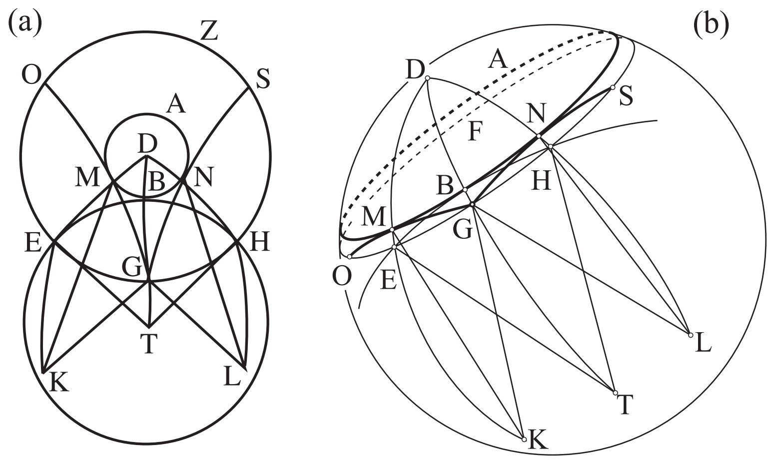

Figure 7: ***Spher.* II 15**: MS Diagram [*T.* 3484, 34], (b) Perspective Reconstruction

In Figure 7, where circle AB is a given lesser circle and G a given point between it and its equal and parallel correlate, Theodosius solves the problem by showing how to draw two great circles, GNS and GMO, passing through G and tangent to AB at points N and M.[69] The first case demonstrates the solution of the problem where arc BG is less than a quadrant.[70] The second case, found in the Greek and in **KL**, demonstrates the solution where arc BG is equal to a quadrant, while the third case, mentioned by Ṭūsī,[71] treats the situation where arc BG is greater than a quadrant. The first case includes a

[68]The second case begins with the phrase "if someone says," فان قال قائل, which is not found elsewhere in the treatise and is clearly the beginning of a scholium [Kraus MS, 47r; *L. or.* 1031, 45r]. The Greek text has the same structure as **KL** and the second case begins with the same phrase, εἰ δέ τις λέγοι (Czinczenheim 2000, 105). (Note that Heiberg's edition is substantially different for this problem (Heiberg 1927, 70–76).)

[69]The construction itself is rather involved and beyond the scope of the present paper. See Sidoli and Saito (Forthcoming) for a discussion of the full solution of this problem.

[70]This is probably the only case that was in the Greek treatise that Theodosius composed.

[71]In fact, this case is also handled in the Greek MS *Paris BN* 2448, but this is a unique 13th century manuscript and its reading need not concern us here (Heiberg 1927, 70–76). See Czinczenheim (2000, 239–258) for a full discussion of this interesting manuscript.

substantial proof following which the second or third case then simply appeal to this proof. Since there are two great circles that solve the problem, a full proof is given for one of them, great circle GNS, and a reduced proof or proof sketch, appealing to the full proof, is given for the other.[72] The proof involves using the auxiliary theorems *Spher.* II 11 & 12 to show that lines LG and TE are equal.

Hence, the overall structure of the Greek and **KL** version is as follows: (1) a proof that one great circle solves the problem using internal, auxiliary lines; (2) a reduced proof that the other great circle solves the problem referring to other internal, auxiliary lines; (3) an interpolated proof that the problem can still be solved when the given point is a quadrant from the given lesser circle.

Al-Ṭūsī reorganized and rewrote the entire proposition, perhaps for two reasons. He probably saw that the second case was an interpolation and he must have easily seen that there was a third case, which could also be solved by the same means as the other two. Nevertheless, Ṭūsī maintained the same proof for the first part, shortening and clarifying it using the same sorts of practices as we saw in the previous section. By these editorial procedures and by using proof sketches for all but the first part, Ṭūsī produced three cases in the same space as the first case in the older version.

The overall structure of Ṭūsī's version is as follows: (1) a proof that one great circle solves the problem using internal, auxiliary lines; (2) a proof sketch that the other great circle solves the problem referring to other internal, auxiliary lines; (3) a proof sketch that the problem can still be solved when the given point is a quadrant from the given lesser circle, (4) a proof sketch that the problem can still be solved when the given point is more than a quadrant from the given lesser circle.

As an example of the way al-Ṭūsī makes the argument more concise by eliminating unnecessary repetition, we may take the example of part (2), again in all three versions.

> ὁμοίως δὴ δείξομεν, ὅτι καὶ ὁ πόλῳ τῷ Κ, διαστήματι δὲ τῷ ΚΓ, κύκλος γραφόμενος ἥξει καὶ διὰ τοῦ Μ σημείου. ἐὰν γὰρ ἐπιζεύξωμεν τὰς ΓΚ, ΘΗ, ἴσαι ἀλλήλαις ἔσονται, καί ἐστιν ἡ ΘΗ τετραγώνου πλευρά· ἐκ γὰρ τοῦ πόλου ἐστὶ μεγίστου κύκλου τοῦ ΕΒΗ· καὶ ἡ ΓΚ ἄρα τετραγώνου ἔσται πλευρά.

[72] The Kraus MS, alone of the versions we consider here, states the problem from the beginning as involving two great circles (نريد ان نرسم دائرتين عظيمتين) [Kraus MS, 46v].

ἀλλὰ καὶ ἡ ΚΜ· ἴση ἄρα ἐστὶν ἡ ΚΜ τῇ ΚΓ· ὁ ἄρα πόλῳ τῷ Κ, διαστήματι δὲ τῷ ΚΓ, κύκλος γραφόμενος ἥξει καὶ διὰ τοῦ Μ σημείου, καὶ ἔσται ὡς ὁ ΓΜΟ, καὶ ἐφάψεται τοῦ ΑΒ κύκλου [καὶ γίγνεται διχῶς τὸ πρόβλημα]. διὰ ἄρα τοῦ δοθέντος σημείου τοῦ Γ, ὅ ἐστι μεταξὺ τοῦ ΑΒ καὶ τοῦ ἴσου τε καὶ παραλλήλου αὐτῷ μέγιστος κύκλος γέγραπται [ὁ ΓΝΞ καὶ ὁ ΓΜΟ].[73]

Similarly, we show that the circle drawn with pole K and with distance KG will pass through point M. For if GK and TH are joined, they will be equal, and TH is equal to the side of the square, but KM is as well, therefore, KM is equal to KG. Therefore, the circle drawn with pole K and with distance KG will also pass through point M. Let it be as GMO, and it will touch circle AB [and the problem will be produced in two ways]. Therefore, through point G, which is between AB and the equal and parallel to it, a great circle has been drawn, [GNS and GMO].

وبمثل ذلك تبين ان الدائرة المخطوطة على قطب كَ وببعد كَجَ تجوز على علامة مَ،[74] لانا ان وصلنا كَجَ طَحَ، يكون خط كَجَ مساوٍ لخط طَحَ، وخط طَحَ[75] ضلع المربع المرسوم،[76] لان مخرجه من قطب دائرة بَحَ العظيمة. فاذاً[77] خط كَجَ ضلع المربع المرسوم،[76] ولكن قد كان تبين ان خط كَمَ ضلع المربع، فاذاً كَمَ تساوي كَجَ، فاذاً[78] الدائرة المرسومة على قطب كَ وببعد كَجَ تجوز على نقطة[79] مَ. فلنرسمها،[80] ولنفرضها دائرة جَمَعَ. فظاهر لنا[81] ان دائرتي[82] اَبَ جَمَعَ متماستان.[83] فاذاً قد رسمنا على علامة جَ دائرتين تماسان دائرة اَبَ.[84]

[73] Czinczenheim (2000, 104–105). Note that Heiberg's text contains a long addition from *Paris BN* 2448, which is not found in the other MSS, and was rightly omitted by Czinczeheim (see note 71). We have changed the bracketing in Czinczeheim's text to facilitate comparison with **KL**. That is, the text we have bracketed does not occur in the **KL** version. The final sentence has some problem with agreement of number, since the statement is about one great circle and then two are named. Since the letter names are also not found in the Arabic versions, they should probably be removed as a gloss that later entered the text.

> In the same way, it is shown that the circle drawn about pole K with distance KG passes through mark M, because if we join KG and TH, line KG will be equal to line TH, and line TH is the side of the drawn square,[85] because its origin[86] is from the pole of great circle BH. Hence, line KG is the side of the drawn square. It was shown, however, that line KM is the side of the square. Hence, KM equals KG. Hence, the circle drawn about pole K with distance KG passes through point M. So let us draw it and let us assume it as circle GMO. So it is obvious to us that circles AB and GMO touch one another. Hence, we have drawn through mark G two circles touching circle AB.

The agreement between these two passages is fairly close. The Arabic passage contains a justification of the claim that KM equals KG, namely "because its [TH] origin is from the pole of great circle BH. Hence, line KG is the side of the drawn square. But it was shown that line KM is the side of the drawn square." This was presumably added by Thābit ibn Qurra in correcting the text, or some other scholar in the Arabic tradition.[87] The

[74]Kraus MS: ن. **KL** uses both علامة and نقطة to refer to points. We translate the former as *mark* and the latter as *point*.

[75]Here, the Kraus MS includes the word مساوٍ, which has then been crossed out with a mark like ـ placed above the word, so used throughout the manuscript.

[76]*L. or.* 1031 omits المرسوم.

[77]*L. or.* 1031: فان.

[78]*L. or.* 1031: فاذن. This spelling for اذاً occurs throughout.

[79]Kraus MS omits نقطة

[80]*L. or.* 1031 omits فلترسمها.

[81]Kraus MS omits لنا.

[82]Kraus MS omits دائرتي.

[83]*L. or.* 1031: متماستين.

[84]Kraus MS, 47r; *L. or.* 1031, 45r. For this passage the two MSS use a different figure with somewhat different labeling. Hence, the letter names naturally disagree. We have followed the diagram and letter names of the Kraus MS and neglected to mention all the variants in the letter names.

[85]The expression "the drawn square," المربع المرسوم, is used as an abbreviation for the side of the square circumscribed by a great circle, which *Spher.* I 16 shows is the pole-distance of a great circle.

[86]Literally, "its place of going out," مخرجه, that is, the place from which it is drawn.

[87]For examples of additions of this sort in the treatise said to be Thābit's correction of

Greek passage, on the other hand, contains the rather more trivial remark that this section constitutes a second solution to the problem, and it ends by explicitly naming the two circles that solve the problem. These are probably late additions.[88]

In contrast to the way in which **KL** fleshes out the argument in the Greek, al-Ṭūsī reduces the entire passage to an essential sketch.

> وبمثل ذلك تبين بعد ان نصل كج طح كم وتبين انها متساوية ومساوية لضلع المربع، ان الدائرة التي ترسم على قطب ك وببعد كج، وهي دائرة جمع، تمر بنقطة ج وتماس دائرة اب.[89]
>
> In the same way, it is then shown that if we join KG, TH, KM and it is shown that they are mutually equal and equal to the side of the square, then the circle drawn about pole K with distance KG, circle GMO, passes through point G and touches circle AB.

This is a typical example of the way Ṭūsī summarizes, and thus simplifies, the text. He briefly points out that if the necessary lines are drawn and shown to be equal to side of the square, then the analogously drawn great circle will also solve the problem. All details of this trivial argument are left up to the reader.

In this case, however, although the argument is merely sketched, al-Ṭūsī may be presumed to have had the same argument in mind. Hence, in *Spher.* II 15, Ṭūsī condenses everything and then briefly adds a third case, as follows.

> As for the case where BT is less than BG, in place of circle AB, we introduce the equal and parallel correlate, so it reverts to the original proof. QED.[90]

This situation should be compared with *Spher.* II 16, in which al-Ṭūsī rewrites the theorem by reorganizing its logical structure, again with the goal of brevity. As mentioned above, *Spher.* II 16 is the culminating theorem of a group of seven propositions and it shows that if a pair of great circles cuts similar arcs from a pair of parallel lesser circles, then the great circles must

Aristarchus's *On the Sizes and Distances of the Sun and the Moon*, also found in the Kraus MS, see Berggren and Sidoli (2007, 235–238 & 241–247).

[88] Czinczenheim (2000, 105) marks this passage as an interpolation.

[89] al-Ṭūsī (1940a, 23).

[90] al-Ṭūsī (1940a, 23).

either both be through the poles of the parallel circles or both be parallel to the same lesser circle, which is parallel to the original two. In all three versions considered here the theorem is demonstrated in three cases, but Ṭūsī structures the logical arrangement of the cases somewhat differently.

In the Greek and **KL** the proof is actually arranged in two cases, of which the second case has two parts. In Figure 8 (a), the first case assumes that one of the great circles goes through the poles and shows, by indirect argument, that the other great circle must then also go through the poles. The second case assumes that one of the great circles does not go through the poles and hence must either be tangent to one of the parallel circles, (b), or be inclined to it, (c).[91] In both parts of the second case it is shown, again by indirect argument, that the other great circle must be tangent to the same parallel circle as the first.

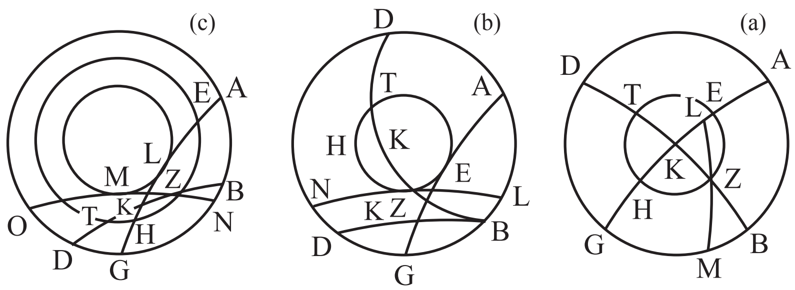

Figure 8: Al-Ṭūsī's diagram for diagram for ***Spher.*** **II 16** [*T.* 3483, 35]

Al-Ṭūsī, on the other hand, asserts that there are in total five possible arrangements of the two great circles and then proceeds to show, in three distinct cases, that in three of these the great circles will not cut the parallel circles in similar arcs. He says,

> We say the two greats are either [1a] simultaneously passing through the two poles of the parallels or [1b] only one of them is through them or [2] not one of them is through them but rather [2a] either they simultaneously touch one of the parallels or [2b] only one of the two of them touches the two of them,[92] [2c] or

[91]The condition that the great circle either touch or intersect the lesser circle is secured by the fact that the great circle is assumed to produce similar arcs on the two lesser circles.

[92]The first "two of them" refers to the two great circles and the second to the parallel circle

> one of the two of them does not touch the two of them.[92] So, these are five classes (اقسام); they have no sixth, and two of them are possible and the remaining three are impossible.[93]

It is not certain why al-Ṭūsī saw the need to reformulate the overall argument of this proof, but we will put forward two reasons that may have weighed in his decision.

In the first place, the actual presentation of the first case in the earlier versions is obscured by the fact that an essential point, the intersection of the great circles, is never introduced. In other words, in Figure 8, these versions set out to prove that some completely unknown point, K, must be the pole of the parallel circles, by showing that the pole is not any other point taken on AEG, say L. Following the Greek,[94] the passage in **KL** reads as follows.

> We prove that circle BTD is likewise through the poles of the parallel circles, namely by this, that mark K is the pole of circles[95] $ABGD$ and $ZHTE$.[96]

This is the first mention of point K in the text and K it is not even included in the letter names of the great circles. Nor is there any further mention of a point K in the Arabic text.[97] Hence, point K as the intersection of the two great circles, is an extreme case of what Netz (1999, 19–21) calls "completely unspecified." The only way to know that K must be the intersection is to look at the diagram and see that it is found there representing the intersection and to realize that the argument, as it is given in the text, would make no mathematical sense if K were not the intersection.

Al-Ṭūsī, for obvious reasons, found this situation unsatisfactory and corrected the text by simply specifying the configuration of K as follows.

> So, let us assume, in the first diagram of the proposition (في الصورة الاولى من الشكل) [Figure 8 (a)], that only the great AEG is passing

now, suddenly, considered as a pair with its correlate equal parallel.

[93] al-Ṭūsī (1940a, 24).

[94] Czinczenheim (2000, 106).

[95] *L. or.* 1031: دائرتي; Kraus MS: دائرة.

[96] Kraus MS, 47v; *L. or.* 1031, 45v.

[97] Point K is mentioned again in the Greek text, but only to say, "Similarly, we might show that it is not any other point, other than K. Therefore, point K is the pole of the parallels" (Czinczenheim 2000, 106). This passage, however, does not serve to specify the mathematical

through their two poles, and let the two greats intersect at K.[98]

Although this clarification is helpful, it only applies to the first case and may not explain why Ṭūsī rewrote the entire proposition. The complete rewrite of *Spher.* II 16 was probably, again, based on al-Ṭūsī's desire to make the individual arguments more concise and thereby bring more clarity to the larger picture.

In the Greek proof, *Spher.* II 16 shows that if one of the great circles passes through the poles, then so does the other, and if it is tangent to a particular lesser circle, then so is the other. Al-Ṭūsī's proof, however, simply eliminates three impossible cases. In this way, his version of the theorem is more clearly the final argument in the overall the claim that two great circles cut lesser, parallel circles in similar arcs if and only if the great circles either both pass through the poles of the parallel circles, or are both tangent to the same parallel circle. *Spher.* II 10 & 13 have already shown that if the great circles pass through the poles, or are tangent to the same parallel, then they cut the lesser circles in similar arcs. In conclusion, Ṭūsī's *Spher.* II 16 enumerates all possible configurations and shows that in all but these the lesser circles are not cut in similar arcs.

As an example of the way al-Ṭūsī restructures the argument we will look at the second part in all three versions.[99]

ἀλλὰ δὴ πάλιν μὴ ἔστω ὁ ΑΗ[100] διὰ τῶν πόλων τῶν παραλ-

relationship of K to the other points in the figure.

[98] al-Ṭūsī (1940a, 24).

[99] Because of a trivial error in the labeling of the diagram or in the transcription of the letter names, the Greek text established by both Czinczenheim (2000, 107) and Heiberg (1927, 78) for this part of the theorem is mathematically inadequate. Ver Eecke (1959, 58) attempted, unsuccessfully, to save the argument by redrawing the diagram. His redrawn diagram, however, would require that the geometer be able to draw a great circle tangent to a given lesser circle through a given point not on the lesser circle and tangent at a given point on the given lesser circle. This construction is, in general, not possible. Hence, in order to restore mathematical sense, we have a made some minor changes to the letter names and added one label to the figure in *Vat. gr.* 204. Although our correction restores sense to the passage with minimal intervention, comparison with the two other parts of the theorem shows that the diagram probably underwent some corruption and the letter names in the text were altered in response to this. Hence, the correction introduced in the **KL** version does a better job of bringing this part into formal agreement with the other two (see note 117).

[100] Both editions, following all the MSS, print ΑΗΓ, but eliminating Γ, here, restores mathematical sense with minimal change to either the text or the diagram.

λήλων. ἤτοι δὴ ἐφάψεται τοῦ ΕΖΘ[101] κύκλου ἢ λοξὸς ἔσται πρὸς αὐτόν.

ἐφαπτέσθω πρότερον κατὰ τὸ Ε ὡς ἔχει ἐπὶ τῆς δευτέρας καταγραφῆς. λέγω, ὅτι καὶ ὁ ΖΒ ἐφάψεται. εἰ γὰρ δυνατόν, μὴ ἐφαπτέσθω, καὶ γεγράφθω διὰ τοῦ Ζ σημείου τοῦ ΕΖΘ ἐφαπτόμενος μέγιστος κύκλος ὁ ΖΓ, ἀσύμπτωτον ποιῶν τὸ ἀπὸ τοῦ ΖΓ ἡμικύκλιον τῷ ἀπὸ τοῦ ΕΑ ἡμικυκλίῳ· ὁμοία ἄρα ἔσται ἡ ΓΑ περιφέρεια τῇ ΕΖ περιφερείᾳ. ἀλλ' ἡ ΕΖ τῇ ΑΒ ἐστιν ὁμοία· καὶ ἡ ΓΑ τῇ ΑΒ ἄρα ἐστιν ὁμοία. καί εἰσι τοῦ αὐτοῦ κύκλου· ἴση ἄρα ἐστὶν ἡ ΓΑ περιφέρεια τῇ ΑΒ περιφερείᾳ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα καὶ ὁ ΒΖ κύκλος τοῦ ΕΖΘ κύκλου οὐκ ἐφάψεται· ἐφάψεται ἄρα.[102]

[Figure 9 (a)] Now again, however, let AH not go through the poles of the parallels. Clearly, it will either touch EZT or be inclined on it.

First let it touch it at E as holds in the second figure. I say that ZB will also be touching. For, if possible, let it not be touching, and let there be drawn through point Z the great circle, ZG, touching EZT, such that semicircle from ZG is not touching the semicircle from EA. Therefore, arc GA is similar to arc EZ [*Spher.* II 13]. EZ, however, is similar to AB, therefore, GA is similar to AB, and they are of the same circle. Therefore, arc GA is equal to arc AB. Which is impossible. Therefore, circle BZ will not be not touching circle EZT. Therefore, it will be touching.

ولنفرض ايضاً دائرة ا ه ج ليس على قطب الدوائر المتوازية،[104] فلا[105] بد من ان يكون اما[106] مماسة لدائرة ه ز ح ط واما مائلة عليها.

فلنفرضها[107] اولاً مماسة لها. ولتكن المماسة على نقطة ه[108] كما هو مرسوم في الصورة الثانية.[109] ونبين ان دائرة ب ز د ايضاً تماس دائرة ه ز ح ط. فنقول انه لا يمكن غير ذلك. فان امكن فلتكن غير مماسة لها، ولنرسم على نقطة

[101] Both editions print EZHΘ, but the majority MS reading is EZΘ, which makes better mathematical sense.

[102] Czinczenheim (2000, 107), Heiberg (1927, 78).

[103] No point H is found in the MSS diagrams. In fact, point H may be located anywhere

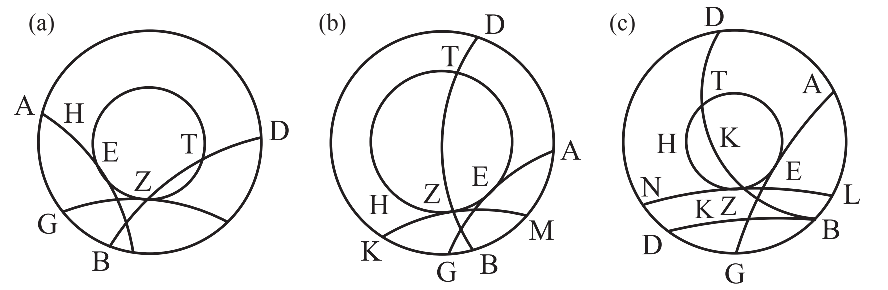

Figure 9: Diagram for comparison of *Spher.* II 16, part 2 [(a) *Vat. gr.* 204, 18r;[103](b) Kraus MS, 47v; (c) *T.* 3483, 35]

ز[110] دائرة عظيمة تماس دائرة هزحط وهي كزم.[111] وليكن نصف الدائرة التي مما يلي زم غير ملاقٍ نصف الدائرة التي مما يلي ها، فتكون قوس ام مشابهة لقوس هز، ولكن[112] قوس هز مشابهة لقوس اب. فاذاً قوس ام مشابهة لقوس[113] اب، وهما من دائرة واحدة. فاذاً قوس ام مساوية لقوس اب، العظمى للصغرى. هذا خلف، لا يمكن. فاذاً ليست[114] دائرة بز غير مماسة لدائرة هزحط. فاذاً هي مماسة لها.[115]

[Figure 9 (b)] Let us again assume that circle AEG is not on the pole of the parallel circles, so it is necessarily either touching circle $EZHT$ or it is inclining on it.

So, let us first assume it is touching it, and let the touching

along the great circle joined through AE.

[104]The essential word ليس is missing from the Kraus MS. The text of *L. or.* 1031 is garbled here but a marginal note gives the full sentence as we print it. The marginal note is followed by صح, denoting it as a correction (Gacek 2001, 82).

[105]Kraus MS: فالا.

[106]Kraus MS: اها.

[107]*L. or.* 1031: فليكن.

[108]*L. or.* 1031: علامة ة.

[109]Kraus MS: كما في الصورة الثانية مرسوم.

[110]*L. or.* 1031: علامة ز.

[111]*L. or.* 1031 دائرة كزم.

[112]Kraus MS: لكن.

[113]Kraus MS omits لقوس.

[114]*L. or.* 1031: ليس.

[115]Kraus MS, 47v; *L. or.* 1031 46r.

> [point] be at point E as drawn in the second diagram. We prove that circle BZD will likewise touch circle $EZHT$. For we say that it is not possible for it to be otherwise. For, if possible let it be not touching it, and let us draw a great circle, KZM, touching circle $EZHT$ at point Z. Let the semicircle that follows from ZM be not meeting the semicircle that follows from EA,[116] so arc AM is similar to arc EZ [*Spher.* II 13]. Arc EZ, however, is similar to arc AB. Hence, arc AM is similar to arc AB, and they are of the same circle, hence, arc AM is equal to arc AB, the greater to the lesser. That is a discrepancy; it is not possible. Hence, circle BZ is not other than touching circle $EZHT$, hence it is touching it.

Although the argument in the **KL** version is essentially the same as that in the Greek, the letter names have been rearranged. This can be contrasted with the previous two example passages, in which the letter names in the Arabic were straightforward transliterations of the Greek. In this passage, Thābit, or someone else in the translation effort, must have recognized the confusion in the Greek version noted above and corrected for it (see n. 99). This correction also led to the diagram being relabeled and the proposition being slightly reworked so that the letter names of the geometric objects are introduced in Arabic abjad order.[117] Despite the need for these changes, the overall text is quite close. This becomes especially clear when we read these two in contrast with the version produced by al-Ṭūsī. Because Ṭūsī has three distinct cases, he has no need to introduce this part as the first part in the second case. He simply states the assumption and proves it false.

> ثم لنفرض في الصورة الثانية ان عظيمة ا ه ج فقط مماسة لمتوازية ه ز ح ط على نقطة ه. ونرسم دائرة ل ز ن العظيمة مماسة لدائرة ه ز ح ط على نقطة ز. فتكون ه ز الشبيهة با ب شبيهة با ل، ويلزم منه تشابه قوسي ا ب ا ل، هذا خلف.[118]
>
> [Figure 9 (c)] Then let us assume, in the second diagram, that

[116]The expression "the semicircle that follows from AB," نصف الدائرة التي مما يلي ا ب, apparently refers to the semicircle that contains arc AB and continues on past its termination.

[117] In the Greek text of Theodosius's *Spherics*, the introduction of the letter names of geometric objects in all but four of the propositions follows Greek alphabetic order. The four non-alphabetic propositions are *Spher.* II 9, 16, 23 & III 4. In **KL**, all but three propositions follow the Arabic abjad order of points. Hence, whoever rewrote *Spher.* II 16, also relabeled the diagram and introduced the objects in abjad order.

[118]al-Ṭūsī (1940a, 24).

> only the great AEG is touching parallel $EZHT$ at point E. We draw great circle LZN touching circle $EZHT$ at point Z. So, EZ, the similar to AB, is similar to AL. From this follows the similarity of the two arcs AB and AL. That is a discrepancy.

Once again, al-Ṭūsī has stripped the argument down to its bare essentials. He relies on the transitivity of similarity, which he asserted in his additional first principles (see page 16), to state the similarity of two different arcs in the same circle in a single sentence. He does not bother to state that these arcs, being in the same circle, must then be equal, because their similarity is also impossible and this is all he needs. Because of the structure of his argument, he is simply proving a conceivable arrangement false, not proving its contrary true.

Al-Ṭūsī's diagram also deserves some comment. As in the Greek and **KL** versions of the argument, Ṭūsī relies fairly heavily on the diagram itself to orient the reader so as to avoid giving a detailed construction. For example, the construction of BZD and the fact that Z is the intersection of circles BZD and $EZHT$ must be taken from the figure. Indeed, the figure contains two cases for the position of BZD, either intersecting $EZHT$ or not, both depicted in the diagram by simply having drawn two BZDs. This is a fairly rare instance of multiple cases being depicted in a figure by means of multiple positions of the same object, carrying the same label.

In this section, we have seen examples of the most frequent means employed by al-Ṭūsī to rewrite the text. In general, his goal appears to have been greater mathematical clarity while still maintaining the overall structure of the original argument. In general, he achieved his aim merely by rewriting individual passages more succinctly so as to make the overall argument more transparent. Thus his goal was to clarify what he believed to be the original argument. Even, in those cases where Ṭūsī rewrote the whole proposition it is possible to find in the older versions some mathematical problem or inconsistency that may have lead Ṭūsī to believe that his sources did not contain the argument as it was originally intended.

Conclusion

These discussions show that al-Ṭūsī's project was to revitalize the text of Theodosius's *Spherics* and make it into something more accessible to his contemporaries. He did this by considering the text firstly as a product of the mathematical sciences, adhering to the internal constraints of mathematical and logical necessity, and secondarily as a historically contingent work, having been transmitted through the centuries by individuals who responded to, and altered, the text itself in various ways.

Ṭūsī's first interest appears to have been to produce a text that was mathematically coherent and independent. By independent, we mean that he saw fit to write a text that could be read as a self-contained argument by a student who had mastered Euclid's *Elements* but need not have had any further mathematical training. To this end, he included a number of additional hypotheses, to give support for steps used in the propositions, and auxiliary lemmas to demonstrate theorems used in the *Spherics* but not shown in the *Elements*. By mathematically coherent, we refer to Ṭūsī's interest in clarifying the underlying mathematical argument by rewriting the work in clearer, more concise prose and reorganizing the proof structure of a few theorems where he presumably thought this was necessary. In this process, Ṭūsī treated the actual words and phrases of the text not as sacrosanct objects that should be preserved in their original form, but as bearers of some more fundamental underlying object; that is, the mathematical argumentation and theory it conveys. Nevertheless, despite the fact that he did not regard the wording of the text as worthy of historical preservation, his very interest in the work must have been motivated by historical appreciation.

Indeed, it was because of its position in the canonical works, that the *Spherics* warranted al-Ṭūsī's critical attention. Although strict textual preservation was not one of his aims, it appears that transmitting other aspects of the work's historicity was an important feature of his scholarship. To this end, he included a brief description of the circumstances of the work's transmission into Arabic as well as a number of historically significant lemmas that were included in his sources, one of which he credited to Thābit ibn Qurra. Moreover, he took care to distinguish his own interventions from the rest of the text, so that the reader would be clear about what parts of the completed treatise were due to al-Ṭūsī himself. Although at first glance, the text has the appearance of an original source which has been commented upon, in fact, Ṭūsī has modified the traditional text, adopting it for his own times and ends and includes some of his own remarks and those of his predecessors. Through these means, the text itself becomes an instantiation of the continuity of an ancient tradition, revealing both the persistence of the tradition and Ṭūsī's participation in it.

Al-Ṭūsī's project in editing the *Spherics* is thus a kind of cultural appropriation. Although in Ṭūsī's case this appropriation does not involve translation from one language, he is modifying a source that was produced in a very different time, under different social and political conditions, to meet the needs of his contemporaries. The finished work has many of the distinctive features of an act of cultural appropriation; it establishes the chain of predecessors and successors and implies that this chain was directed naturally and properly to Ṭūsī himself, the rightful heir of the tradition. In this regard, Ṭūsī is laying claim to his cultural heritage, not doing historical scholarship. It is presumably because he was not engaged in historical scholarship that he was willing to introduce changes to the presentation of the text.

The fact that al-Ṭūsī was willing to rewrite the text itself – in contrast to, say, a commentary on a work in the religious or poetic tradition – gives us some indications of how he understood the relationship between predecessors and successors in the mathematical tradition. The object that the predecessor imparts is not an original revelation or some set of carefully crafted words, but an arrangement of diagrams and arguments that formulate a theory. Hence, the detailed preservation of the words and drawings is less important than the presentation of the arguments and mathematically coherent diagrams. Moreover, in the mathematical tradition, the predecessor does not simply impart knowledge which the successor then receives. Ṭūsī's practices in editing the text make it clear that he considered himself an active participant in the tradition, fully capable of understanding the work that his predecessors had done, making advances on this and correcting and improving the received texts on this basis.

Acknowledgments

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Appendix: Translation of the scholium to *Spher.* III 11

Let ABG be a right triangle, and let some AD be drawn through it. To prove that BG to BD has a greater ratio than angle ADB to angle AGB. For let DE be drawn through D, parallel to AG. Now, since DE is greater than BD – because it subtends the greater angle, being right – while the [angle] E is acute, therefore angle AED is obtuse, therefore AD is greater than ED. Therefore, the circle drawn with center D and distance DE cuts AD and extends beyond BD. Let it be drawn as ETZ. Therefore the triangle AED has to the sector EDZ a greater ratio than the triangle EBD to the

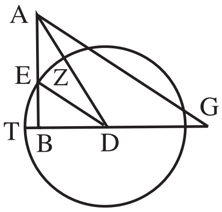

Figure 10: Diagram for the scholium to *Spher.* III 11

sector EHD. And alternatively, the triangle AED has to the triangle EBD a greater ratio than the sector EDZ to the sector EHD. But the triangle AED is to the triangle EBD as line AE to line BE, and the sector EDZ to the sector EHD is as angle ZDE to the angle EDB, and combining, line AB to line BE has a greater ratio than angle ZDH to angle EDB. But angle EDB is equal to angle AGB because ED is parallel to one of the sides of triangle ABG, AG. Therefore, AB has to BE a greater ratio than angle ZDB to angle AGB. Therefore GB has to BD a greater ratio than angle ZDB has to angle EDB, for ED cuts the sides proportionally, and it will be as AB to BE so GB to BD.[119]

Bibliography

Aghayanī-Chavoshī, J. A., intro. (2005), *Nasir al-Din at-Tusi: Taḥrīr-e Mutawassīṭāt*, facsimile of Tabriz National Library MS 3484, Tehrān.

Berggren, J. L. (1976), Spurious Theorems in Archimedes' Equilibrium of Planes: Book I, *Archive for History of Exact Sciences*, 16, 87–103.

Berggren, J. L. (1991), The Relation of Greek Spherics to Early Greek Astronomy, in A. Bowen, ed., *Science & Philosophy in Classical Greece*, New York, 227–248.

Berggren, J. L. and N. Sidoli (2007), Aristarchus's *On the Sizes and Distances of the Sun and the Moon*: Greek and Arabic Texts, *Archive for History of Exact Sciences*, 61, 213–254.

Czinczenheim, C. (2000), *Edition, traduction et commentaire des* Sphériques *de Théodose*, Lille. (These de docteur de l'Universite Paris IV.)

[119]Czinczenheim (2000, 435).

Dold-Simplonius, Y. (1995), The *Book of Assumptions* by Thābit ibn Qurra (836–901), in J. W. Dauben, M. Folkerts, E. Knobloch, W. Wussing, eds., *History of Mathematics: States of the Art*, 207–222.

Fournarakis, P. and J. Christianidis, (2006), Greek geometrical analysis: A new interpretation through the "givens" terminology, *Bollettino di Storia delle Scienze Matematiche*, 26, 33–56.

Gacek, A. (2001), *The Arabic Manuscript Tradition: a glossary of technical terms and bibliography*, Leiden.

Gutas, D. (1998), *Greek Thought, Arabic Culture*, New York.

Heiberg, J. L., ed., (1898–1903), *Claudii Ptolemaei syntaxis mathematica*, 2 parts, Claudii Ptolemaei opera quae exstant omnia, vol. 1, Leipzig.

Heiberg, J. L., ed. (1910-1915), *Archimedis opera omnia cum commentariis Eutocii*, Leipzig.

Heiberg, J. L., ed. et int. (1927), *Theodosius Tripolites Spaerica*, Berlin.

Hultsch, F. (1876), *Pappi Alexandrini collectionis quae supersunt*, Berlin.

Kheirandish, E. (1999), *The Arabic Version of Euclid's* Optics *(Kitāb Uqlīdis fī Ikhtilāf al-manāẓir)*, New York.

Knorr, W. (1978), Archimedes and the Pre-Euclidean Proportion Theory, *Archive internationales d'histoire des sciences*, 28, 183–244.

Knorr, W. (1985), Ancient Versions of Two Trigonometric Lemmas, *Classics Quarterly*, 35, 362–391.

Kraus, H. P. (1974), *Monumenta codicum manuscriptoirum. An Exhibition Catalogue of Manuscripts of the* 6^{th} *to the* 17^{th} *Centuries from the Libraries of the Monasteries of St. Catherine, Mount Sinai, Monte Cassino, Lorsch, Nanantola*, New York.

Krause, M., her. und über., (1936b), *Die* Sphärik *von Menelaos aus Alexandrien in der Verbesserung von Abū Naṣr Manṣūr b. ʿAlī b. ʿIrāq mit Untersuchungen zur Geschichte des Texte bei den islamischen Mathematikern*, Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen, Philosophisch-Historische Klasse, Dritte Folge, Nr. 17, Berlin.

Krause, M. (1936a), Stambuler Handschriften islamischer Mathematiker, *Quellen und Studien zur Geschichte der Mathematik Astronomie und Physik*, Abteilung B, Studien 3, 437–532.

Lorch, R. (1996), The Transmission of Theodosius' *Spaerica*, in M. Folkerts, her., *Mathematische Probleme im Mittelalter: Der lateinische und arabische Sprachbereich*, Wiesbaden, 159–183.

Lorch, R. (2001), *Thābit ibn Qurra On the Sector Figure and Related Texts*, Islamic Mathematics and Astronomy 108, Frankfurt am Main.

Mendell, H. (2007), Two Traces of Two-Step Eudoxan Proportion Theory in Aristotle: a Tale of Definitions in Aristotle, with a Moral, *Archive for History of Exact Sciences*, 2007, 3–27.

Nadal, R., A. Taha and P. Pinel (2004), Le contenu astronomique des *Sphériques* de Ménélaos, *Archive for History of Exact Sciences*, 59, 381–436.

Netz, R. (1999), *The Shaping of Deduction in Greek Mathematics*, Cambridge.

Rashed, R. (1996), *Les mathématiques infinitésimales du IX^e au XI^e siècle: Vol. 1: Fondateurs et commentaterus*, London.

Rome, A. (1931), *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste: Tome I, Commentaire sur les livres 5 et 6 de l'Almageste*, Roma.

Schimdt, O. (1943), On the Relation between Ancient Mathematics and Spherical Astronomy, Ph.D. Thesis, Department of Mathematics, Brown University.

Sidoli, N. and K. Saito (Forthcoming), The role of geometric construction in Theodosius's *Spherics*, *Archive for History of Exact Sciences*. 30 pages.

al-Ṭūsī N. al-D. (1940a), *Kitāb al-Ukar li-Thā'ūdhūsiyūs*, Hyderabad.

al-Ṭūsī, N. al-D. (1940b), *Kitāb Mānālā'us*, Hyderabad.

al-Ṭūsī N. al-D. (1940c), *Kitāb Arisṭarkhus fī jirmay al-nayyirayn wa-bu'dayhimā*, Hyderabad.

Ver Eecke, P. (1959), *Les Sphériques de Théodose de Tripoli*, Paris.