Al-Harawī's Version of Menelaus' *Spherics*

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Abstract
This paper is a study, based on four medieval manuscripts, of al-Harawī's edition of Menelaus' *Spherics*, the oldest version of this treatise in our manuscript sources. We provide a critical edition and translation of a number of key passages. We show that this version of the text can be used to elucidate some of Menelaus' concerns as a mathematician, to study al-Harawī's work as an editor and as a mathematical scholar, and to understand the early efforts of the ʿAbbāsid translators of technical works into Arabic.

Keywords: Ancient and medieval geometry, Arabic and Syriac translation from Greek, Spherical geometry, Spherical astronomy, *Spherics*, Menelaus, al-Harawī, al-Māhānī.

1. Introduction

The author of this revision of Menelaus' *Spherics* is called Aḥmad ibn Abī Saʿd (or Saʿīd) al-Harawī,[1] and he is usually thought to be the Abū al-Faḍl al-Harawī mentioned by al-Bīrūnī.[2] If these names do, indeed, refer to the

[1] The name is Saʿd in three of our manuscripts, but Saʿīd in one, **B**; see Section 1.3.2.

[2] Ali [1967, 67], who translated al-Bīrūnī's discussion of the observations, reads the name as al-Hirawī. For discussions of the little that we know about al-Harawī, see Krause [1936, 32–34], Sezgin [1974, 329], and Rosenfeld and İhsanoğlu [2003, 101].

same individual, then—on the basis of various comments made, and astronomical observations attributed to him, by al-Bīrūnī—he can be dated to between about 930 and 990 CE. If they are different men, then the author of our treatise lived sometime between around 883 CE, the death of Abū ʿAbdallāh Muḥammad ibn ʿĪsā ibn Aḥmad al-Māhānī, whose work he edited, and 1144 CE, the date of our earliest manuscript.

Al-Harawī made his revision of Menelaus' *Spherics* on the basis of al-Māhānī's revision of a lost translation probably made through Syriac,[3] and another unnamed source in the same tradition,[4] but in apparent ignorance of a more recent translation by Isḥāq ibn Ḥunayn,[5] which may have been revised by Thābit ibn Qurra. As we will argue below, despite the fact that al-Harawī discusses Thābit's comments on the *Almagest*, he seems not to have known, or to have been uninfluenced by, the more specialized work by Thābit on the Sector Theorem—often called the Menelaus Theorem—and compound ratio that would have been useful to his project.[6] Hence, this source gives us valuable insight into mathematical activity in the medieval Islamic world, probably in Iran, which was taking place in relative isolation from the presumably more active center in Baghdad.

To date, the only study of this complete treatise is that provided by M. Krause [1936, 34–42] in his masterly study of the medieval traditions of Menelaus' *Spherics*.[7] Krause worked with a single manuscript, **L** (see Section 1.3.1), providing a summary of the whole work, comparisons with other medieval versions, and translations of certain historically interesting sections. This paper complements, and largely confirms the findings of Björnbo [1902], Krause [1936], Hogendijk [1996], and Lorch [2001a]—whose work we refer to in many places below—but also supplements them with a study of all the relevant manuscript sources now known.

[3] This suggestion was made by Krause [1936, 85], and supported by Hogendijk [1996, 26] on the basis of the letter names of geometric objects. We add further evidence for Hogendijk's position on the basis of a new manuscript below; see Section 1.3.2.

[4] This source is discussed by Krause [1936, 34], and see also page 165, below.

[5] Krause [1936, 39] shows that al-Harawī was uninfluenced by Isḥāq's translation.

[6] These works have been edited and translated by Lorch [2001a].

[7] Lorch [2001a] also studied two manuscripts of this treatise, **LA**, in connection with al-Harawī's treatment of the Sector Theorem, including editing and translating some passages.

1.1 *Sketch of the Early Medieval Versions of Menelaus'* Spherics

In order to facilitate the discussions below, we provide a brief overview of what is currently understood about the early medieval transmission of Menelaus' work.[8]

No known Greek manuscripts contain Menelaus' text, although his terminology is discussed by Pappus, and passages of the text can be extracted from Theon's *Commentary to the Almagest*.[9]

It is not known when the oldest Arabic translation, $\mathbb{U}$ (= Krause's $\ddot{\mathrm{U}}_1$), was made, but it is thought to have been based on a Syriac translation, or to have been made by Syriac speakers.[10] This translation was apparently quite poor and a number of "corrections" were made based on it. One of these, $\mathbb{Ma}$, was made by al-Māhānī, probably in the middle of the 9th century.[11] Al-Harawī used this version when he studied the text and as his primary source for compiling his edition, telling us that al-Māhānī stopped working on it at a proposition that we will call *Māhānī's Terminus*, Prop. H.II.10 (N.III.5).[12] Al-Harawī states that he also made use of another correction, $\mathbb{O}$, which was apparently itself in rather bad shape. A marginal note in some of the manuscripts of al-Ṭūsī's recension of the text suggests that al-Harawī made use of a correction by Ibn Yusif of a different translation by al-Dimishqī, which some scholars have associated with $\mathbb{O}$.[13] Since, however, al-Harawī mentions no names associated with $\mathbb{O}$, it is probably best to admit that we do not know anything about this other correction.

Another translation, $\mathbb{bH}$ (= Krause's bḤ), was made by Isḥāq ibn Ḥunayn, in the second half of the 9th century, directly from at least one Greek manuscript. Although his name is not mentioned in this regard in our sources, Thābit ibn Qurra may also have been involved in the production of this text. Isḥāq and Thābit worked together on a number of other canonical treatises in the Greco-Roman exact sciences,[14] and the difficulty of this treatise, its

[8] This section can be taken as an update to a previous survey of the text history by one of us [Sidoli 2006, 46–52]. This previous survey was probably too optimistic in its claims to certainty.

[9] Björnbo [1902, 22–27] compiled the Greek fragments and compared them with the corresponding passages in Gerard's Latin translation.

[10] See note 3, above.

[11] Al-Māhānī's lost correction is discussed by Krause [1936, 24–32].

[12] See Section 1.2.1 , for a discussion of our conventions with regard to naming propositions.

[13] See, for example, Krause [1936, 35] and Taha and Pinel [1997, 153, n. 10].

[14] A useful overview of the Greco-Roman sciences in Arabic translation is given by Lorch

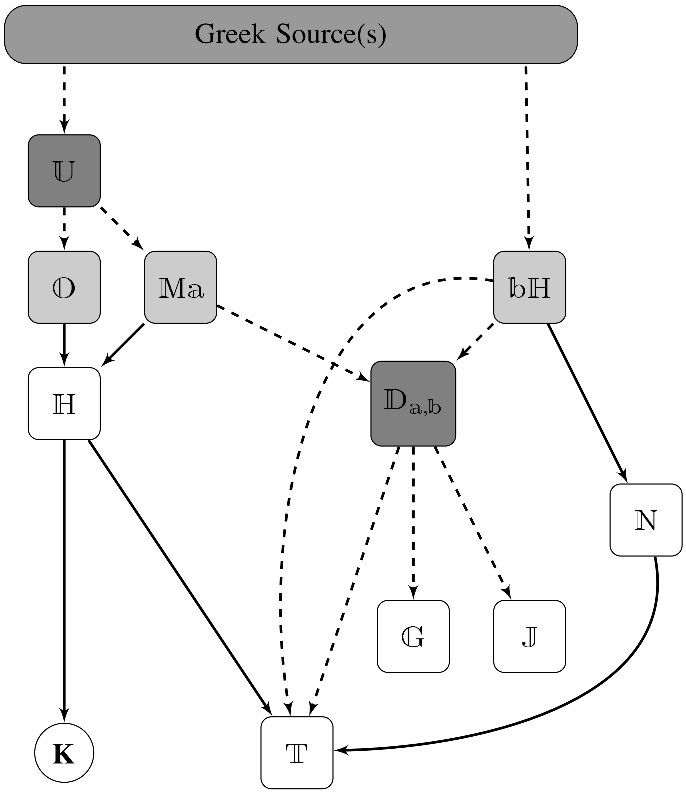

Figure 1: Stemma of the early medieval transmission of Menelaus' *Spherics*. Open-face letters ($\mathbb{A}$) indicate a text tradition; bold-face letters (**A**) indicate a manuscript. White items are found in our extant sources; light gray items are directly attested by these; dark gray items are supposed by scholars on the basis of evidence drawn from extant sources. Solid lines indicate dependancies that are stated or can be shown; dashed lines indicate dependancies that are assumed, conjectured or doubtful.

late position in mathematics curricula, and Thābit's demonstrated interest in its contents make it plausible that he had an active role in its translation.[15] It

[2001b].

[15] Thābit's treatises *Sector Theorem* and *Composition of Ratios* deal directly with material relevant to the final part of Menelaus' *Spherics*, after the introduction of the Sector Theorem, the so-called Menelaus Theorem. These treaties have been edited, translated, and studied by

is often assumed that this was a straightforward translation of a single, unproblematic Greek manuscript, but it is also possible that this version was the result of a long-term research project on a linguistically and mathematically challenging source. At any rate, it was probably completed fairly late in Isḥāq's life, towards the end of the 9th century, since it was apparently unknown to al-Māhānī. Indeed, the motivations for making this new translation may have included al-Māhānī's realization that the earlier translation was unmanageably corrupt.

No manuscripts of any of these versions—$\mathbb{U}$, $\mathbb{O}$, $\mathbb{Ma}$ or $\mathbb{bH}$—are extant. The earliest tradition for which we have any manuscript sources is the edition of the text made by al-Harawī around the middle of the 10th century, $\mathbb{H}$, which forms the subject matter of this paper. Al-Harawī seems to have been unaware of $\mathbb{bH}$, which may be an indication that $\mathbb{bH}$ was not well known.

Further evidence that $\mathbb{bH}$ did not circulate widely can be drawn from a reconstructed amalgamation of $\mathbb{Ma}$ and $\mathbb{bH}$ that we call $\mathbb{D}_{\mathbb{a},\mathbb{b}}$ (= Krause's D). $\mathbb{D}_{\mathbb{a},\mathbb{b}}$ was apparently drawn from these two sources such that $\mathbb{D}_{\mathbb{a}}$, consisting of Props. H.I.1–61 (N.I.1–II.17), came from $\mathbb{Ma}$, while $\mathbb{D}_{\mathbb{b}}$, consisting of Props. H.I.50–III.11 (N.II.9–III.25), came from $\mathbb{bH}$.[16] Although $\mathbb{D}_{\mathbb{a},\mathbb{b}}$ is also lost, its basic structure, and the key fact that part of it came from $\mathbb{Ma}$ and the rest from $\mathbb{bH}$, can be drawn from a Latin translation of it made by Gerard of Cremona, $\mathbb{G}$, a Hebrew translation made by Jaqob ben Māḥir, $\mathbb{J}$, and excerpts drawn from it by Ibn Hūd al-Muʾtaman for his *Perfection* (*Istikmāl*).[17] All of this suggests that $\mathbb{D}_{\mathbb{a},\mathbb{b}}$ was made somewhere in the western part of the Islamic sphere, possibly from an early draft of $\mathbb{bH}$.

Sometime in the first half of the 11th century, a revision of $\mathbb{bH}$ was made by Abū Naṣr Manṣūr ibn ʿAlī ibn ʿIrāq, $\mathbb{N}$, who added comments relevant to astronomical application. This is the only version of the text that has yet been critically edited.[18] The final phase of the early history of the text was the production by Nasīr al-Dīn al-Ṭūsī, $\mathbb{T}$, which was based on a number of different versions—particularly, $\mathbb{H}$ and $\mathbb{N}$—and contains historical, textual,

Lorch [2001a].

[16] Note that Props. H.I.50–61 (N.II.9–17) are repeated. It used to be believed that Prop. H.II.5 (N.III.1) was part of $\mathbb{D}_{\mathbb{a}}$, but this can no longer be maintained; see Lorch [2001a, 332–334] and Sidoli [2006, 50].

[17] The argument for the structure of $\mathbb{D}_{\mathbb{a},\mathbb{b}}$ was made by Krause [1936, 10–20]. Hogendijk [1996] showed that the Men. *Spherics* material in al-Muʾtaman's *Perfection* was also drawn from this source.

[18] Krause [1936] gives the Arabic edition and a German translation.

mathematical and astronomical commentaries. This version has been printed in the Hyderabad [1940] series of canonical texts of Islamic societies. There were also versions of the treatise made after al-Ṭūsī, but these will not concern us here.

It should also be noted that the different traditions probably underwent some crossover and it is clear that individual parts of the work, such as diagrams and individual propositions sometimes circulated separately.[19] Hence, we have tried to distinguish clearly between traditions of manuscripts, which we denote with open-face letters ($\mathbb{A}$) and individual manuscripts, which we denote with bold-face letters (**A**).

Textual criticism is usually based on the tenuous assumption of a single, authoritative original followed by an unbroken chain of reproductions, the goal of each one of which was textual fidelity to its source. But whether or not this actually is what happened is a question that must be decided on the basis of the extant sources. In the case of the early history of the Menelaus texts, however, this is not possible. For example, we cannot now determine whether or not all of the perceived superiorities of the sources that depend on $\mathbb{bH}$ over those that depend on $\mathbb{U}$ are due to the fact that $\mathbb{U}$ was based on a corrupt source manuscript while $\mathbb{bH}$ was based on a fine one. Indeed, they may also be due to mathematical scholars in Isḥāq's circle extensively reworking the $\mathbb{bH}$ translation, or, most likely, some now undecidable combination of both. Just as modern critical editors produce uniformity out of sometimes conflicting manuscript sources, we often find that medieval editors had a tendency to standardize any diversity that they found in their sources. This was especially true in the mathematical sciences, in which scientific content and intelligibility were often considered as important as authorial intention. Thus, the notion of a text tradition is a substitute for a number of manuscripts, extant or lost, that bear some, often unspecified, relation to one another.

1.2 *Conventions*

In this section, we introduce a number of conventions that we employ in the hope of making this material more accessible.

[19] For an example of the diagrams circulating separately see Section 1.3.2, below. As for individual propositions, in the case of the famous Sector Theorem, one of us has argued that a number of the medieval traditions of the Men. *Spherics* contain a version of this theorem that has been adopted from Thābit ibn Qurra's *Sector Theorem*, based on his reading of Ptolemy's *Almagest* and perhaps Theon's commentary on it; see Sidoli [2006].

1.2.1 Naming Conventions

There is a bewildering diversity in the way propositions are named in the various surviving medieval versions of Menelaus' *Spherics*. Even the four known manuscripts of the al-Harawī version have different numbers. In Appendix A, we present a concordance of these manuscripts and the Abū Naṣr version, $\mathbb{N}$, since this is the only critically edited text.

For the purposes of this paper, we refer to propositions from Menelaus' *Spherics* using the numbers in the Istanbul manuscript, **A** (see Section 1.3.3), followed by the number for the equivalent proposition in the Abū Naṣr version, since this is the one that readers will be most readily able to consult, and which is most often used by scholars. For example, Men. *Spherics* H.III.1 (N.III.14) refers to the first proposition in Book III of the al-Harawī text, which covers the same material as the fourteenth proposition of Book III in Abū Naṣr's version.

Since Menelaus often uses propositions from Theodosius' *Spherics* in his arguments, it is also necessary to refer to these. As a critical edition of one of the Arabic versions of this text is now available,[20] we refer to this Arabic version but include also the name of the relevant proposition in the Greek editions. In this way, Theo. *Spherics* A.I.18 (G.I.17) refers to the eighteenth proposition in Book I of the Arabic version, which has the same mathematical content as the seventeenth proposition in Book I in the Greek.

1.2.2 Editorial Conventions

The texts in Appendix B are edited following fairly standard procedures. All major variants have been noted in the critical apparatus. We have not, however, noted differences in dotting, often silently correcting, or supplying, the gender of verbs, and so on. With respect to the letter names of the geometric objects we have usually not noted variants in dotting or the difference between ح and جـ, unless the alternate reading can be understood to make some mathematical sense.

For the orthography of the Arabic text, we have generally followed the practice of the Brown school of the history of mathematics, which is itself an attempt to follow the medieval sources. Since, however, the manuscripts do not always agree, this can be somewhat arbitrary. In general, we have not noted initial hamzas, unless they are essential for the sense of the passage, we

[20] See Kunitzsch and Lorch [2010].

have neglected the shaddas of sun letters, and so forth. We have not noted every variation of spelling in the apparatus—for example, we omit noting هاذا for هذا, مساويان for مساوياً, and so on. In order to contribute to readability, however, we have punctuated the text.[21] Since there is almost no punctuation in the manuscripts, this process is necessarily somewhat subjective and is often influenced by the punctuation of the translation.

In the translation, material that is not found in the Arabic text appears in square brackets, [ ], while English expressions that appear in parentheses, ( ), translate Arabic expressions that we take to be parenthetical to the train of thought.

1.2.3 Graphical Conventions

Since the ancient and medieval diagrams of solid geometry and spherics have their own internal logic,[22] but are sometimes difficult for modern readers to interpret, we have supplied diagrams based on the manuscript figures for the texts and diagrams using techniques of linear perspective for the translations.

1.3 *The Manuscript Sources*

In this section, we discuss the following four manuscript sources, which are the basis of our study of the text:

L: Leiden, Universiteit *Leiden Or.* 399, sec. 2, ff. 82b–105b. 539 AH (1144 CE),

B: London, *British Library Or.* 13127. 55 ff. 548 AH (1153 CE),

A: Istanbul, Saray, *Ahmet III* 3464, sec. 5, ff. 74b-103a. Early 7th c. AH (early 13th c. CE),

K: Private collection, sold by H.P. *Kraus*, sec. 5, ff. 71v-94r. Late 7th c. AH (late 13th c. CE).

[21] Dallal [1999, 67] makes a good case for punctuation.

[22] Neugebauer [1975, 751–755] made a case for studying the diagrams as they appear in the manuscripts, which has been done, by, among others, Malpangotto [2010] and Le Meur [2012]. Saito and Sidoli [2012, 148–152] provide a brief overview of the characteristics of the manuscript diagrams for solid geometry and spherics.

1.3.1 The Leiden Manuscript, **L**

This is the oldest extant manuscript containing the treatise. Our treatise is the second in the manuscript, following the famous commentary and text of Euclid's *Elements* compiled by Abu al-ʿAbbās al-Faḍl ibn Ḥātim al-Nayrīzī. The manuscript was copied by a certain Abū Saʿd Muḥammad al-Bayhaqī al-Barzuhī (f. 81a) in a slightly cramped naskh script, in black ink, of generally 29 lines per page. There is some red highlighting, such as at the beginning of books and propositions. There are a few marginal scholia and a number of corrections that were made when the text was copied. The bottom portion of the pages have been affected by water damage that starts at about a third of the way up the central binding and slopes towards the corners, rendering some passages illegible.

The propositions are labeled continuously from 1 to 91 in abjad numerals, written in the margins. Despite this, however, the text contains internal references to a system of numbering by book (for example, see f. 102b). As well as the original labeling, a later hand has labeled some of the propositions from 71 and on, in black ink, with numbers that correspond to those in some other manuscript of the al-Harawī text (see Appendix A). Despite the fact that the proposition numbers are continuous, the second book is introduced as such. The break between the second and third book, however, which is found in the other manuscripts, is not found in this manuscript, so that it has only two books. Indeed, the final proposition, 91, is followed by the statement, "The end of the second book of the treatise of Menelaus" (f. 105b).

The diagrams are well-drawn in red ink with black letter names, placed in boxes left for them when the text was copied.

1.3.2 The London Manuscript, **B**

This manuscript, which is just a bit younger than **L**, is almost exclusively the al-Harawī text, followed by a set of diagrams from an uncorrected version of the Menelaus *Spherics*, and finally a few lines of scholastic theology and a table of measures written much later, in Cairo, dated 1509/10.[23] According to the colophon, the al-Harawī text was written by a certain Ismaʿīl, who may have been a student of the mathematician and astronomer Najm al-Dīn Abū al-Futūḥ Aḥmad ibn Muḥammad ibn al-Sarī ibn al-Ṣalāḥ from whose copy

[23] We are grateful to Dr. Bink Hallum for bringing this manuscript, which had been incorrectly described in the British Museum catalog, to our attention.

of the text we are told this transcription was made (f. 51a). The script is a large, legible naskh, in black ink, now faded to brown, at 17 lines per page. There is no red highlighting by the original hand.[24] There are a number of longer scholia in the margins, some of which are attributed to Ibn al-Ṣalāḥ and some of which are largely illegible.[25] There are also marginal corrections that were written in the process of correcting the copy. The manuscript has water damage on both the top and the bottom, which takes the form of dark bands around an unaffected island in the middle of some pages and renders the text illegible due to fading in a number of places. The folio numbers of the codex were apparently added in 1968, since the same hand that has written them in European style Arabic numerals wrote "55 ff" followed by "21.6.1968" on the final page, above a small red stamp for the British Museum.

One set of proposition numbers in the margin are continuous, but starting from book two, there is a second set of numbers, apparently in the same hand, directly below the first, which count the propositions according to three books (see Appendix A). Both the second and third books are introduced and closed in the text, so the second set of proposition numbers clearly corresponds to the numbering of the propositions in these books.

The diagrams of this manuscript are historically quite interesting. From Prop. 1 to Prop. 51, the diagrams for the text are well-drawn in faded black ink using a fine-tipped pen with black letter names, in boxes left for them when the text was copied. Starting from Prop. 52 to the end of the treatise, however, the diagrams are drawn with a thick pen in unfaded black ink, with black letter names, sometimes very crudely, in boxes that were probably previously left blank. The black ink of this second set of diagrams has bled over onto the facing pages, indicating that these figures were probably drawn after the manuscript was put into codex form—although this may have happened later due to excess moisture. The second set of diagrams has been supplied from a different source, which is preserved at the end of the same manuscript.

Starting from folio 52, there are four folia containing seven pages of diagrams, from which one folio, containing around twenty diagrams (numbers

[24] In the table of dry measures on the final page, written in 1509/10, we find a short passage and numerals in red ink.

[25] One of the scholia attributed to Ibn al-Ṣalāḥ appears below, see pages 189 (n. 127) and 207.14.

46–65),[26] has gone missing.[27] The second set of diagrams are well-drawn, using a fine-tipped pen in faded black ink, with black letter names, and the whole set is labeled "Figures (اشكال) of the *Treatise of Spheres* by Menelaus, transcribed from a copy that was not corrected, but was translated based on the first composition" (f. 52a). These figures, which we will call the *uncorrected diagrams*, are, indeed, different from those in the al-Harawī text, sometimes significantly so. Moreover, they show some interesting features, such as early usage of ي, ط, and ق, which Hogendijk [1996, 26] found in al-Muʾtaman's Men. *Spherics* material, and which he pointed out is evidence for a transmission through Syriac. The figures are numbered so as to agree with the proposition numbers in the text.

It is clear that the later, poor diagrams that are found in the text were copied from these, often with little mathematical understanding, since the diagrams are sometimes in error and the letter names in the figures often do not correspond to those in the text. We cannot be certain whether or not the diagrams for the lemmas at the beginning of Book II were in the uncorrected edition, because they would have been on the missing folio, but a consideration of the diagrams that are found in the text makes it likely that they were not. The three diagrams for the lemmas in **B** are all different from those in **L** and have an interesting relationship to those in **A**.[28] The first two are very poorly drawn and indicate that the copyist had no exemplar and did not understand the mathematics. The third is quite different from that in **L** and seems to have been drawn from a failed attempt to reconstruct the figure from the text. In **A**, the first two drawings are essentially the same as those in **L** but these are quite simple and could have been correctly redrawn by anyone who followed the mathematical argument. The third diagram for the lemmas in **A** is essentially similar to that in **B**, although that in **A** is better and more complete. It appears that these three diagrams in **B** were haphazardly redrawn based on a superficial reading of the text.

The change in the diagrams, along with the fact that the 28 folia containing the original diagrams contain almost three times as many marginal scholia as the 23 folia containing the uncorrected, often unusable, diagrams, make it clear that the manuscript was probably not seriously studied in its entirety.

[26] It is not certain whether or not the diagrams for the lemmas to Book II were included, although it seems unlikely.

[27] This folio must have gone missing before 1968, because the codex is numbered continuously.

[28] The **K** manuscript does not contain the lemmas (see Section 1.3.4, below).

1.3.3 The Istanbul Manuscript, **A**

This is a collection of seventeen treatises, mostly of the *Middle Books*, of which our treatise is the fifth.[29] The manuscript is written in a few different hands, and a number of the individual treatises have dates in the colophons. Our treatise is not dated, but the fact that those that are were all written in the early 13th century makes it likely that the al-Harawī text was copied around the same time.

The al-Harawī treatise was copied by two hands, both of which are different from the hands of most of the other works in the manuscript. Hand 1, which makes up most of the text is a clear, midsize naskh, in black ink, with little dotting and almost no other diacritical marks, of 23 lines per page. There are relatively few red highlights, especially in comparison to the other treatises in **A**. Hand 2, which takes over for short stretch from the middle of 96a to the middle of 97a (Prop. H.II.4–7), is a sloppy naskh, with little dotting or diacritical marks. The little red highlighting in this short section was probably added later, after Hand 1 took over again. There are some marginal comments and corrections, but fewer than in **L** and **B**. The manuscript is in good condition and legible.

The proposition numbers are written in red ink in the margins. All three books are introduced and closed in the text and the proposition numbers correspond to the numbers in the individual books. We have used these numbers as the basis of our numbering system for the text (See Appendix A).

The diagrams are well-drawn in red ink with black letters, placed into boxes left for them. The diagrams for that part of the text written by Hand 2 were also drawn by a different hand, again fairly hastily. The ink for these six figures is darker than those of the other diagrams and the diagrams of Prop. H.II.4 were drawn in the place left for those of Prop. H.II.5, and visa versa. Notes in red ink make it clear, both verbally (مؤخر, مقدم) and numerically (٥, ۴), that this mistake was noticed (f. 96a,b).[30]

It appears that **A** was drawn from an exemplar much closer to **B** than to **L**. As an examination of the critical apparatus of the edited text in Appendix B shows, **BA** generally agree against **L**, and most significantly in omitting, or including, the same words and phrases. Furthermore, a number of scholia found in **B** are also in **A**, and, in one case, a scholia in the margin of **B** has found its way into the text of **A** (see page 200.9). This relationship is also

[29] A full list of the treatises in the manuscript is given by Lorch [2001a, 22–23].

[30] See Sidoli and Li [2013, 49–50], for a discussion of these details.

confirmed by the diagrams for the lemmas. The diagram for the final lemma, H.II.Lemma.3, although rather odd, is structurally similar between **B** and **A**, although that in **A** is more competently done (see page 205, below). It appears that a copyist in the **A** tradition looked at a diagram like that in **B** and tried to rework it based on a careful reading of the text.

1.3.4 The So-called Kraus Manuscript, **K**

This is a collection of the *Middle Books* that is of high historical value, because the colophons of many of the treatises report that they are the versions of Qusṭā ibn Lūqā or Thābit ibn Qurra.[31] It is now privately owned and its name derives from the bookseller H.P. Kraus, from whom it was purchased.[32] Our study is based on a black and white reproduction and we have seen only a single page in color.

It was copied by a certain al-Shaykh Abī ʿAlī al-Mushhūr, who E. Kheirandish [1999, xxvii] has cogently argued should be identified as Sharaf al-Dīn Abū ʿAlī al-Ḥasan ibn ʿAlī ibn ʿUmar al-Marrākushī, who worked in Cairo in the latter part of the 13th century. At any rate, the text often offers mathematically superior readings when compared to other manuscripts. Likewise, the diagrams of the whole manuscript are competently done and generally mathematically sound. Both of these observations support the claim that this manuscript was composed by a mathematically competent scholar.

The Men. *Spherics* material contained in this manuscript is not, in fact, the al-Harawī text itself, but is an epitome that is clearly based on al-Harawī's work and has no other obvious sources.[33] The argument for this is as follows. Although many of al-Harawī's interventions have been stripped out, **K** still refers to al-Harawī in a number of critical places, includes material that al-Harawī explicitly claimed as his own, and, importantly, **K** does not contain any significant deviations from the mathematical structure of the arguments in al-Harawī's version.[34] Below we will see a number of places where **K** contains material that is attributed to al-Harawī, and which is differentiated from material due to "Menelaus," in agreement with **LBA**. Finally, **K** offers

[31] See Lorch [2001a, 28] for a list of the treatises in the manuscript.

[32] We are grateful to the owner of this manuscript for making images of it available to scholars.

[33] One of us, following Lorch [2001a, 333-334], previously claimed, incorrectly, that this version of the Men. *Spherics* predated al-Harawī's correction; see Sidoli [2006, 49].

[34] Krause [1936, 35–36] drew up a list of places where the argument in al-Harawī's version agrees with, or differs from, other medieval versions of the treatise.

no information, material or overall proofs that are not also found in **LBA**, despite the fact that the mathematical arguments are sometimes slightly better in **K**. Hence, we take **K** to be an indirect witness to the al-Harawī edition.

The script is a highly legible, well-pointed Maghribī naskh, in black ink, of 25 lines per page. The diagrams are competently drawn in black ink, with red letters, and where they differ from the diagrams in **LBA**, this can usually be explained on the basis of mathematical considerations. The proposition numbers are written in the margins. All three books are introduced and closed in the text and the proposition numbers correspond to the numbers in the individual books.

The text has been subjected to a number of structural changes. The prefaces for Books I and II have been taken out, as well as all the enunciations of the individual propositions. Although the latter may seem an odd choice, because the verbal enunciations of propositions in ancient spherics are sometimes so involved as to be almost incomprehensible, this may have been done in order to make the treatise more readable.[35]

The prose is often locally different from **LBA**. There are minor shifts in vocabulary and some arguments have been tidied up. A number of examples of these types of changes can be seen in the critical apparatus to the edited passages (see Appendix B). Individual expressions have often been rewritten to be clearer, more concise, and more in line with contemporary Arabic usage. By comparing these four manuscripts, we can see how the Arabic prose was made more natural over time. For example, verbs that could be taken as either personal-active or impersonal-passive, were eventually taken as active, although they are sometimes surrounded by sentences in which the forms of the nouns required passive readings. The syntax, which originally reflexed the source language was changed to be more natural in the target language.[36]

The critical apparatus to those parts of the text that we have edited show that **K** is somewhat closer to the **BA** transmission than to that of **L**. Nevertheless, since **K** sometimes shares readings with **L** against **BA**, this relationship is not as clear as that between **B** and **A**. Full clarification of these relationships will have to wait for a complete study of all manuscripts.

The overall impression is that this epitome was made in order to facilitate

[35] By way of example, we ask the reader to compare the intelligibility of the relatively simple enunciation of Prop. H.I.37 with that of the exposition that follows (page 180). Many of the enunciations in the text are much more convoluted than this.

[36] For an example of these sorts of changes, see Section 4.2, below.

mathematical work with this rather difficult text. Anything unnecessary to the mathematical argument has been stripped out and both the text and the diagrams have been altered in numerous ways that seem to have been guided by the mathematical sense.

2. Al-Harawī's Editorial Interventions

Because we no longer have al-Māhānī's edition of the *Spherics*, nor the other correction that al-Harawī used, it is difficult to be certain about many aspects of al-Harawī's work as an editor. For example, when al-Harawī's text differs from our reconstruction of $\mathbb{D}_{\mathbb{a}}$, we cannot know whether this difference is due to al-Harawī himself or to one of his sources, from which the unknown editor of $\mathbb{D}_{\mathbb{a},\mathbb{b}}$ deviates. Nevertheless, there are some cases where we can be fairly sure that al-Harawī is relying on his sources, such as when his text agrees with $\mathbb{D}_{\mathbb{a},\mathbb{b}}$, or where, for example, his attribution of certain obfuscating abbreviations to Menelaus indicates that they were found in at least one of his sources (see Section 3.1.1.1, below).

We do not know for certain if the overall division of the text into books and the numbering of propositions is al-Harawī's, but the fact that he explicitly calls Māhānī's Terminus the tenth proposition of the second book suggests that the numbering did, indeed, derive from his sources (see Section 3.1.3, below). Hence, we have followed the numbering by books found in **A** in preference to the continuous numbering found in **L**.

Finally, there are a number of cases where we can be certain that particular features of the text are due to al-Harawī himself: the first and second books begin with prefaces that he explicitly claims, the second of which contains a number of lemmas; and he gives three alternative proofs, two of which he claims as his own. We turn to this material now.

2.1 *The First Preface*

Al-Harawī begins by praising Menelaus' originality and making a few programatic and historical remarks. Here and elsewhere, al-Harawī is attentive to what we could describe as Menelaus' philosophy of mathematics. He claims that Menelaus believed that the science of spherics has its own "fundamental principles" (قانون), which involved avoiding the use of straight lines, cutting planes and their intersections. He will later, in the course of Book

I and in the preface of Book II, also explain that Menelaus avoided indirect arguments.

He then states that geometers found the *Spherics* difficult, and this along with “the wretchedness of the translation” (رداءة النقل) led to it being neglected for some time. This state of affairs persisted until al-Māhānī corrected the first book and part of the second.

Al-Harawī then makes some remarks about his view of the historical relationship between Menelaus and Ptolemy, particularly with regards to the Sector Theorem. He says[37]

> We find Ptolemy dealing with this treatise, especially in the second book of the *Almagest*, on the subject of angles and the triangles that result from the intersection of great circles. As for the Sector Theorem, which is the one that the treatise the *Almagest* relies on, it is due to this man. For, he provided it as a lemma for many propositions, for he secured propositions with it, and he resolved propositions into it. We find him demonstrating the two parts of this proposition, namely that in which the lines meet in it and in which they do not meet,[38] as one finds in what Thābit ibn Qurra corrected of the *Almagest*.[39] We will demonstrate the cases of these propositions when we come to the second book.

It appears that al-Harawī had no special historical knowledge of these matters but is merely basing his remarks on his reading of the texts.[40] This is true also of his understanding of the function of the Sector Theorem in Menelaus’ *Spherics*, as we will argue below.

Following these remarks, al-Harawī explains that he was urged to work on this treatise by a certain Ustādh Abū ʿAlī Muhammad ibn Aḥmad ibn al-Faḍl, who, considering the praise that al-Harawī bestows on him, was probably his own teacher. To this end, al-Harawī studied al-Māhānī’s correction of the treatise, which, because he found it defective, he says he corrected in “word”

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[37] The Arabic text for the following passage is given in Appendix B, page 197.

[38] These are the two cases given in this text, based on whether or not two of the internal lines are parallel. The parallel case is also demonstrated in $\mathbb{G}$ and $\mathbb{N}$—both of which are thought to go back to $\mathbb{D}_{\mathbb{b}}$—and in Thābit’s *Sector Theorem*, but neither in the *Almagest* nor in Theon’s commentary on it.

[39] This may be a reference to Thābit’s revision of the *Almagest*. For a recent study of a Latin translation of this work see Grupe [2012].

[40] See Section 2.2.1, below, for a discussion of our current understanding of the historical relationship between Ptolemy and Menelaus.

(لفظ), "sense" (معنى), and "proof" (برهان) until he reached the proposition at which al-Māhānī gave up—Māhānī's Terminus, Prop. H.II.10 (N.III.5).

He goes on to state that he then found another "correction" that was far from correct, made by some modern authors (ووجدت ايضاً اصلاحاً بعيداً من الصلاح لبعض المحدثين). This version, apparently, corrected parts of the text and left parts of the text untouched, but was, at any rate, unimpressive to al-Harawī. Nevertheless, he appears to have used it to supplement al-Māhānī's correction and to complete his own version of the text. There is no indication in his description of these sources that he had more than one manuscript of either source, nor that he thought that this second text was based on a different translation than that of the al-Māhānī correction.

After some remarks on the difficulty of this treatise and its place in the science of *hay'a* (cosmography, or structural astronomy), he proceeds to Menelaus' introduction. The text reads as follows:[41]

> This is the beginning of Menelaus' treatise. Menelaus, the geometer, said:
>
> Oh King Alādhyā,[42] I found an approach—deductive, reputable, astonishing—to the properties of spherical figures. So, from the obscurity of this science, many things were established by me, which I do not believe occurred to anyone before me. I arranged the preliminaries and the proofs suitably, and with it the advancement becomes easy for the lovers of science, and the attainment of sublime, universal theorems (علوم كلّيّة شريفة). I address you with what I say—Oh King, due to my learning, in your time, you delight in the knowledge of the difficulty of this science. But brevity is best.

This is followed immediately by the definitions.[43]

[41] The text for this passage is given in Appendix B, page 198.

[42] As Krause [1936, 117, n. 3] points out, here and below, this vocative is probably a misguided attempt to translate the proper name "Basileides." Based on the reading in $\mathbb{N}$, Krause plausibly restores the name to "Basileides Helladios."

[43] Menelaus' definitions, in al-Harawī's version, have been translated by Krause [1936, 36–37].

2.2 *The Second Preface*

The second time that al-Harawī intervenes editorially at length in the text is with the preface to Book II, which is longer than that for Book I. It begins:[44]

> Aḥmad ibn Abī Saʿd[45] al-Harawī said: Of the difficulty of this science, Menelaus overcame what was not possible for any besides him, but, with his mastery of it and the sublimity with which he carried it out, he did not mention many lemmas that are necessary for whoever would contemplate this book and is not of the rank of Menelaus.
>
> We see that he considers Theodosius inadequate in his treatise *On Spheres* and thinks that the method he followed is other than satisfactory, since there is difficulty in it, and the setting out of many lines, and he did not adhere, in it, to the properties of the figures that occur on the sphere, namely the conditions of the angles that arise from the intersection of the circles. Upon my life, Menelaus has easily shown everything that Theodosius proved in that book and he intends that the proof be by the direct method[46] without using straight lines.

The first part of this passage indicates that al-Harawī did not find any lemmas in his sources that were attributed to Menelaus—a conclusion that is supported by the evidence of the diagrams in the **B** manuscript (see page 159, above).

Following the discussion of Menelaus' abilities and the lack of lemmas, al-Harawī again compares the approach of Menelaus to that of Theodosius, coming down in favor of Menelaus. There seem to be two main points.

The first of these is that Menelaus showed everything using direct argumentation. In the text, al-Harawī has already noted this feature of Menelaus' work, in the comments to Prop. H.I.37 (see page 184, below), and it is not simply his own interpretation based on his reading of the work. The opening

[44] The text is found in Appendix B, page 202. It was also edited and translated by Lorch [2001a, 330, 339].

[45] Again, **B** reads "Saʿīd."

[46] This is literally "straight way," but the expression denotes a direct, as opposed to indirect, argument.

remarks that are attributed to Menelaus following al-Harawī's preface, in the opening of Book II are as follows:[47]

> Since, we have shown, by way of exposition, the preliminaries[48] that are required, then, let us now turn to what Theodosius desired to come to grips with, for he proved it stating the complete opposite if he obtains the absurd.[49] So, we show his error and we correct what he marred.

It is clear from this that Menelaus took one of his primary accomplishments to be the production of proofs by direct argument for propositions that Theodosius had shown indirectly. The idea that Menelaus sought to establish foundations of mathematics that eschewed indirect arguments is supported by a passage in Proclus' *Commentary on Elements Book I*, in which he attributes to Menelaus a proof of *Elements* I.25 by direct argument—in two triangles with two sides respectively equal, the greater angle subtends the greater base.[50] Although Proclus neither points out the fact that this argument is direct, nor mentions Menelaus' motivation for producing an alternative proof, it becomes apparent when we consider that Euclid used an indirect argument for *Elements* I.25 by assuming the contrary to the thesis and showing that it leads to a contradiction with *Elements* I.24—in two triangles with two sides respectively equal, the greater base subtends the greater angle. In light of this comment in the *Spherics*, it is clear that Menelaus was interested in rewriting the foundations of geometry in order to do away with indirect arguments, perhaps in his lost *Elements of Geometry* or *Triangles*.[51]

The second point that al-Harawī makes about Menelaus' philosophy of mathematics concerns a belief about how spherics should be pursued—namely, by dealing directly with the objects that lay on the surface of the

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[47] The text is in Appendix B, page 205.

[48] This could also mean "lemmas," but since al-Harawī's discussion makes it fairly clear that there were no lemmas in the older sources, it is likely that Menelaus is referring to the elementary methods of spherical geometry, based on spherical triangles, that he developed as preliminary to the study of more advanced topics in spherics, to which he now turns.

[49] The text might also be read as "stating the complete opposite, that he obtains the absurd." The expression is awkward but the meaning is clear. If a contraction was reached, Theodosius then asserted the opposite of what he had assumed.

[50] For this proof, see Freidlein [1873, 345–346], or Morrow [1970, 269–230].

[51] See Hogendijk [1999/2000] for a recent discussion of these lost works and some material drawn from the former found in Arabic sources.

sphere, such as the properties of spherical figures and the angles between circles, and an avoidance of the use of cutting planes and straight lines, which was a persistent feature of Theodosius' approach. Since a statement to this effect is not found in either of the prefaces that al-Harawī attributes to Menelaus, it may be his own interpretation of the mathematical approach of the *Spherics*. Nevertheless, given Menelaus' other interests in foundations of mathematics, it is a plausible claim.

Although Menelaus makes much use of propositions by Theodosius that involve straight lines, such as Theo. *Spherics* A.II.11 (G.II.11,12) or A.III.1,2 (G.III.1,2), he himself does not construct straight lines, and when he requires a straight line in an argument he uses indirect expressions such as "the line drawn from A to B," without actually depicting this line in the figure.[52] In fact, the only proposition in the treatise in which Menelaus produces a straight line is the Sector Theorem, to which we now turn.

2.2.1 The Sector Theorem

Next, al-Harawī turns to a discussion of the Sector Theorem, covering Ptolemy's treatment of the theorem, and comparing this to the way that Menelaus dealt with it. Before we discuss al-Harawī's comments, however, it may be useful to go briefly over the historical difficulties involved with this theorem.

In Greco-Roman antiquity, the theorem makes two primary appearances.[53] First, it was used by Menelaus, in his *Spherics*, as the starting point of a new approach to spherical trigonometry, focusing on the properties of spherical triangles, and opening up the prospect of a new field of mathematics. Second, it was used by Ptolemy, in his *Almagest*, to provide numerical solutions to problems in spherical astronomy, but with no reference to Menelaus, nor to any of the more interesting work that the latter had done in spherical trigonometry. Ptolemy's approach was commented upon by Theon of Alexandria, with no additional historical or mathematical material and again no mention of Menelaus.

In the medieval period, however, Islamicate mathematicians became excited about the promise of a new approach to spherical trigonometry offered by the final section of Menelaus' *Spherics*, and they studied the treatise

[52] This terminology can be found in Prop. H.I.37; see page 181, below.

[53] See Van Brummelen [2009, 56–82], for a recent summary of the Sector Theorem and spherical trigonometry in Greco-Roman antiquity.

intensively—even after the mathematics of the Sector Theorem were well understood and even while they were simultaneously developing theorems that would eventually replace the Sector Theorem as a basis for spherical trigonometry.[54] From this period on, the Sector Theorem was attributed to Menelaus and eventually became known as the Menelaus Theorem. The complicating issue is that all of our documents containing Menelaus' approach to the Sector Theorem come from this same time period in which the Sector Theorem itself, and the spherical trigonometric methods founded on it, were active areas of current research. Hence it has been difficult to determine how Menelaus himself handled this theorem.[55]

Considering the stemma in Figure 1, it is clear that the three versions of the theorem that are most likely to be close to the original are those found in $\mathbb{H}$, $\mathbb{D}_{\mathbb{a},\mathbb{b}}$ (as attested in $\mathbb{G}$ and $\mathbb{J}$), and $\mathbb{N}$.

An examination of the text of the theorem in $\mathbb{N}$, however, allows us to rule this version out for consideration as that which Menelaus included in his text.[56] It consists of three cases, with one subcase: *disjunction* (M.T.II) under the assumption that (1a) an internal line meets a diameter of the sphere in one direction, or (1b) the other direction, or that (2) the internal line is parallel to the diameter, and (3) *conjunction*.[57] The proof of (2), however, contains an indirect argument which obviously was not produced by Menelaus, and which is also found in Thābit's proof of the Sector Theorem in his eponymous treatise.[58] Moreover, the case of *conjunction*, which is not found in any of the other early versions of Menelaus' *Spherics*, also contains the same argument as that found in Thābit's treatise. Hence, the overall structure and the internal arguments of this version of the theorem lead us to the conclusion that it was put together from what was found in the Menelaus text and from supplementary material drawn from Thābit's *Sector Theorem* or his redaction of the *Almagest*, presumably by Abū Naṣr, but possibly by Thābit himself.[59]

[54] For an overview of these developments, see Van Brummelen [2009, 173–192].

[55] The medieval history of the Sector Theorem has previously been studied by one of us, Sidoli [2006], although some of the conclusions in this work can no longer be maintained.

[56] The text of the Sector Theorem in $\mathbb{N}$ is given by Krause [1936, 62–64 (Arabic)].

[57] For the purposes of this paper, we will not enter into a full discussion of the details of the Sector Theorem. See Lorch [2001a, 154–156] or Sidoli [2006, 44–46] for a discussion of the terminology and its mathematical meaning. See also Section 3.1.1, below.

[58] The text of the theorem in Thābit's *Sector Theorem* is provide by Lorch [2001a, 50–58].

[59] The version of this theorem in $\mathbb{D}_{\mathbb{a},\mathbb{b}}$ is different from that in $\mathbb{N}$, but it is possible that $\mathbb{D}_{\mathbb{a},\mathbb{b}}$ derives from an early draft of $\mathbb{b}\mathbb{H}$, in which case $\mathbb{N}$ would have derived from a "corrected" draft.

The version in $\mathbb{H}$, however, can also be dismissed, because it is merely a proof sketch.[60] It deals with cases (1a) and (2), providing a diagram for each, but merely outlining the proof. Perhaps the argument was meant to appeal to the lemmas that al-Harawī provided, but this is not made explicit in the proposition as it stands. Moreover, as we will discuss below, al-Harawī did not provide all of the lemmas necessary to the argument.

Hence, the variant of the Sector Theorem in $\mathbb{D}_{\mathbb{a},\mathbb{b}}$ emerges as the most likely candidate for a version close to what Menelaus actually wrote, and an examination of its contents supports this assessment. This version must come from $\mathbb{D}_{\mathbb{b}} = \mathbb{b}\mathbb{H}$ and its basic structure can be found in $\mathbb{G}$.[61] It contains three cases: (1a), (2), and a final case, (4), which is again *disjunction* but, trivially, takes the inverse of the compound ratio shown in (1a) and (2).[62] Significantly, in this variant of the theorem, the mathematics contained in the lemmas provided by Ptolemy, and others following him, have been incorporated into the proof itself, so that there is no need for auxiliary lemmas, which were apparently not found in the Menelaus text.

With this as background, we now turn to what al-Harawī has to say about the Sector Theorem in his second preface. Immediately following the passage of al-Harawī's second preface that we quoted above, the text continues as follows:[63]

> In this book, he produced the theorem that Ptolemy calls the *Sector [Theorem]* and he based many propositions upon it.
>
> Ptolemy uses many propositions from this treatise in the second book of the *Almagest*, without attributing it to anyone or showing anything from it.[64] Everything that is used in the angles that arise from the intersection of the ecliptic,[65] the horizons, and so on is only made clear with this treatise. The lemmas that are necessary for it are the very ones that Ptolemy furnished, as the

[60] The text for this theorem is given by Lorch [2001a, 340–342].

[61] For the text of the theorem in Gerard's version, see Sidoli [2006, 72–74].

[62] The final case, (4), is probably a late interpolation.

[63] The text is in Appendix B, page 202. Our text and translation are slightly different from that of Lorch [2001a, 330–331, 339–340].

[64] By this he presumably means that Ptolemy did not use the Sector Theorem to demonstrate any of the more useful theorems of spherical trigonometry that Menelaus developed in his *Spherics*.

[65] Literally, "the orb of the signs."

> intersection of two lines between two lines and the composition of the ratios that are made up from them.
>
> In this book, we perhaps see Menelaus shifting, transitioning to the *Sector Theorem* without setting forth his method, as he makes for it neither a preface (مقدّمة) nor a study (رسالة), nor does he make it the starting point of a book. So, the lemmas that belong to this proposition were either facts generally known to them, or have dropped out of the book. These are the lemmas...

As al-Harawī remarks, Ptolemy uses a number of theorems that are found in Menelaus' *Spherics* in *Almagest* II, but these are all from the early parts of the book, and none of them involve the spherical trigonometric methods that Menelaus developed on the basis of the Sector Theorem. Moreover, the lemmas that al-Harawī believes the theorem requires, and which he did not find in his sources for the Menelaus text, are among those found in *Almagest* I.13. As we will see below, al-Harawī includes two of Ptolemy's lemmas in his introductory material.

Finally, al-Harawī claims that Menelaus' approach to the Sector Theorem is remarkable, insofar as he does not seem to make any special fuss about it, neither introducing it in any way, making it the focus of a specialized study, as Thābit had done, nor making it the starting point of a book. As Krause [1936, 39] points out, this passage makes it clear that al-Harawī was unfamiliar with the 𝕓ℍ translation, since in this version the Sector Theorem is the first theorem of Book III.

A full consideration of the Sector Theorem in the various editions of Menelaus' *Spherics* reveals that this theorem has a number of characteristics that differentiate it from the rest of the text. (1) In all editions, it is the only proposition that has no enunciation; it begins directly with an exposition through specific letter names. (2) It is the only theorem in the treatise that does not treat spherical triangles, which were the stated object of study—indeed, it concerns a spherical convex quadrilateral. (3) It is the only proposition that directly constructs internal, straight lines and includes these in the diagram. As al-Harawī points out, one of Menelaus' foundational goals appears to have been an avoidance of straight lines in spherical figures. All of this indicates that in Menelaus' approach the Sector Theorem acted as a lemma, which was probably either well-known or taken over from a previous work.[66] Whatever the case, it is clear that Menelaus intended to mark the Sector Theorem as an outlier.

2.3 *Other Historical Remarks*

As well as the foregoing remarks by al-Harawī in his prefaces, his version contains an important historical remark that appears to go back to Menelaus at the end of Prop. H.III.8 (N.III.22). Although this remark was not included in $\mathbb{N}$, it appears to have been included in Isḥāq's translation, $\mathbb{bH}$, because it is found in $\mathbb{G}$.[67] Since these remarks were contained in both of the translations, they almost certainly go back to Menelaus. Al-Harawī's text reads as follows:[68]

> Menelaus said: This proposition shows the difference in how he proceeds and Theodosius in the third book of his treatise *On Spheres*, since he looks to prove that *GH* has a ratio to *DE* less than that which is as the diameter of the sphere to the diameter of circle *DA*. Apollonius uses this in his treatise *On the Complete Art* (في الصناعة الكلّيّة), and that makes it clear, once again, that this is extremely useful, insofar as Apollonius uses it—namely, he demonstrates that ratio *GH* to *DE* is greater than a certain ratio and less than a certain ratio.

In this comment, Menelaus situates his work in the context of that of two of his predecessors. The reference to Theodosius is clearly to his *Spherics*. Björnbo [1902, 117] and Krause [1936, 239, n. 1] were of the opinion that the reference to Apollonius was to a *Universal Treatise* (ἡ καθόλου πραγματεία) referred to by Marinus in his *Commentary on Euclid's* Data.[69] This is possible, but not certain. At any rate, the only thing we know about this *Universal Treatise* is that it contained a definition of the concept of mathematically *given*.

Whatever the case, from this passage we can see that Menelaus was trying to justify what he must have believed his readers would perceive as an ob-

[66] A more historical argument for this position has already been presented by one of us; see Sidoli [2006].
[67] Krause [1936, 239, n. 1] gives the text for Gerard's translation.
[68] The text is in Appendix B, page 209.
[69] For Marinus' discussion see Menge [1896, 234], or Taisbak [2003, 242].

scure approach by relating it to canonical authors. As usual, he simply points out that his approach is better than that of Theodosius, a fairly recent author. He then goes on to explain that this material is important because Apollonius made use of it—or rather, could have made use of it. In this sense, it is taken for granted that everything related to the work of Apollonius must be important. Here we see Menelaus as a typical intellectual of the Roman imperial period, explicitly laying the motivation for his work in the interests and activities of his great predecessors.

3. Al-Harawī's Mathematical Interventions

As well as the historical and philosophical prefaces, al-Harawī provides a number of new mathematical arguments in his edition. Most of these take the form of a group of lemmas to material in his Book II, but he also provides a number of alternative proofs.

3.1 *Lemmas*

At the end of the preface to Book II, al-Harawī provides a number of different lemmas that he considers necessary to the argument. The first group concern the Sector Theorem, the next two deal with compound ratios, and the final one he intends to use to complete his own argument for Māhānī's Terminus.

3.1.1 Lemmas for the Sector Theorem

Immediately following the passage quoted above, al-Harawī demonstrates five different forms of the plane version of the Sector Theorem. Considering Figure 2, he shows the following:

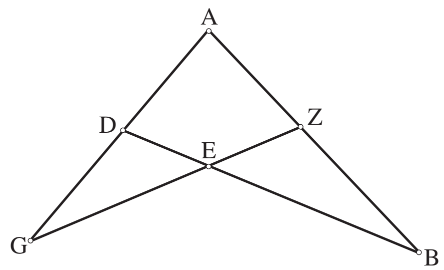

Figure 2: Simplified figure for Men. *Spherics* H.II.Lemma.1.

Lemma 1.1: $\frac{BZ}{ZA} = \frac{BE}{ED} \times \frac{DG}{GA}$,

Lemma 1.2: $\frac{BA}{AZ} = \frac{BD}{DE} \times \frac{EG}{GZ}$,

Lemma 1.3: $\frac{BE}{ED} = \frac{BZ}{ZA} \times \frac{AG}{GD}$,

Lemma 1.4: $\frac{ZA}{AG} = \frac{ZB}{BE} \times \frac{ED}{DG}$, and

Lemma 1.5: $\frac{AZ}{DE} = \frac{ZG}{GE} \times \frac{AB}{BD}$.

The first two of these, Lemma 1.1,2, are those shown by Ptolemy in his *Almagest*,[70] and are plane versions of what Thābit, in his *Sector Theorem*, calls "the aspect of disjunction" (جهة التفصيل) and "the aspect of conjunction" (جهة التركيب), respectively.[71] The next two, Lemma 1.3,4, are simply rearrangements of the terms of Lemma 1.1, while Lemma 1.5 is a rearrangement of the terms of Lemma 1.2.

Ptolemy proves just one version of each of *disjunction* and *conjunction* and leaves it up to the reader to infer how these can be manipulated to form the other versions that he occasionally uses. Thābit, relying on the lemmas in the *Almagest*, proves one of each type from the geometry of the figure, and then shows that many other combinations can be found by carrying out ratio manipulations on these two. In his *Composition of Ratios*, Thābit exhaustively shows all of the combinations that hold when a compound ratio is given.

Al-Harawī, for his part, proves each of the five lemmas in about the same space directly from the geometry of the figure but provides neither a general discussion of the different cases nor any explanation of how they are organized. Nevertheless, his presentation of these lemmas probably has pedagogical and foundational implications. His statements of the compound ratios themselves—involving consecutive uses of the same letter (as "*AB* to *BG*") and moving through certain geometric patterns on the figure—were probably meant to make the lemmas easier to remember. His derivation of each of the lemmas from the geometry of the figure may have been meant to show the reader that they follow directly from the geometric configuration and do not need to be shown using ratio manipulation.

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[70] For Ptolemy's treatment of this material, see Heiberg [1898–1903, 69–76], or Toomer [1984, 64–69].

[71] See Lorch [2001a, 48].

Following the compound ratios that he gives, he notes that[72]

> Many cases are made up from the composition of these ratios; nevertheless, most of what is required in this book are these cases we have mentioned.

On the whole, al-Harawī's approach seems somewhat disorganized. Although he mentions a commentary to the *Almagest* by Thābit, and presumably read it, he seems not to have carefully studied Thābit's works directly treating the Sector Theorem and compound ratios, if indeed these are different from the commentary he mentions. He gives no justification of why he chose to provide just those lemmas that he has, and gives no indication of where in the text these particular lemmas are used. Finally, he does not provide either of the lemmas from the *Almagest* that allow us to establish the spherical Sector Theorem on the basis of its plane counterpart by equating a ratio of two chords of double arcs with a ratio of two segments that are related to these arcs, which are also found in Abū Naṣr's edition.[73] He may have intended that his readers consult the *Almagest*, or should have mastered its contents, but if so he does not explicitly state this.

3.1.1.1 *Comments on Abbreviations*

Directly following the foregoing remarks about the various lemmas, al-Harawī explains certain abbreviations that occurred in his sources for the text. He says[74]

> It should be known that when he mentions the drawing or says "as what is in the plane" or "as what is in the drawing" or "due to what is in the drawing," it simply means these cases of the intersection of these lines, which we have mentioned. When he says "the ratio of an arc to an arc," it simply means the ratio of the chord of its double to the chord of the double of the arc that is related to it. He uses this expression for expedience.

The way that al-Harawī expresses this indicates that he found these expressions in his sources, not that he introduced them himself. Since, however,

[72] The text is in Appendix B, page 203.
[73] Toomer [1984, 65–67] calls these two lemmas *Almagest* I.13.3,4.
[74] The text is in Appendix B, page 203.

the latter part of $\mathbb{Ma}$ was not included in $\mathbb{D}_{\mathbb{a},\mathbb{b}}$, it is now no longer possible to be sure when these abbreviations originated.

The second abbreviation is particularly haphazard. By dropping the "chord of double" from the expression "chord of double arc AB," this abbreviation makes it difficult to appreciate any mathematical difference between an arc and the chord that subtends twice that arc. Hence, the abbreviation renders different mathematical objects linguistically indistinguishable, and makes all of the material following the Sector Theorem—involving both "arcs" and "chords of double arcs"— somewhat difficult, and the final theorems (following Prop. H.III.1 (N.III.14))—involving both "ratios of arcs" and "ratios of chords of double arcs"—almost incomprehensible. Since these second abbreviations are found in neither $\mathbb{N}$ nor $\mathbb{G}$, however, they are almost certainly not due to Menelaus himself.

3.1.2 Lemmas for Compound Ratios

The comments on abbreviations are immediately followed by two lemmas on compound ratio. For these lemmas on ratio manipulation, we adopt a notation of circled numerals. This is fairly true to the sense of the medieval Arabic. Because the text denotes the terms of the proportions by the expressions "first," "second," "third," etc., along with the Arabic letters *alif*, *bāʾ*, *jīm*, etc., which could also have been read as abjad numerals, by translating them as numbers we hope to convey to a modern reader a better sense of how a medieval reader would have understood the flow of the theorem than if we transliterated them in purely alphabetic form.

Considering Figure 3, which is not needed for the argument, the lemmas are as follows:

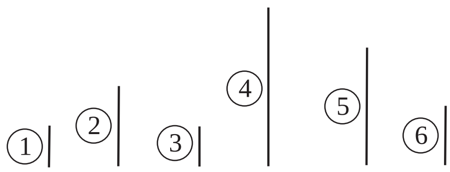

Figure 3: Men. *Spherics* H.II.Lemma.2.

Lemma 2.1: $\frac{\textcircled{1}}{\textcircled{2}} = \frac{\textcircled{3}}{\textcircled{4}} \times \frac{\textcircled{5}}{\textcircled{6}}, \textcircled{1} = \textcircled{3} \implies \frac{\textcircled{4}}{\textcircled{2}} = \frac{\textcircled{5}}{\textcircled{6}}$ and

Lemma 2.2: $\frac{\textcircled{1}}{\textcircled{2}} = \frac{\textcircled{3}}{\textcircled{4}} \times \frac{\textcircled{5}}{\textcircled{6}} \implies \frac{\textcircled{1}}{\textcircled{2}} = \frac{\textcircled{3}}{\textcircled{6}} \times \frac{\textcircled{5}}{\textcircled{4}}.$

The first lemma, Men. *Spherics* H.II.Lemma.2.1, is demonstrated, while the second, *Spherics* H.II.Lemma.2.1, is simply stated.

In order to appreciate how far removed al-Harawī's procedure is from an algebraic one—in which the lemma becomes too trivial to warrant a full argument—it may be useful to look at his argument. His text proceeds as follows:[75]

> Again, in the composition [of ratios], he uses when the ratio of the first to the second is composed of the ratio of the third to the fourth and of the ratio of the fifth to the sixth, and the third is equal to the first, then the ratio of the fourth to the second is as the ratio of the fifth to the sixth. So, because the ratio ① to ② is composed of the ratio ③ to ④ and of the ratio ⑤ to ⑥, and if ③ is equal to ①, then the ratio ④ to ② is as the ratio ⑤ to ⑥, and that is obvious because when we make ④ a mean between ① and ②, [then] the ratio ① to ② is composed of the ratio ① to ④ and of the ratio ④ to ②. But, the ratio ① to ② is composed of the ratio ① to ④ (that is of ③ to ④) and of the ratio ⑤ to ⑥. Hence, the ratio ④ to ② is as the ratio ⑤ to ⑥.

The argument, which is based on the ancient operational concept of compound ratio—mathematically equivalent to $a : b \implies a : x = x : b$, for any x—is as follows:

By the definition of compound ratio,

$$\frac{\textcircled{1}}{\textcircled{2}} = \frac{\textcircled{1}}{\textcircled{4}} \times \frac{\textcircled{4}}{\textcircled{2}}.$$

But since $\textcircled{3} = \textcircled{1}$, by hypothesis,

$$\frac{\textcircled{1}}{\textcircled{2}} = \frac{\textcircled{1}}{\textcircled{4}} \times \frac{\textcircled{5}}{\textcircled{6}}.$$

Hence, we can conclude the proposition from the structure of the two statements.

[75] The text is in Appendix B, page 203.

3.1.3 A Lemma for *Māhānī's Terminus*

Following the lemmas for compound ratios, al-Harawī turns to Mahanhi's Terminus, stating:[76]

> The tenth proposition of this book is that at which al-Māhānī ended, and he did not go beyond it. It requires a lemma, which is this...

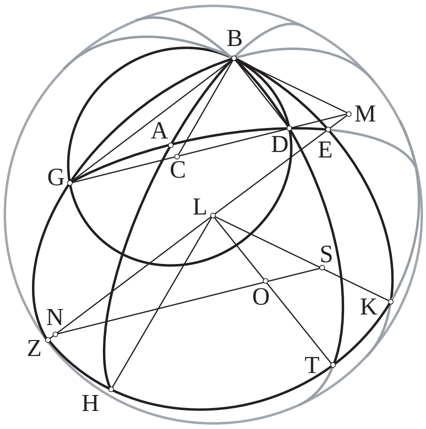

Figure 4: Men. *Spherics* H.II.Lemma.3.

Considering Figure 4, the lemma shows that where *A* is the pole of circles *BGD* and *ZHTK*, and where $BG \parallel LN$, $BC \parallel LH$, $BD \parallel LT$, and $BM \parallel LK$, then

$$\frac{NS}{SO} = \frac{Crd(2GE)}{Crd(2DE)}.$$

The proof that al-Harawī gives for this proposition is valid; however, as we will see below, it is insufficient for showing Māhānī's Terminus.

The argument relies on one of the lemmas that Ptolemy proves in the *Almagest*, which allows us to equate a ratio between two chords subtending double arcs with a ratio between two segments that depend on these arcs.[77] This lemma is not explicitly referred to by al-Harawī, but this may be another

[76] The text for the following passage, and for the lemma itself, is given in Appendix B, page 204.

[77] Toomer [1984, 67] calls this lemma *Almagest* I.13.4.

indication that he expected his readers to either have the *Almagest* to hand, or to have mastered its material.

3.2 *Al-Harawī's Alternative Proofs*

As well as these lemmas, al-Harawī provides a number of alternative proofs. One of these appears to have been by al-Māhānī, but the other two are certainly his own.

3.2.1 Al-Harawī's treatment of Men. *Spherics* H.I.8,9 (N.I.8)

Al-Harawī's version counts individually two arguments for this theorem, which all other versions count together as the eighth proposition. The theorem is the spherical analog of *Elements* I.24 and its converse I.25—namely that in two triangles if two sides are respectively equal then the greater angle between these two sides is subtended by the greater base (I.24), and the greater base is subtended by the greater angle (I.25).

Prop. H.I.8 simply notes that "the proof of this and its converse is as the proof according to the straight lines" (وبرهان هذا وعكسه كالبرهان على الخطوط المستقيم), which is either a reference to the proofs of *Elements* I.24 and 25, or a reference to the proofs of these theorems that Menelaus himself wrote. As we pointed out above, Section 2.2, Menelaus wrote an alternative proof to *Elements* I.25 that used a direct argument.

Following this, Prop. H.I.9 begins "And this is sometimes known by another proof" (وقد يعلم ذلك ببرهان آخر) and proceeds to give an argument using construction of two lesser circles and the definition of equal inclination found in the prefatory material. This argument must be due to al-Māhānī, because it is also found in $\mathbb{G}$,[78] along with a similar argument for another case, which assumes a different side is longer than the other. Its use of two lesser circles is particularly unsuitable to Menelaus' approach, which hardly ever employs lesser circles.

The proof found in $\mathbb{N}$ may be a better candidate for what Menelaus wrote since it only uses great circles and is analogous to a proof in the plane. It also involves two cases, depending on which of the lines is assumed to be longer, but this was a practice found already in the ancient commentaries to the *Elements* and may have appealed to Menelaus.[79] The argument for the

[78] See Krause [1936, 27–28] for a translation of the the proof in Gerard's version.

[79] See, for example, Proclus' *Commentary to Euclid's Elements I* [Freidlein 1873, 336–344].

converse in this text, however, cannot be Menelaus' because it simply states "The converse to this is proved through the method of contradiction."[80]

Here, as often occurs when studying this text, we run up against a difficulty in every one of our early versions of the Men. *Spherics* and we must affirm, once again, that none of our sources can be assumed unproblematically to be whatever Menelaus actually wrote.

3.2.2 Al-Harawī's treatment of Men. *Spherics* H.I.37 (N.I.36)

The next substantial, mathematical intervention that al-Harawī makes in the text is an alternative proof to Prop. H.I.37 (N.I.36), for which he explicitly takes credit. Since this is a good opportunity to see al-Harawī at work as a mathematician, we quote this text in its entirely. It reads as follows:[81]

> [I.]37 When the sum of two unequal sides of a triangle are less than a semicircle, and an arc (of a great circle) is produced from the angle that they enclose, bisecting the base, such that when a point is marked on that arc inside the triangle, and two arcs (of great circles) are produced to it from the endpoints of the base, then they enclose, with the two unequal sides, two unequal angles, the greater [angle] being with the lesser side and the lesser [angle] with the greater side.
>
> Example of it: *BG* is greater than *BA*, and their sum is less than a semicircle. *BD* (of a great circle) is produced, bisecting *AG* at *D*. Point *E* is marked on *BD*,[82] and *AE* and *EG* (of great circles) are produced. I say that angle *BAE* is greater than angle *BGE*.
>
> Proof of it: *BD* bisects *AG*, so angle *ABD* is greater than angle *GBD*.[83] Angle *GBE* is less than a right [angle],[84] and angle *AGB* is less than angle *BAG*,[85] and they are less than two right angles,[86] so angle *BGD* is less than a right [angle]. So, the arc

[80] See Krause [1936, 7 (Arabic)].

[81] The text is in Appendix B, page 198.

[82] **K** reads "supposed" in place of "marked."

[83] Proved in the second part of Men. *Spherics* H.I.34 (N.I.33).

[84] Since, by Definition 1, all angles are less than 180°, $\angle GBA < 180°$, and since, as just noted, $\angle ABE > \angle GBE$, therefore $\angle GBE < 90°$.

[85] Men. *Spherics* H.I.10 (N.I.9).

[86] Because the sum of arcs $GB + BA$ is less than a semicircle.

produced from *E* at right angles to arc *BG* cuts between points *B* and *G*. So let it be produced, and let it be arc *EZ*. What was produced from *E* to *AB* at a right angle falls between points *A* and *B*, or it does not fall like that.

[Case 1] So, first, let it fall. And let that arc be *EH*. So, angle *BHE* is a right [angle], and angle *BZE* is a right [angle]. Angle *HBE* is greater than angle *EBZ*, and *BE* is common to each of the triangles,[87] so *HE* is greater than *EZ*.[88] So, let *HT* be equal to *EZ*,[89] and we produce *AT* (of a great circle). And *AB*[+]*BG* is less than a semicircle, and *AB* is less than *BG*, so *AB* is less than a quadrant, so *AH* is less than a quadrant.

And *AT* is greater than *AH*, because angle *AHT* is a right [angle], and arc *AH* is less than a quadrant, and arc *AH* intersects with arc *EH* at right angles.[90] So, the line produced from *A* to *H* is less than all of the lines produced from *A* to arc *EH*, and the nearer to it is less than the farther.[91] So, the line produced from *A* to *T* is less than the line produced from *A* to *E*. So, arc *AT* is less than arc *AE*. And arc *AE* is less than arc *GE*, because *AD* is equal to *DG*. And *DB* is common, and *BG* is greater than *BA*, so angle *BDG* is greater than angle *BDA*.[92] And likewise when we make *DE* a common,[93] *GE* is greater than *AE*, and *AE* is greater than *AT*, so *GE* is greater than *AT*.

And *AT* is greater than *TH* (because angle *H* is a right [angle]),[94] and arc *TH* is equal to arc *EZ*, so arc *AT* is greater than arc *EZ*. So, we can produce from *E* to arc *ZG* an arc equal to arc *AT*, falling between *Z* and *G*. Let it be produced, and let it be *EK*. So, because *ZE* is equal to *HT*, and right angle *Z* is equal to

[87] That is spherical triangles *BHE* and *BEZ*.

[88] Men. *Spherics* H.I.36 (N.I.35).

[89] This is a construction. See Sidoli and Saito [2009], for a discussion of these sorts of constructions in ancient spherics.

[90] The Arabic could more literally be translated as "arc *AH* is an intersectant with arc *EH*."

[91] Theo. *Spherics* A.III.1 (G.III.1).

[92] Men. *Spherics* H.I.8,9 (N.I.8).

[93] It is not clear what this means, especially as the next two steps repeat the previous two.

[94] Following this, the more recent manuscripts, **AK**, add "and *TH* is less than a quadrant." **A** as a marginal note, and **K** in the text. See the critical lemma to the text, page 199.14, below.

angle *H*, and *EK* is equal to *AT*, so the angle *HAT* is equal to angle *ZKE*.[95] So, angle *HAE* is greater than angle *ZKE*.[96] And *GE*[+]*EK* is less than a semicircle,[97] so angle *ZKE* is greater than angle *ZGE*, so angle *HAE* is greater than angle *ZKE*, therefore it is much greater than angle *ZGE*.

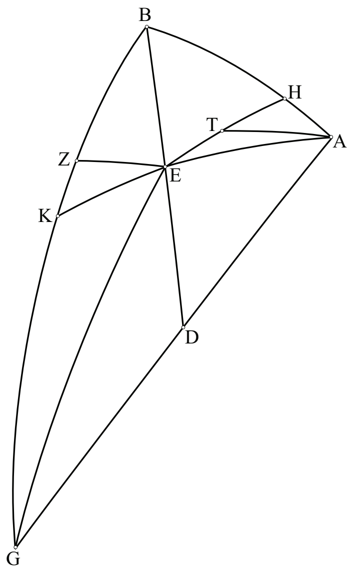

Figure 5: Men. *Spherics* H.I.37, Case 1.

[Case 2.1] Again, if the arc produced from *E* to *AB* at right angles is such that it falls outside, as in the second diagram, like arc *EH*, and we extend *HA* and *HE*, meeting at *K*, then *KAH* is a semicircle.[98] As for when *AH* is less than a quadrant, the proof of this is as before.

[95] This is a form of SSA congruency, shown in Men. *Spherics* H.I.14 (N.I.13).
[96] Since, by the geometry of the figure, $\angle HAE > \angle HAT$.
[97] The argument could be fleshed out a bit. By hypothesis, $GB + BA < 180°$, but, by the geometry of the figure, $GE < BG$ and $EK = AT < BA$, so $GE + EK < 180°$.
[98] By construction and Theo. *Spherics* A.I.12 (G.I.11).

[Case 2.2] And if it is not less than a quadrant, so that *AK* is less than a quadrant, then I say that angle *BAD* is greater than a right [angle].

> For if not, [2.2a] it is a right [angle],[99] and angle *K* is a right [angle],[100] then point *T* is the pole of circle *KAH*, and *AT* is a quadrant and it is less than half of *AG*,[101] so *AG* is greater than a semicircle. So, angle *BAG* is not a right [angle].[102] [2.2b] And if it is less than a right [angle], and *BAL* is made a right [angle],[103] then point *L* is a pole of circle *KAH*,[104] so *AL* is quadrant. And, *AT* is greater than *AL*,[105] and it is less than half of *AG*, so *AG* is much greater than a semicircle. Therefore, angle *BAD* is greater than a right [angle].

And angle *ABD* is less than a right [angle], as what became clear before this,[106] so angle *EBH* is obtuse, so *EH* is greater than a quadrant. And, *EK* is less than a quadrant, and *AK* is less than an quadrant, and angle *K* is a right [angle], so angle *EAK* is acute, so angle *EAB* is obtuse. And it was shown that angle *EGB* is acute.[107] Therefore, angle *BAE* is greater than angle *BGE*. And that is what we wanted to show.

The argument for this theorem is difficult to follow, as the many variants in the manuscripts attest. It divides into two cases, the second of which has subcases. As al-Harawī himself, in the following passage, asserts, the only

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[99] The Arabic appears to say "because it is not a right [angle]," but the structure of the mathematical argument requires "because if it is not, it is either a right angle or less than a right angle."

[100] By construction.

[101] Because *D* is the midpoint of *AG*.

[102] This follows from considering the initial conditions, namely that $GB + BA < 180°$ and $GB > AB$.

[103] We can use Men. *Spherics* H.I.1 (N.I.1) to construct an angle equal to the angle at *H* or *K*.

[104] Theo. *Spherics* A.I.14 (G.I.13).

[105] Because $AK < 90°$, $AL = 90°$, and *AT* extends out of the spherical triangle *ALK*.

[106] This appears to be a reference to the argument in Case 1.

[107] Again, this probably refers to Case 1.

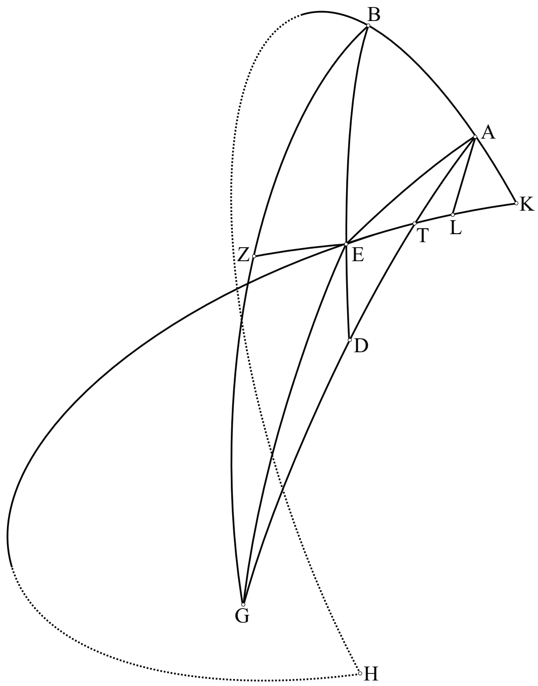


Figure 6: Men. *Spherics* H.I.37, Case 2. The dotted lines are the continuation of arcs *KB* and *KE* onto the back surface of the sphere.

part of this theorem that is actually his own is Case 2. Case 1 is essentially the same in both $\mathbb{D}_{\mathbb{a}}$ and $\mathbb{H}$, and is similar to that in $\mathbb{N}$. Case 2, however, has an indirect argument, and cannot be due to Menelaus. Hence, the version of Case 2 that appears in $\mathbb{N}$ should be taken as closer to Menelaus' approach.

Following his treatment of the theorem, al-Harawī includes the mangled remains of an argument that he says is the best he can make out of what has been attributed to Menelaus for the second case of the theorem. The text reads:[108]

> Al-Harawī said: The proper proof for the second part of this proposition is that which we use. As for Menelaus, he intends that all of his proofs be by the direct method, and what appears to

[108] The Arabic is given in Appendix B, following the text for the foregoing theorem; see page 201.

> us of his arguments for this, despite the corruption of the translation and its distance from what is taught, is this:[109]
>
> [Case 2.1] If arc *AH* is less than a quadrant, then the arrangement in it is one such that it is clear that angle *BAE* is greater than angle *BGE*. [Case 2.2] And if arc *AK* is less than a quadrant, then arc *EH* is at right angles on arc *AH*, and *EB* is less than a quadrant. So, angle *EBA* is less than a right [angle], and *EH* is greater than a quadrant, and *AH* is greater than a quadrant, and *EK* is less than a quadrant, and *AL* is less than a quadrant, so angle *EAL* is less than a right [angle]. So, angle *EAB* is obtuse and angle *BGE* is acute. And that is what we wanted to show.

Again, the argument is in two cases, but neither of them are complete, nor make much sense. Krause [1936, 35] argued that the presence of such passages in al-Harawī's edition shows that he must have had access to the original translation, $\mathbb{U}$. But there is no reason to assume this. On the one hand, al-Harawī mentions that he worked with another version produced by unnamed, recent scholars, which was corrected in parts and left uncorrected in parts and which may have contained the remains of Menelaus' arguments. On the other hand, al-Māhānī's correction may also have contained these references to Menelaus' proof, just as al-Harawī's does. The fact that they do not appear in $\mathbb{D}_{\mathbb{a},\mathbb{b}}$ may be because they were taken out by whoever composed that later edition. At any rate, there is no compelling reason to suppose that al-Harawī worked with more sources than those he explicitly discusses.

The fact that al-Harawī refers to his argument as the correct one makes it clear that he, like the other Islamicate mathematicians, merely took note of Menelaus' avoidance of indirect argument, but neither thought that it was worth following, nor tried to reconstruct Menelaus' thought along such lines.

3.2.3 Al-Harawī's treatment of Men. *Spherics* H.II.10 (N.III.5)

Following the lemmas, discussed above, the next major mathematical change that al-Harawī introduces to the text is his attempt to prove Māhānī's Terminus, using the final lemma from his second preface. As we will see below, his argument was not successful.

[109] In the **K** manuscript this paragraph reads: "That proof is al-Harawī's and it is by the method of contradiction. As for Menelaus' proof, it is by the direct method, and its substance is like this."

Prop. H.II.10 (N.III.5) reads as follows:[110]

[II.]10 When two angles of two triangles are right, and two angles are acute and mutually equal and each of the sides containing the mutually equal, acute angles are less than a quadrant, then the ratio of [*Crd* 2] the sum of the two sides containing the acute angle to [*Crd* 2] the excess of one of them over the other is as the ratio of [*Crd* 2] the sum of the two sides containing the angle equal to it, to [*Crd* 2] the excess that one of them has over the other.[111]

Let angles *B* and *M* of triangles *ABE* and *MLO* be two right [angles], and let angles *A* and *L* be acute and mutually equal.

I say that the ratio of [*Crd* 2] the sum of *BA* and *AE* to [*Crd* 2] the excess of *AE* over *AB* is as the ratio of [*Crd* 2] the sum of *ML* and *LO* to [*Crd* 2] the excess of *LO* over *LM*. Then, we make *AG* equal to *AB* and *AD* equal to *AB*, and likewise *LN* and *LS* to *ML*. I say that the ratio [*Crd* 2] *GE* to [*Crd* 2] *DE* is as the ratio [*Crd* 2] *NO* to [*Crd* 2] *OS*.

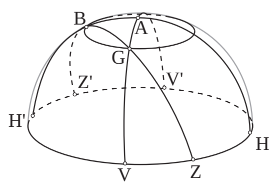
Proof of it: We make *A* a pole, and we describe circle *ZHTK* with the side of the square,[112] and we extend *BGZ*, *BAH*, *BDT* and *BEK*. So, because point *A* is the pole of *ZHTK* and arcs *AG*, *AB* and *AD* are mutually equal, then when we make point *A* a pole of a circle that is parallel to circle *ZHTK* with distance *AG*, therefore, arc *BG* with twice *GZ* is a semicircle.[113] So, when we make *ZP* equal to *ZG* and we extend *AH*, it is clear that it will

[110] The text is in Appendix B, page 206.

[111] Although the text omits an expression for "double the chord," as discussed in the passage in Section 3.1.1.1, we have supplied this as "*Crd* 2" to make the argument more intelligible.

[112] The expression "the side of the square" was standard in ancient spherics and refers to the side of a square inscribed in a great circle of the sphere. Theo. *Spherics* A.I.17,18 (G.I.16,17) implies that a circle drawn with a point as a pole and this side as distance will be the great circle about the given pole; see Sidoli and Saito [2009, 593, n. 44].

[113] The argument can be fleshed out as follows:
Where great circles AGV and HAH' meet at the pole, A, shared by parallel circle BG and great HZV, and some other great circle BGZ intersects the parallel circle at B and G, then $GZ = BZ'$ and $GZ + GB + BZ'$ is a semicircle.



meet it at *P*.[114] We extend *AG* and *HZ*, meeting one another at *V*. Then, because the pole of circle *ZHT* is on *DAG*, angle *V* is a right [angle]. Likewise, angle *H* is a right [angle]. So, angles *H* and *V* in the two triangles *GZV* and *HZP* are right [angles], and the two angles [at] *Z* are mutually equal,[115] but, both *HP* and *GV* are less than a quadrant, so *HZ* is equal to *ZV*.[116] We extend *AZ* (of a great circle), and *AH* is equal to *AV*,[117] and *HZ* is equal to *ZV*, so angle *HAZ* is equal to angle *VAZ*,[118] so angle *HAG* has been bisected by arc *AZ*. Likewise, angle *HAD* is bisected by arc *AT*.[119] And, the two angles *GAH* and *DAH* are equal to two right [angles], so half of them is a right [angle]. So, angle *ZAT* is a right [angle].[120] Again, because angle *H* is a right [angle], and angle *B* is a right [angle], then point *K* is the pole of circle *BAH*, and therefore *T* is the pole of circle *BGZ*,[121] so *TZ* is a quadrant, and *KH* is a quadrant.

We proceed with triangle *MLO* from the production of *MNF*, *MLC*, *MSQ* and *MOX*, and great circle *FCQ* with pole *L*, and the production of *LF* and *LQ*. So, it is clear that angle *FLQ* is a right [angle], and arc *FQ* is a quadrant, and likewise *XC*. So, *HZ* is equal to *FC*, and the sum *ZK* is equal to the sum *FX*.[122] So, according to what we prefaced,[123] the ratio [*Crd* 2] *GE* to [*Crd* 2] *DE* is as the ratio [*Crd* 2] *NO* to [*Crd* 2] *OS*. And that is what we wanted to show.

The proof in the text does not appear to justify the proposition. Although it is not certain how al-Harawī intended to proceed, even with the lemma, the

[114] That is, we construct the spherical opposite of triangle $BZ'H'$ (Figure in note 113).
[115] That is, the vertical angles.
[116] If *HP* and *GV* were quadrants, circle *GZP* would be perpendicular to *HZV* and the arcs *HZ* and *ZV* could have any relation to one another.
[117] They are both quadrants.
[118] Men. *Spherics* H.I.30 (N.I.29).
[119] That is, by the same type of construction and argument.
[120] Since $2\angle HAZ + 2\angle HAT = 2\mathbf{R}$, and $\angle ZAT = \angle HAZ + \angle HAT$.
[121] **K** states that *T* is a pole of circle *AZ*.
[122] Since *HZ* and *FC* are arcs of great circles subtending equal angles while *KH* and *XC* are quadrants.
[123] This is presumably a reference to his lemma, Men. *Spherics* H.II.Lemma.3, see Section 3.1.3.

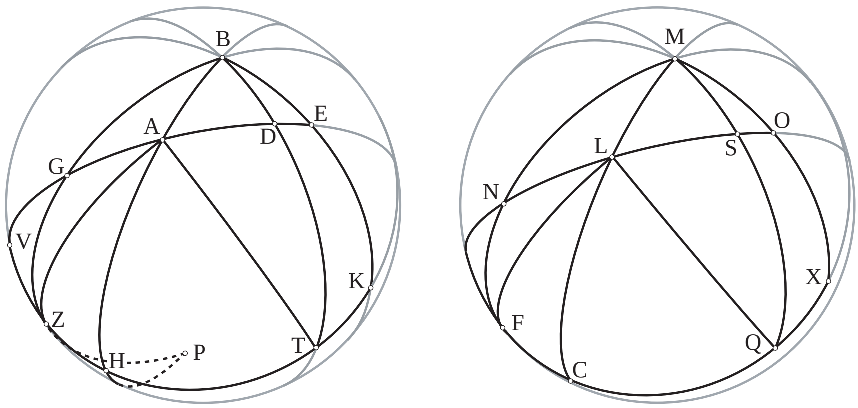


Figure 7: Men. *Spherics* H.II.10.

argument is incomplete. In the step just before al-Harawī invokes his lemma, he stated that $ZK = FX$, in which case, also $TK = QX$, which suggests that he may have intended to argue that since

$$\frac{Crd(2GE)}{Crd(2DE)} = \frac{Crd(2ZK)}{Crd(2TK)} \text{ and } \frac{Crd(2NO)}{Crd(2OS)} = \frac{Crd(2FX)}{Crd(2QX)}, \qquad (*)$$

then

$$\frac{Crd(2GE)}{Crd(2DE)} = \frac{Crd(2NO)}{Crd(2OS)}.$$

The lemma provided, however, does not sufficiently show that (*) holds. It seems that, although the lemma gives a valid argument, it is not what is required in the proof.

For the proof, we need to show that

$$\frac{Crd(2GE)}{Crd(2DE)} = \frac{Crd(2ZK)}{Crd(2TK)},$$

which, considering Figure 4, would, by *Almagest* I 13.4,[124] hold under the assumption that N coincides with Z, O coincides with T, and S is extended outside great circle ZTK. Al-Harawī, however, makes no such stipulation, nor does he argue that that triangle NSL is similar to a triangle drawn through points L, Z and the intersection of lines LK and ZT, extended.

[124] See note 73, above.

Whatever route al-Harawī intended to take, given the current state of the text, it is clear that the lemma provided is insufficient to prove the theorem.

Following his attempted proof, he gives what remains of Menelaus' argument. The text reads:[125]

> Al-Harawī said:[126] This is the proof that I made for this proposition, and which Menelaus indicates, as became clear in the preliminaries, and moreover, because he said that ZA and AT, when extended, bisect the two angles DAH and GAH respectively, but he does not prove this.[127] Then he mentions the equality of the two arcs ZK and FX, ZH and FC, HT and QC.[128] Then he said the ratio [$Crd\,2$] GE to [$Crd\,2$] ED, when we make [$Crd\,2$] AG a mean, is composed of the ratio [$Crd\,2$] EG to [$Crd\,2$] GA and of the ratio [$Crd\,2$] GA to [$Crd\,2$] ED, that is, [$Crd\,2$] DA to [$Crd\,2$] ED.[129] He said: The ratio [$Crd\,2$] EG to [$Crd\,2$] GA is as the ratio [$Crd\,2$] KZ to [$Crd\,2$] ZH, and the ratio [$Crd\,2$] DA to [$Crd\,2$] DE is as the ratio [$Crd\,2$] TH to [$Crd\,2$] TK, and the ratio [$Crd\,2$] KZ to [$Crd\,2$] ZH is the ratio [$Crd\,2$] HT to [$Crd\,2$] TK. It is necessary, likewise, in the second figure, of the ratio [$Crd\,2$] NO to [$Crd\,2$] NL and the ratio [$Crd\,2$] LS to [$Crd\,2$] SO.[130] And that is what we wanted to show.

[125] The text for this passage is given in Appendix B; see page 207.

[126] The opening of this section is introduced in **K** as follows: "This proposition is al-Harawī's theorem (علم), as for Menelaus, his theorem is this."

[127] **BA** gloss this passage with the following marginal note: "Aḥmad ibn al-Sarī says that the demonstration of this is obvious from the converse of proposition 30 of 1. And this is because we extend BH and BT until they meet. So, DA and AH and the whole of AH, extended to the meeting, is a semicircle. And arc AT is a quadrant, and two sides of all of the triangles are different, and they are a semicircle, and the arc that is extended from the angle contained by them at the base is a quadrant, so it bisects the angle and the base. So, angle DAT is equal to DAH."

[128] Again, because KH, TZ, XC and QF are quadrants while TH and QC are arcs of great circles subtending the equal angles TAH and QLC, while HZ and CF are arcs of great circles subtending the equal angles HAZ and CLF.

[129] By the definition of compound ratio and since $GA = DA$.

[130] Here, **L** includes the following sentence, which was probably originally a gloss: "This is correct because it was proved in the lemmas, which we mentioned in the first preface, and as for the bisection of the two angles GAH and NLC, what we said above proves it." (Note that the manuscript reading of the second angle is obscure. NLC is our best reading based on the mathematical sense. See the critical lemma to the text, page 208.4, below.)

The statements in the second part of the argument are not mathematically sound. What we have is a sort of proof sketch which has been corrupted over time. In Abū Naṣr's version, the steps of the proof are given in somewhat more detail, and correctly, but the justifications for the steps of the argument are still not clear.[131] Hence, Abū Naṣr follows the proof in his source text with a series of arguments for each of the steps in the original that remain unclear. As Björnbo [1902, 96–99] has shown, it is also possible to derive this theorem directly from the Sector Theorem, but given the textual evidence, it does not seem that Menelaus proceeded in that way.

3.3 *Other Mathematical Comments*

In some of the manuscripts we find two other trivial comments of a mathematical nature attributed to al-Harawī, which we only mention here.

The first of these follows Prop. H.III.1 (N.III.14).[132] This is the first of a series of propositions at the end of the treatise that concerns both ratios of arcs and ratios of the chords of double arcs. Because of the abbreviation that al-Harawī adopts, see Section 3.1.1.1, these theorems are difficult to parse. Al-Harawī refers to supporting theorems in the first book and tries to explain the difference between Menelaus' argument involving ratios of arcs and his argument involving ratios of the chords of double arcs, whereas, in fact, this difference is not clear in the text itself.

The final comment, found only in the oldest manuscript, follows immediately after the historical remarks attributed to Menelaus following Prop. H.III.8 (N.III.22), see Section 2.3.[133] Al-Harawī remarks briefly on the difference between Theodosius' approach and Menelaus' use of the chords of double arcs.

4. Conclusion

This close study of the text allows us to draw a number of general conclusions along the following lines.

[131] The text for Menelaus' sketchy argument and Abū Naṣr's supplements are given by Krause [1936, 69–72 (Arabic)].

[132] We have not edited this comment. It is found in **L** 103a, **B** 46a, **A** 100b.

[133] The text, which we have not edited, is found in **L** 105a.

4.1 Menelaus' *Spherics*

Although the original text has been lost, a study of the various surviving versions allows us to say something about Menelaus' mathematical interests. As al-Harawī notes, Menelaus had a strong interest in what we could call the foundations of mathematics.

This led him to reject indirect argumentation and to pay careful attention to the types of constructions that he permits himself. That is, Menelaus' approach to the foundations of mathematics is found directly in the way he organizes his mathematical works, which accords with what we know of the foundational approaches of other ancient mathematicians [Acerbi 2010]. In particular, as we saw in Prop. H.I.37, above, Menelaus is careful to distinguish between objects that are actually drawn on the sphere using constructive "problems" and those whose construction is introduced conceptually in order to make a proof possible [Sidoli and Saito 2009]. The distinction between constructions that solve problems and constructions that prove theorems is clear in Theodosius' *Spherics*, but is also evident in earlier authors, such as Euclid (see, for example, *Elements* III.1).

These considerations serve to isolate the Sector Theorem, often known as the Menelaus Theorem, from the rest of Menelaus' work. In this way, Menelaus' *Spherics* appears as a work in three parts. The first introduces the geometry of spherical triangles focusing on great circles that can be constructed on the surface of the sphere, although unconstructed internal objects, such as straight lines, can be invoked in the proofs. The second part introduces the Sector Theorem, as a lemma directly using straight lines, to develop a metrical geometry of spherical triangles, again avoiding internal constructions and focusing on great circles. The final part uses these theorems to develop a theory of spherical astronomy, now involving both lesser circles and great circles.[134]

4.2 The Source Translation

The four manuscripts of al-Harawī's text can be used as a way to say something about the source translation of Menelaus' *Spherics* as an example from which we may draw some general reflections about the early efforts to trans-

[134] A discussion of the astronomical aspects of Menelaus' project is given by Nadal, Taha and Pinel [2004].

late technical works into Arabic, the use of Syriac in this regard, and the way that these early translations were reworked over time.

It appears that the source translation, $\mathbb{U}$, was made through some use of Syriac and employed an Arabic syntax that was sometimes unnatural, but which adhered fairly closely to the syntax of the original Greek. We should be wary of believing, however, that the first Arabic translation was produced directly from a complete Syriac version that had been in circulation beforehand. As D. Gutas [1998, 20–22] has stressed, the production of a Syriac translation was often a component of the early ʿAbbāsid translation efforts.

Menelaus' *Spherics* is not a text for beginners. In order to understand it, one must have first mastered Euclid's *Elements* and Theodosius' *Spherics* and it is not clear that anyone before the time of al-Māhānī and Thābit ibn Qurra appreciated the real significance of the work. Hence, the scarcity of early Syriac texts in the exact sciences makes it difficult to imagine how any serious scholar could have understood this treatise in a purely Syriac context.[135] A more likely scenario is that the Syriac version was made as part of an initial effort to understand this challenging text by scholars who had more facility with Syriac than with Arabic.

As the direct accounts make clear, the translation of rare and difficult works, such as Galen's *Therapeutic Medicine* or Apollonius' *Conics*, sometimes took many years of repeated and varied efforts (Rosenthal 1975, 20–21, Toomer 1990, 621–629). The traces of a transmission through Syriac that we now find in our sources for Menelaus' *Spherics* are probably the result of an initial effort to render the text more comprehensible by putting it into a Semitic language into which Greek could be more literally translated. This appears to have then been translated into Arabic in such a way that some instances of unnatural syntax remained.

When we look at the four manuscripts of the al-Harawī version, we see that Arabic copyists who were farther removed from the translation process had a difficult time reading these expressions and tended to garble the prose. An example may help make this clear.

[135] The small list of known Syriac works in mathematics and astronomy is summarized by Duval [1907, 277–282, 283–284]. Takahashi [2010, 32–35] provides a list of known Syriac works relating to Greek philosophy and natural sciences with a tentative chronological ordering.

In the exposition of Prop. H.II.10, **L** reads[136]

فليكن مثلّثى ابه ملع زاويتى بم قائمتين وزاويتى ا ل حادتين متساوسين

while **BA** reads

فلىكن مثلّثا ا ب ه م ل ع زاويتا ب م قائمتان وزاويتا ا ل حادتان ومتساويتان

Both of these passages are grammatically problematic, which indicates that by this point in the text most copyists had given up much hope of understanding and simply copied what they thought they saw. Here, the syntax of مثلّث early in the passage, although strange in Arabic, probably reflects a Greek expression in the genitive (see for example *Elements* I.5, or Theo. *Spherics* I.6). Likewise, as in Greek, the second verb is supplied, so that the grammar should remain the same, requiring a nominative noun, followed by an accusative.

A similar passage from earlier in the text can be used to help us clarify both the meaning and the grammar. For example, the setting out of Prop. H.I.13 (**L** 85a, **B** 7a, **A** 77a) reads as follows:

فليكن زاويتا [**L**: زاويتى] ا د من مثلثى ا ب ج د ه ز قائمتين وزاويتا ج ز متساويتين وليستا بقائمتين وضلع ب ج مساوٍ لضلع ه ز

Let the two angles *A*, *D* in triangles *ABG*, *DEZ* be right [angles], and [let] the two angles *G*, *Z* be equal to one another and they are not right [angles], and [let] side *BG* be equal to side *EZ*.

The author of **K**, having followed the mathematical argument, and having seen a number of passages such as that for Prop. H.I.13 was then able to correct the setting out for Prop. H.II.10, as follows:

فلىكن زاويتا ب م من مثلثى ابه م ل ع قائمتان[١٣٧] وزاويتا ا ل حادتان متساويتان

Let the two angles *B*, *M* in triangles *ABE*, *MLO* be right [angles], and let the two angles *A*, *L* both be acute [angles], equal to one another.

[136] This text is edited in Appendix B, see page 206.6.

[137] This is probably a typo for قائمتين.

In this way, we can see the author of **K** as a mathematically competent scholar who sought to understand the al-Harawī treatise not just as a piece of mathematics but also as a historically significant document that preserved an older way of talking and thinking about spherical geometry and astronomy.

4.3 Al-Harawī as Editor

For much of the details, it is difficult to be certain to what extent al-Harawī's editorial work changed his source documents. Following the lead of his sources for the earlier part of his text, he presumably set the letter names of the geometric objects into abjad order for the later sections of the treatise. The overall numbering and division of the propositions may have been his own, or may include some of his decisions. The two introductions with their historical and philosophical concerns are clearly his own, as are the mathematical lemmas in the second preface. Including lemmas to the Sector Theorem may have been his innovation, in which he was then followed by Abū Naṣr and al-Ṭūsī. On the other hand, the decision to include the garbled remains of Menelaus' arguments was an editorial decision in which he was evidently not followed. A final, and important, aspect of al-Harawī's edition was the production of new arguments to replace what he took to be flawed in his sources. In this, however, he was not particularly successful.

4.4 Al-Harawī as Mathematician

It is not possible to form a high assessment of al-Harawī's mathematical creativity, and one is led to the conclusion that he probably should have followed al-Māhānī's lead in abandoning this version of the text as too obscure to untangle. Al-Harawī's proof for the second part of Prop. H.I.37 is longwinded, and indirect, but basically sound. He provides a number of lemmas for the Sector Theorem that are all correct, but the organization is somewhat scattered. Moreover, two lemmas that are also needed are absent. His proof of the first lemma on compound ratios is circuitous, although correct. His proof for the lemma he will invoke to try to argue for Māhānī's Terminus is valid, but it is insufficient to prove the theorem. His proof of Māhānī's Terminus is unsatisfactory, for this reason. Finally, his reliance on abbreviating language in the part of the treatise following the Sector Theorem makes it increasingly difficult to follow the mathematical train of thought.

4.5 Al-Harawī's Version of the *Spherics*

The difficulties presented by this treatise show that al-Harawī's edition of the *Spherics* cannot be taken as a reader's text. The strained language, use of obscuring abbreviations and al-Harawī's failure to completely master the mathematical contents all result in a text that becomes progressively more impenetrable as one works one's way through it. Nevertheless, because of al-Harawī's historical interests, the result is a text that preserves a number of different historical strata.

Hence, from a purely historical perspective, the text is quite interesting. It is a remnant of an early stage of the translation of technical works into Arabic, which has been preserved, and repeatedly reworked, by scholars who appear to have taken an almost reverential interest in the text. Unlike the editorial activities of al-Ṭūsī, who produced reader's texts for the canonical works in exact sciences,[138] the editors of this branch of the tradition of Menelaus' *Spherics* were interested in preserving aspects of the source documents with all of their problems, along with their own revisions of mathematical and textual details. The manuscripts of this tradition provide us with another example of the way in which medieval Islamicate mathematical scholars responded to the ancient mathematical tradition, as both a source of historical, even antiquarian, value and as a living canon worthy of continuous study.

Appendix A: Concordance of Proposition Numbers

In this appendix, we give the proposition numbers found in the manuscripts of the al-Harawī edition, along with those for Abū Naṣr's edition. This serves as a complement, and minor correction, to the concordance made by Krause [1936, 6–9]. We note only numbers that are found in the manuscripts, even where the missing numbers can easily be supplied. An illegible mark is printed as "?," although, again, it can be easily determined. Hence, a number of the parenthetical numbers in **L** that Krause includes are missing from our table.

As Krause [1936, 36] noted, there are various ways of numbering the propositions of the al-Harawī text. His $Z_1 = \{91\}$ and $Z_2 = \{61, 30\}$ are both contained in **L**, and may indicate that a reader of **L** had access to another manuscript numbered $\{61, 30\}$, but might also simply be the result of a

[138] See Sidoli and Kusuba [2008] for a discussion of al-Ṭūsī's treatment of Theodosius' *Spherics*, or Suzuki [2010] for a discussion of al-Ṭūsī's version of Hypsicles' *Ascentions*.

reader counting the propositions in the second book and labeling them sporadically. In his introductory material, al-Ṭūsī discusses the different numbering systems that he found in his sources and mentions a $\mathbb{T}_1 = Z_2 = \{61, 30\}$ numbering and a $\mathbb{T}_2 = Z_3 = \{61, 18, 12\}$ numbering.[139] This indicates that $\mathbb{T}_1 = \{61, 30\}$, recorded as a secondary system in **L**, was, indeed, found in some manuscripts. Al-Ṭūsī's second system, $\mathbb{T}_2$, is not found in any of our manuscripts.

A	**L**	**B**	**K**	$\mathbb{N}$	**A**	**L**	**B**	**K**	$\mathbb{N}$	**A**	**L**	**B**	**K**	$\mathbb{N}$
I.1	1	1	I.1	I.1	I.31	31	31	I.31	I.30	I.61	61	61	I.61	II.17
I.2	2	2	I.2	I.2	I.32	32	32	I.32	I.31	II.1	62	62(1)	II.1	II.18
I.3	3	3	I.3	I.3	I.33	33	33	I.33	I.32	II.2	63	63(2)	II.2	II.19
I.4	4	4	I.4	I.4	I.34	34	34	I.34	I.33	II.3	64	64(3)	II.3	II.20
I.5	5	5	I.5	I.5	I.35	35	35	I.35	I.34	II.4	65	65(4)	II.4	II.21
I.6	6	6	I.6	I.6	I.36	36	36	I.36	I.35	II.5	66	66(5)	II.5	III.1
I.7	7	7	I.7	I.7	I.37	37	37	I.37	I.36	II.6	67	67(6)	II.6	III.2a
I.8	8	8	I.8	I.8	I.38	38	38	I.38	I.37	II.7	68	68(7)	II.7	III.2b
I.9[a]	9[a]	9[a]	I.9[a]	—	I.39	39	39	I.39	I.38	II.8	69	69(8)	II.8	III.3
I.10	10	10	I.10	I.9	I.40	40	40	I.40	I.39	II.9	70	70(9)	II.9	III.4
I.11	11	11	I.11	I.10	I.41	41	41	I.41	—	II.10	71(10)	71(10)	II.10	III.5
I.12	12	12	I.12	I.11	I.42	42	42	I.42	II.1	II.11	72(11)	72(11)	II.11	III.6
I.13	13	13	I.13	I.12	I.43	43	43	I.43	II.2	II.12	73(12)	73(12)	II.12	III.7
I.14	14	14	I.14	I.13	I.44	44	44	I.44	II.3	II.13	74(13)	74(13)	II.13	III.8
I.15	15	15	I.15	I.14	I.45	45	45	I.45	II.4	II.14	75	75(14)	II.14	III.9
I.16	16	16	I.16	I.15	I.46	46	46	I.46	II.5	II.15a	76	76(15)a	II.15a	III.10a
I.17	17	17	I.17	I.16	I.47	47	47	I.47	II.6	II.15b	77(15)	76(15)b	II.15b	III.10b
I.18	18	18	I.18	I.17	I.48	48	48	I.48	II.7	II.16	78	77(16)	II.16	III.11
I.19	19	19	I.19	I.18	I.49	49	49	I.49	II.8	II.17	79	78(17)	II.17	III.12
I.20	20	20	I.20	I.19	I.50	50	50	I.50	II.9	II.18	80	79(18)	II.18	III.13
I.21	21	21	I.21	I.20	I.51	51	51	I.51	II.10	III.1	81	80(1)	III.1	III.14
I.22	22	22	I.22	I.21	I.52	52	52	I.52	II.11	III.2a	82	81(2)	III.2	III.15
I.23	23	23	I.23	I.22	I.53	53	53	I.53	II.12	III.2b	83(22)	82(3)	III.3	III.16
I.24	24	24	I.24	I.23	I.54	54	54	I.54	II.13	III.3	84	83(4)	III.4	III.17
I.25	25	25	I.25	I.24	I.55	55	55	I.55	—	III.4	85	84	III.5	III.18
I.26	26	26	I.26	I.25	I.56	56	56	I.56	—	III.5	86	85(6)	III.6	III.19
I.27	27	27	I.27	I.26	I.57	57	57	I.57	—	III.6	87	26(7)	III.7	III.20
I.28	28	28	I.28	I.27	I.58	58	58	I.58	II.14	III.7	88	27(?)	III.8	III.21
I.29	29	29	I.29	I.28	I.59	59	59	I.59	II.15	III.8	89	88(9)	III.9	III.22
I.30	30	30	I.30	I.29	I.60	60	60	I.60	II.16	III.9	90	89(10)	III.10	III.23
										III.10	91(30)	90(11)	III.11	III.24[b]

[139] Krause [1936, 36], in discussing this passage, prints "16," but this must be a typo for 12, since he refers to his table and the table, indeed, has 12. At any rate, al-Ṭūsī says "twelve" [Hyderabad 1940, 2–3].

[a] Prop. H.I.9 is an alternative proof, probably due to al-Māhānī, for Prop. H.I.8. It probably should not have been numbered separately, but is found so numbered in all our manuscripts.

[b] The $\mathbb{N}$ version contains a final proposition, Prop. N.III.25, not found in the al-Harawī text, which is another case related to the previous three propositions.

All of this can be summarized in the following list:

A	$\{61, 18, 10\}$
$\mathbf{L}_1$ $(= Z_1)$	$\{91\}$
$\mathbf{L}_2$ $(= \mathbb{T}_1 = Z_2)$	$\{61, 30\}$
$\mathbf{B}_1$	$\{90\}$
$\mathbf{B}_2$	$\{61, 18, 11\}$
$\mathbf{K}$ $(= \mathbf{B}_2)$	$\{61, 18, 11\}$
$\mathbb{T}_2$	$\{61, 18, 12\}$

Nevertheless, it is unlikely that any of these was the numbering system used by Menelaus when he composed his work. The numbering system used by $\mathbb{G}$ and $\mathbb{J}$—probably going back to $\mathbb{D}_{\mathbb{b}}$ and to $\mathbb{b}\mathbb{H}$—seems the most likely candidate for something close to what Menelaus used, since it reckons cases of the same configuration under the same proposition number.

Appendix B: Edited Texts

In this section we supply edited texts for the passages of al-Harawī's version of Menelaus' *Spherics* that we have translated at length in the paper. Each passage is referenced by the folio number where it begins in the manuscripts.

Excerpt from the first preface on the relation between Menelaus and Ptolemy, **L** 82b, **B** 2a, **A** 74b:

وقد نجد بطلميوس يقول على هذا الكتاب في المقالة الثانية خاصةً من المجسطي في امر الزوايا وما يحدث من تقاطع دوائر عظام من المثلّثات. فامّا الشكل القطاع، وهو الذي اليه يستند كتاب المجسطي، فهو لهذا الرجل. فانّه يورده مقدّمةً لاشكال كثيرة، فيركّب عليه اشكالاً، ويحلّل اليه اشكالاً. وقد نجده يبيّن قسمي هذا الشكل، اعني الذي يلتقي فيه الخطوط والذي لا يلتقي، كما يوجد ذلك فيما اصلحه ثابت بن قرّة من المجسطي. وسنبيّن مواقع تلك الاشكال اذا صرنا الى المقالة الثانية.

---

4 فيركّب] وركب **A**. وقد] و **L**. يبيّن] سن **A**. 5 الشكل] ~~الكتاب~~ الشكل **B**. 6–7 بن قرّة ... المقالة الثانية] (–) **B**، بن قرّة (–) **A**.

Menelaus' introduction to Book I, **L** 82b, **B** 2a, **A** 75a:

وهذا مبدأ كتاب منالاوس. قال منالاوس المهندس:

ايّها الملك الاذيا، انّي وجدت ضرباً برهانياً فاضلاً عجيباً في خواصّ الاشكال الكرية. فقد أبدأ لي اشياء كثيرة من عويص هذا العلم ما لا اظنها سنحت لأحدٍ قبلي. وقد رتبت المقدّمات والبراهين ترتيباً، يهون به النهوض على محبّي العلم والوصول الى علوم كلّيّة شريفة. وانا اخاطبك بما اقول، ايّها الملك، لعلمي بآنك تسرّ بمعرفة العويص من هذا العلم. وتجب الاختصار.

The text of Men. *Spherics* H.I.37 (N.I.36), **L** 91b, **B** 20b, **A** 85b, **K** 80a:

لز اذا كان مجموع ضلعين مختلفين من مثلّث اصغر من نصف دائرة، واخرج من الزاوية التي يحيطان بها قوس من دائرة عظمى يقسم القاعدة بنصفين، فانّه اذا تعلّم على تلك القوس نقطة في داخل المثلّث، واخرج اليها من طرفي القاعدة قوسان من دائرتين عظيمتين، فانّهما يحيطان مع الضلعين المختلفين بزاويتين مختلفين، تكون العظمى مع الضلع الاصغر والصغرى مع الضلع الاعظم.

مثاله انّ ب ج اعظم من ب ا، ومجموعهما اصغر من نصف دائرة، واخرج ب د من دائرة عظمى يقسم ا ج بنصفين على د، وتعلّم نقطة ه على ب د، واخرج ا ه ه ج من دائرتين عظيمتين. اقول انّ زاوية ب ا ه اعظم من زاوية ب ج ه.

برهانه انّ ب د يقطع ا ج بنصفين، فزاوية ا ب د اعظم من زاوية ج ب د. وزاوية ج ب ه اصغر من قائمة، وزاوية ا ج ب اصغر من زاوية ب ا ج، وهما اصغر من زاويتين قائمتين، فزاوية ب ج د اصغر من قائمة. فالقوس المخرجة من ه على زاوية قائمة على قوس ب ج تقطع بين نقطتي ب ج. فلتخرج، ولتكن قوس ه ز، والذي

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1 منالاوس[1]] مانالاوس **L**. منالاوس[2]] مانالاوس **L**. 2 الاذيا] الادنا **A**، (–) **K**. 3 أبدأ لي] بدا لي **L**، ابدا الى **A**. لا اظنها] لا ظنها **A**، اظنها **K**. 5 والوصول] فالوصول **B**. لعلمي] (–) **B**. 6 الاختصار] الاقتصار **A**. 7 اذا] *enunciation* (–) **K**. 8 قوس] قوساً **LBA**. عظمى] عظيمة **L**. 10 دائرتين عظيمتين] دائرة عظمى **L**. مع الضلعين المختلفين] (–) **L**. 12 مثاله انّ] ليكن **K**. واخرج] ونخرج **K**. 13 عظمى] عظيمة **K**. وتعلّم] وفرضت **K**. 14 زاوية ب ج ه] بجه **K**. 15 زاوية ج ب د] جبد **K**. 15–17 وزاوية ج ب ه ... زاويتين قائمتين] (–) **L**، وزاوية ج ب ه: وجبه **K**، وزاوية ا ج ب: واجب **K**، وزاوية ب ا ج: وباج **K**، زاويتين قائمتين: قائمتين **K**. 17 فزاوية ب ج د] فزاوية جبد **L**، فبجد **K**. المخرجة] التى خرجه **B**. 17–18 على زاوية قائمة على قوس ب ج] الى بج على زاوية قائمة **K**. 18–199.1 بين نقطتي ... او لا يقع كذلك] بين بد ج وهى هز والقوس المخرجة من ه الى اب تقع بين نقطتي اب **K**.

يخرج من ة الى ا ب على زاوية قائمة يقع بين ا ب، او لا يقع كذلك.

فليكن اولاً يقع، وليكن ذلك قوس ه ح. فزاوية ب ح ه قائمة، وزاوية ب ز ه قائمة. وزاوية ح ب ه اعظم من زاوية ه ب ز، وب ه مشترك للمثلّثين جميعاً، فح ه اعظم من ه ز. فليكن ح ط مثل ه ز، ونخرج ا ط من دائرة عظمى. وا ب ب ج اصغر من نصف دائرة، وا ب اصغر من ب ج، فا ب اصغر من ربع دائرة، فا ح اصغر من ربع دائرة.

وا ط اعظم من ا ح، لانّ زاوية ا ح ط قائمة، وقوس ا ح اصغر من ربع دائرة، وقوس ا ح قاطعة لقوس ه ح على زوايا قائمة، فالخطّ الخارّج من ا الى ح اقصر من كلّ الخطوط المخرجة من ا الى قوس ه ح، والاقرب اليه اقصر من الابعد. فالخطّ الخارّج من ا الى ط اقصر من الخطّ الخارّج من ا الى ة، فقوس ا ط اصغر من قوس ا ه. وقوس ا ه اصغر من قوس ج ه لانّ ا د مثل د ج، ود ب مشترك، وب ج اعظم من ب ا، فزاوية ب د ج اعظم من زاوية ب د ا. وكذلك اذا جعلنا د ه مشتركاً، كان ج ه اعظم من ا ه، وا ه اعظم من ا ط، فج ه اعظم من ا ط.

وا ط اعظم من ط ح، لانّ زاوية ح قائمة، وقوس ط ح مثل قوس ه ز، فقوس ا ط اعظم من قوس ه ز. فمكّن ان نخرج من ة الى قوس ز ج قوساً مساوية لقوس ا ط يقع بين ز وج. فلتخرج وليكن ه ك. فلانّ ز ه مثل ح ط، وزاوية ز قائمة مساوية لزاوية ح، وه ك مثل ا ط، فزاوية ح ا ط مساوية لزاوية ز ك ه. فزاوية ح ا ه

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[2] فليكن اولاً يقع، وليكن ذلك قوس ه ح] فلتقع اولاً مثل ه ح **K**. فزاوية ب ح ه] فزاوية ب ج ه **L**، فب ج ه **K**. وزاوية ب ز ه] و بزه **K**. [3] وزاوية ح ب ه] وزاوية ج ب ه **L**، وجبه **K**. زاوية ه ب ز] هبز **K**. جميعاً] (–) **K**. فح ه] فجه **KL**. [4] وا ب] فا ب **A**. [7] وا ط] فا ط **B**، وامّا *in margin* ا ط **A**. وقوس ا ح] وا ح **L**، بوفى هز بط (؟) **B**، وا ط **K**. [8] وقوس ا ح] وا ح **KL**. لقوس ه ح] لهج **K**. [8–10] فالخطّ الخارّج من ا … الخطّ الخارّج من ا الى ة] فخط اط اقصر من خط اه **K**. [9] المخرجة] الخارجة *in margin, later hand* **L**. اقصر] اصغر **A**. [10] الخارّج] احارح **B**. اصغر] اقصر **A**. [11] قوس ا ه] ا ه **K**. وقوس ا ه] وا ه **K**. قوس ج ه] ج ه **K**. [12] وكذلك] ولذلك **K**. [13] كان] (–) **B**، يكون *in margin* **A**. [14] قائمة] (+) وط ح اصغر من ربع دايرة *in margin* **A**. وقوس ط ح] وط ح **K**. قوس ه ز] هز **K**. [14–15] فقوس ا ط] فا ط **K**. [15] قوس ه ز] هز **K**. قوس ز ج] ز ح **K**. مساوية لقوس] مثل **K**. [16] ز وج] زج **K**. فلتخرج] (–) **K**. وزاوية ز] وز **K**. [17] لزاوية ح] لح **K**. ح ا ط] ج ا ط **KL**. مساوية لزاوية] مثل **K**. ز ك ه] ركح *in margin* زكه **L**. فزاوية ح ا ه] فزاوية جاه **L**، فجاه **K**.

اعظم من زاوية ز ك ه. وجه ه ك اصغر من نصف دائرة، فزاوية ز ك ه اعظم من زاوية ز ج ه، فزاوية ح ا ه اعظم من زاوية ز ك ه، فهي اذاً اعظم كثيراً من زاوية ز ج ه.

وايضاً فإن القوس المخرجة من ه الى ا ب على زاويا قائمة إن وقعت خارجاً، كما في الصورة الثانية، مثل قوس ه ح، ونخرج ح ا وح ه يلتقيان على ك، فكا ح نصف دائرة. وا ح امّا إن يكون اصغر من ربع دائرة، فالبرهان فيه كما يقدّم.

وإن لم يكن اصغر من ربع دائرة، فانّ ا ك اصغر من ربع دائرة، فاقول انّ زاوية ب ا د اعظم من قائمة، لانّها لو كانت قائمة، وزاوية ك قائمة، لكانت نقطة ط قطب دائرة ك ا ح، وكان ا ط ربع دائرة وهي اصغر من نصف ا ج، فا ج اعظم من نصف دائرة. فليس زاوية ب ا ج قائمة. وإن كانت اصغر من قائمة، وجعلت ب ا ل قائمة، لكانت نقطة ل قطباً لدائرة ك ا ح، فا ل ربع دائرة. وا ط اعظم من ا ل، وهو اصغر من نصف ا ج، فا ج اعظم كثيراً من نصف دائرة. فاذاً زاوية ب ا د اعظم من قائمة. وزاوية ا ب د اصغر من قائمة، كما يتبين قبل هذا، فزاوية ه ب ح

[1] زاوية ز ك ه] زكه **K**. وجه ه ك] وجه ك **A**. فزاوية ز ك ه] فزكه **K**. [2] زاوية ز ج ه] زاوية ا جه **L**، زجه **K**. فزاوية ح ا ه] فزاوية جا ه **LA**، فجاه **K**. زاوية ز ك ه] ز ك ه **K**. [2–3] زاوية ز ج ه] زجه **K**. [4] المخرجة] التي **K**. إن وقعت خارجاً] ان وقعت خارجة *in margin* **A**، خارجاً: خارج **L**. [5] مثل] ~~كما~~ مثل **B**. قوس ه ح] ه ح **K**. ح ا وح ه] جا وجه **LAK**. [6] فالبرهان] والبرهان **BA**. [7] فانّ ... ربع دائرة] فان اك اصغر من ربع دايرة فان اك اصغر من ربع دايرة **B**. [8] لانّها] لانه **K**. وزاوية ك] وك **K**. نقطة ط] ط **K**. [9] قطب دائرة] قطب **K**. ك ا ح] كاح وكان ا ط مساوية لط ح لطريق منالاوس في الانجاز بان يقال قوس ا ك اصغر من ربع دايرة وقوس ا ه ايضاً اصغر من ربع دايرة لانّها اصغر من ه ج وهما جميعاً اصغر من نصف دايرة وزاوية ك قايمة فلان مثلّث ا ه ك زاوية ك منه ليست باصغر من قايمة وكل واحد من ضلعي ك ا ا ه وهما المحيطان باحدي الزاويتين الباقيتين اصغر من ربع دايرة فكل واحدة من الزاويتين الباقيتين وهما ك ا ه ك ه د اصغر من قايمة بحسب شكل كو فيبقي زاوية ه ا ب اعظم من قايمة وزاوية ه ج ب اصغر من قايمة فزاوية ه ا ب اعظم من زاوية ه ج ب وذلك ما اردنا ان نبين **A**، *in margin (partly illegible)* **B**، لط ح: كضح *in margin* **A**، ك ه د: كها *in margin* **A**. وكان ا ط] وا ط **K**. [10] فليس] فليست **K**. زاوية ب ا ج] زاوية باد **A**، باد **K**. وإن] فان **LAK**. [11] ب ا ل] باك **K**. نقطة ل] ل **K**. قطباً لدائرة] قطب **K**. فا ل] وكان ا ه **L**. [12] نصف ا ج] نصف دايرة ا ج **K**. [12–13] كثيراً من نصف دائرة. فاذاً زاوية ب ا د اعظم] (–) **K**. [13] وزاوية ا ب د] فابد **K**. يتبين قبل هذا] بينا **K**. فزاوية ه ب ح] فهبح **K**.

منفرجة، فهح اعظم من ربع دائرة. وهك اصغر من ربع دائرة، واك اصغر من ربع دائرة، وزاوية ك قائمة، فزاوية ه ا ك حادة، فزاوية ه ا ب منفرجة. وقد كان يبيّن انّ زاوية ه جـ ب حادة. فاذاً زاوية ب ا ه اعظم من زاوية ب جـ ه. وذلك ما اردنا ان نبيّن.

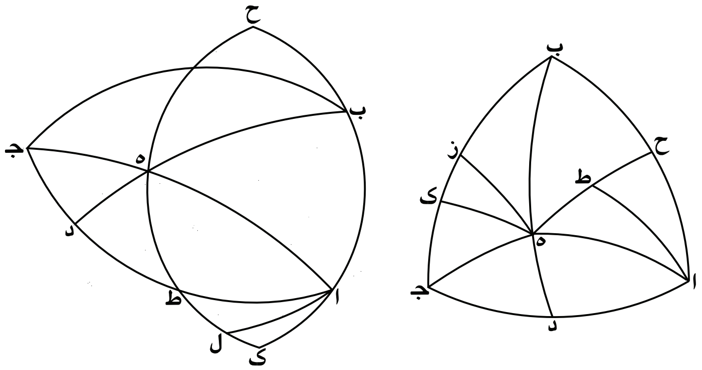

Diagram for Men. *Spherics* H.I.37 (N.I.38). In **B**, the diagrams are given in two different places, each following the corresponding text. In **K**, the horizontal position of the two diagrams is alternated. In **A**, line ط ا has been drawn in twice, once incorrectly, once incorrectly and then erased.

قال الهروي: البرهان القريب على القسم الثاني من هذا الشكل هو هذا الذي استعملنا. فامّا مانالاوس فانّه يتوخي ان يكون براهينه كلّها بطريق الاستقامة، والذي يترأيا لنا من براهينه لهذا مع فساد النقل وبُعده من ان يُفهم هو هذا:

إن كانت قوس ا ح اصغر من ربع دائرة، فالتدبير فيه واحد حتّى يظهر انّ زاوية ب ا ه اعظم من زاوية ب جـ ه. وإن كانت قوس ا ك هي اصغر من ربع دائرة، فقوس ه ح على زوايا قائمة على قوس ا ح، وه ب اصغر من ربع دائرة. فزاوية

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[1] فه ح] فه ب ح **LAK**. وه ك اصغر من ربع دائرة] (–) **KL**، وه ك: فه د **B**. [2] وزاوية ك] وك **K**. فزاوية ه ا ك] فهاك **K**. فزاوية ه ا ب] فهاب **K**. كان] (–) **K**. [3] زاوية ه جـ ب] بهجـ **K**. فاذاً زاوية ب ا ه] فان زاوية باه **L**، فاذ زاوية باه **B**، فباه **K**. زاوية ب جـ ه] بجه **K**. [4–3] ان نبيّن] بيانه **K**. [7–5] قال ... يُفهم هو هذا] (–) **A**، هاذا البرهان للهروي وهو بطريق الخلف واما برهان منالاوس فهو بطريق الاستقامة وصفته هاكذا **K**، القسم: ازلقسم **L**، **B** *in margin*، هو هذا الذي: وها **B**، لهذا: (–) **B**، وبعده: وبعُدِهِ **B**. [8] إن] وان **A**. [9–8] زاوية ب ا ه] باه **K**. [9] ا ك] اجـ **B**. [10] قوس ا ح] ا ح **K**. [202.1–10] فزاوية ه ب ا] وزاوية ب د **L**، وزاوية دا **B**، فهبا **K**.

ه ب ا اصغر من قائمة، وه ح اعظم من ربع دائرة، وا ح اعظم من ربع دائرة، وه ك اصغر من ربع دائرة، وا ل اصغر من ربع دائرة، فزاوية ه ا ل اصغر من قائمة، فزاوية ه ا ب منفرجة وزاوية ب جه حادة. وذلك ما اردنا ان نبيّن.

Excerpt from the second preface on the relation between Menelaus and Theodosius, and on Menelaus' treatment of the Sector Theorem, **L** 97b, **B** 34a, **A** 93b:

قال أحمد بن ابي سعد الهروي ان منالاوس قد ذلّل من صعب هذا العلم ما لم يتيسر لغيره، فهو مع تمكّنه منه وجلالة ما يتصّرف فيه يُعرِض عن مقدّمات كثيرة يحتاج اليها من يتأمّل هذا الكتاب ويعجز عن مرتبة منالاوس. ونراه يستقصر ثاوذوسيوس في كتابه في الأكر ويرى انّ طريقته التي سلكها غير مرضية، اذ فيها تعسّر واخراج خطوط كثيرة وانّه لم يلزم فيها خواصّ الاشكال التي تقع في الكرة، اعني احوال الزوايا التي يحدث من تقاطع الدوائر. لعمري انّ جميع ما برهنه ثاوذوسيوس في ذلك الكتاب قد بيّنه منالاوس بسهولة، وتوخى ان يكون البرهان بطريق الاستقامة من غير ان يستعمل الخطوط المستقيمة.

وعمل في هذه المقالة الشكل الذي يسمّيه بطلميوس القطاع، وبنى عليه اشكالاً كثيرةً. وبطلميوس يستعمل من هذا الكتاب اشكالاً كثيرةً في المقالة الثانية من كتاب المجسطي من غير ان ينسبه الى احد او يبيّن شيئاً منه. وانّ جميع ما يستعمله في الزوايا التي يحدث من تقاطع فلك البروج والآفاق وغير ذلك انّما يتبين بهذا الكتاب. والمقدّمات التي يحتاج اليها هي التي اوردها بطلميوس بعينها من تقاطع خطين بين خطين وتأليف النسب التي تتركب منها. وقد نجد منالاوس ينتقل في هذه المقالة الى الشكل القطاع انتقالاً ليس يلتق بطريقته اذ لم يجعل له

[1] وه ك] وهط **K**. [2] وا ل اصغر من ربع دائرة] (—) **K**. [2–3] فزاوية ه ا ب] فهاب **K**. [3] وزاوية ب جه] وبجه **K**. ان نبيّن] بيانه **K**. [4] سعد] سعيد **B**. منالاوس] مانالاوس **B**. [5] منه] منها **L**. فيه] (—) **BA**. [6] منالاوس] مانالاوس **B**. [7] اذ] (—) **BA**. فيها] (—) **A**. [12] القطاع] (—) **A**. عليه] علها **A**. [14] ينسبه] تنسبَهُ **L**. وانّ] فان **L**. [16] التي اوردها بطلميوس بعينها] التي اورد بطلميوس بعضها **L**. [17] وتأليف النسب] وبالف والنسب **L**. منالاوس] منالا *in margin* وس **B**. [18] يلتق] (—) **A**. اذ] اذا **BA**.

مقدّمة ولا رسالة ولا جعله مبدأ مقالة. فإمّا ان تكون المقدّمات التي لهذا الشكل كانت معلومة عندهم معروفة او قد سقطت من المقالة. والمقدّمات هي هذه.

Excerpt from the second preface on the various lemmas for the Sector Theorems, abbreviations and the first lemma on compound ratio, **L** 98a, **B** 35a, **A** 94b:

وقد تتركّب من تأليف هذه النسب وجوه كثيرة الّا انّ جلّ ما يحتاج اليه في هذا الكتاب هي هذه الوجوه التي ذكرنا.

وينبغي ان نعلم انّه متى ذكر الرسم او قال كما في السطح وكما في الرسم ومن اجل الرسم فانّما يعني هذه الوجوه التي ذكرناها من تقاطع هذه الخطوط. ومتى قال نسبة قوس الى قوس فانّما يعني نسبة وتر ضعفه الى وتر ضعف القوس الذي ينسبه اليه، ويستعمل هذا اللفظ للتّخفيف.

ويستعمل في التأليف ايضاً متى كانت نسبة الاوّل الى الثاني مؤلّفة من نسبة الثالث الى الرابع ومن نسبة الخامس الى السادس وكان الثالث مثل الاوّل، فانّ نسبة الرابع الى الثاني كنسبة الخامس الى السادس. فلانّ نسبة ا̅ الى ب̅ مؤلّفة من نسبة ج̅ الى د̅ ومن نسبة ه̅ الى ز̅، وإن كان ج̅ مساوياً لا̅، فانّ نسبة د̅ الى ب̅ كنسبة ه̅ الى ز̅، وذلك ظاهر لانّا متى جعلنا د̅ وسطاً بين ا̅ وب̅، كانت نسبة ا̅ الى ب̅ مؤلّفة من نسبة ا̅ الى د̅ ومن نسبة د̅ الى ب̅، ولكن نسبة ا̅ الى ب̅ مؤلّفة من نسبة ا̅ الى د̅، اعني من ج̅ الى د̅، نسبة ه̅ الى ز̅. فاذاً نسبة د̅ الى ب̅ كنسبة ه̅ الى ز̅.

1 رسالة] رسالة **B**. المقدّمات التي لهذا الشكل] المقدّمات الّتي لهذا الشكل **A**، مقدّمات هذا الشكل **L**. 2 سقطت] سقط **L**. 4 ذكرنا] ذكرناها **A**. 5 او قال] وقال **A**. 7 نسبة قوس] قوس **B**. 9 التأليف] النالف **A**. ايضاً] (–) **A**. 11 فلانّ] فليكن **L**. 12 من نسبة] (–) **L**. وإن كان] وان **L**، فان كان **B**. فانّ] اقول ان **L**. 13 ا̅ وب̅] ا̅ب̅ **L**. 14 ا̅ الى د̅] (+) اعني من ج̅ الى د̅ **L**. 15 اعني من ج̅ الى د̅] (–) **L**. نسبة] (–) **BA**.

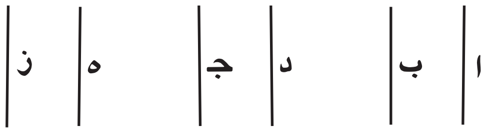

Diagram for Men. *Spherics* H.II.Lemma.2. In **A**, the numerals 1–6 have been included, following the abjad ordinal sequence of the letter names.

Excerpt from the second preface on Māhānī’s Terminus and al-Harawī’s lemma in this regard (H.II.Lemma.3), **L** 98b, **B** 35a, **A** 94b:

والشكل العاشر من هذه المقالة هو الذي انتهى اليه الماهاني ولم يتجاوزه، ويحتاج الى مقدّمة هي هذه.

دوائر ب جـ ز ب ا ح ب د ط ب ه ك تتقاطع على نقطة ب، وقد قطعت بسطحين متوازيين، وهما ب جـ د ز ح ط، ونقطة ا قطب دائرة ز ح ط، ومركز الكرة نقطة ل، وقسيّ ا ب ا جـ ا د متساوية. فلانّ ا قطب دائرة ز ح ط ك، فا ح قائم على ز ح ط ك على زوايا قائمة، فسطح ب جـ د اذاً قائم على سطح ب ا ح على زوايا قائمة. فالفصول المشتركة التي لدوائر ب جـ ز ب ا ح ب د ط ب ه ك ولسطحي ب جـ د ز ح ط ك متوازية. لكن الفصول المشتركة التي لهذه الدوائر ولسطح ز ح ط ك هي اقطار الدوائر المخرجة من نقط ز ح ط ك. والفصول المشتركة لهذه الدوائر ولسطح ب جـ د هي خطوط ب جـ ب د ب ص ب م، وهي موازية لاقطار الدوائر التي ذكرنا، ب جـ يوازي ل ز، وب ص يوازي ل ح، وب د يوازي ل ط، وب م يوازي ل ك. فزاوية جـ ب م مساوية لزاوية ز ل ك، وجـ ب ص لـز ل ح، وص ب د لـح ل ط، ود ب م لـط ل ك. ونصل جـ د، ونبعده ليلقي ب م على م. وظاهر انّه يلقاه لانّهما في سطح دائرة ب جـ د. ونخرج ل ه. فظاهر انّه يلقي ب م على م لانّ ل ه في سطح دائرة ب ه ك وفي سطح دائرة جـ ا ه وب م هو الفصل المشترك

3 ب ه ك] ب ه ط L. ب] اب B. 4 ب جـ د] حدر حط L. 5 فا ح] وا ح L. ز ح ط ك]
ز ح ط BA. 6 ب جـ د] ب جـ ز L. 7 ولسطحي] ولسطح A. 8 ز ح ط ك متوازية] ر ط ك متوازية
B. 9 نقط] نقطة L، (–) L. ز ح ط ك] ز ح ط ك LA. 10 موازية] متوازية A. 11 يوازي
ل ح] ل ح L. يوازي ل ط] ل ط L. 12 يوازي ل ك] ل ك L. 13 ونصل] وفصل L. 14 يلقاه]
يلقي A. فظاهر] وظاهر A. ب م] به L. 15 ب ه ك] ه ط B.

لسطح دائرة ب ه ك ولسطح دائرة ب جـ د، فنقطة م هي في سطح دائرة ب ه ك وفي سطح ب جـ د وفي سطح جـ ا ه. فهي على الفصل المشترك لهذه السطوح. ونفصل ل ن مساوياً لـب جـ ول س مساوياً لـب م، ونصل ن س. فمثلّث ن ل س مثل مثلّث جـ ب م، فنسبة جـ م الى م د كنسبة ن س الى س ع. لكنّ نسبة جـ م الى م د هي كنسبة وتر ضعف جـ ه الى وتر ضعف ه د. فنسبة ن س الى س ع كنسبة وتر ضعف جـ ه الى وتر ضعف ه د. وذلك ما اردنا ان نبيّن.

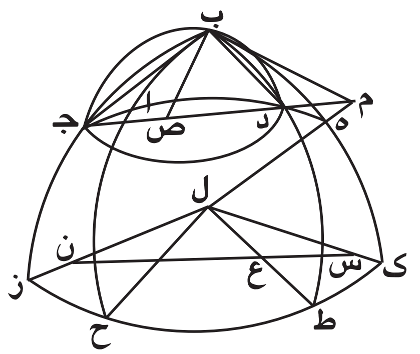

Diagram for Men. *Spherics* H.II.Lemma.3. The overall structure of this figure is based on that in **L**. In **L**, arc جـ ب د is not drawn, and in **A**, arc جـ ا د ه is not drawn. Moreover, in both cases the one circle has to serve both roles, which causes various labeling problems. The diagrams in **BA** are quite different from what we find in **L**. See Sidoli and Li [2013, 45] for a reproduction of the diagram in **A**. The diagram in **B** is peculiar. It is drawn on its own page in what seems to be a different hand than the other diagrams in the second set of figures for this manuscript (see Section 1.3.2); it is incomplete and missing labels. The diagram in **A**, which is similar, may have been reworked from such a figure following a close reading of the text.

Menelaus' introduction to Book II, **L** 99a, **B** 36b, **A** 95a, **K** 86b:

فاذ قد بيّنا على شرح المقدّمات التي يحتاج اليها، فلنعطف الآن على ما رام ثاودوسيوس تبيّنه، فبرهن عليه يقول كلّيّ من غير إن يتسلم المحال. فنبيّن خطأه، ونصلح ما افسده.

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1 دائرة ب جـ د] ب جـ د **L**. 3 لـب م] لهم **L**. 3–4 ن ل س مثل] ن لس مثل ملب ن لس مثل **A**. 5 وتر ضعف] ضعف ه د **L**. ن س] لس **L**. 7 فاذ] قال منالاوس فاذ **L**. على شرح المقدّمات] المقدّمات **L**، المقدّمات والبراهين **K**. رام] اراد **K**. 8 ثاودوسيوس] ثادسيوس **K**. تبيّنه] تبيينه **L**. يتسلم] يتسلم فيها **LK**.

Text of Men. *Spherics* H.II.10 (N.III.5), **L** 100a, **B** 41a, **A** 97b, **K** 89a:

ي اذا كان زاويتان من مثلّثين قائمتين وزاويتان حادتان ومتساويتان وكلّ واحد من الاضلاع المحيطة بالزوايا المتساوية الحادة اصغر من ربع دائرة، فانّ نسبة مجموع الضلعين المحيطين بالزاوية الحادة الى زيادة احدهما على الآخر كنسبة مجموع الضلعين المحيطين بالزاوية المساوية لها الى الزيادة التي لاحدهما على الآخر.

فليكن مثلّثي اب ه م ل ع زاويتا ب م قائمتين، وزاويتا ا ل حادتين ومتساويتين. اقول انّ نسبة مجموع ب ا اه الى زيادة اه على اب كنسبة مجموع م ل ل ع الى زيادة ل ع على ل م. فنجعل اج مساوياً لاب واد مساوياً لاب، وكذلك ل ن ل س لم ل. واقول انّ نسبة جه الى ده كنسبة ن ع الى ع س.

برهانه ان نجعل ا قطباً، ونديرٍ بضلع المربع دائرة ز ح ط ك، ونخرج ب ج ز ب ا ح ب د ط ب ه ك. فلانّ نقطة ا قطب ز ح ط ك وقسيّ اج اب اد متساوية، فاذا جعلنا نقطة ا قطباً لدائرة ببعد اج كانت موازية لدائرة ز ح ط ك، فلذلك قوس ب ج مع ضعف ج ز نصف دائرة. فاذا جعلنا ز ت مثل ز ج واخرجنا ا ح، فظاهر انّه يلقاه على ت. ونخرج اج وح ز ويلتقيان على غ. فلانّ قطب دائرة ز ح ط على د ا ج، فزاوية غ قائمة. ولذلك زاوية ح قائمة. فمثلّثي ج ز غ ح ز ت زاويتا ح غ منهما قائمتان، وزاويتا ز متساويتان، وكلّ واحدة من ح ت ج غ اقل من

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1 ي] عا *glossed* ي **L**. اذا] *enunciation*: (–) **K**. حادتان ومتساويتان] حادتين ومتساويتين **L**.
6 فليكن ... قائمتين] لىكن زاوىتا ب م من مثلّثي ا به م ل ع قايمتان **K**. مثلّثي] مثلثا **BA**. زاويتا] زاوىتى **BA**. قائمتين] قائمتان **A**. وزاويتا] وزاويتى **L**. حادتين ومتساويتين] حادتان ومتساويتان **BA**، حادتين متساويتين **LK**.
7 ب ا اه] ب ا الى اه **A**.
8 فنجعل اج] فنجعل اد اح **BA**. مساوياً لاب][1] مثل اب **K**. واد] فاد **BA**. مساوياً لاب][2] مثل اب **K**.
8–9 ل ن ل س لم ل] لن لس **B**، كن كس **A**، لن نس لم **K**.
9 واقول] اقول **BA**.
10 ان] انا **K**. نجعل ا] ا **K**. ز ح ط ك] د ح ط ك **L**، د ح ط ك **K**.
11 نقطة] (–) **K**.
12 جعلنا نقطة ا] جعلنا م نقطة ا **L**، جعلنا ا **K**. لدائرة ز ح ط ك] لز ح ط ك **K**.
13 جعلنا] *in margin* **L**. ز ت] ج ب **K**. ز ج] ز ح **K**.
14 وح ز] جز **K**. ز ح ط] ز ح ط ك **BA**.
15 فمثلّثي] فمثلثا *in margin* **B**، فمثلثا **AK**. ح ز ت] حرب **K**.
15–16 زاويتا ح غ] زاويتي ح غ **L**، ح ع **K**.
16 وزاويتا] وزاويتي **L**. وكلّ] فكل **K**. واحدة] واحد **A**. من ح ت] من قوسي ج ب **K**.

ربع دائرة، فحز مثل زغ.

ونخرج از من دائرة عظمى، واح مثل اغ، وحز مثل زغ، فزاوية حاز مساوية لزاوية غاز، فزاوية حاجـ قد انقسمت بنصفين بقوس از. وكذلك زاوية حاد تنقسم بنصفين بقوس اط. وزاويتا جاح داح مساويتان لقائمتين، فنصفهما قائمة، فزاوية زاط قائمة. وايضاً لانّ زاوية ح قائمة وزاوية ب قائمة، فنقطة ك قطب دائرة باح، ولذلك ط قطب دائرة بجز، فطز ربع دائرة وكح ربع دائرة.

ونعمل بمثلّث مل ع من إخراج من ف مل ص مس ق مع ش، ودائرة ف ص ق العظمى بقطب ل، وإخراج ل ف ل ق. فيظهر انّ زاوية ف ل ق قائمة، وقوس ف ق ربع دائرة وكذلك ش ص. فحز مثل ف ص، وجميع زك مثل جميع ف ش. فبحسب ما قدّمنا، تكون نسبة جـه الى ده كنسبة ن ع الى ع س. وذلك ما اردنا ان نبيّن.

قال الهروي: هذا هو البرهان الذي عملته لهذا الشكل والذي يومئ اليه منالاوس، يتبيّن بمقدّمات، اكثر من هذا لانّه قال انّ زا اط اذا أخرجا قسما زاويتي داح جاح بنصفين نصفين، ولم يبيّن ذلك. ثمّ ذكر من تساوي قوسي زك ف ش، وزح ف ص، وحط ق ص. ثمّ قال نسبة جـه الى هد اذا جعلنا اجـ وسطاً مؤلّفة

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[2] عظمى] عظيمةٍ **K**. واح] فاح **LBAK**. وحز] حز **A**. [3–2] حاز مساوية لزاوية غاز] جاد مثل عاز **K**. [3] فزاوية حاجـ] فحاجـ **K**. بنصفين] بقسمين **K**. زاوية حاد] زاوية داح **A**، داح **K**. [4] وزاويتا] وزاويتى **L**، وزايتا **K**. داح] داط **K**. مساويتان لقائمتين] مثل قايمتين **K**. فنصفهما] فنصفاهما **K**. [5] فزاوية زاط] فزاط **K**. وزاوية ب] وب **K**. فنقطة ك] فك **K**. [6–5] دائرة باح] دايرة اح **A**، باح **K**. [6] ط] (–) **B**. دائرة بجز] از **K**. [7] من ف] منق **K**. مل ص] منص **K**. مع ش] مس ع **BA**. [8] بقطب ل] (–) **K**. ف ل ق] فاق **K**. [9] فحز] وحر **LA**، وحد **K**. [11–10] وذلك ما اردنا ان نبيّن] (–) **BA**، وذلك ما اردنا بيانه **K**. [13–12] قال الهروي: هذا ... اكثر من هذا لانّه قال انّ] هاذا الشكل هو عمل الهروي واما منالاوس فانه عمل هاذا **K**، يتبيّن: س **A**. [13] زا] دا **K**. [14] يبيّن] قال احمد بن السري بيان ذلك ظاهر من عكس شكل ل من ا وذلك بان نخرج ب ح بط حتى تلتقيا فيكون دا اح وتمام اح المخرج الى تلقا نضف دايرة وقوس اط ربع دايرة وكل مثلث يكون ضلعاه مختلفين وهما نضف دايرة وتكون القوس المخرجة من الزاوية التى تحيطان بها الى قاعدة ربع دايرة فانها تقسم الزاوية والقاعدة بنصفين فزاوية داط مثل زاوية طاح *in margin* **BA**، *partly illegible* **B**، احمد بن السري: سربا رصى الله عن (؟) **B**. من] (–) **BA**. [15] وزح] ودح **K**. ق ص] مق **L**.

من نسبة هج الى جا ومن نسبة جا الى هد، اعني دا الى هد. قال: ونسبة هج الى جا كنسبة كز الى زح، ونسبة دا الى ده كنسبة طح الى طك، ونسبة كز الى زح هي نسبة حط الى طك. ويلزم في الشكل الثاني، كذلك من نسبة نع الى نل ونسبة لس الى سع. وذلك ما اردنا ان نبيّن.

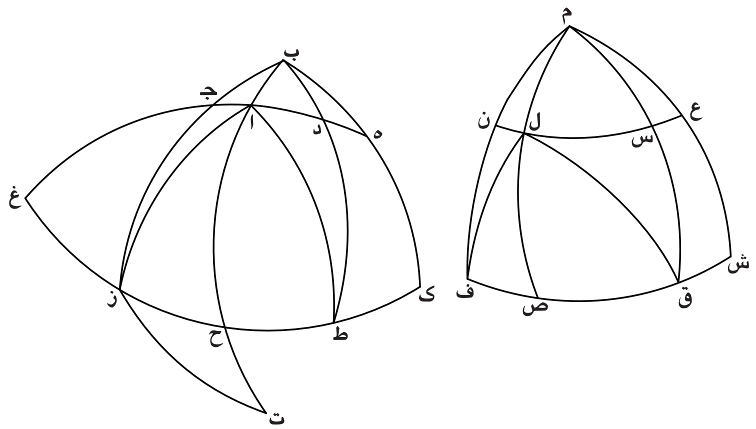

Diagram for Men. *Spherics* H.II.10 (N.III.5). We have based our diagram on that in **L**. In **B**, the diagram has been taken from the uncorrected diagrams and is problematic— arcs زجه, كز, بز, and بح do not extend beyond the spherical triangle كبز, and the letter names do not correspond to the text. In **A**, circle هاج curves upward, so as to appear parallel to circle كطهز, while arcs هجغ and بحت do not appear to be continuations of the same arc, but rather appear to be broken at ج and ح. In **K**, arc علن is curved downward and extends beyond spherical triangle شمف so that it meets شف extended at an unnamed point, غ is labeled as ع along with the actual ع. (It is referred to as ع in the text as well.) In **BA**, there is an additional diagram for the version attributed to Menelaus. In **B**, this is a single figure, labeled in a way that does not correspond with the present text. For the diagram in **A**, see Sidoli and Li [2013, 51].

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1 من نسبة هج] نسبتي هج **K**. ومن نسبة جا] وجا **K**. اعني دا الى هد] (–) **B**. قال] (–) **K**. 2 زح] دح **K**. ونسبة دا الى ده] فنسبة دل الى دح **K**. كنسبة طح] طح **B**. 3 هي نسبة حط الى طك] هي نسبة كز الى زح هي نسبة حط الى طك **K**، هي نسبة حط: هي كنسبة حط **BA**. 4 نع] لع **K**. سع] سع وهذا صحيح لكنه يتبيّن بمقدّمتين والذي ذكرنا بمقدّمة واحدة فامّا اقسام زاويتي حاح نص بنضفين فانّه يتبيّن ما قدّمنا **L**. اردنا ان نبيّن] اردنا بيانه **K**.

Menelaus' historical remark, **L** 105a, **B** 49b, **A** 102b, **K** 93a:

قال منالاوس قد تبيّن هذا الشكل على خلاف ما ذهب اليه ثاوذوسيوس في المقالة الثالثة من كتابه في الكريّات حيث رام ان يبيّن انّ جح له نسبة الى ده اصغر من التي لقطر الكرة الى قطر دائرة دا. ابلونيوس يستعمل هذا في كتابه في الصناعة الكلّيّة والذي يتبيّن من بعد هو نافع جدّاً فيما يستعمله ابلونيوس، وهو ان يبيّن انّ نسبة جح الى ده اعظم من نسبة ما واصغر من نسبة ما.

Acknowledgement

We would like to thank Sonja Brentjes and the anonymous referees of this journal for a number of suggested improvements. The early stages of our collaboration was made possible by grants from the Japan Society for the Promotion of Science and Waseda University.

[1] قد] فقد **BK**. [2] الكريّات] الأكر **K**. جح] ~~نسبة~~ جح **K**. [3] دا] زا **K**. ابلونيوس] ثاوذسيوس **L**. [4] هو] هاذا **K**. ابلونيوس] ثاوسيوس **L**. [5] ما][1] ما **L**. ما][2] ما **L**، نا **A**.

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