

The manuscript diagrams of al-Harawī's version of Menelaus' *Spherics*

Research report for “Databasing the manuscript diagrams of sources in
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Introduction

This report presents the medieval diagrams of al-Harawī's version of Menelaus' *Spherics*. The images are .eps files that were produced using the program DRaFT, based on the following manuscripts:

L: Leiden, Universiteit Leiden Or. 399, sec. 2, ff. 82v-105v. 539 AH (1144 CE).

A: Istanbul, Saray, Ahmet III 3464, sec. 5, ff. 74v-103r. 625 AH (1227 CE).

K: Private collection, sold by H. P. Kraus, sec. 5, ff. 71v-94r. 7th c. AH (13th c. CE).

The **L** and **A** manuscripts both contain copies of al-Harawī's version of Menelaus' *Spherics*, while the **K** manuscript contains a text that is closely based on this. There is textual evidence that the author of **K** had access to at least two sources, one of which was similar to **L** and the other of which was similar to **A**.¹

It would be interesting to know if it is possible to show a closer relationship between **K** and either of **L** or **A** on the basis of the diagrams alone. This is, in principle, an important question because it raises the issue of how one traces relationships of dependency between manuscript diagrams. We will argue, however, that, in the case of these three manuscripts, it is not possible to be certain that the author of **K** actually saw figures similar to those in both **L** and **A**. Nevertheless, it will also become clear that the author

*The official title is 古代中世数学文献の図版のデータベース化.

¹The full argument for this will be published by N. Sidoli and T. Kusuba, who are currently studying the relationship between these three manuscripts.

of **K** did not simply see the figures in one of the other two traditions, and copy these. Indeed, there were so many independent choices made by the author of **K**, that, given such a small set of manuscripts, it is no longer possible, on the basis of the diagrams alone, to determine any relationships of strict dependence.

First of all we will divide all cases in which **K** agrees with, or differs from, the other manuscripts into two types of relations, *mathematical* and *visual*, on the tentative assumption that the copyists would have tried to preserve the mathematical relations but may have been more ambivalent about the need to preserve purely visual features. Unfortunately, even this basic division is somewhat problematic. For example, in I 5, **K** has an extra line, which is a well represented type of visual relation. In this case, however, there may be a valid mathematical reason for introducing this new line, since it is a, perhaps more, mathematically sound way of representing one of the objects in the figure (see Fig. ??). Another example of these difficulties may be shown by a consideration of Fig. ?? (I 57). In this case, **K** mostly agrees with **L** in the labeling and visual layout, since the diagram in **A** is fairly sloppy and some letter names are missing. The ordering of the points on the base, however, is different in each of the figures, and the ordering of the points could be regarded as a mathematical feature of the diagram. So, in this regard, we would say that **K** is mathematically independent of the other two. A final example is Fig. ?? (II 3). In this case, all three of the diagrams are different, showing a set of four arcs either intersecting in a single point (**K**), not intersecting at all (**A**), or three of them intersecting pairwise with the fourth (**L**). These might be construed as mathematical differences, but since the points are unnamed and do not enter into the mathematical argument, the scribes may have considered these as purely visual differences. Nevertheless, despite these various ambiguities, we will use this classification to describe the relations between these three manuscripts.

Within the category of mathematical relations, we can recognize certain general types, such as labeling, ordering of points, positioning of lines, and differentiation into cases. In terms of labeling, we have one case where **K** agrees with **A** (I 58), and one case where it agrees with **L** (II 1). There is one clear case in which **K** has been labeled twice, such that it includes the divergent labeling of both **L** and **A** (I 18). This may be the only direct piece of diagrammatic evidence that can be used to argue that the author of **K** was working with manuscripts in two traditions similar to **L** and **A**. There is also one ambiguous case, in which a point in **K** has been labeled twice, but the corresponding point is missing in **L**, so it is not certain that **K** is meant to represent two manuscript traditions (II 12). With regards to the ordering of points, we have two cases where **K** is independent (I 40, I 50), but in neither of these is the ordering found in **K** unacceptable from a mathematical perspective. As for the positioning of lines, we have one case where **K** agrees with **A** (I 51), and one case where it agrees with **L** (II 17), but in both cases the other diagram is incorrect, so this does not show any necessary dependence, since the author of **K** could have simply corrected the diagram. There is one case in which the diagram in **K** is independent and is correct (I 53), and four cases in which it is independent but incorrect (I 55, I 60, II 5, II 9).

The category of visual representation is more vague and there is little reason to believe that the scribes of medieval texts saw their task as one of accurate visual reproduction. We can divide the visual relations into general layout or orientation, direction of curvature,

and the presence of extra, unnecessary arcs. In seven cases **K** has the same orientation as **L** (I 10, I 12, I 15, I 22, II 6, II 7, II 13), in four cases it has the same orientation as **A** (I 4, I 11, I 32, I 48), and in six cases it has an independent orientation (I 6, I 8, I 14, I 16, I 31). In six cases **K** has the same direction of curvature as **L** (I 12, I 34, II 8, II 16 (but **A** was changed), II 18, III 3), in two cases it has the same curvature as **A** (I 20, III 8), and in three cases it is independent (I 10, I 28, I 30). Finally, in five cases, **K** has extra, unnecessary lines (I 38, I 39, I 40, I 53, II 10). The difficulty with judging the significance of these relations lies in the fact that it is not possible to establish any particular choice as more or less justified, so that any similarities or differences can simply be attributed to the whim of the copyists.

The overall impression of this analysis is (1) that, in terms of the diagrams alone, **K** shows a high degree of independence, both correctly and incorrectly, (2) that **K** is not definitively more closely related to either one of the other two manuscripts although it has a tendency to be closer to **L**, and (3) that although it is possible that the diagrams in **K** was made on the basis of traditions similar to both **L** and **A**, this is not demonstrably the case.

Scale

The relative scale of each individual manuscript has been preserved in our reproduction. That is, if one compares two figures from, say, **L** they will have the same sizes relative to one another in our reproduction as they do in the original manuscript. We have made no attempt, however, to preserve the relative scale across different manuscripts. Instead, we have made the first diagram in all three manuscripts about the same size in our reproduction (see Fig. ??).

Conventions

In the manuscripts, lines are generally drawn with an instrument such as a compass or ruler. In **A**, the lines are in red ink, with the lettering in black. In **K**, the lines are in black ink, while the lettering is in red. In **L**, both the lines and the lettering are in the same black ink as the text. In our reproduction, we use only black ink.

Standard lines: We use a standard line weight to represent lines that are drawn with a drawing tool, such as a compass.²

Thick lines: We use a line of double weight to represent lines that are drawn by hand, probably at a later time and usually in the same color ink as the text.

Dotted lines: We use dotted lines to represent lines that have been erased. This erasure may have been done by the original copyist, or at a later date.

²In Menelaus' text, the only diagram that includes straight lines, is that for II 5. Al-Harawi's version, however, contains a number of figures employing straight lines in the preface to book II.

Dots: A point is sometimes drawn using the brown or black ink of the letter names. We have used a small dot to represent this (see, for example, Fig. ?? (a)).³

Letter names of points

The points of the diagrams are labeled by letter names, following the usual convention. The following variation on the Arabic abjad order is used: ا, ب, ج, د, هـ, ز, ح, ط, ك, ل, م, ن, س, ع, ف, ق, ش, ت, غ — although it should be noted that only one diagram, II 10, has letter names beyond ق.

Some letters are occasionally omitted: د, II 3; ز, I 5, I 36, II 1, II 14; ط, I 19, I 20, I 21aL, I 21, I 53; and ن, II 3, II 4. The figure for II 1 is missing either د or ز, depending on which manuscript one follows. Occasionally, a letter is added: ر, I 21A; ش, I 21aL. The final letters are sometime out of order. In I 55, although there is no س, ع or ص, the final two letters are ف and ق. In I 56, we find ص in place of ع; in II lemma 3, ص in place of ف; and in II 9 and 10, م in place of ل.

The placement of the letter names in our reproduction is rather loose. This is primarily due to the fact that the scribes exercised a fair degree of freedom with regards to the shape of each individual letter, whereas we have employed a standardized form for each occurrence. Hence, deciding where, exactly, to place the letter is sometimes arbitrary.

Proposition numbers

The manuscripts contain a few extraneous figures. Hence the figure numbers do not generally agree with the proposition numbers. Moreover, in a few places, in order to save space on the page, we have printed the figures in the wrong order.

³A dot is generally used to denote a point that is not found at an intersection, such as in Figs. 5 (a), 11 (b), or 43 (a) and (b). In A, however, dots are also used in the three dimensional diagram for II 5, to differentiate between lines which are actually intersecting and lines which only appear to be intersecting. The use of dots is then continued for the next three propositions — II 4, II 6 and II 7 (in A, II 4 is drawn after II 5) — until the scribe apparently lost interest.

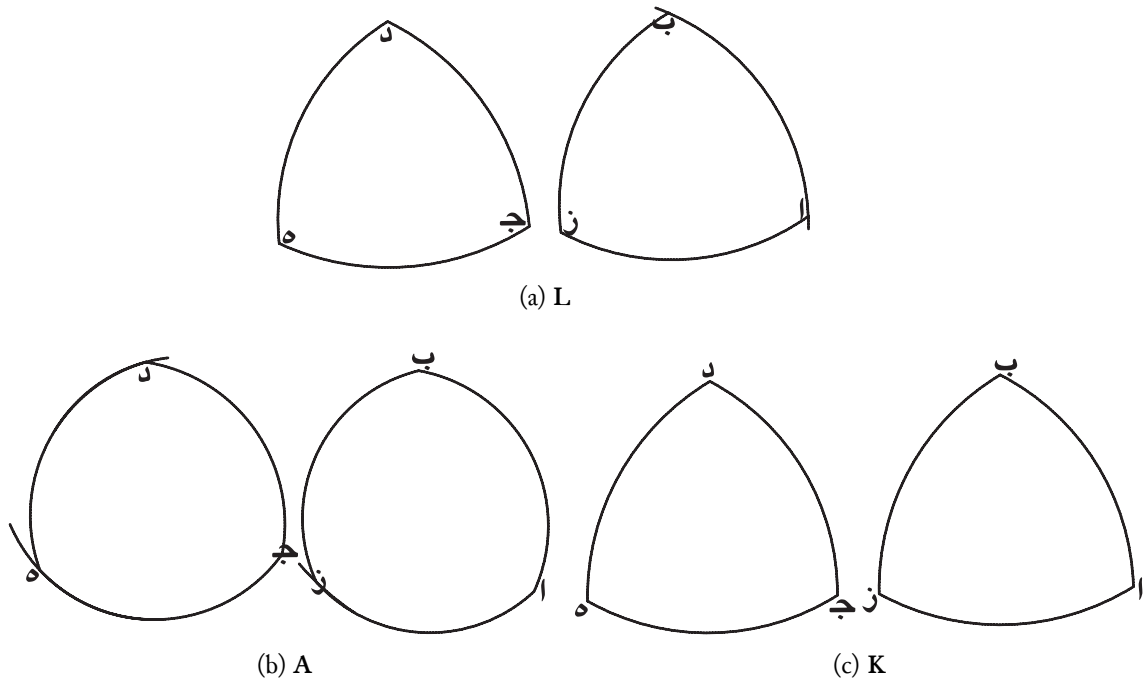


Figure 1: Book I, Prop. 1

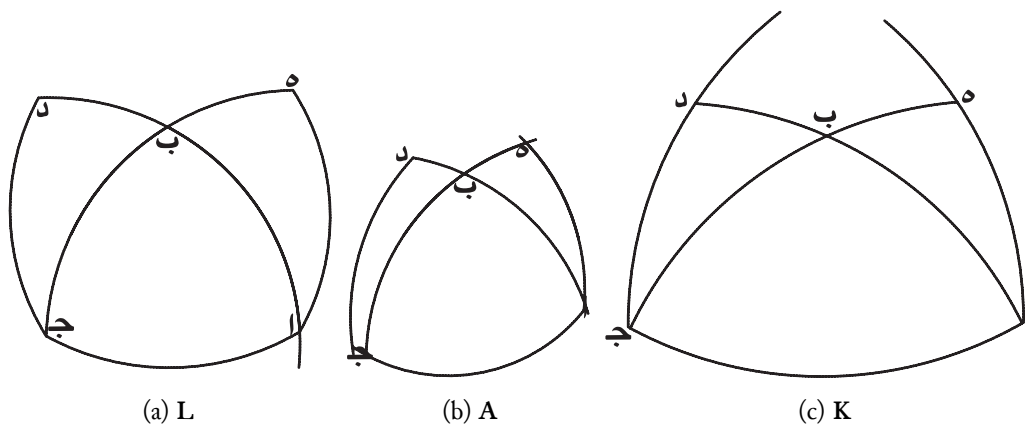


Figure 2: Book I, Prop. 2

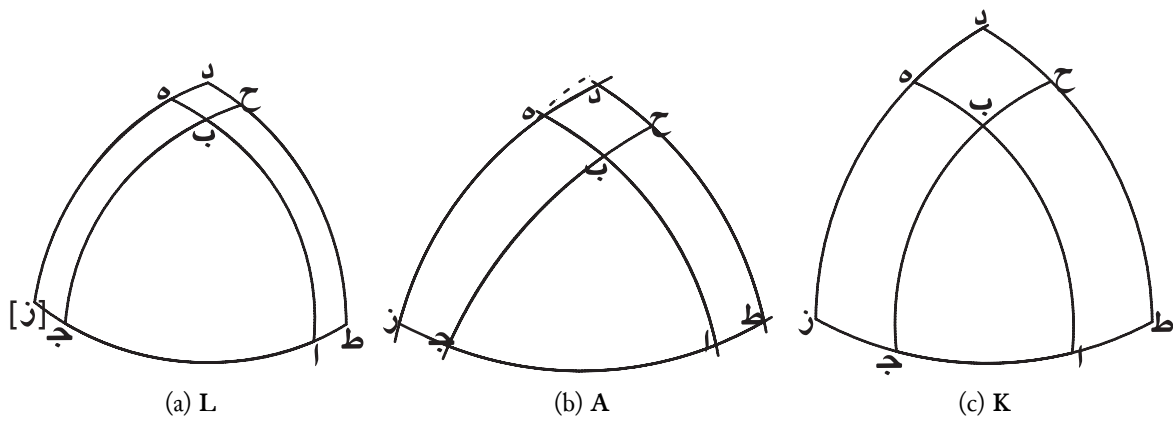


Figure 3: Book I, Prop. 3

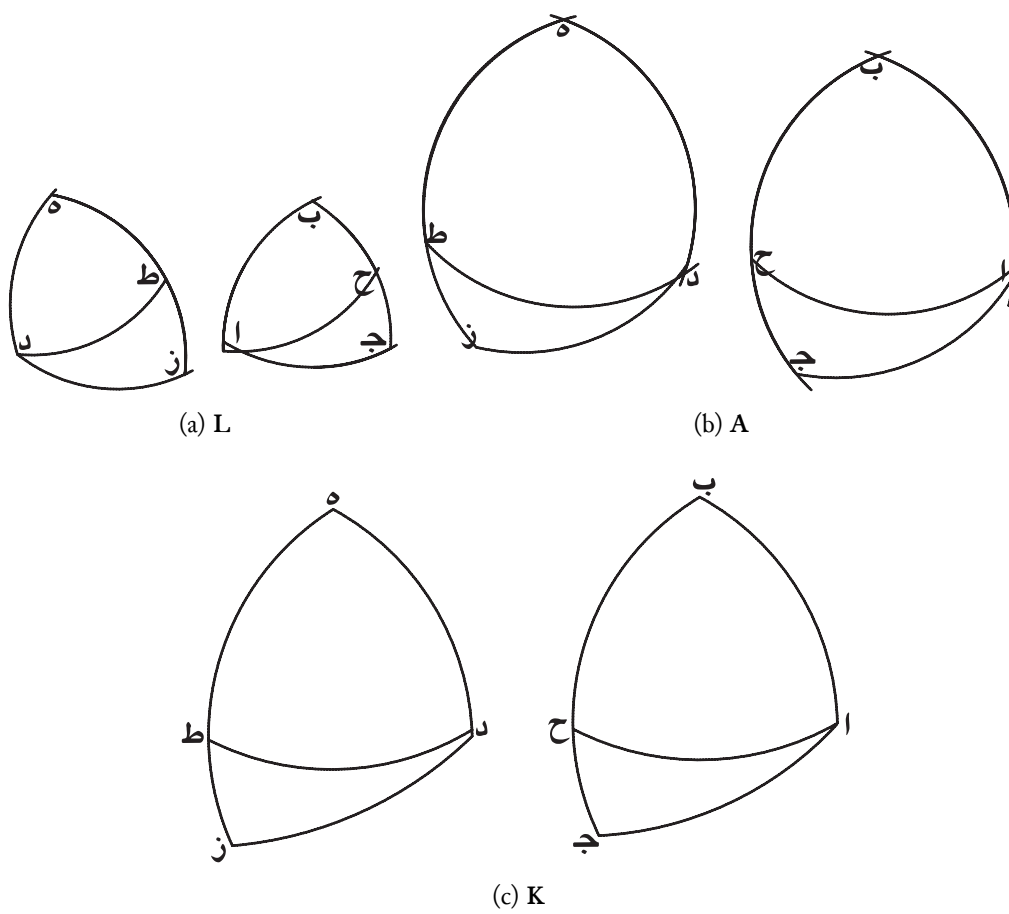


Figure 4: Book I, Prop. 4

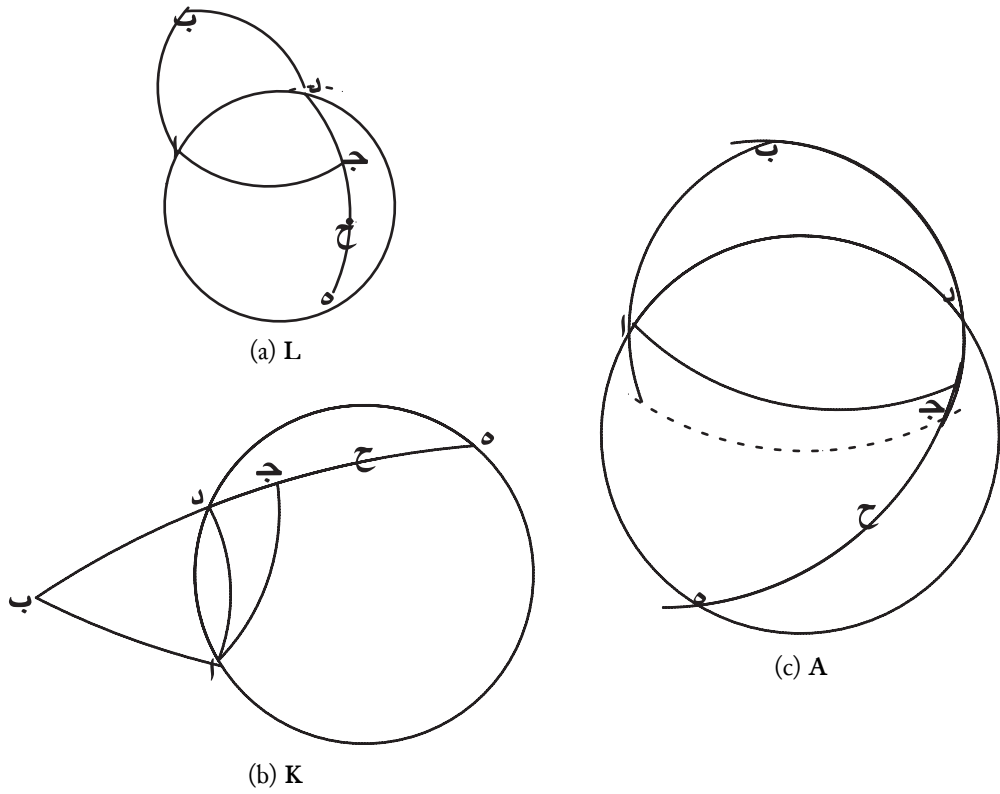


Figure 5: Book I, Prop. 5

The arc $\overline{\alpha\beta}$, in **K**, is probably not an error. By construction, circle $\overline{\alpha\beta}$ is drawn about pole $\overline{\alpha\beta}$, but it is difficult to see this because the circle has been rotated out and set in the plane of the figure. Hence, the copyist of **K** probably included arc $\overline{\alpha\beta}$ as another image of circle $\overline{\alpha\beta}$ — one which will look more accurate although it will not display the arrangement of points needed by the proof.

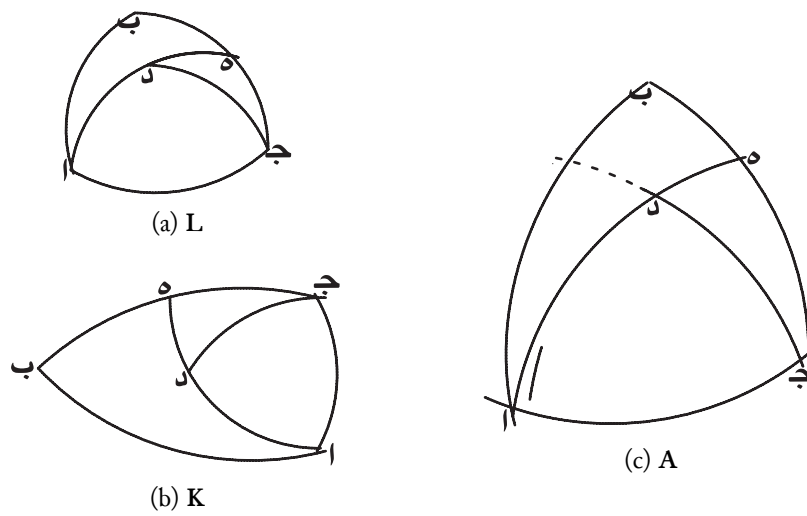


Figure 6: Book I, Prop. 6

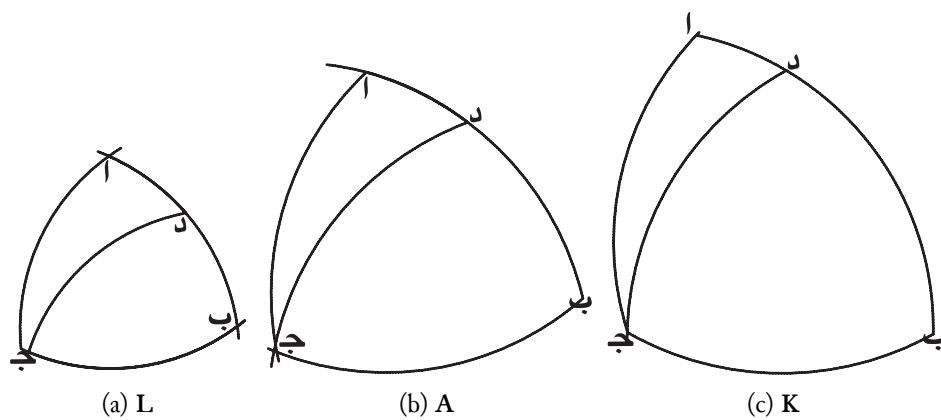


Figure 7: Book I, Prop. 7

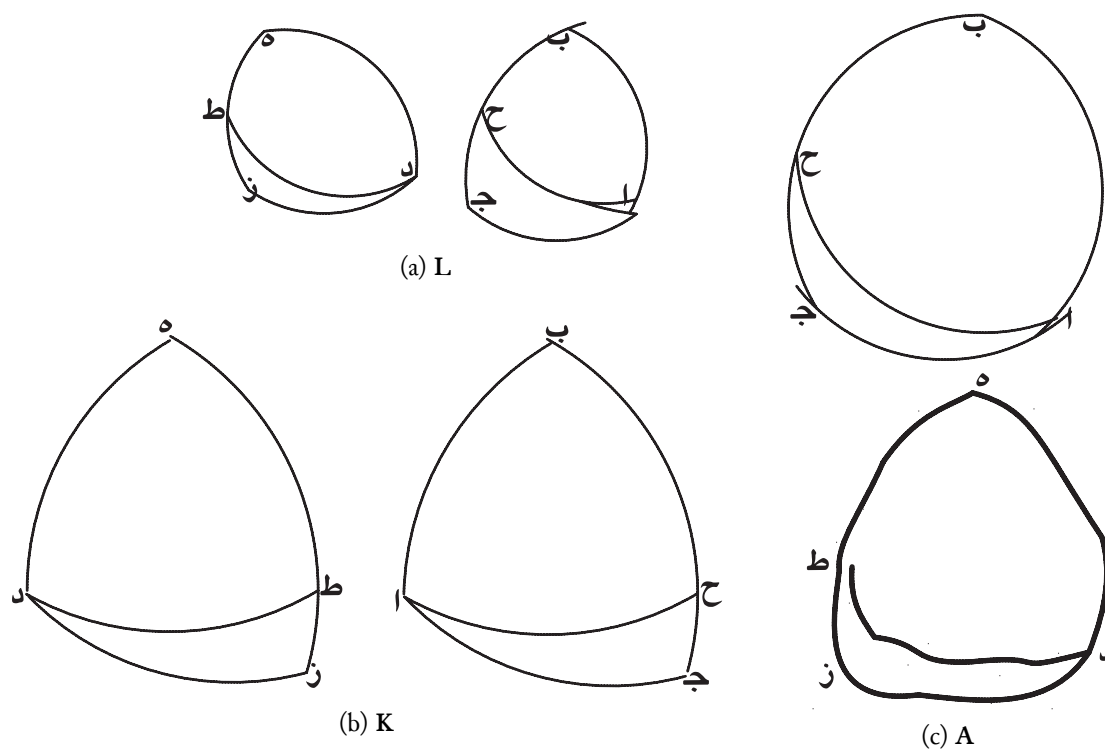


Figure 8: Book I, Prop. 9

Diagram (c) A has been drawn in brown ink by a later hand, extending into the lower margin of the page.

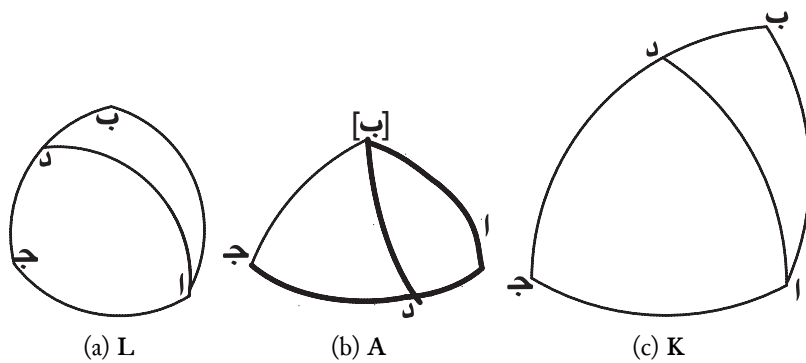


Figure 9: Book I, Prop. 10

Diagram (b) A has been drawn in brown ink by a later hand, squeezed into the top margin of the page.

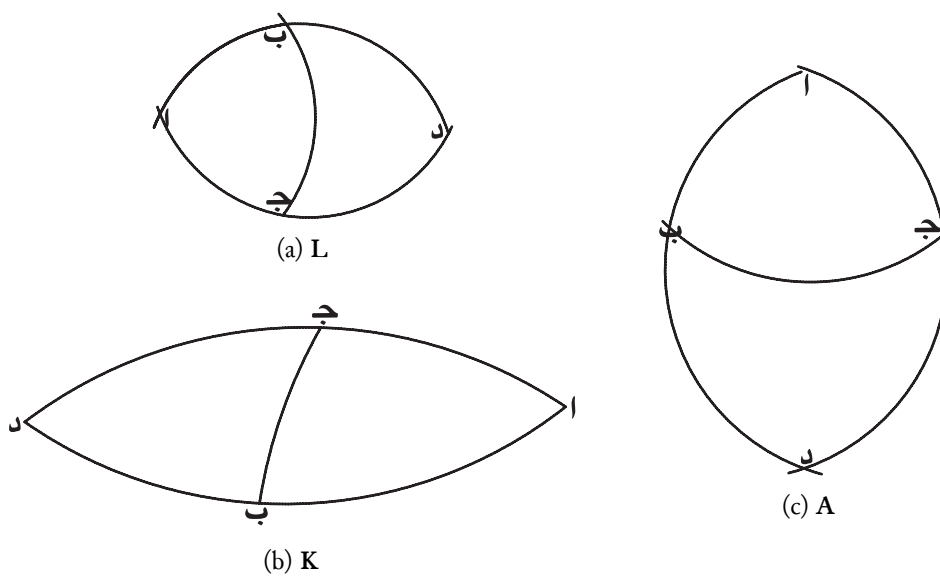


Figure 10: Book I, Prop. 11

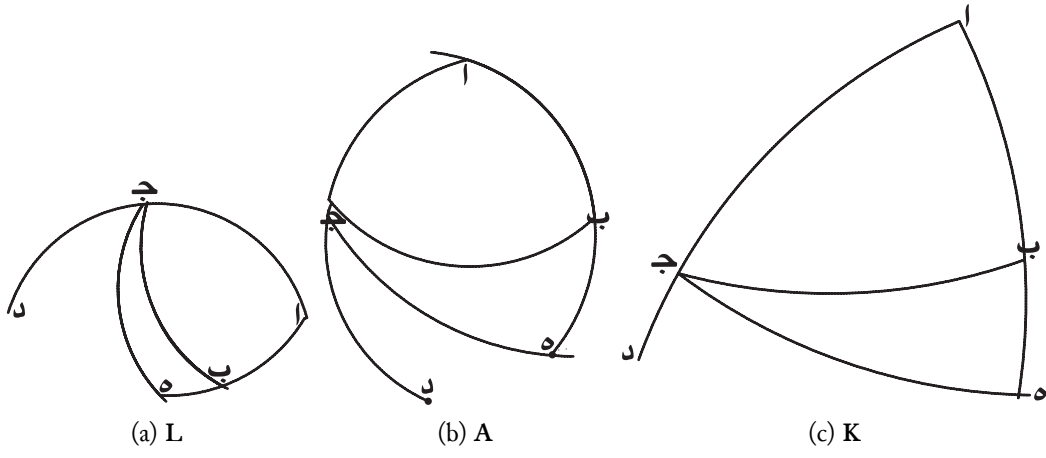


Figure 11: Book I, Prop. 12

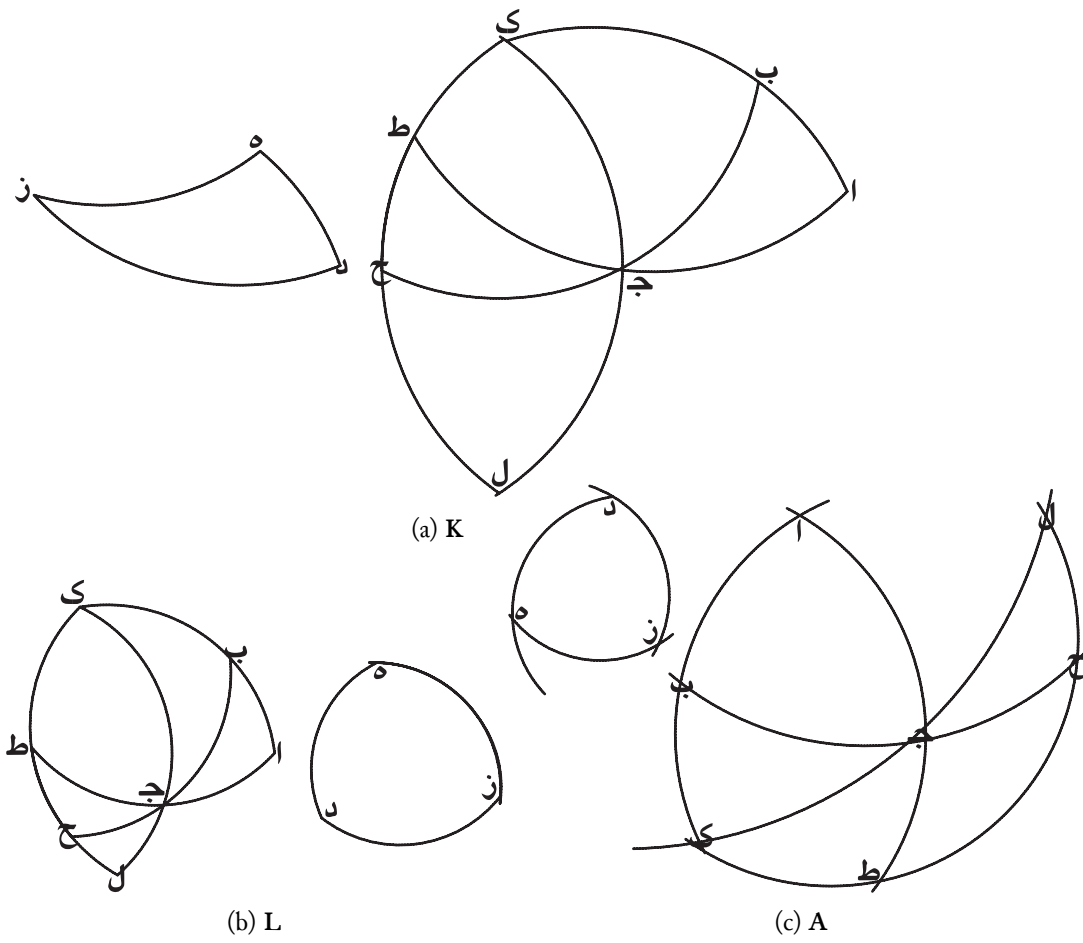


Figure 12: Book I, Prop. 13

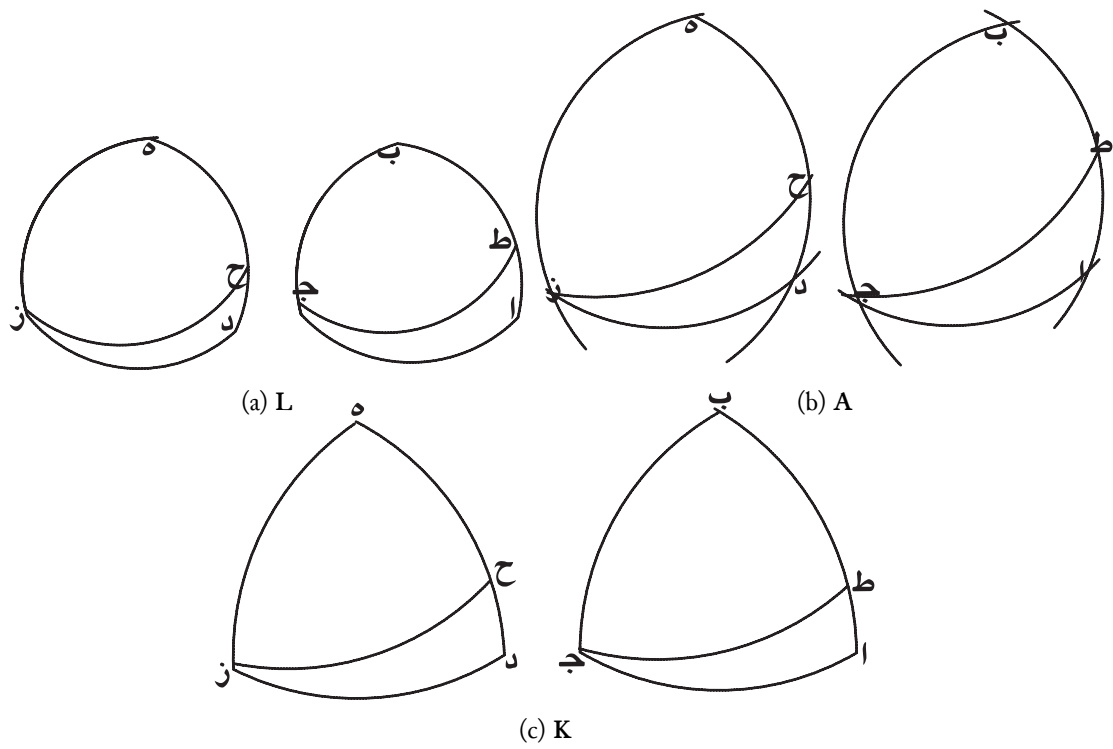


Figure 13: Book I, Prop. 14

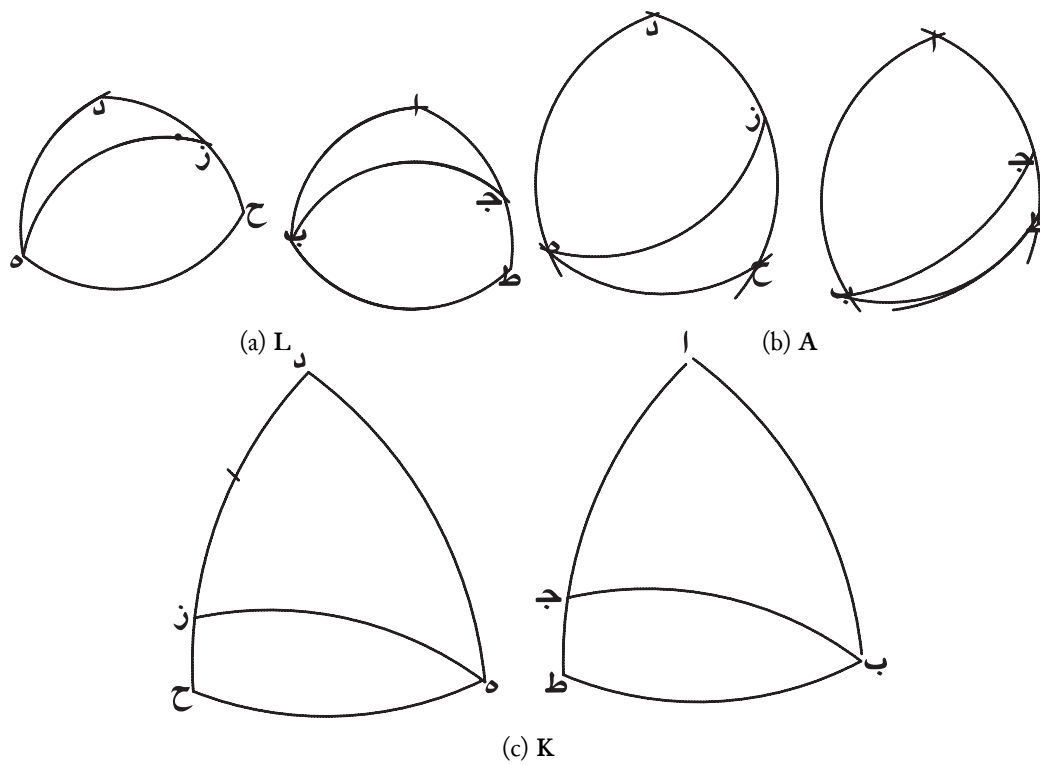
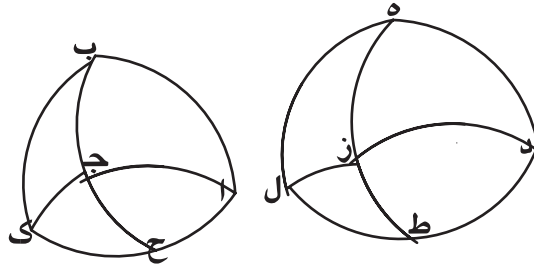
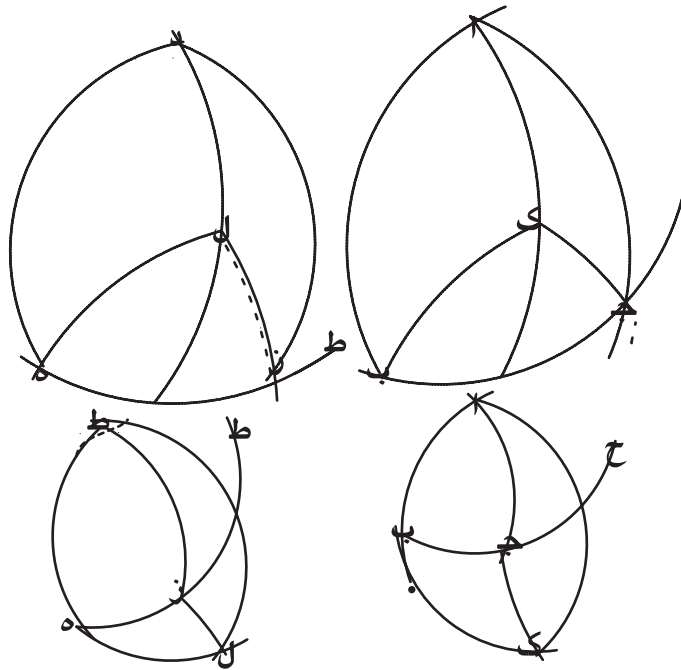


Figure 14: Book I, Prop. 15

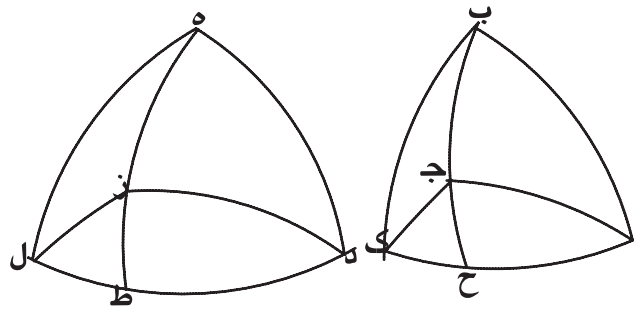
Notice that the figure in **K** has a different orientation then that in either of the other manuscripts.



(a) L



(b) A



(c) K

Figure 15: Book I, Prop. 16

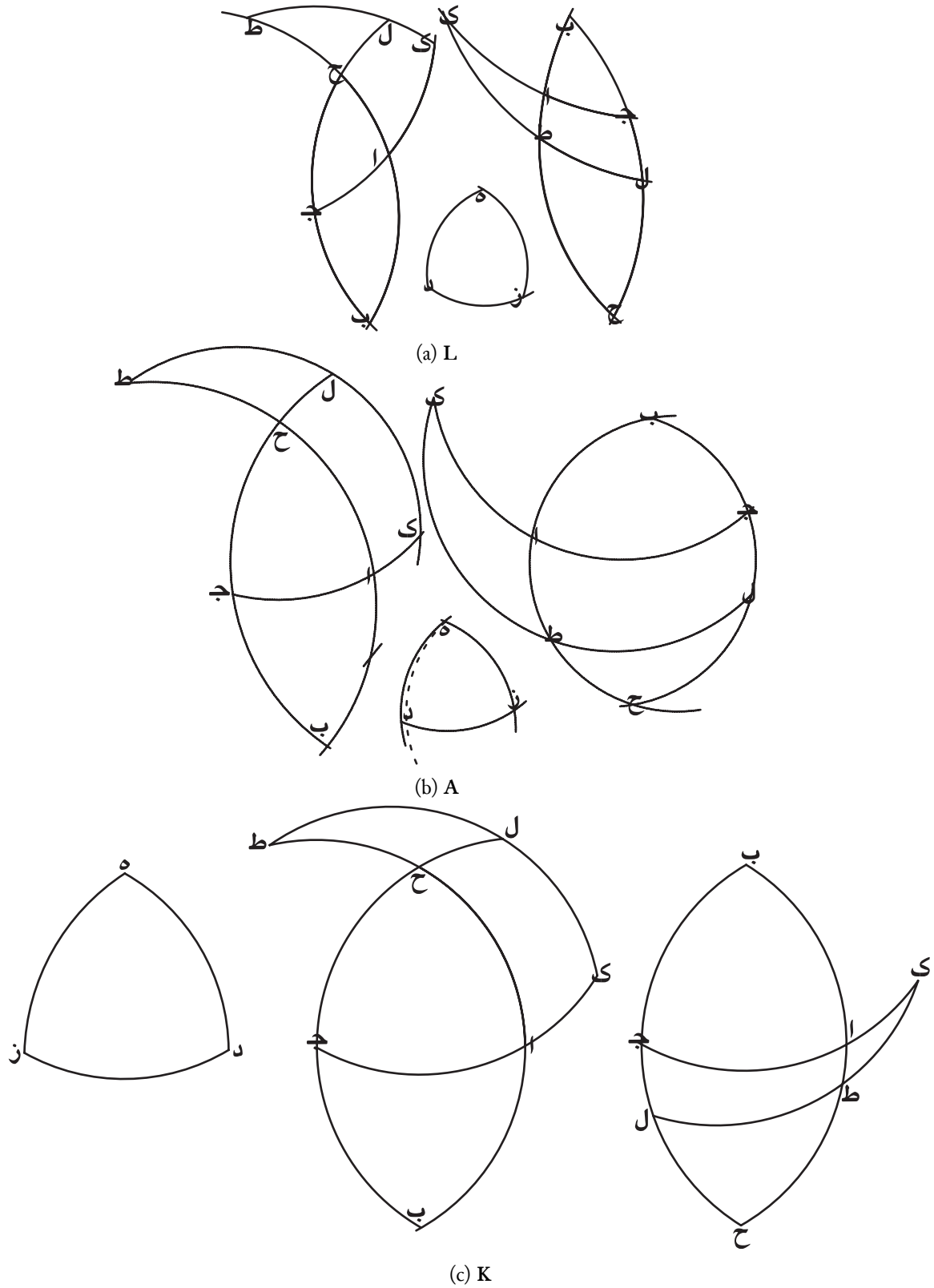
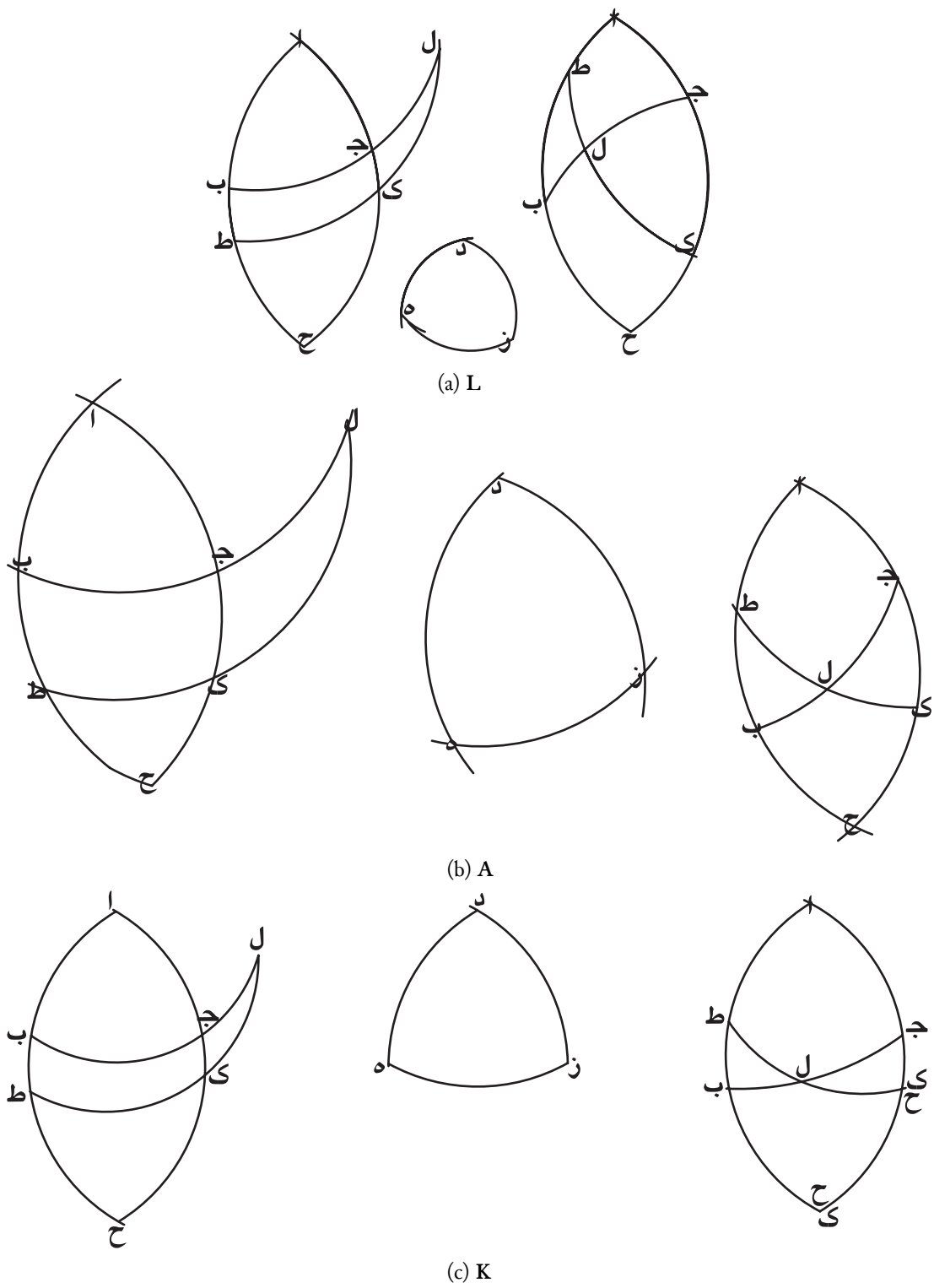
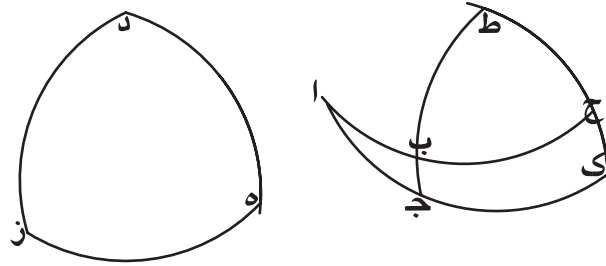


Figure 16: Book I, Prop. 17

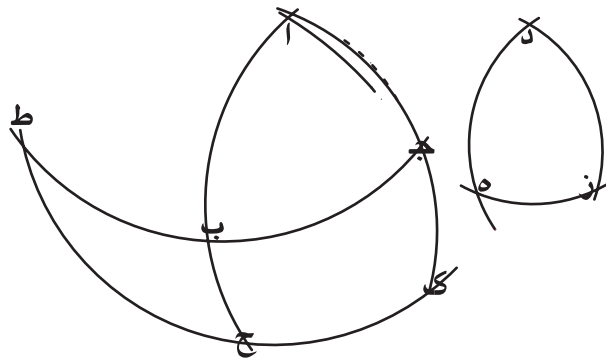
In L, the lower part of the figure on the right contains the word “first”
 (الاولى) and that on the left contains “second” (الثانية).



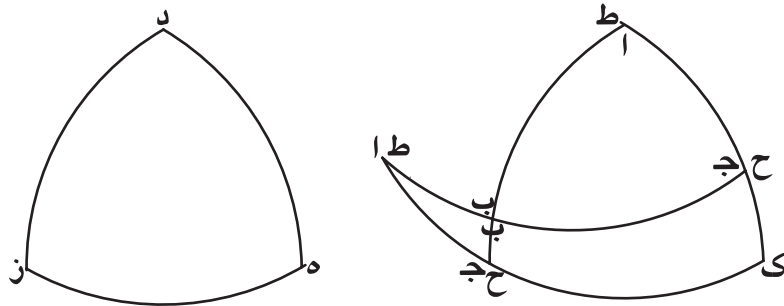
(c) K
Figure 17: Book I, Prop. 18



(a) L



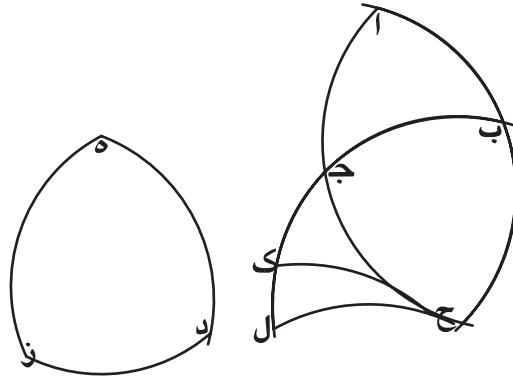
(b) A



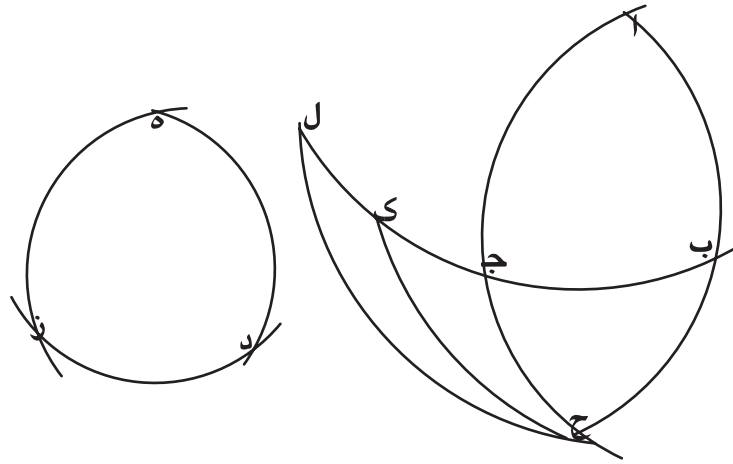
(c) K

Figure 18: Book I, Prop. 19

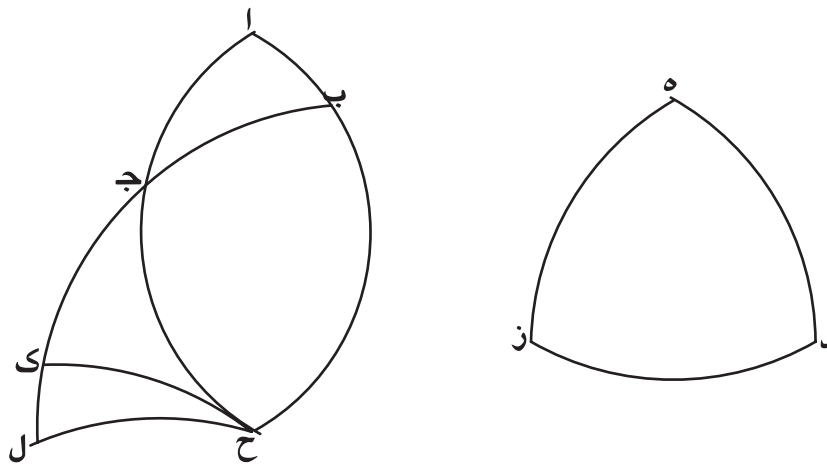
The copyist of **K** appears to have had access to two figures, one of which was labeled as that in **L** and the other as that in **A**. Another possibility is that the labeling was corrected independently so as to agree with the text, but this would not explain why φ had been labeled twice.



(a) L

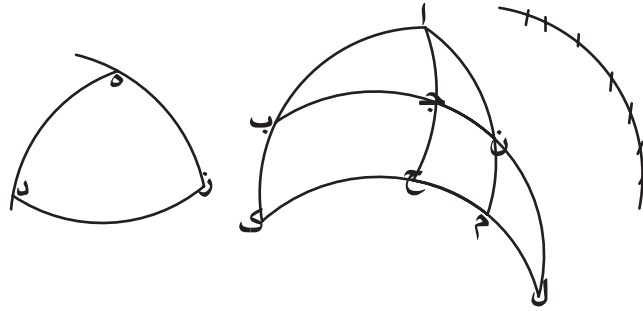


(b) A

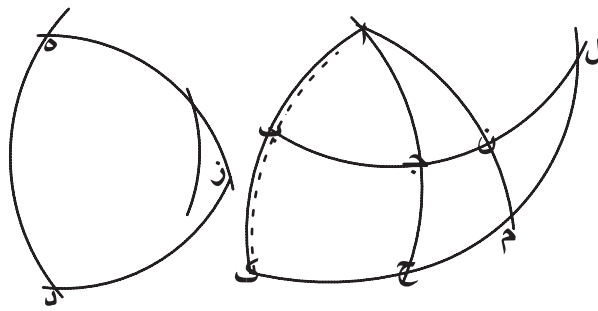


(c) K

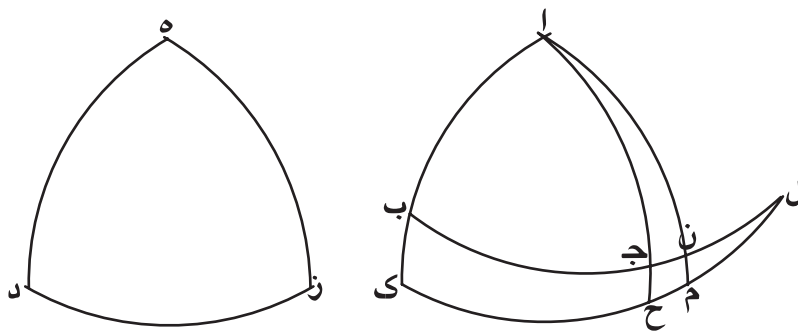
Figure 19: Book I, Prop. 20



(a) L



(b) A



(c) K

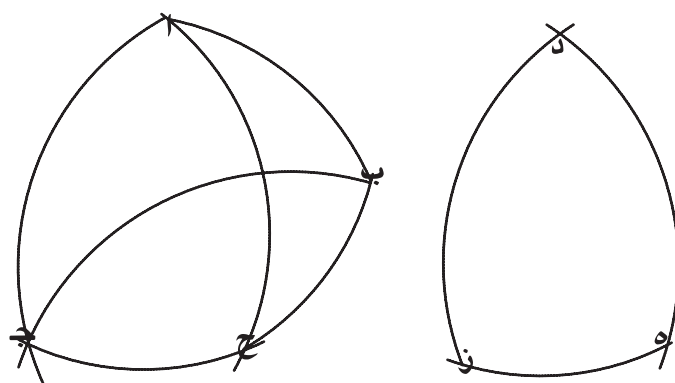
Figure 20: Book I, Prop. 21



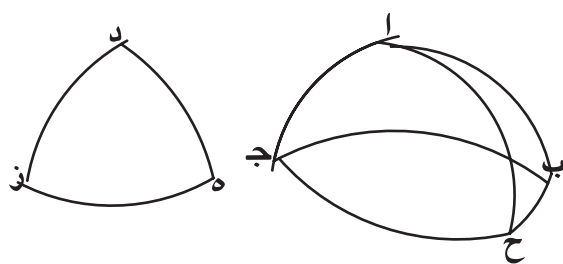
In L, there is an extra figure with the words “this figure is the interpretation, 22” (هذا الشكل الترجمة ٢٢), written between the two drawings. The unusually large spacing between the two parts of the diagram, which gives ample room for this critical remark, may be an indication that the remark was added at the same time as the figures were drawn.



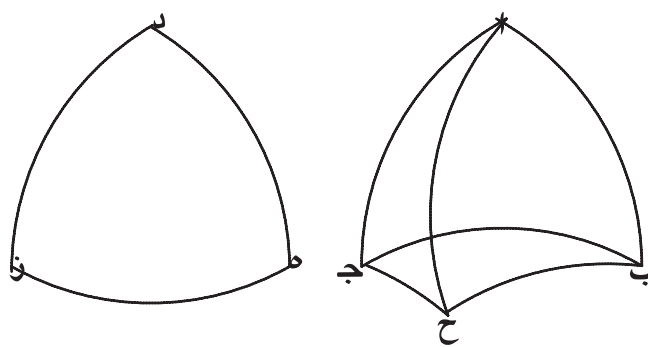
18



(a) A



(b) L



(c) K

Figure 23: Book I, Prop. 23

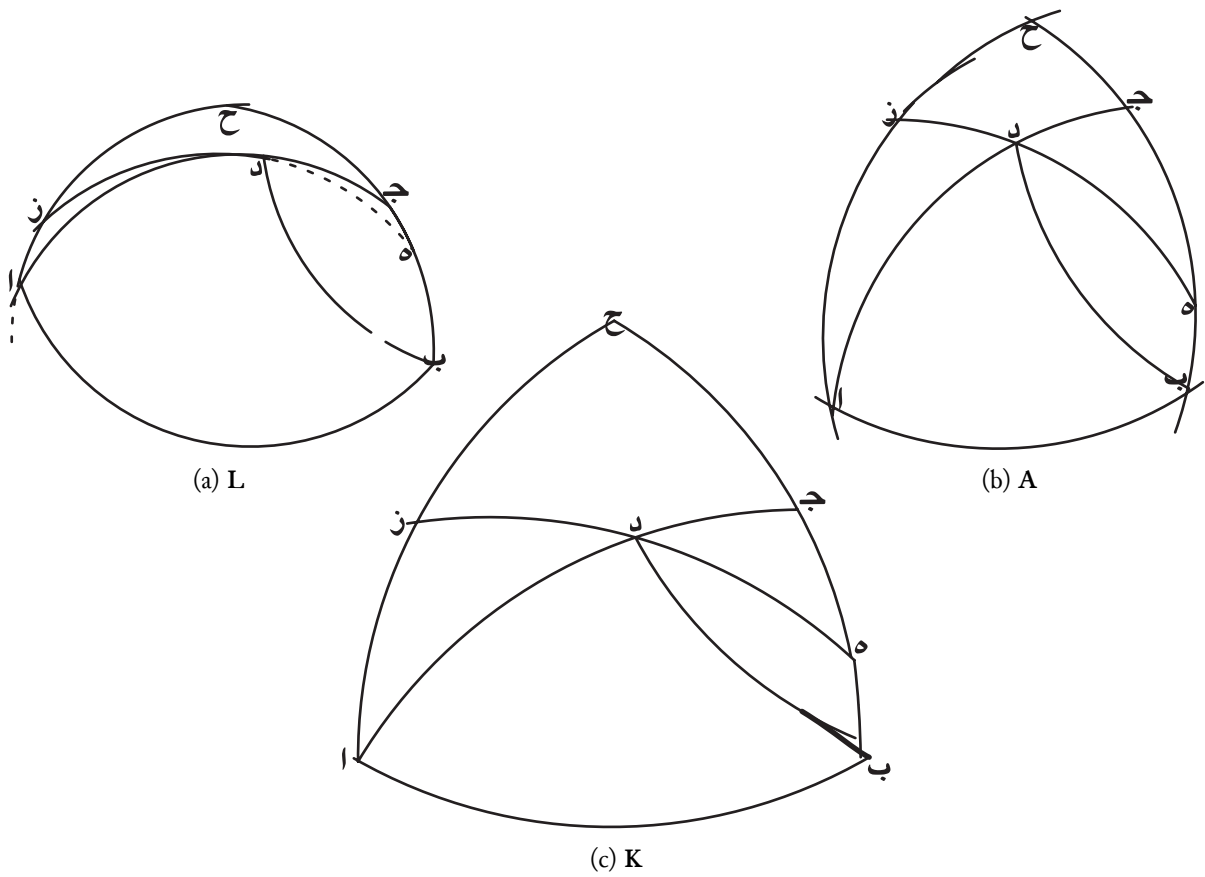


Figure 24: Book I, Prop. 24

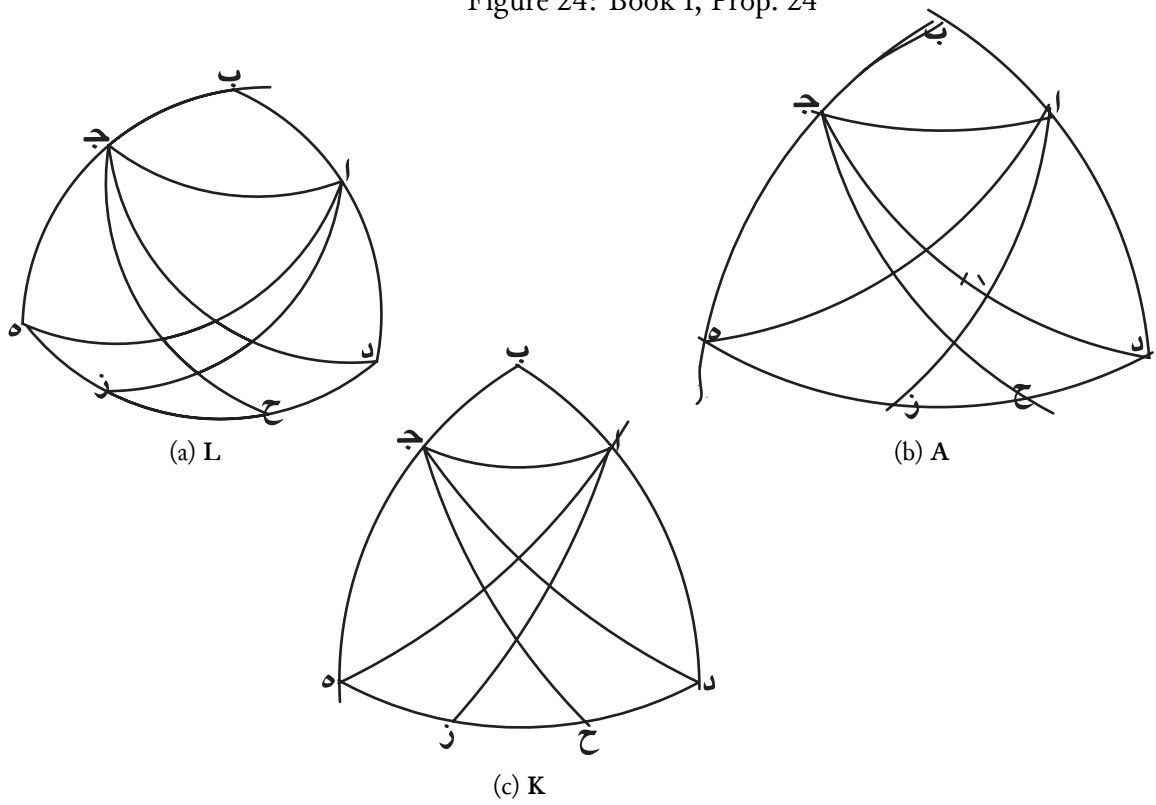


Figure 25: Book I, Prop. 25

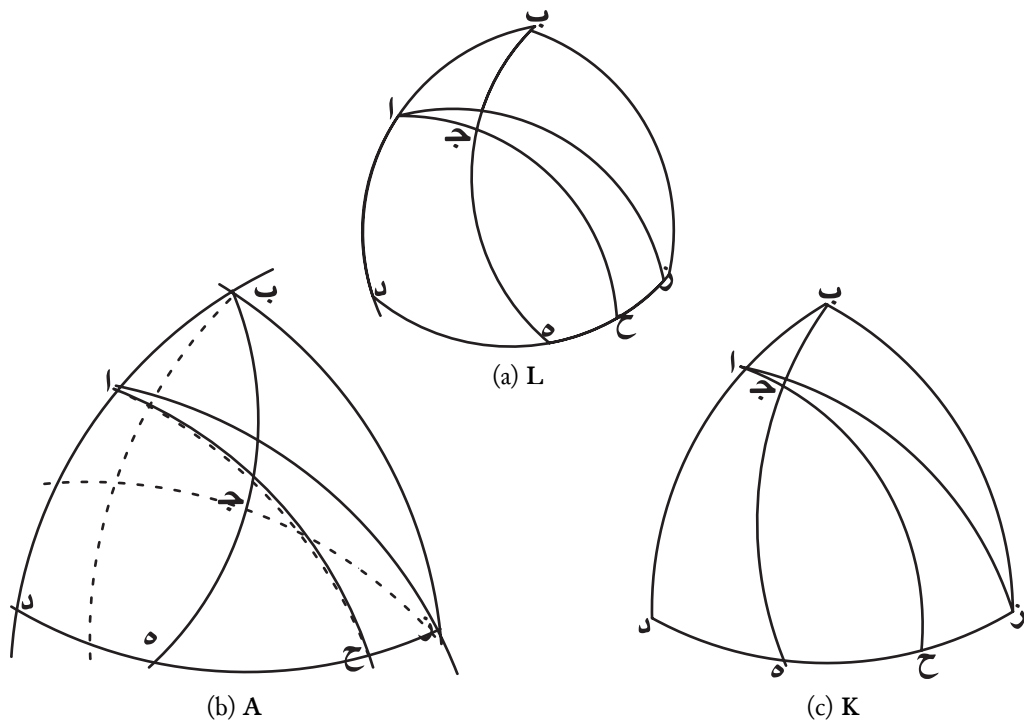


Figure 26: Book I, Prop. 26

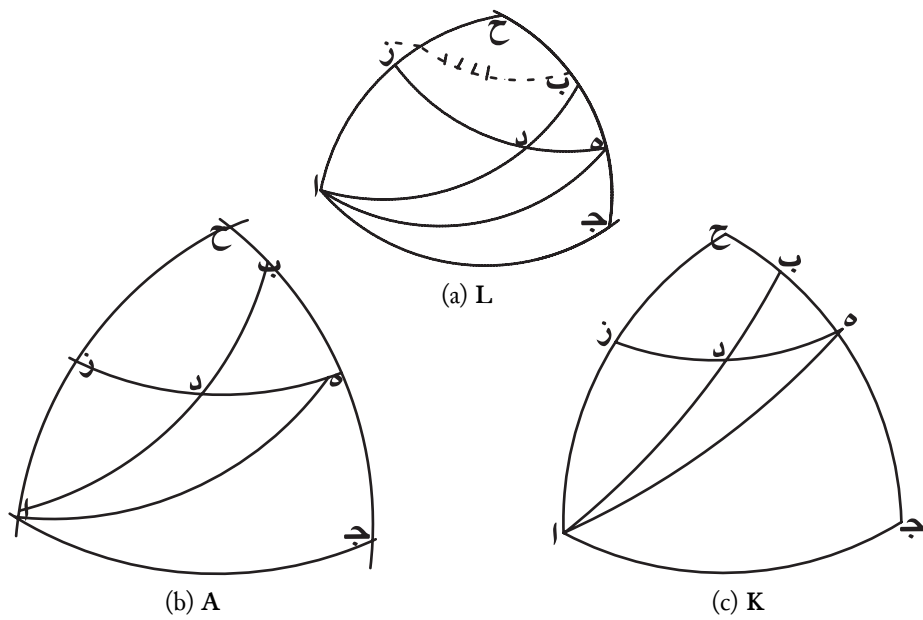


Figure 27: Book I, Prop. 27

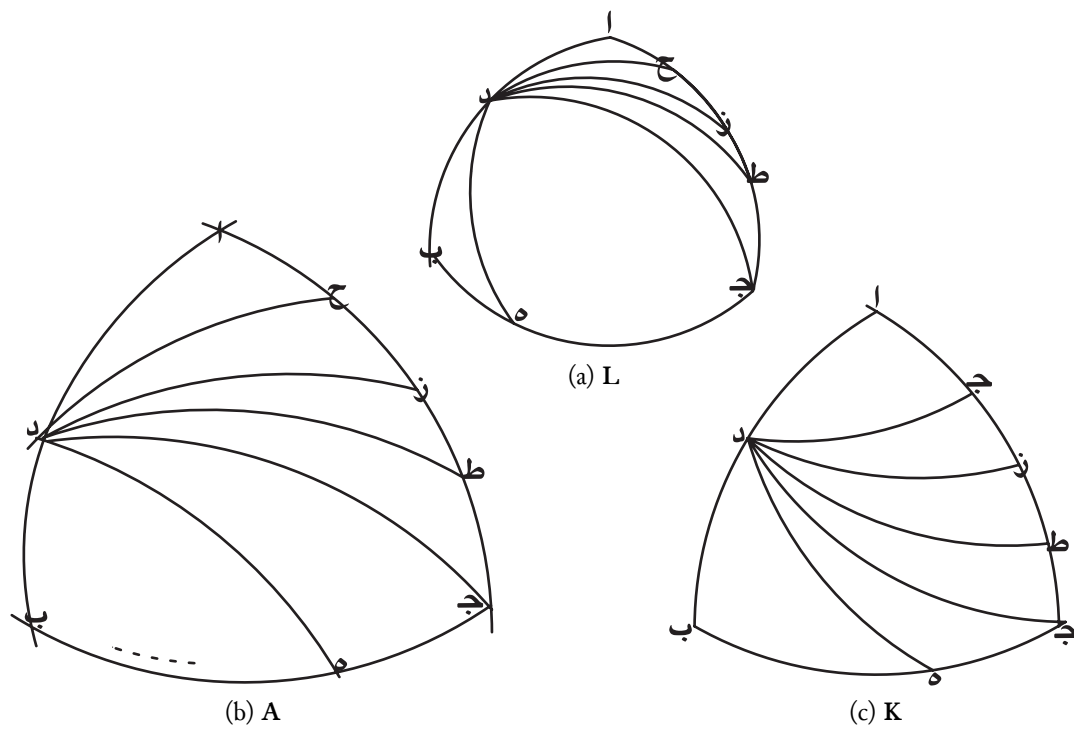


Figure 28: Book I, Prop. 28

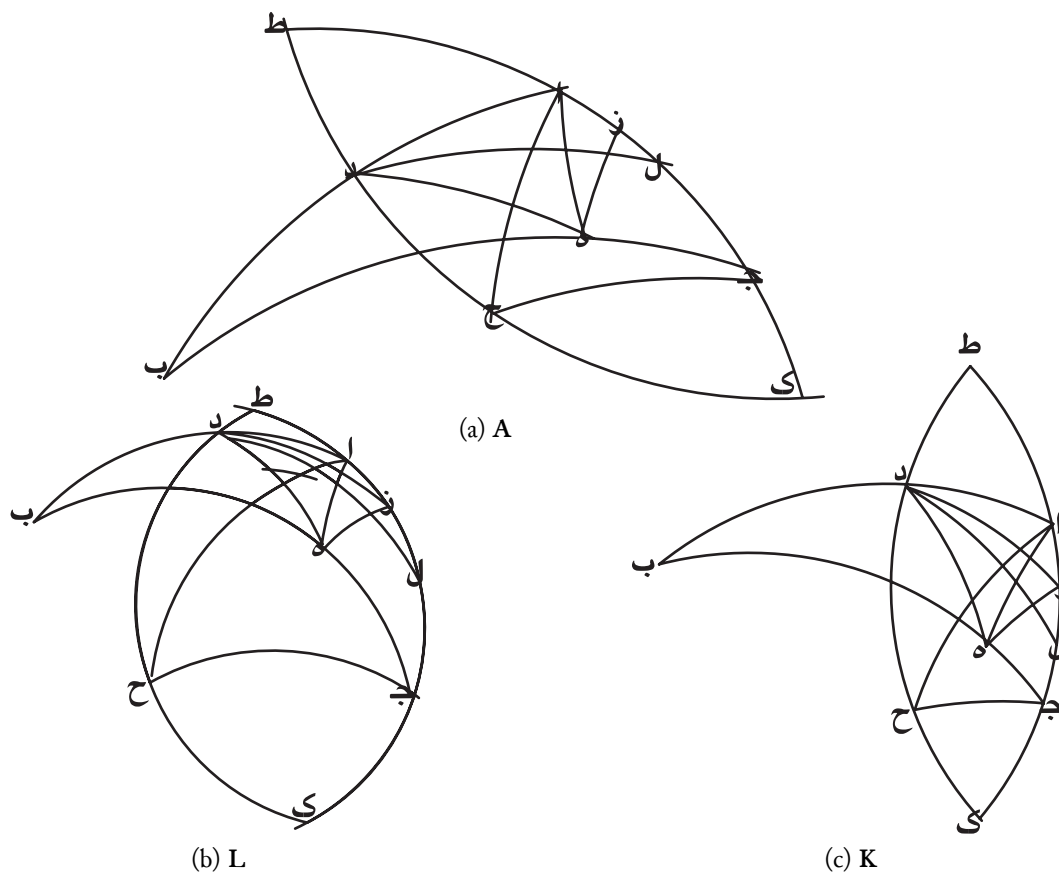


Figure 29: Book I, Prop. 29

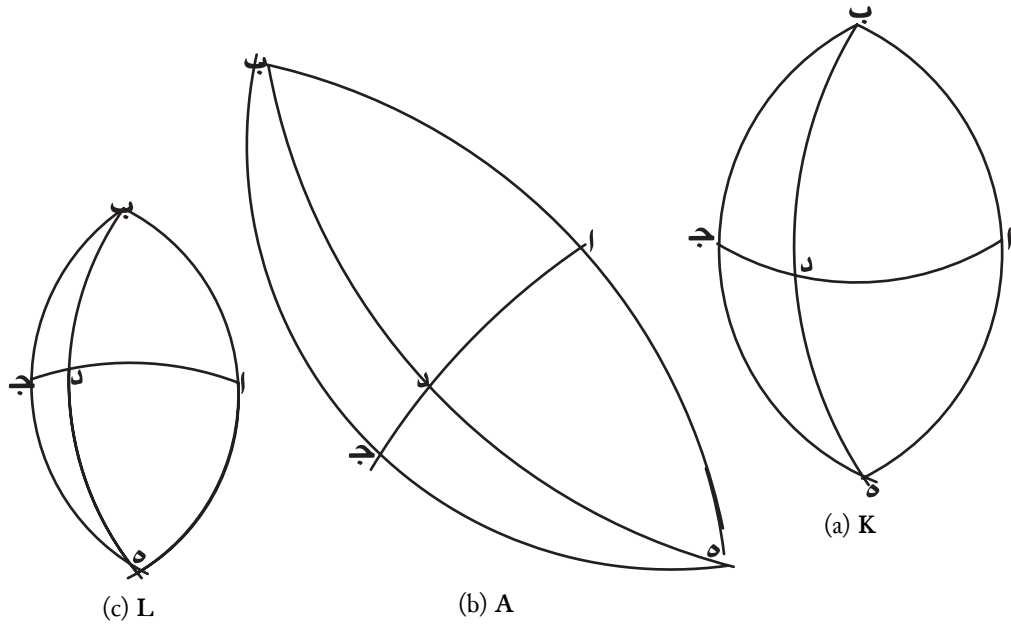


Figure 30: Book I, Prop. 30

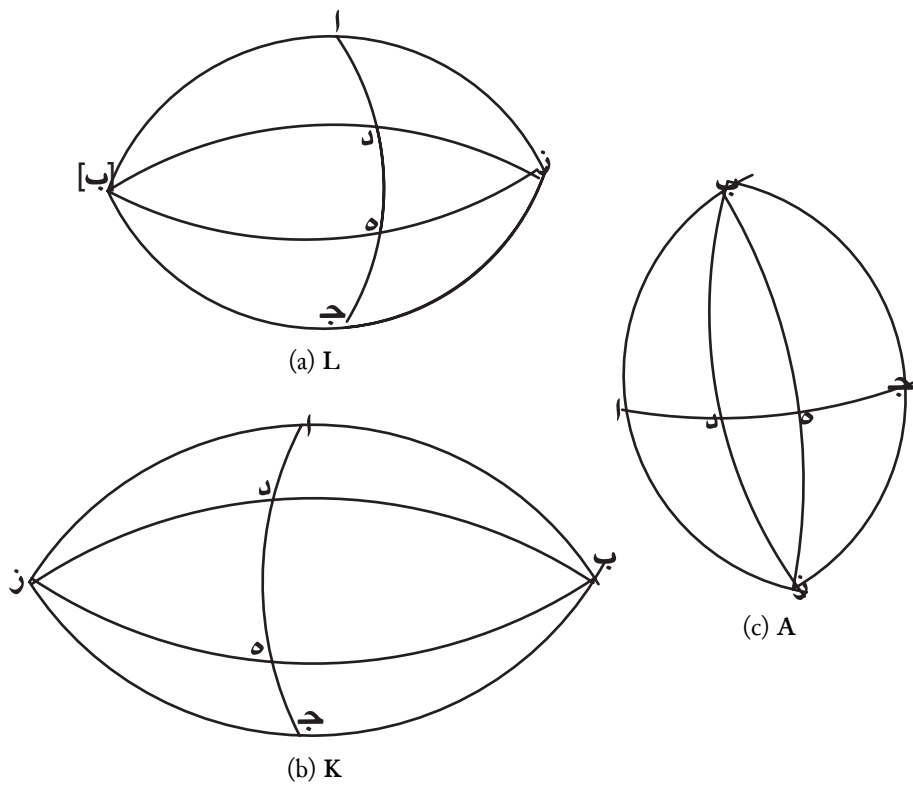


Figure 31: Book I, Prop. 31

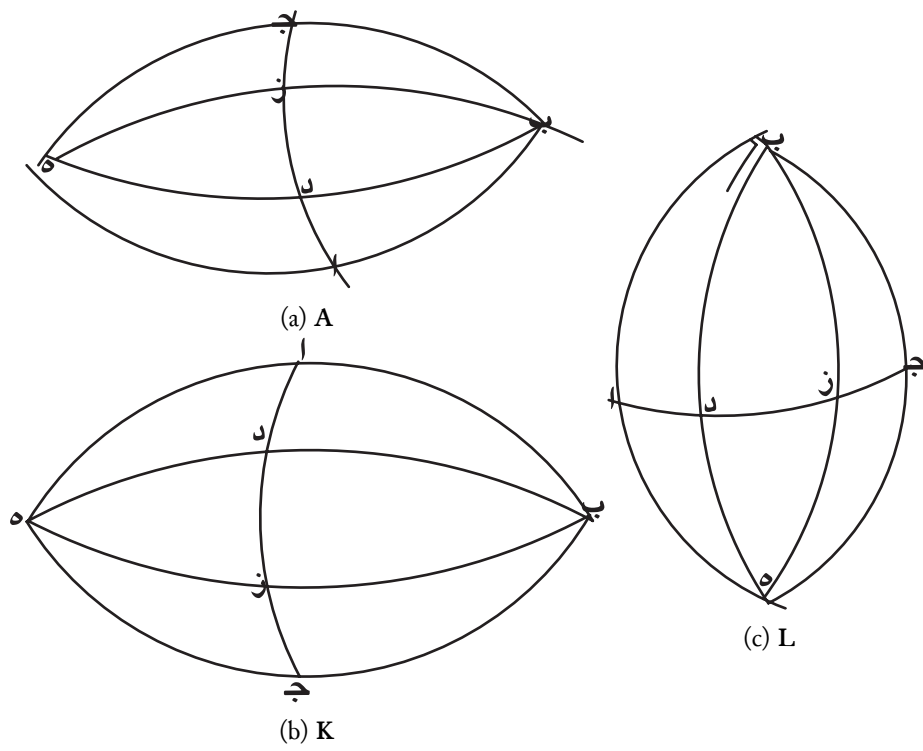


Figure 32: Book I, Prop. 32

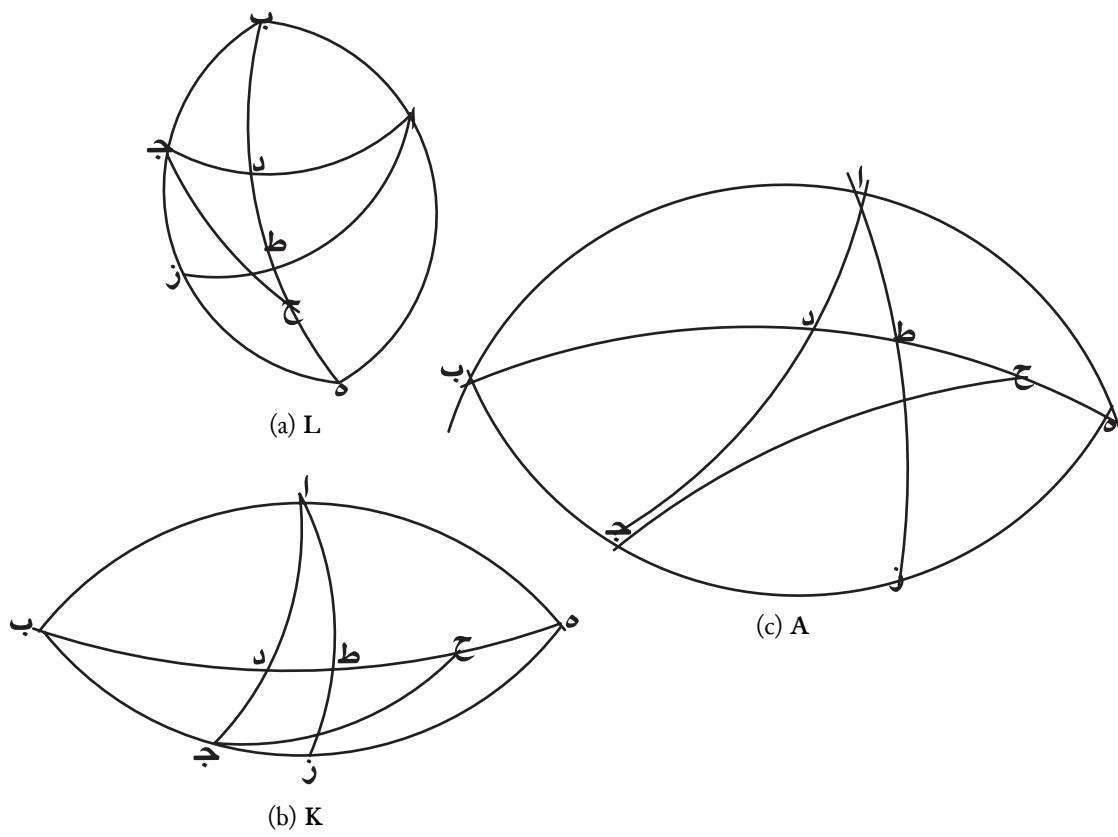


Figure 33: Book I, Prop. 33

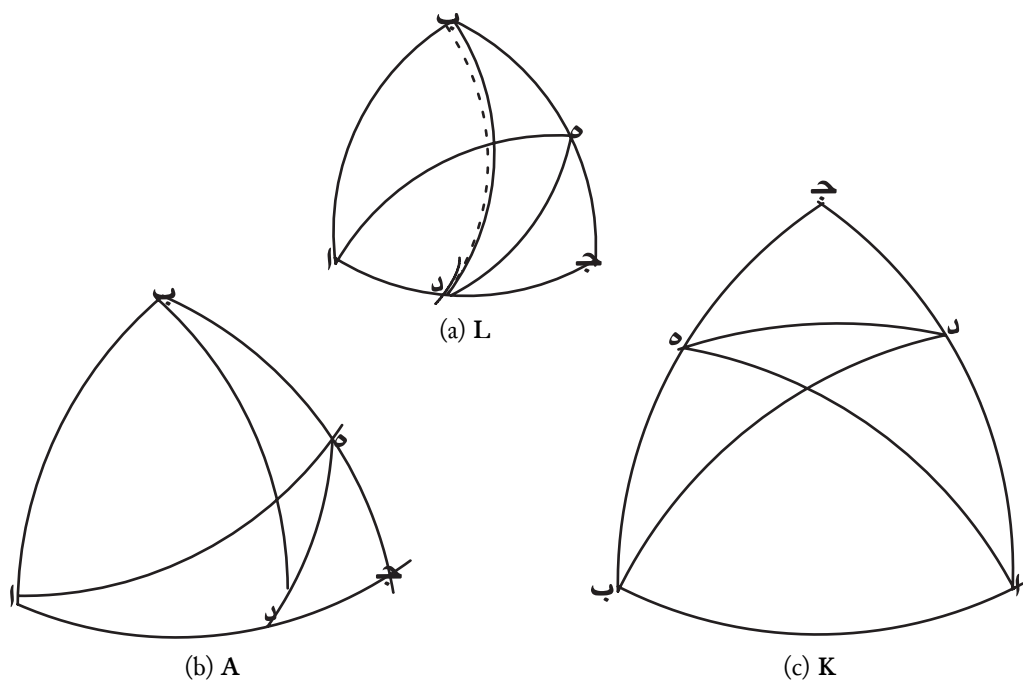


Figure 34: Book I, Prop. 34

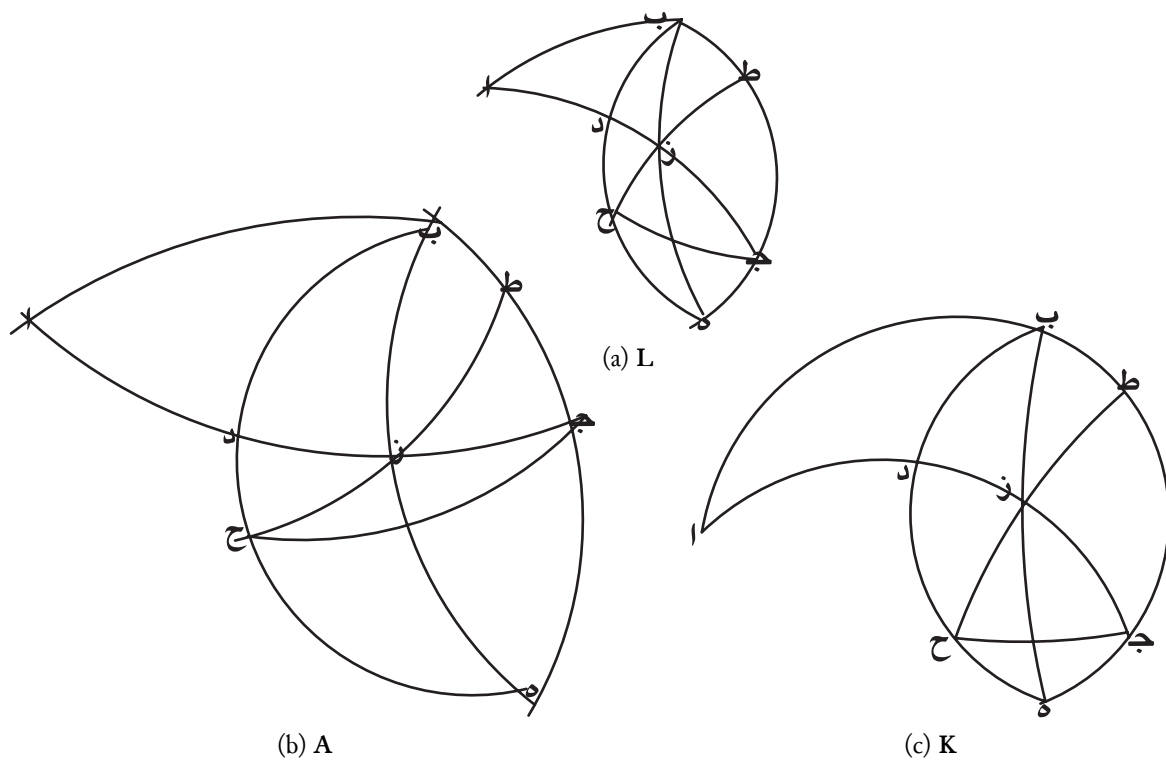


Figure 35: Book I, Prop. 35

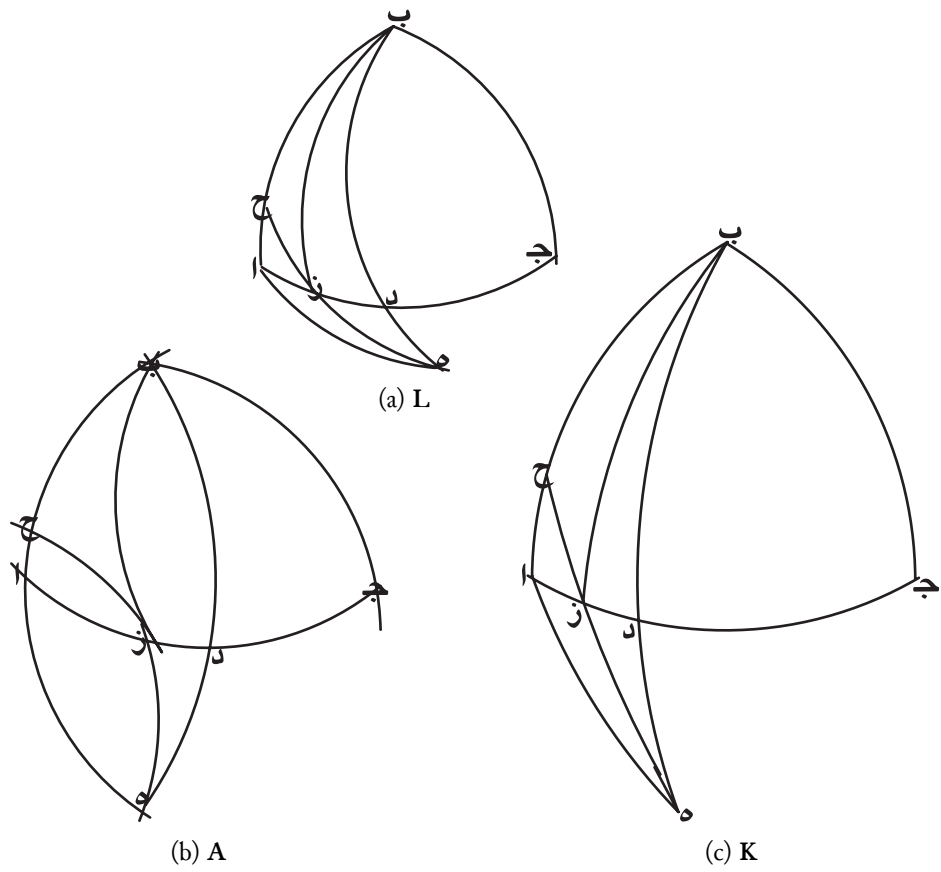


Figure 36: Book I, Prop. 36

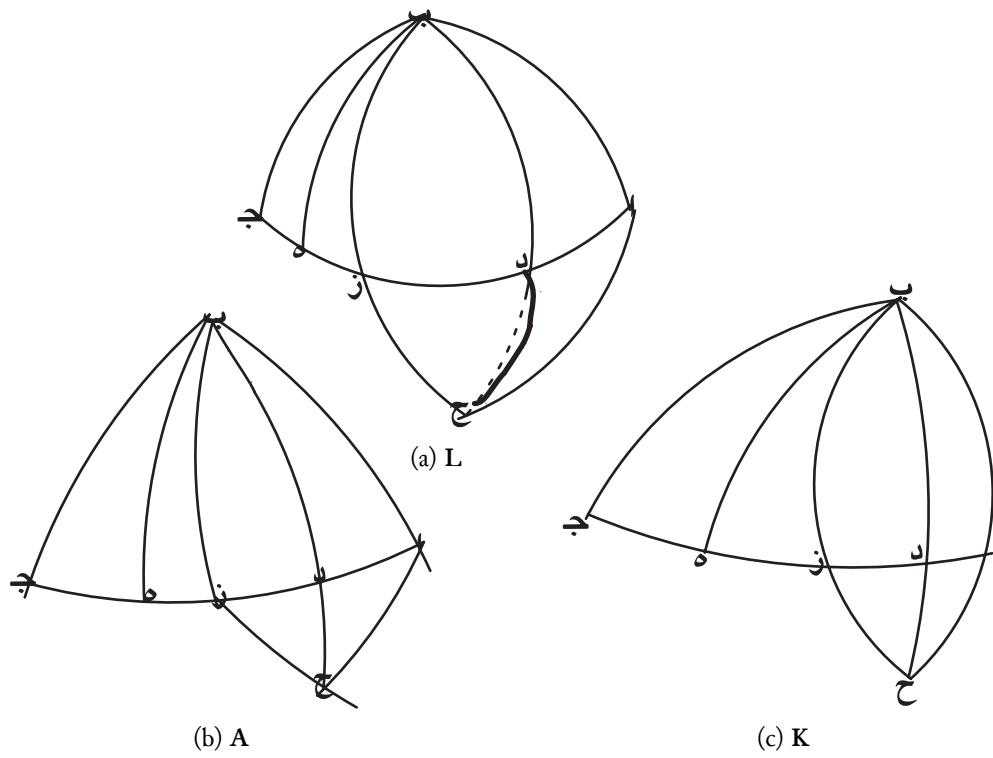
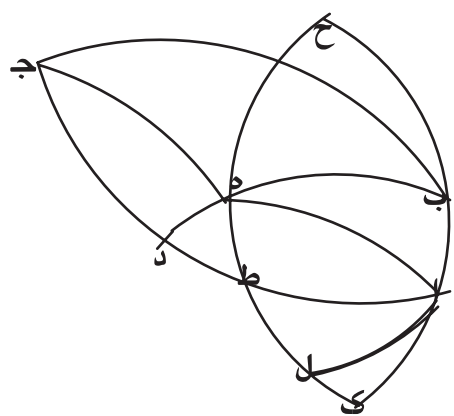
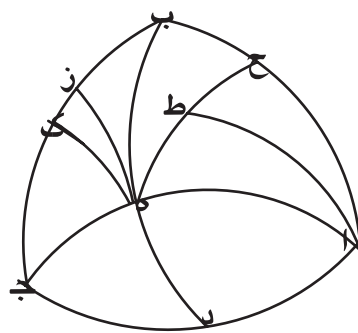


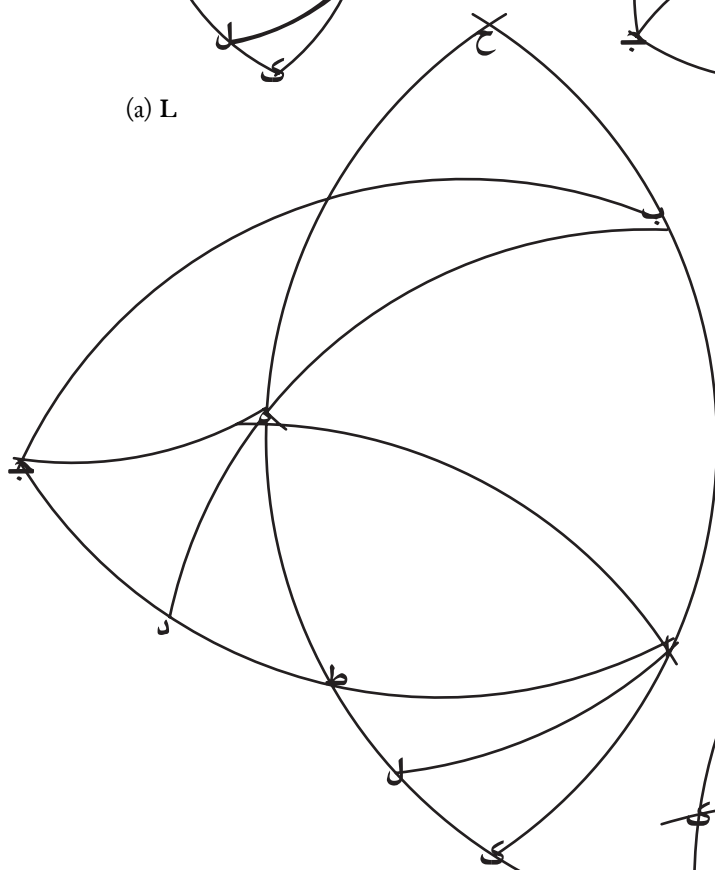
Figure 37: Book I, Prop. 38



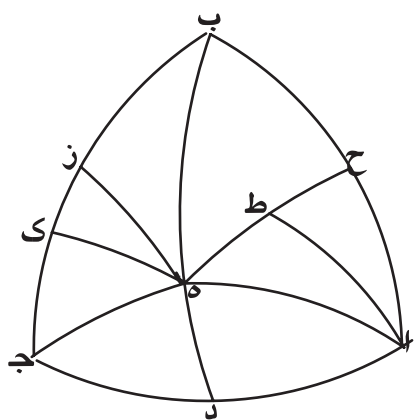
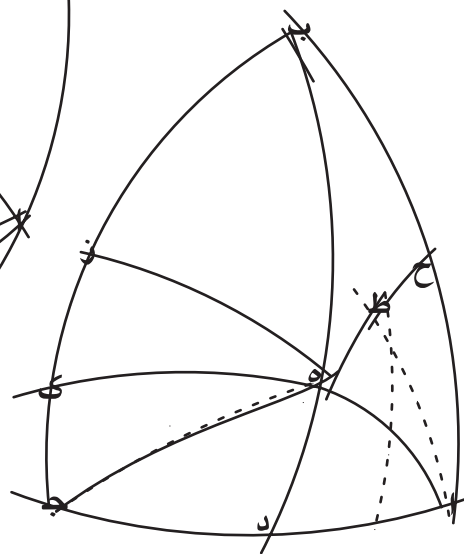
(a) L



(b) L



(c) A



(d) K

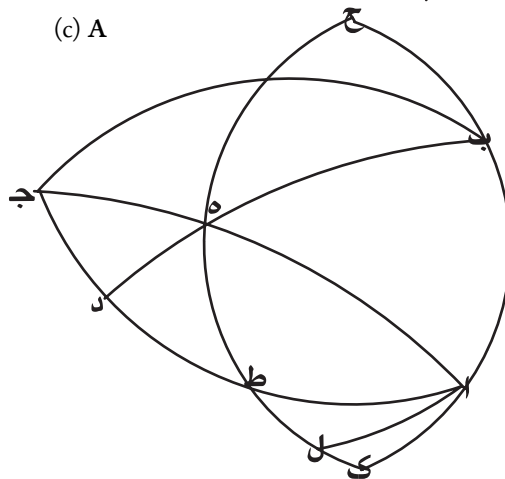


Figure 38: B27k I, Prop. 37

In L, the two figures are separated, placed on facing pages.

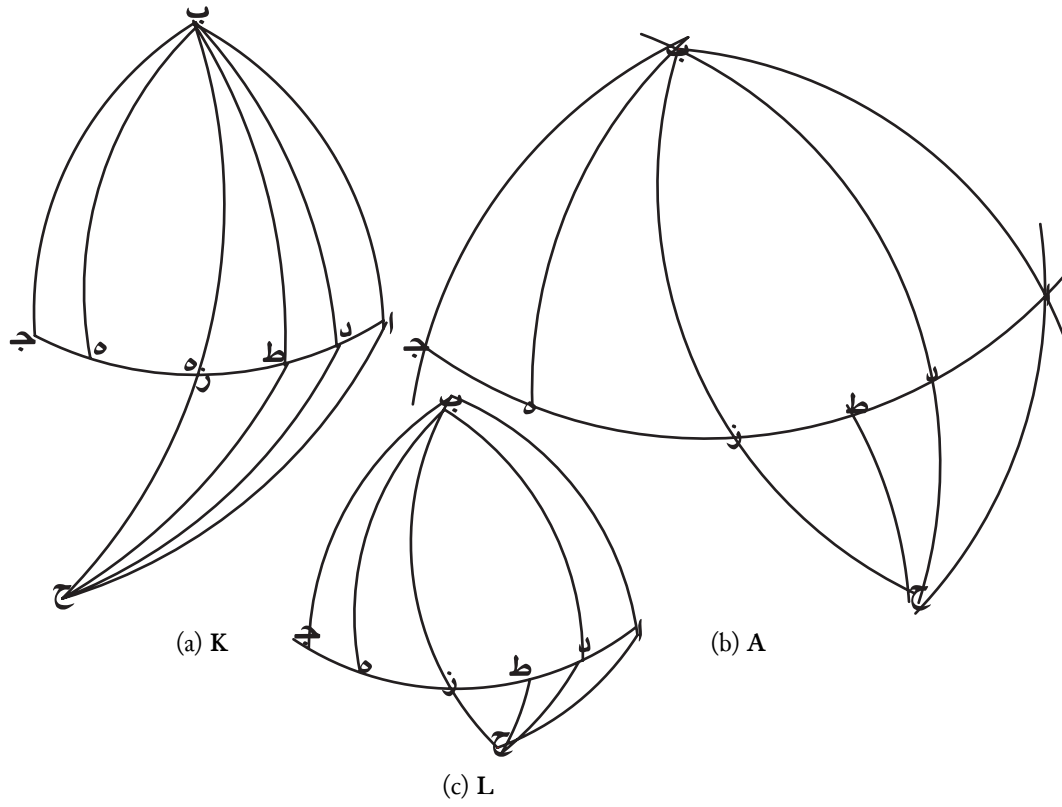


Figure 39: Book I, Prop. 39

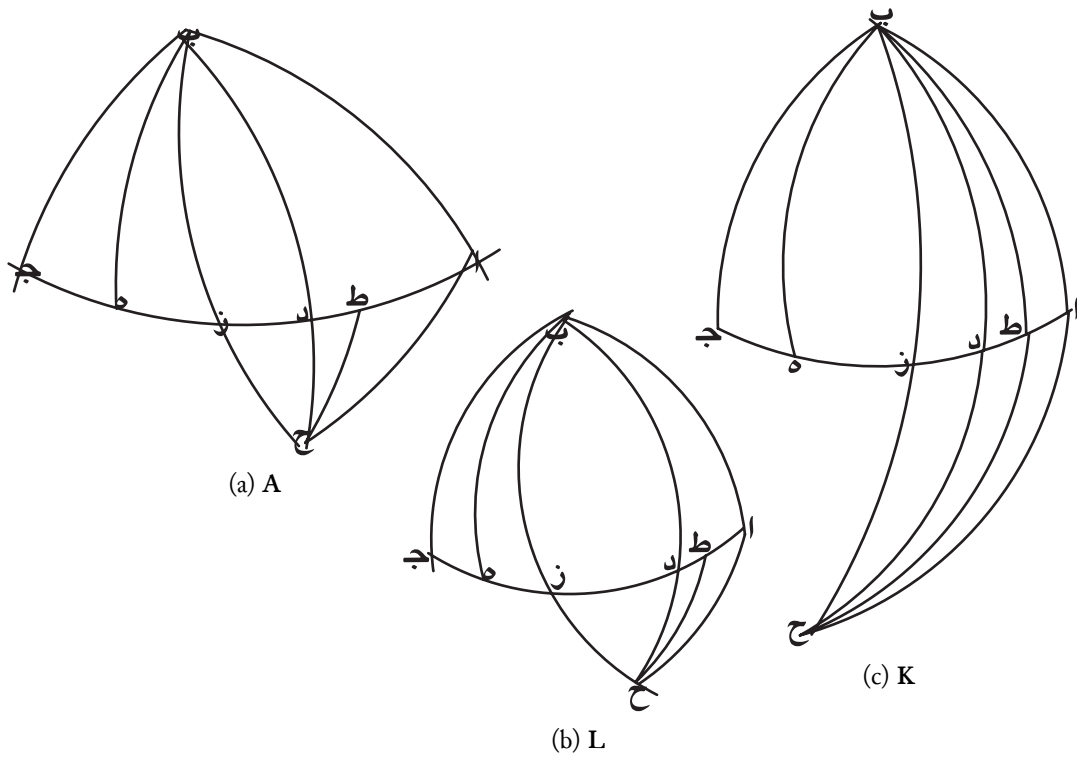


Figure 40: Book I, Prop. 40

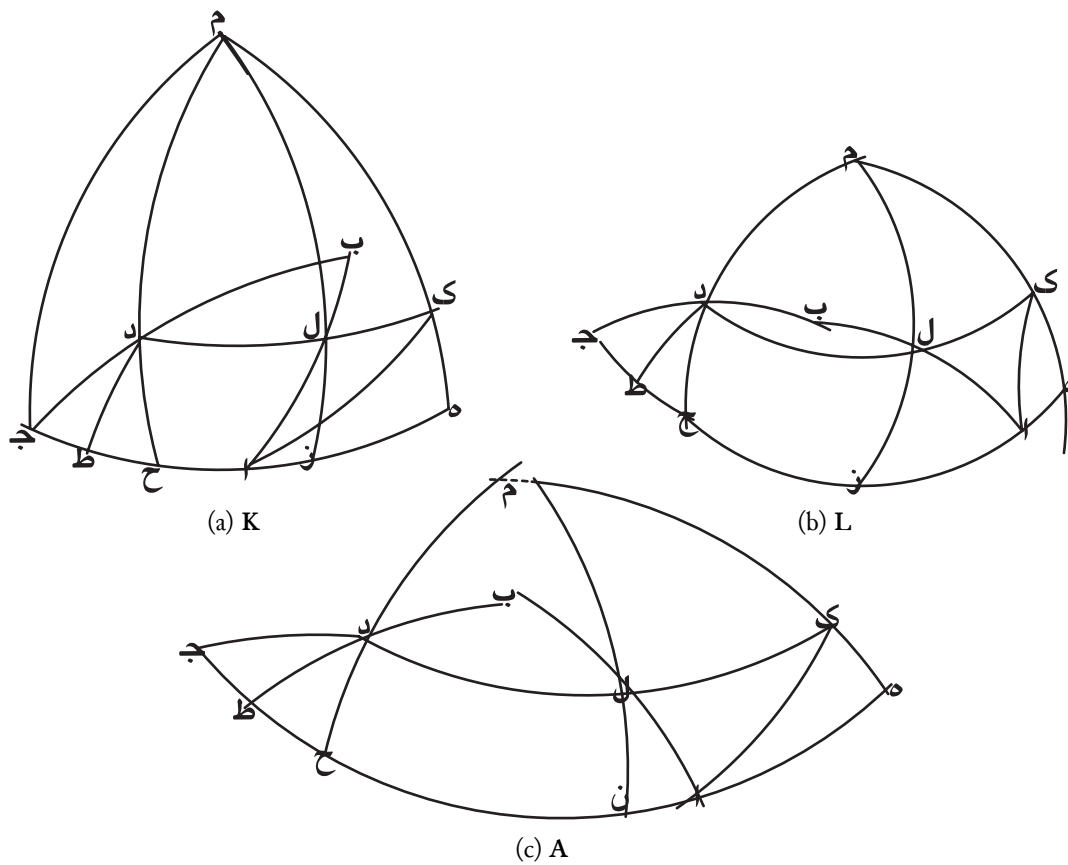


Figure 41: Book I, Prop. 41

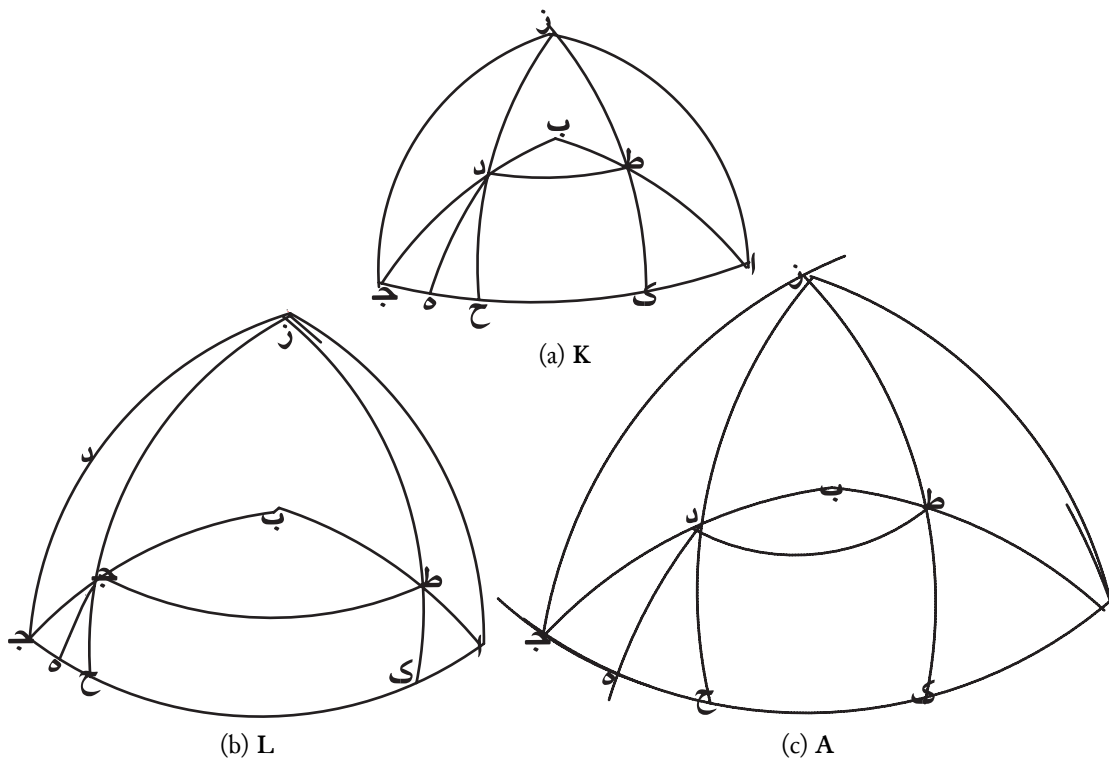


Figure 42: Book I, Prop. 42

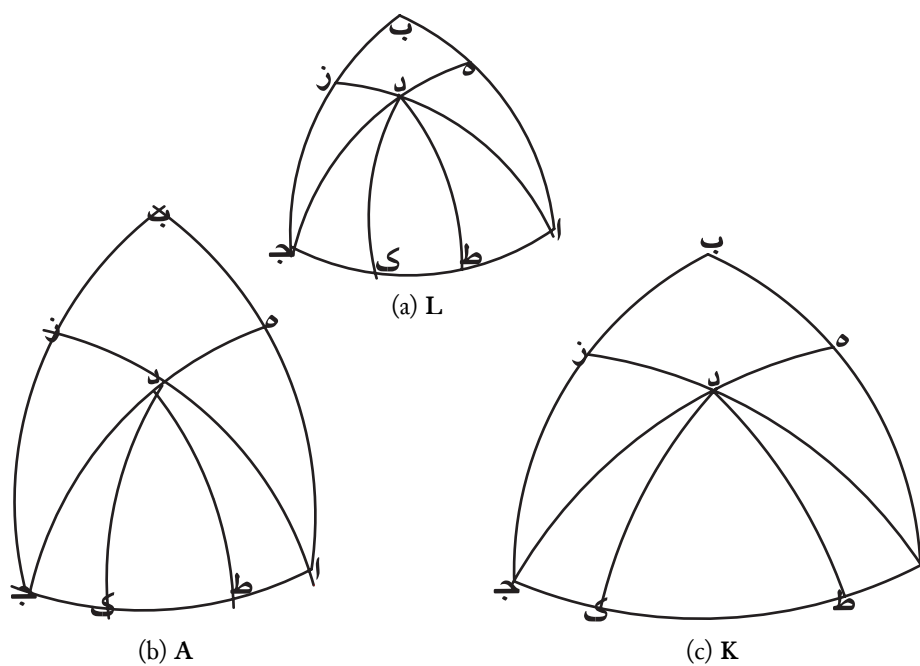


Figure 43: Book I, Prop. 43

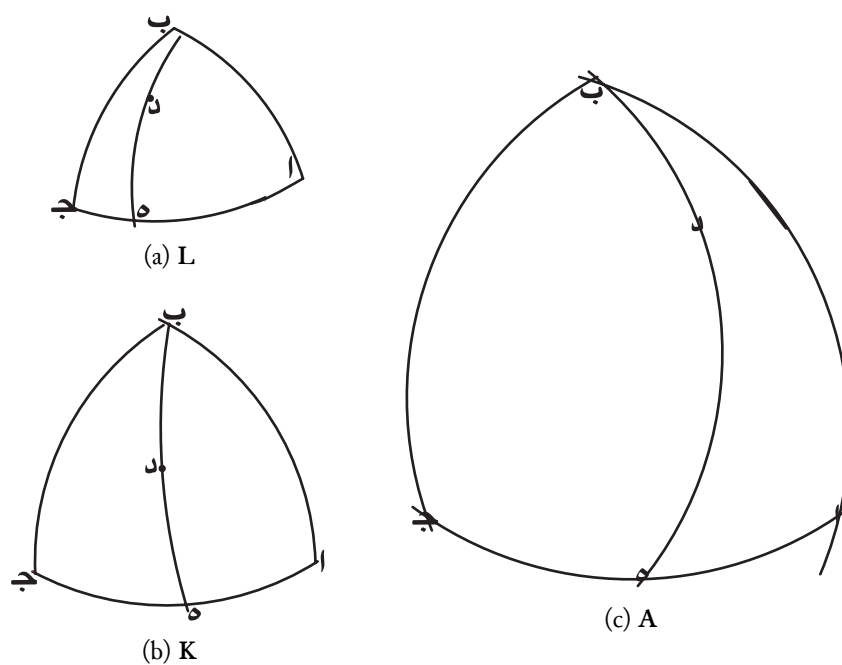
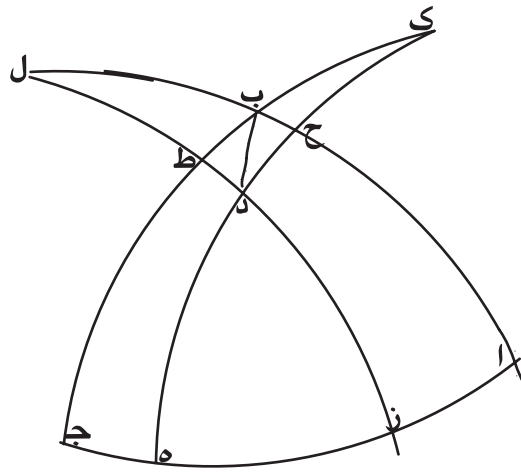
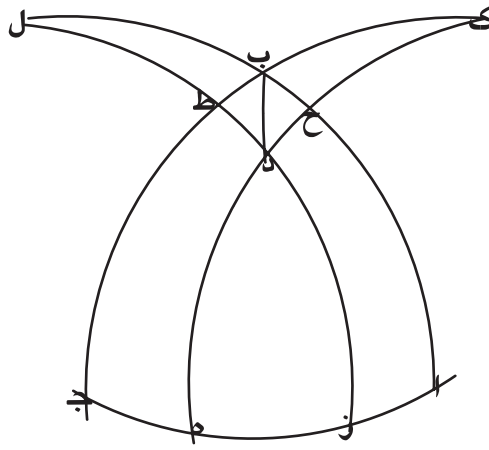


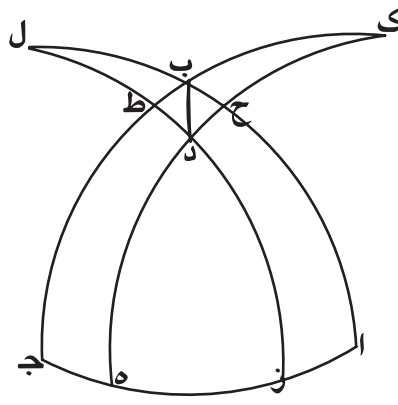
Figure 44: Book I, Prop. 44



(a) L

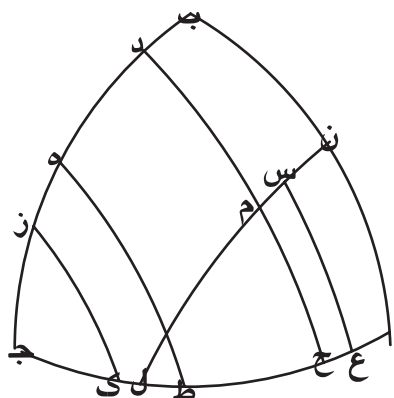


(b) A

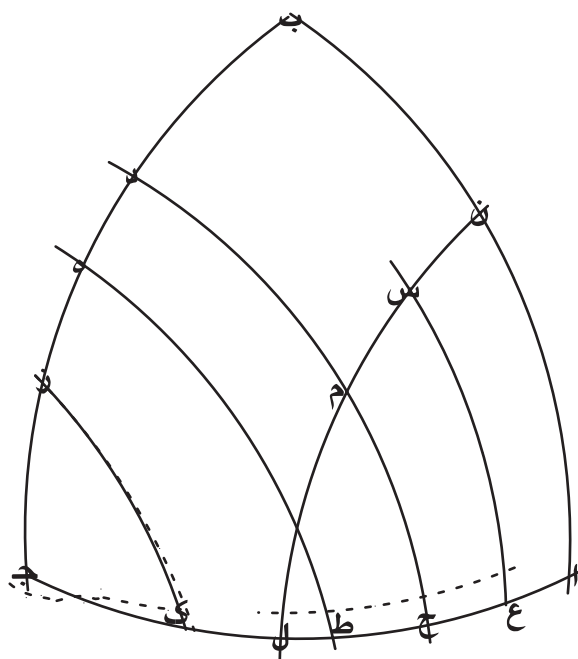


(c) K

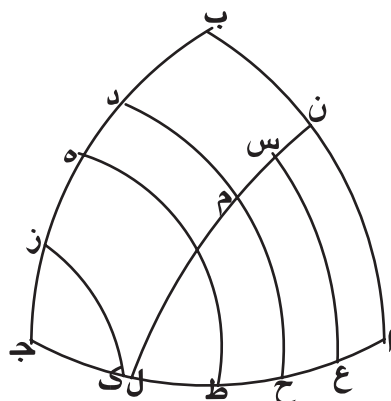
Figure 45: Book I, Prop. 45



(a) L

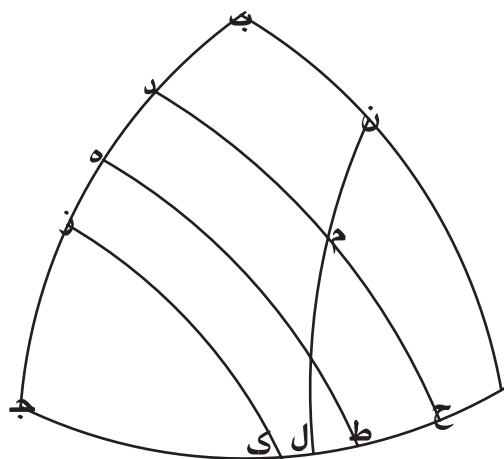


(b) A

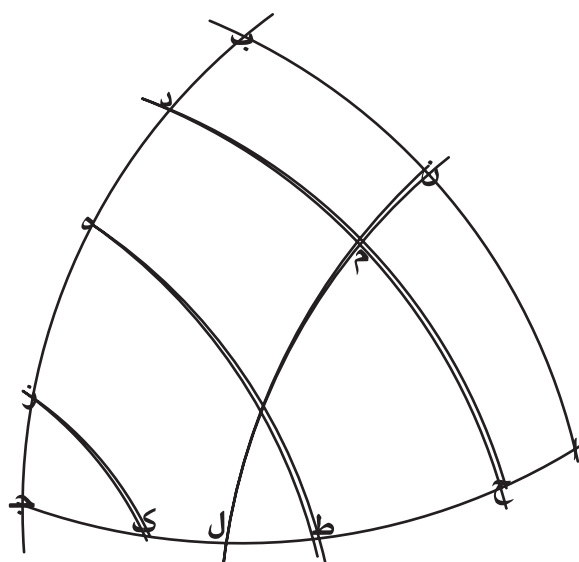


(c) K

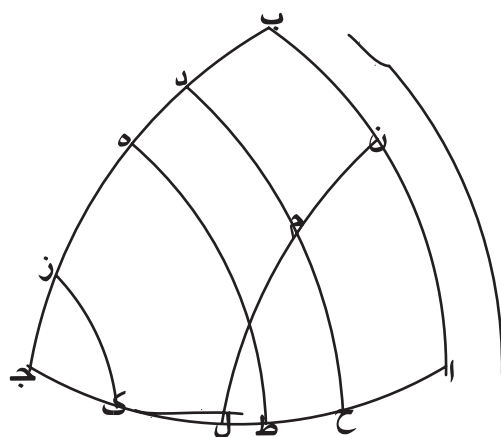
Figure 46: Book I, Prop. 46



(a) L



(b) A



(c) K

Figure 47: Book I, Prop. 47

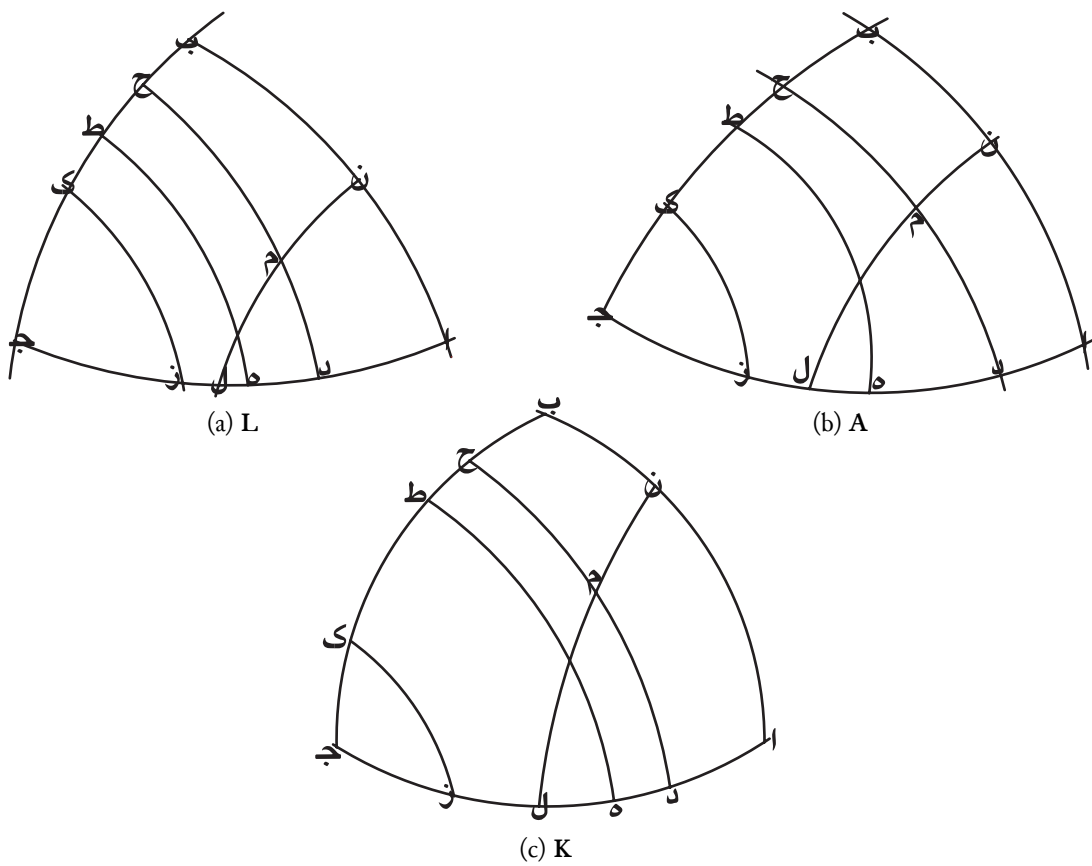


Figure 48: Book I, Prop. 48

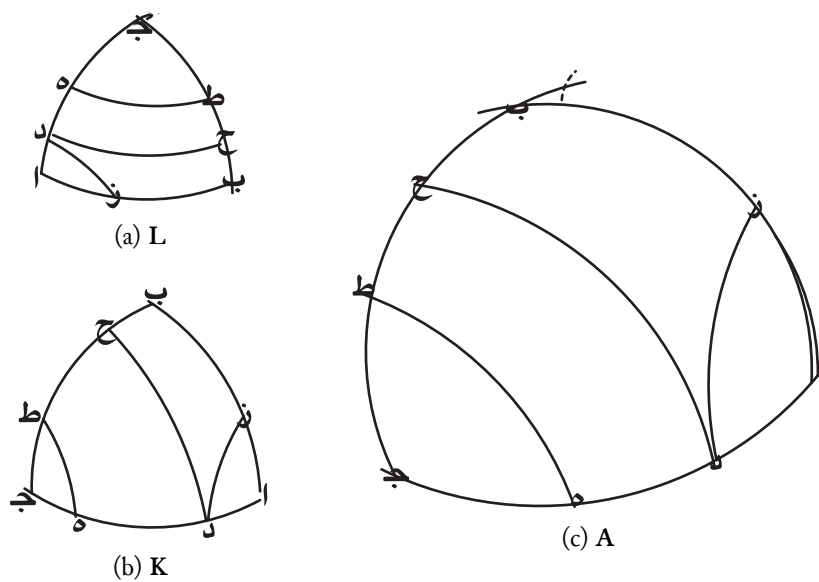
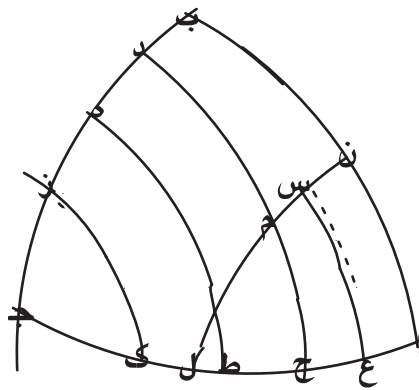
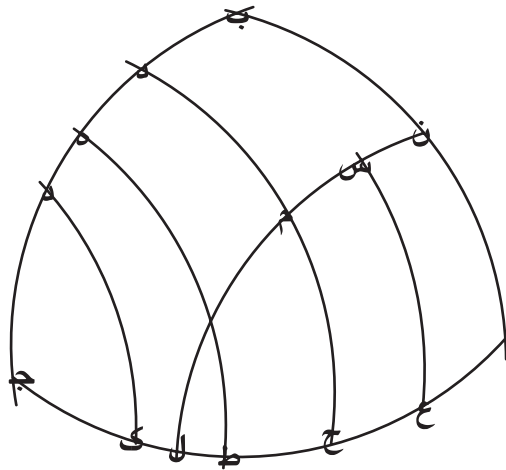


Figure 49: Book I, Prop. 49

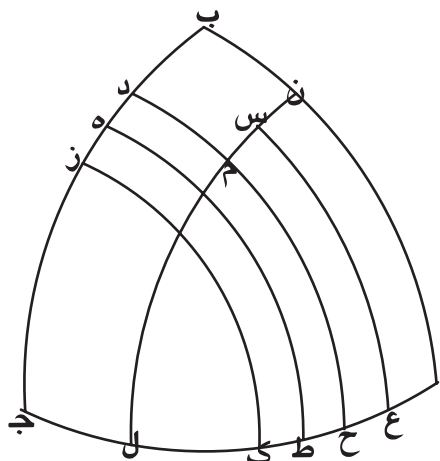
In L, the diagram has no proper place, but is squeezed into into the margin.



(a) L

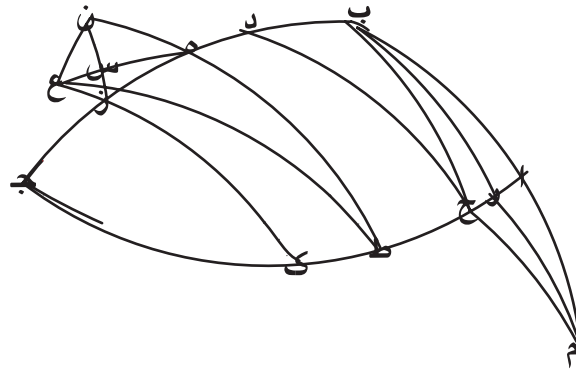


(b) A

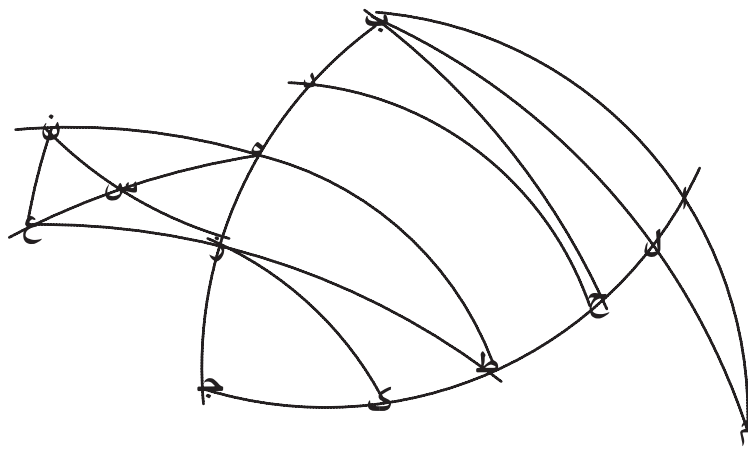


(c) K

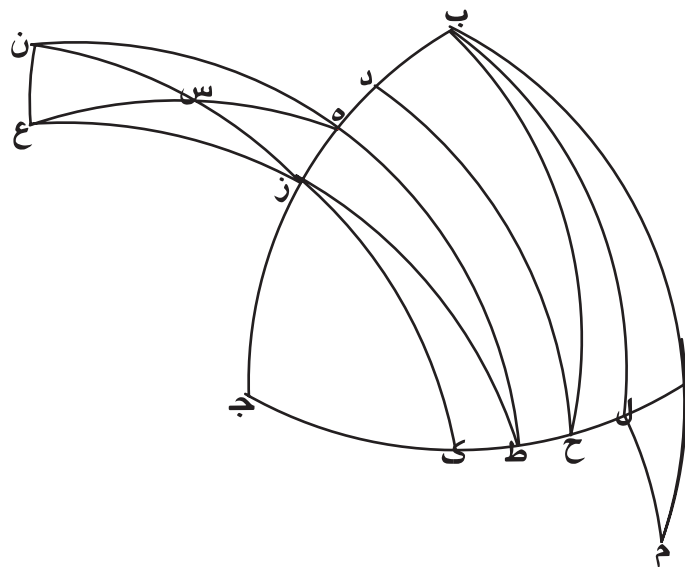
Figure 50: Book I, Prop. 50



(a) L



(b) A



(c) K

Figure 51: Book I, Prop. 51

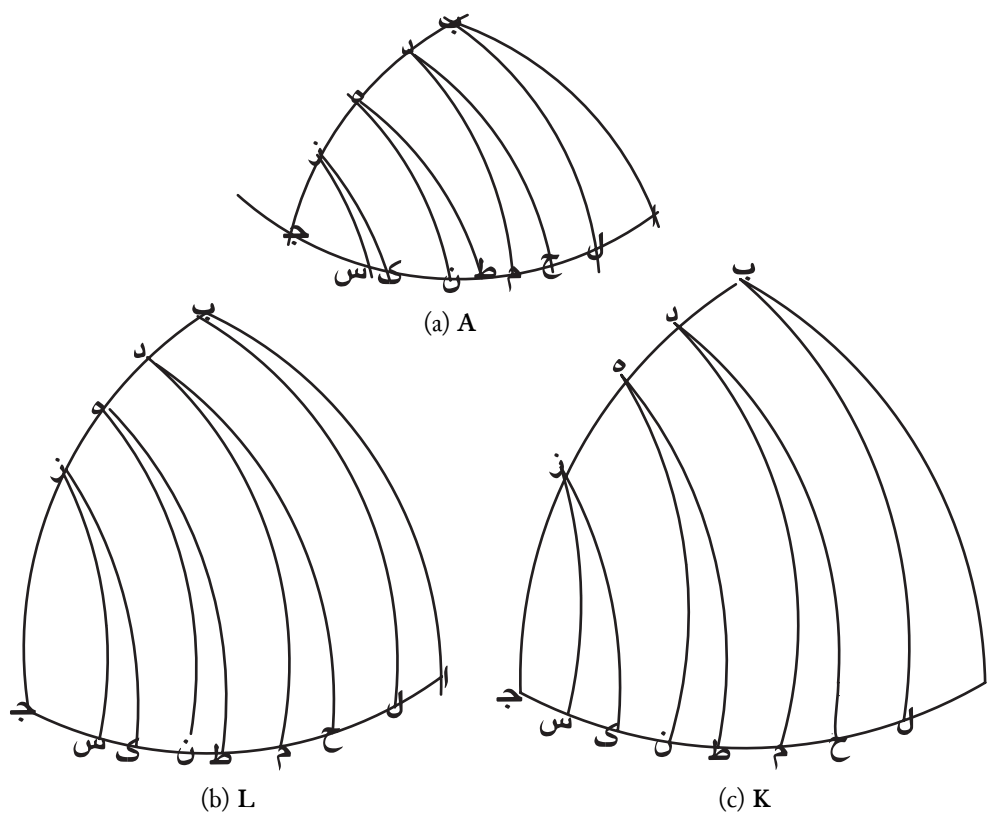


Figure 52: Book I, Prop. 52

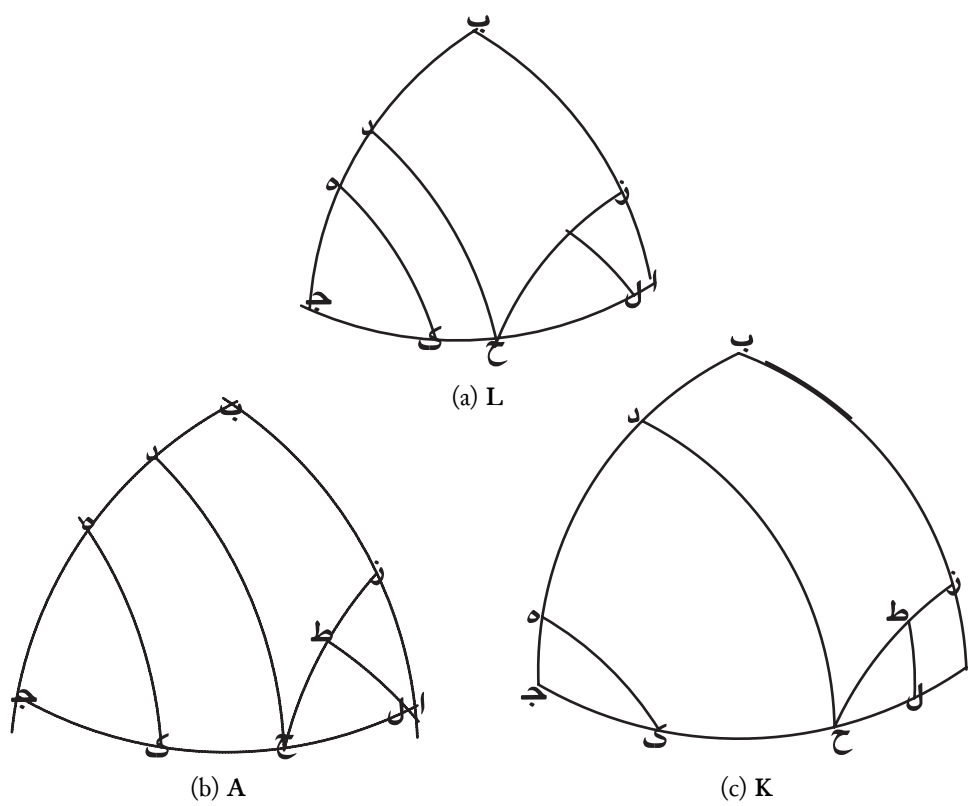
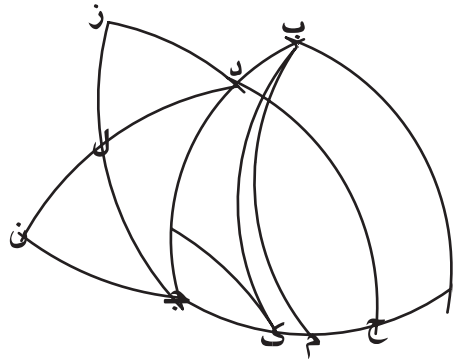
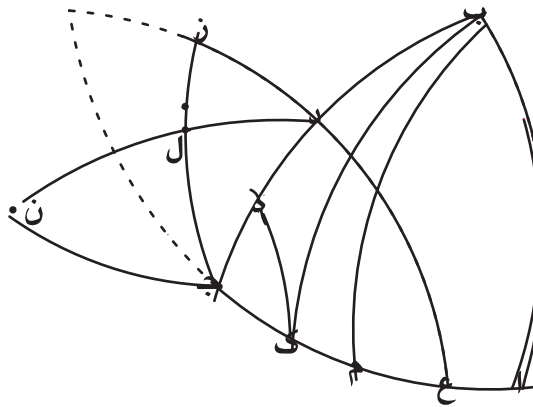


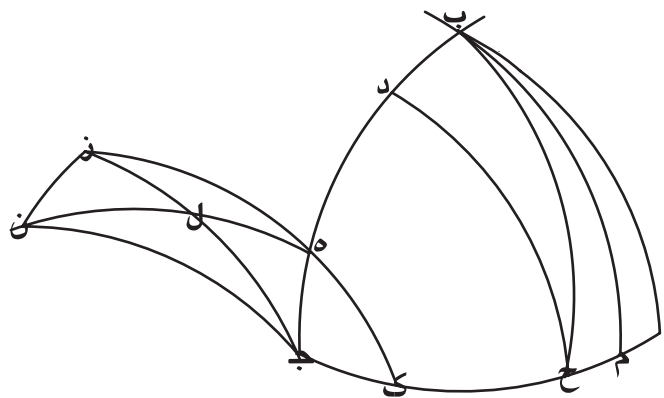
Figure 53: Book I, Prop. 53



(a) L



(b) A



(c) K

Figure 54: Book I, Prop. 54

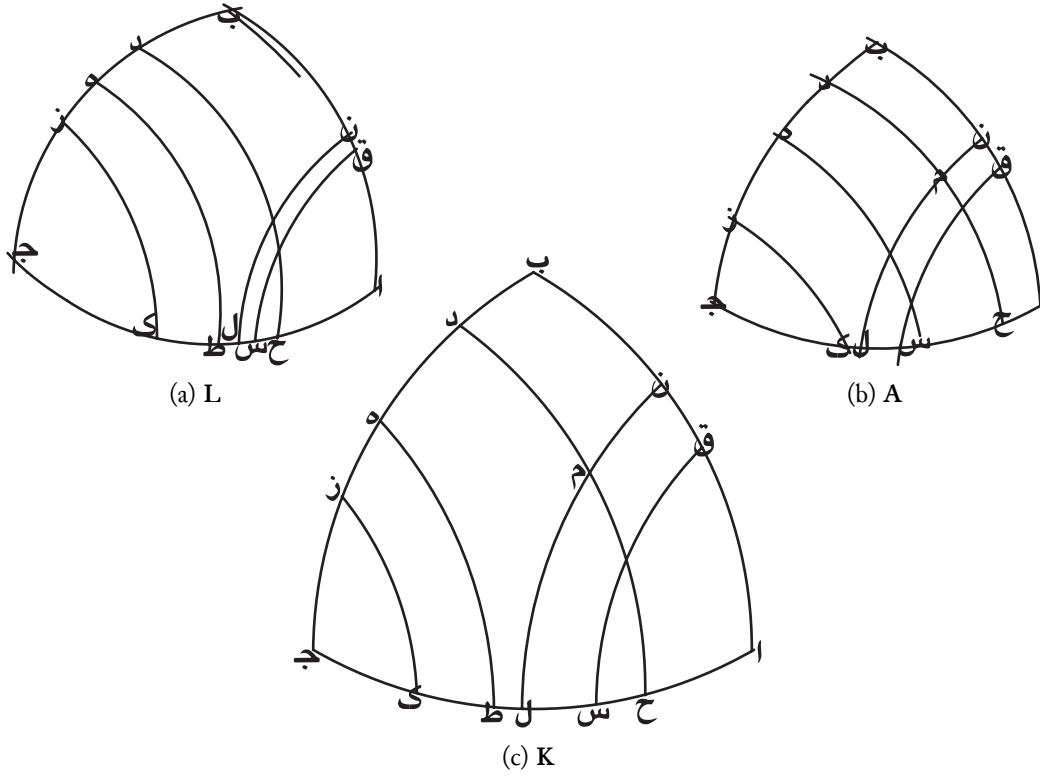


Figure 55: Book I, Prop. 55

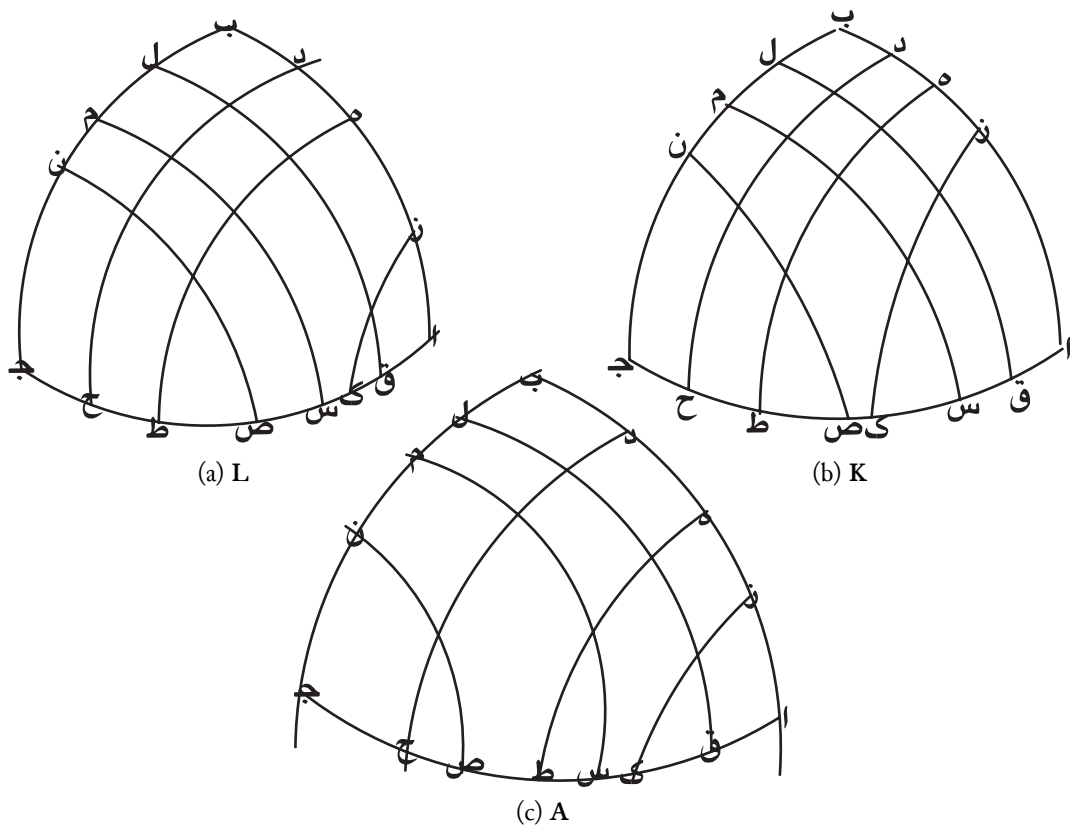
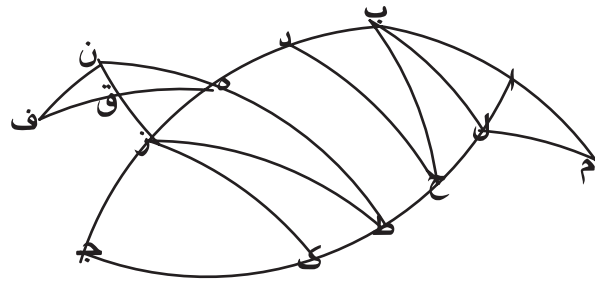
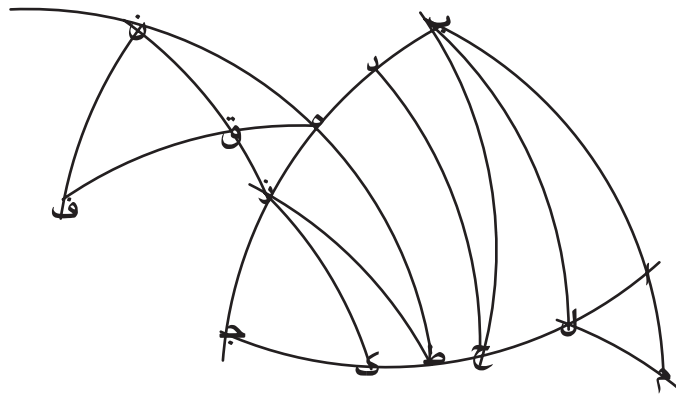


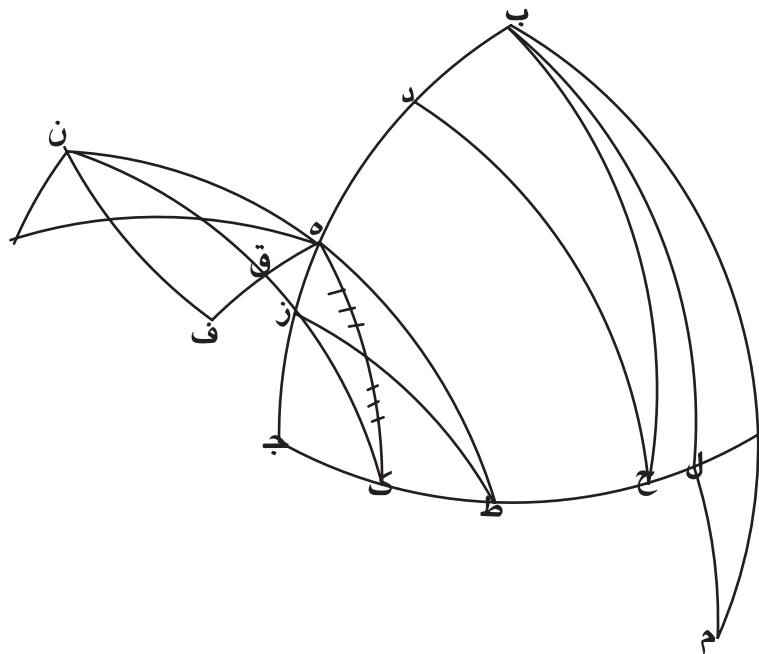
Figure 56: Book I, Prop. 57



(a) L

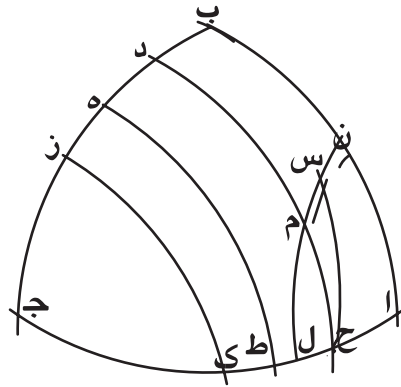


(b) A

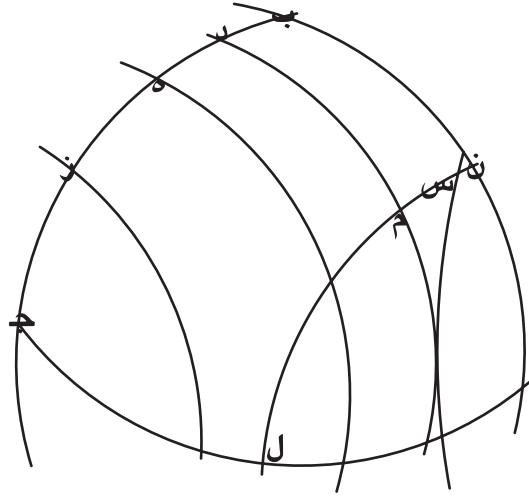


(c) K

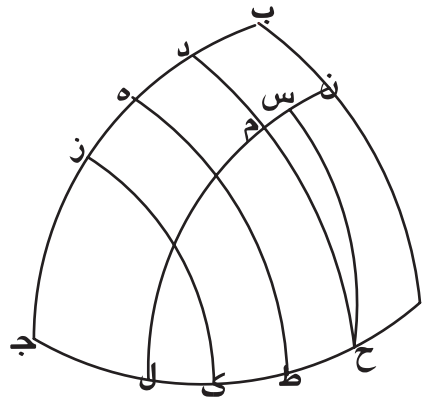
Figure 57: Book I, Prop. 57



(a) L



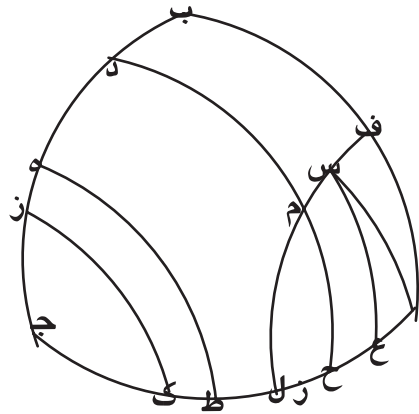
(b) A



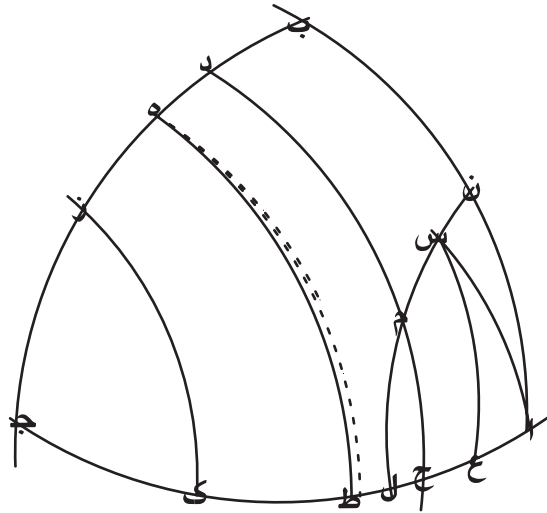
(c) K

Figure 58: Book I, Prop. 58

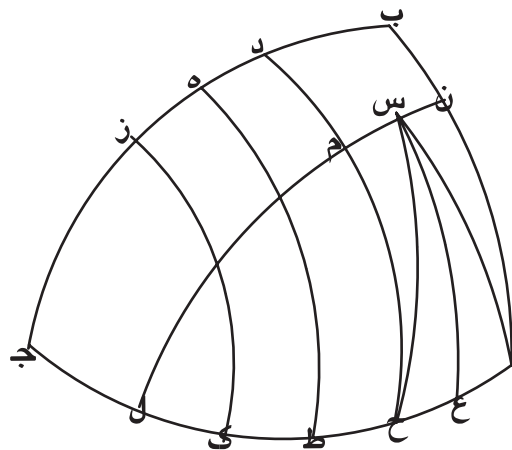
In general, **K** appears to follow **L**; however, in **K**, arc $\overline{ل\ د}$ intersects all four of the internal arcs, whereas in **L**, it only intersects two of them. Moreover, the diagram in **A** is defective, so the diagram in **K** may have simply been redrawn.



(a) L

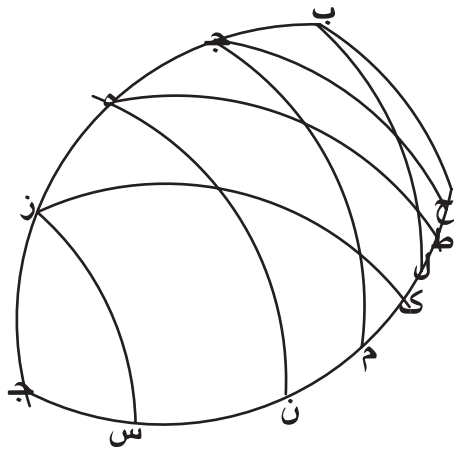


(b) A

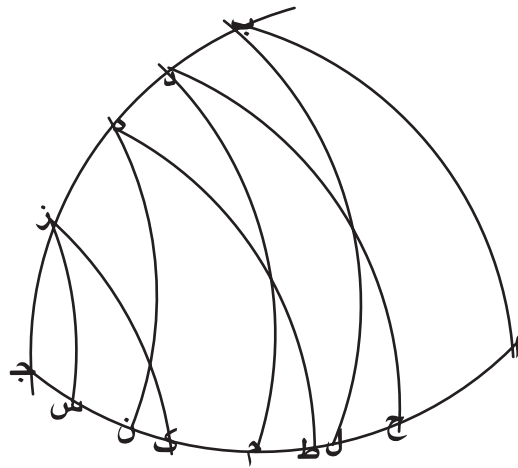


(c) K

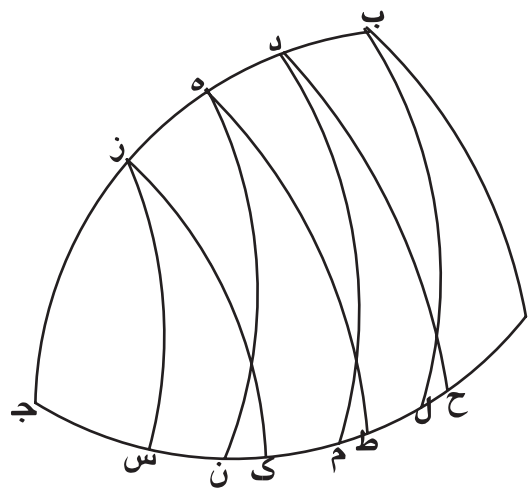
Figure 59: Book I, Prop. 59



(a) L



(b) A



(c) K

Figure 60: Book I, Prop. 60

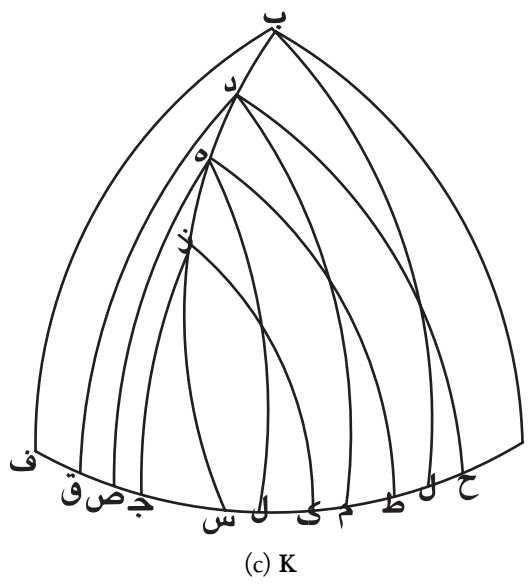
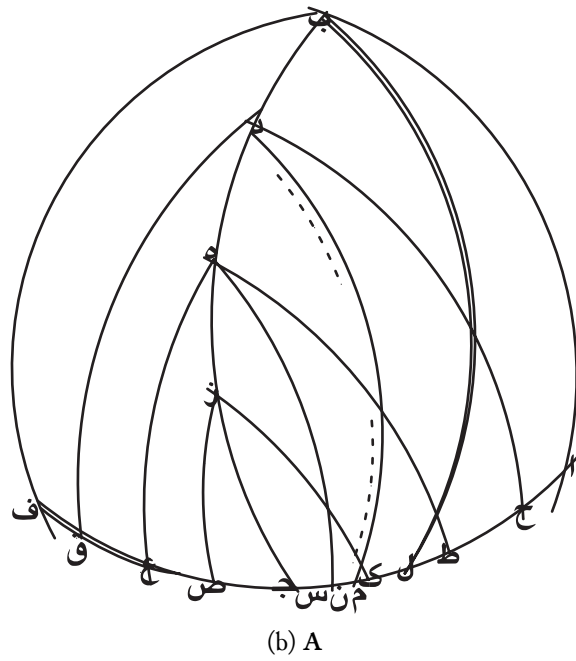
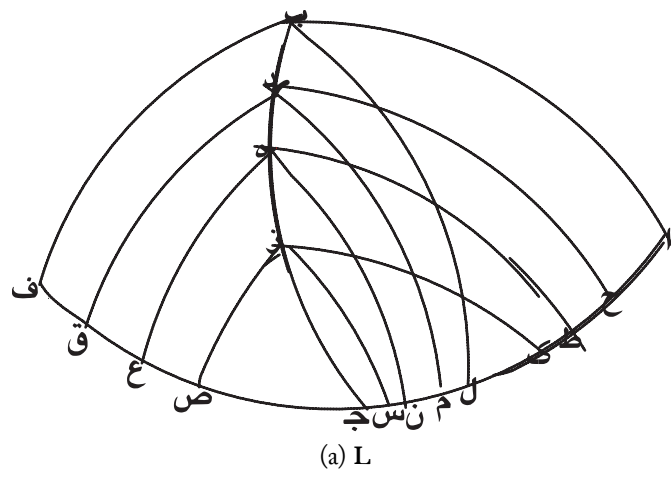


Figure 61: B44k I, Prop. 61

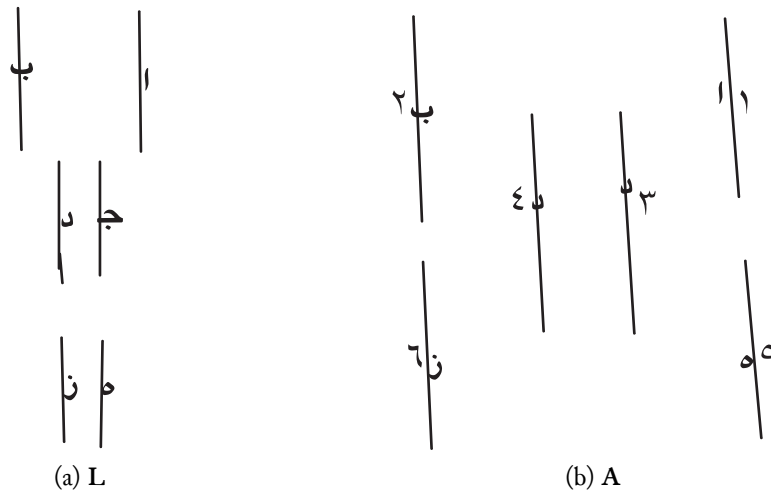


Figure 62: Book II, Preface, Lemma 1

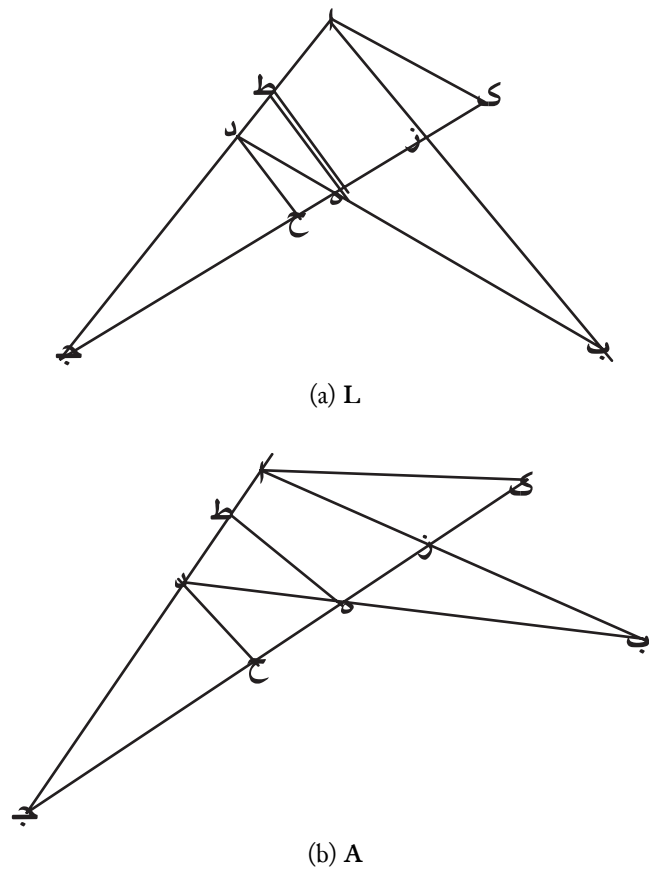
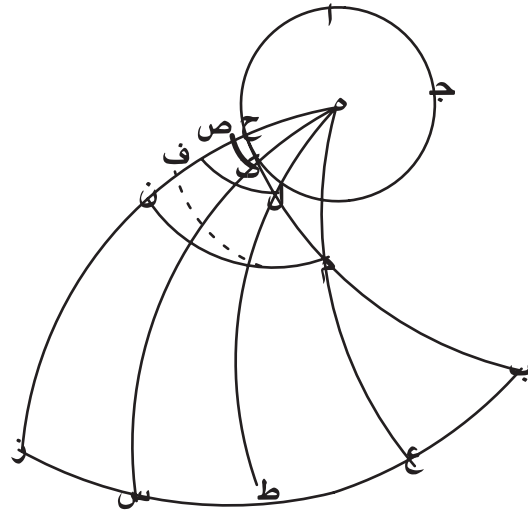


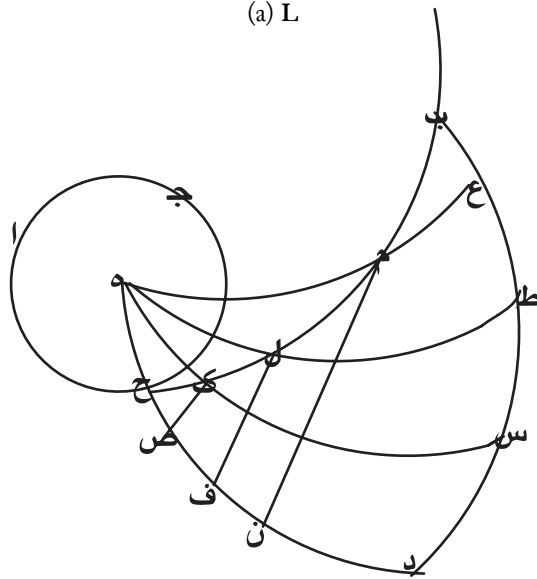
Figure 63: Book II, Preface, Lemma 2



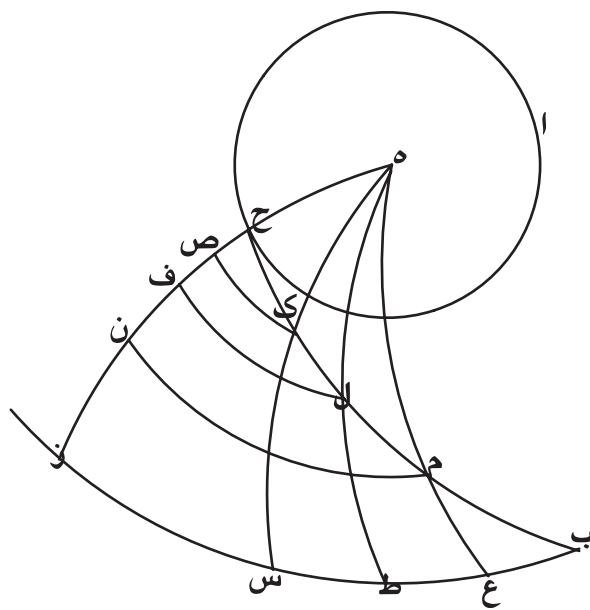
Figure 64: Book II, Preface, Lemma 3



(a) L



(b) A



(c) K

Figure 65: Book II, Prop. 1

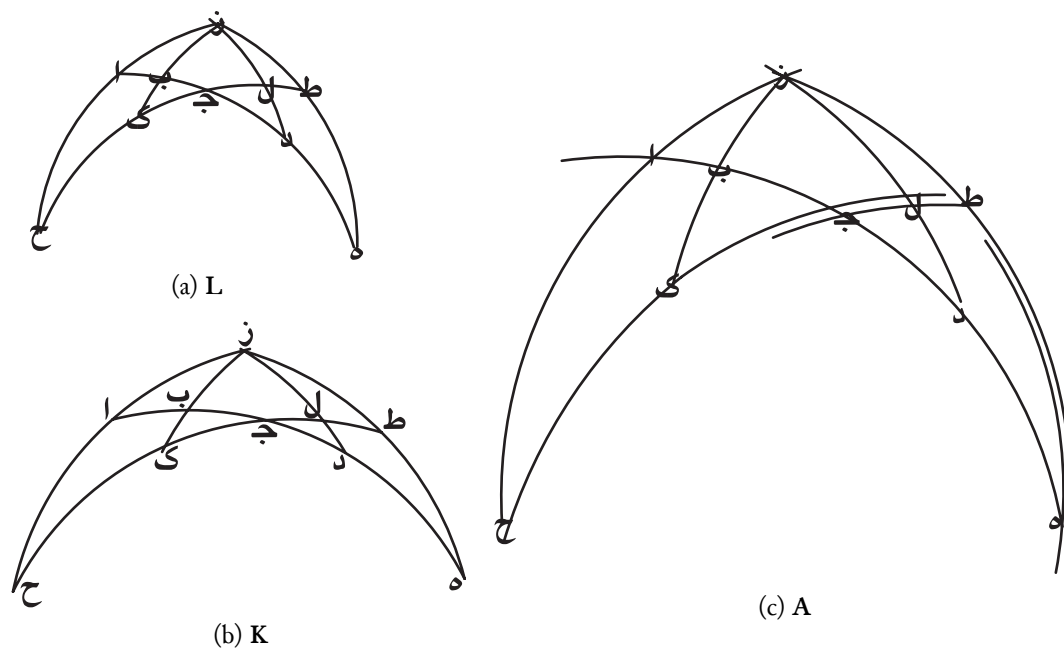


Figure 66: Book II, Prop. 2

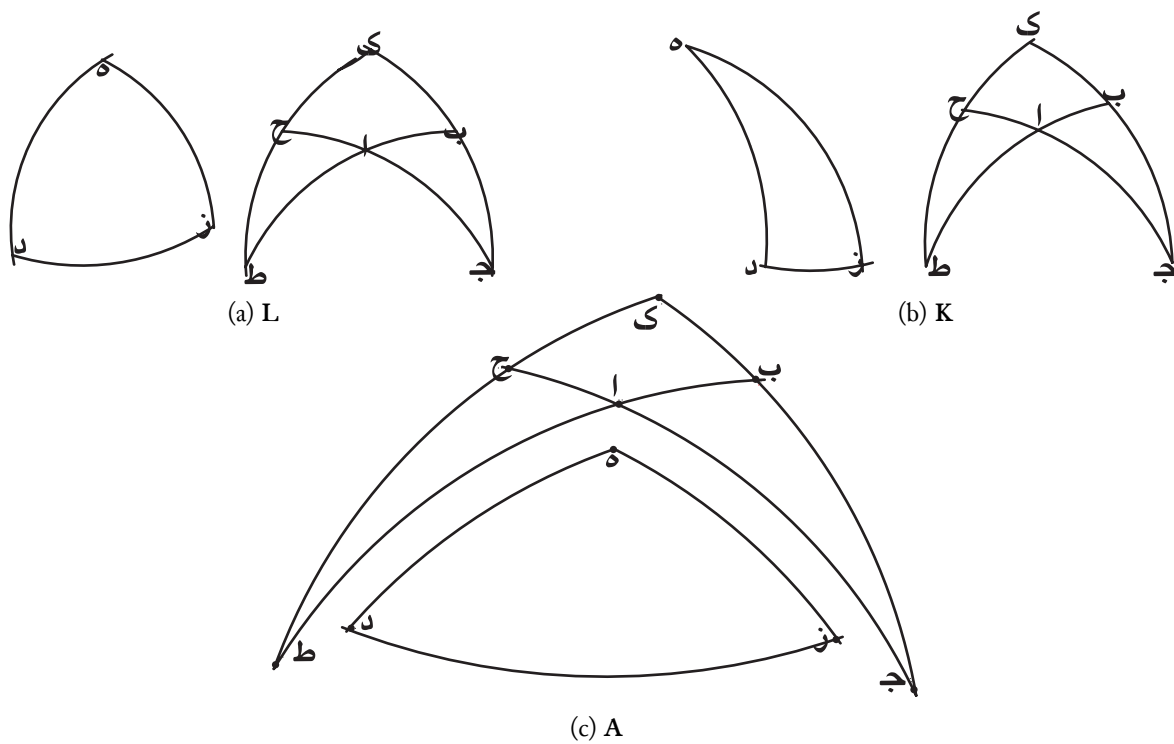


Figure 67: Book II, Prop. 6

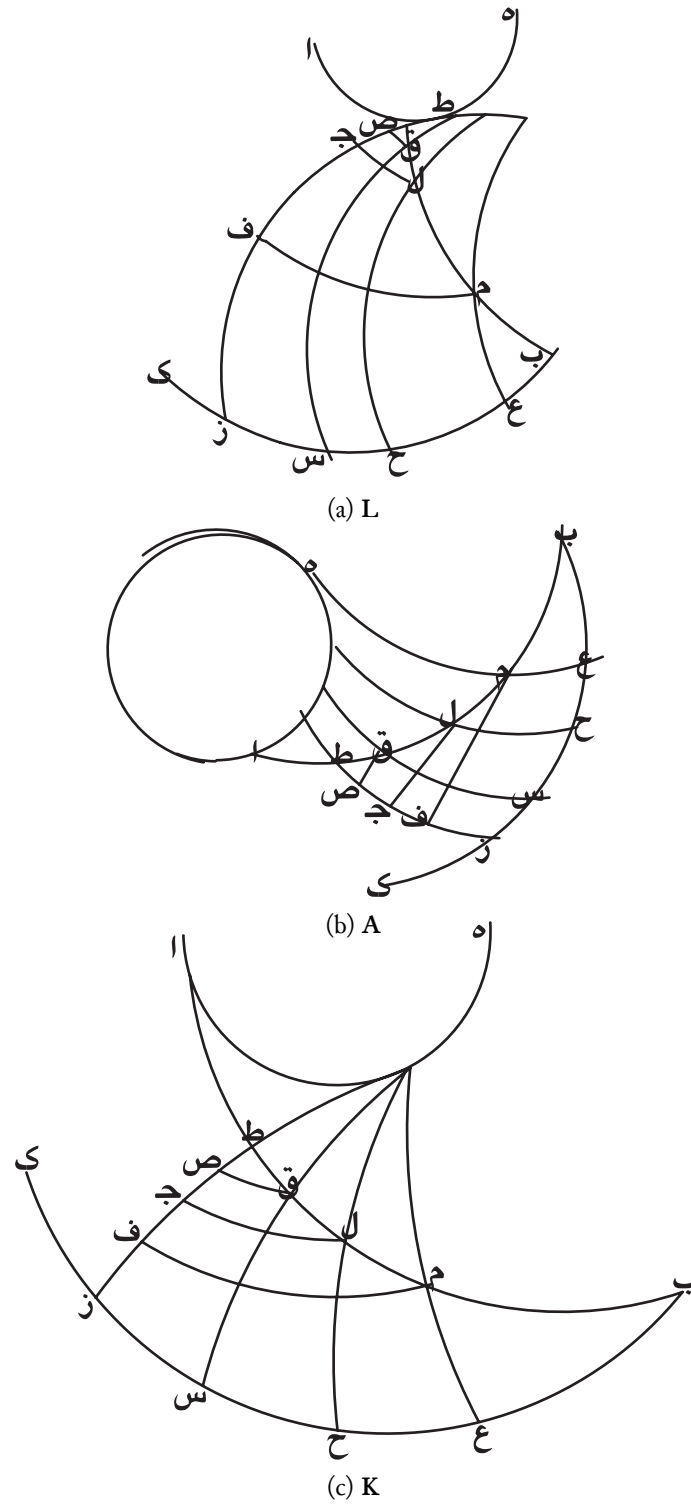
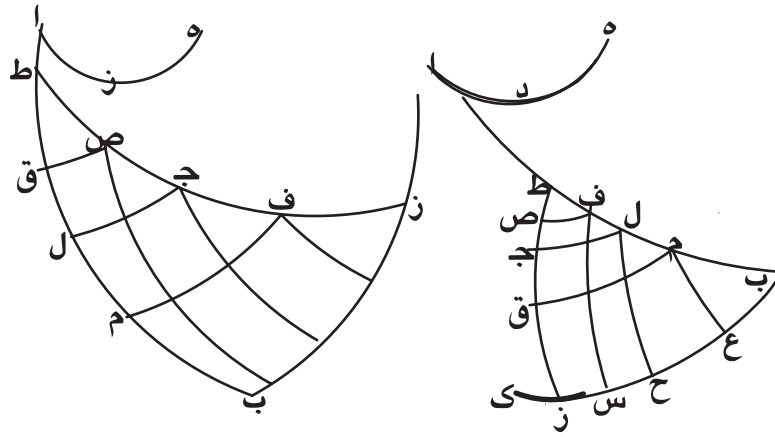
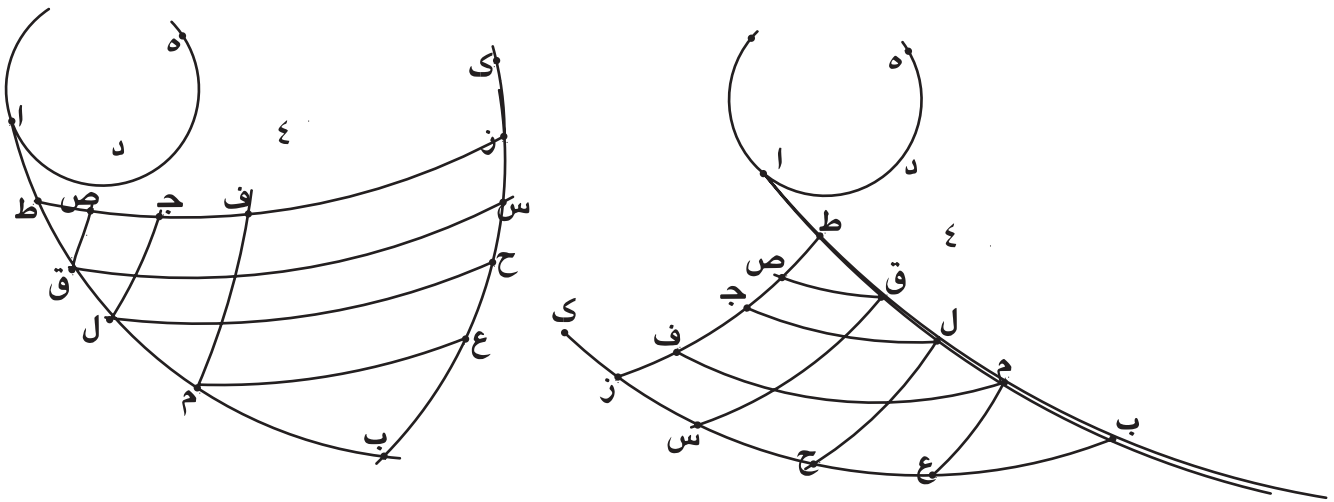


Figure 68: Book II, Prop. 3

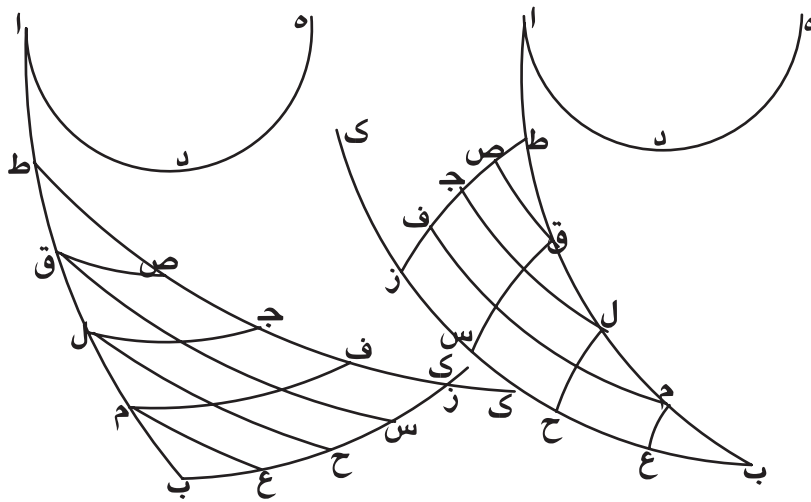
The tangency of circle $\overline{ا ه}$ with each of arcs $\overline{ط ز}$, $\overline{ق س}$, $\overline{ل ح}$ and $\overline{م ع}$ is depicted differently in the three figures, and none of them seems ideal. In L, only $\overline{ط ز}$ appears as tangent, and the other three arcs all appear to intersect this arc. In A, only $\overline{م ع}$ appears as tangent, at point $\overline{ع}$, and the other three arcs appear to cut circle $\overline{ا ه}$, which is not the case. Finally in K, arcs $\overline{ط ز}$, $\overline{ق س}$, $\overline{ل ح}$ and $\overline{م ع}$ appear to meet at a single point, which is not the case.



(a) L



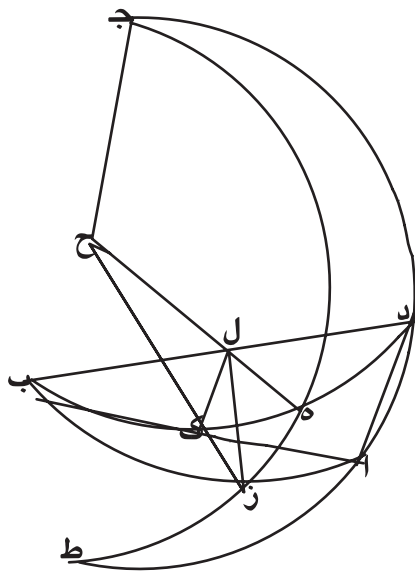
(b) A



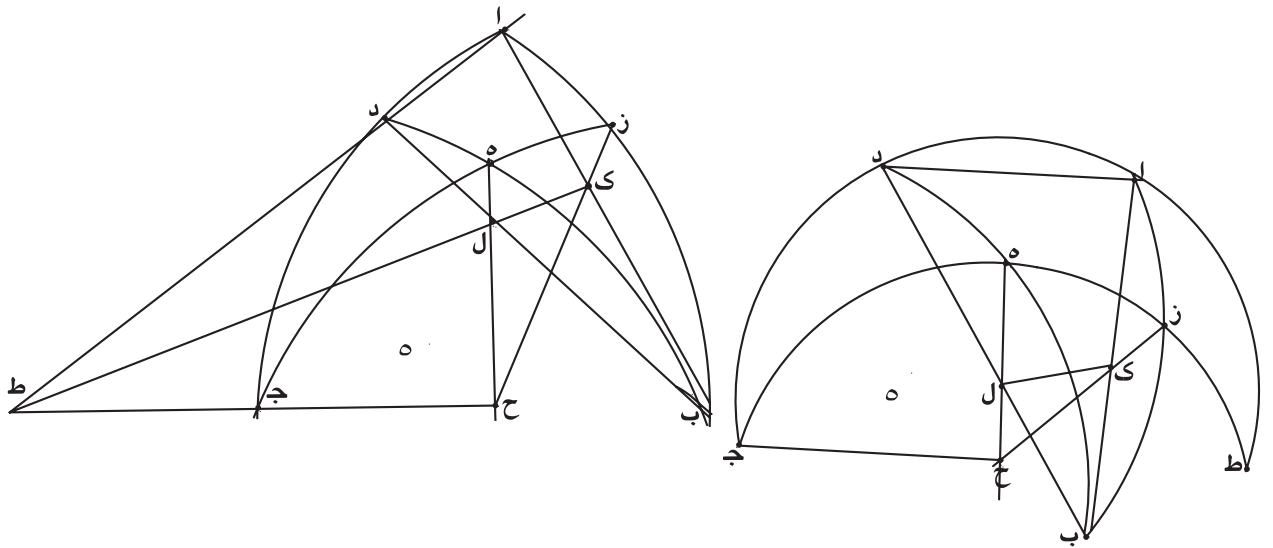
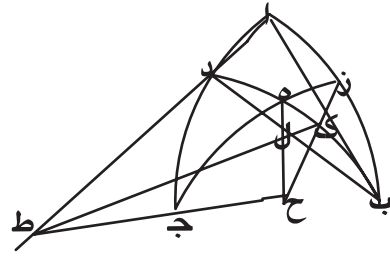
(c) K

Figure 69: Book II, Prop. 4

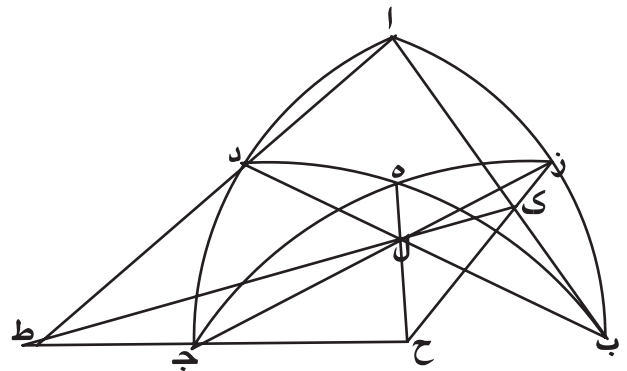
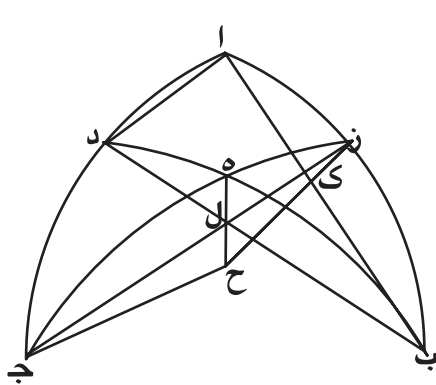
In A, the diagrams for II 4 and 5 are interchanged. This is noted, in red ink, with the numeral 4 as shown, and the word “previous” (مقدم), placed between the two figures.



(a) L



(b) A



(c) K

Figure 70: Book II, Prop. 5

In A, the diagram is quite large. The ⁵¹image has been reduced by 80% from its normal size to fit on the page. The interchange with II 4 has been noted, again, with the numeral 5 and the word “latter” (مؤخر), both in red ink.

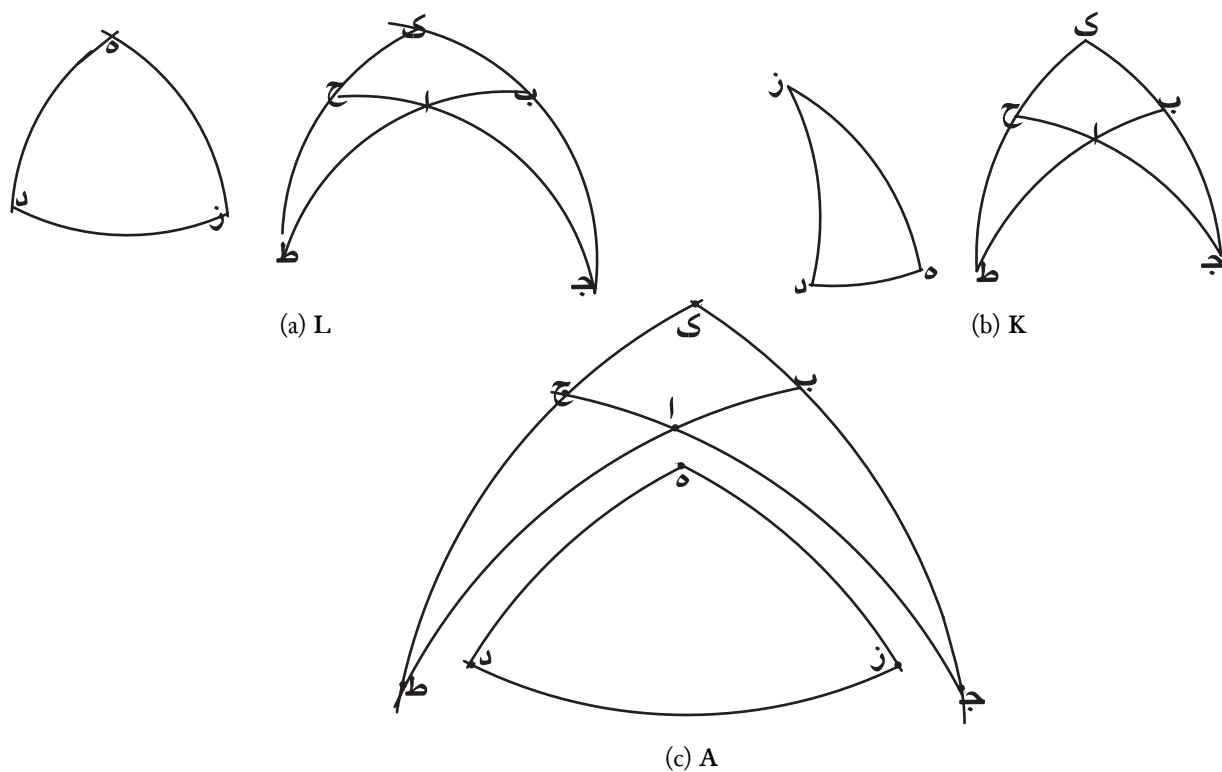


Figure 71: Book II, Prop. 7

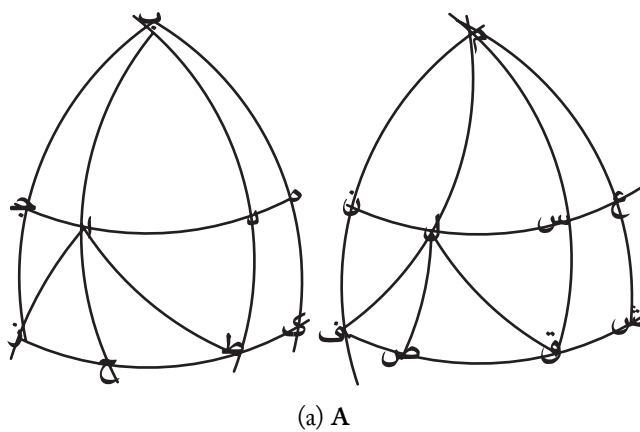
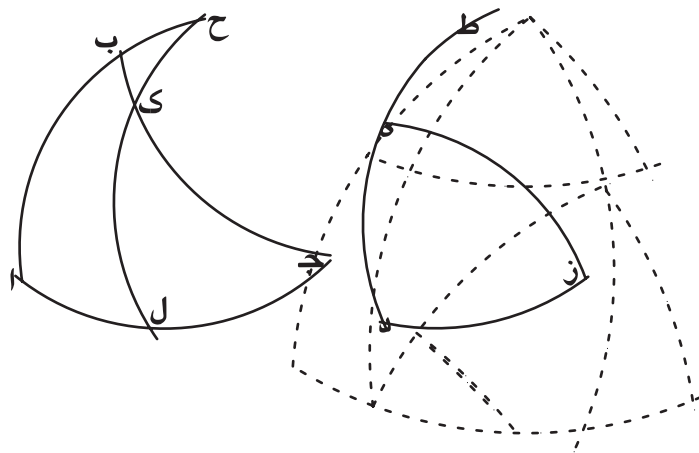
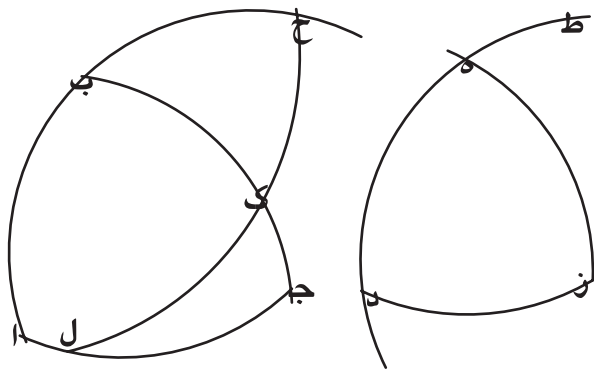


Figure 72: Book II, Prop. 10, Commentary

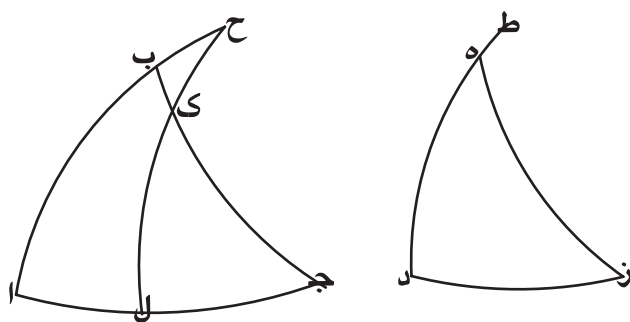
A includes an extra figure for the commentary following II 10, which attributes a second, defective argument to Menelaus. (Actually, both arguments are defective.)



(a) L

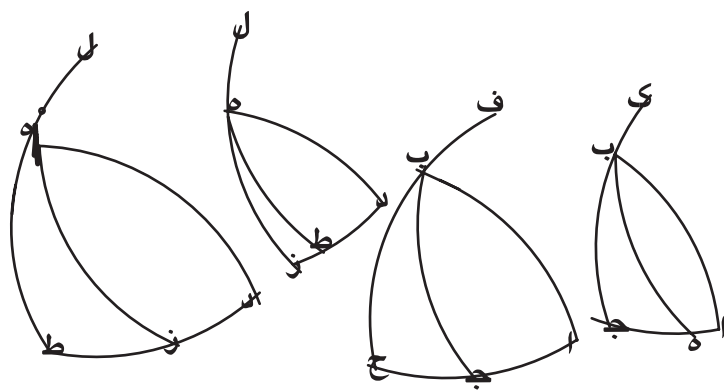


(b) A

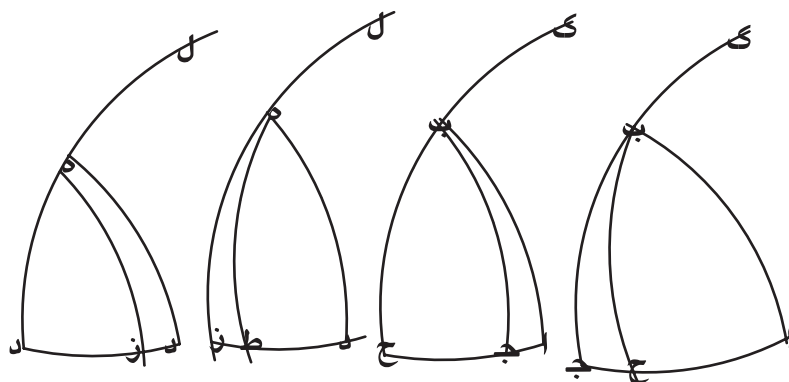


(c) K

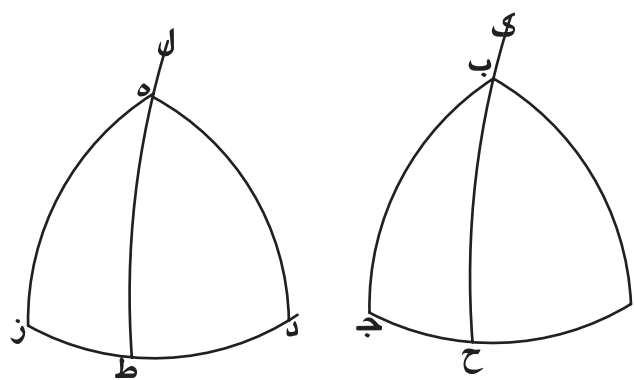
Figure 73: Book II, Prop. 8



(a) L



(b) A



(c) K

Figure 74: Book II, Prop. 9

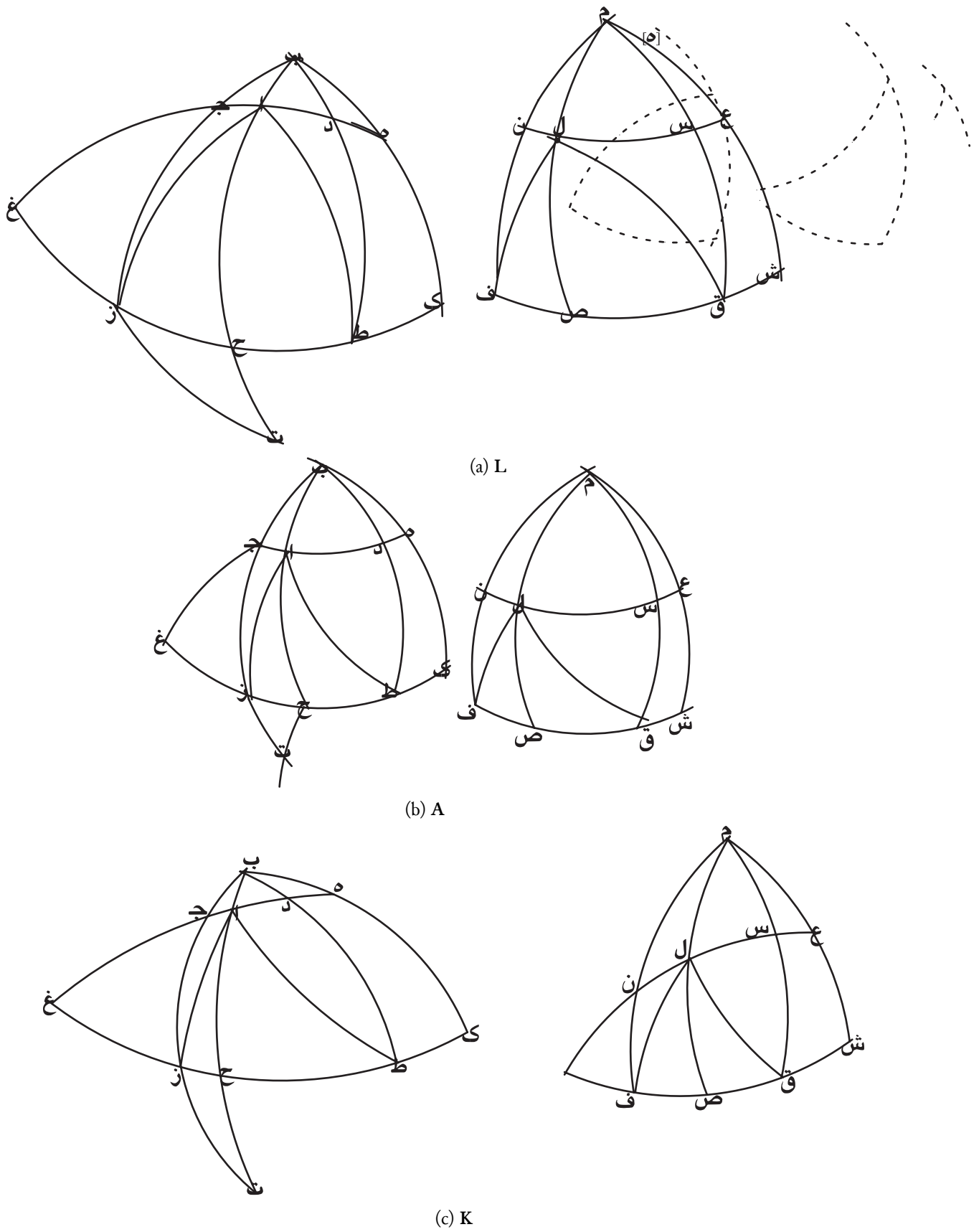


Figure 75: Book II, Prop. 10

A also includes an extra figure for the version attributed to Menelaus, see Fig. ??.

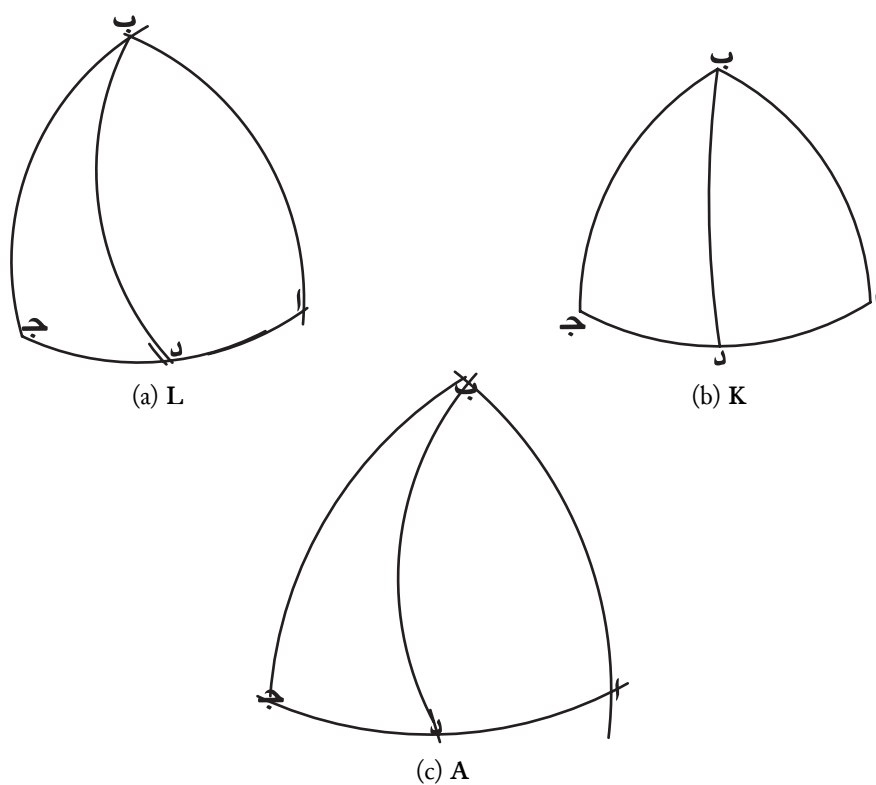


Figure 76: Book II, Prop. 11

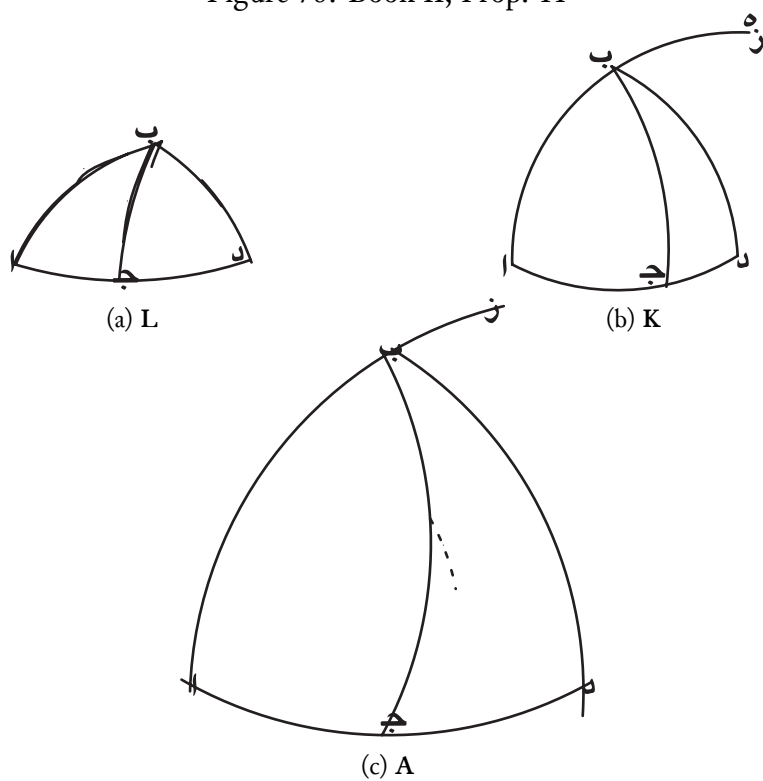


Figure 77: Book II, Prop. 12

In L, the figure is squeezed into the margin, as a sort of afterthought.

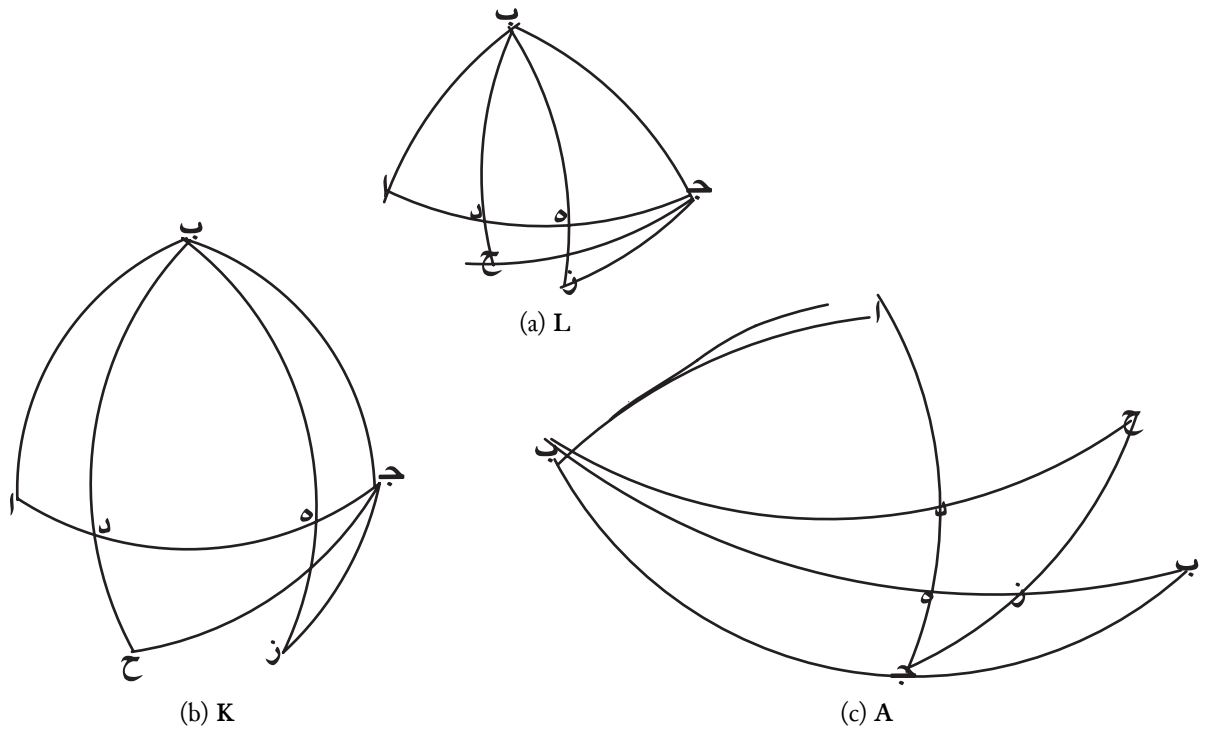


Figure 78: Book II, Prop. 13

In A, the labels $\bar{\text{ا}}$, $\bar{\text{ب}}$, $\bar{\text{ج}}$ and $\bar{\text{د}}$ were written by someone orientated to the right of the page.

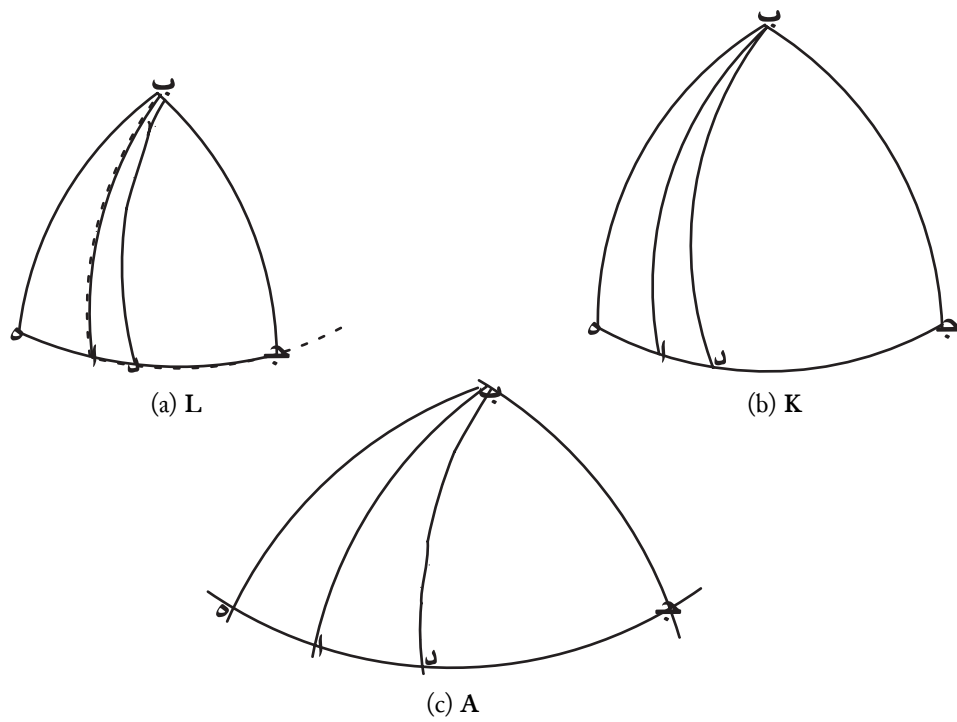
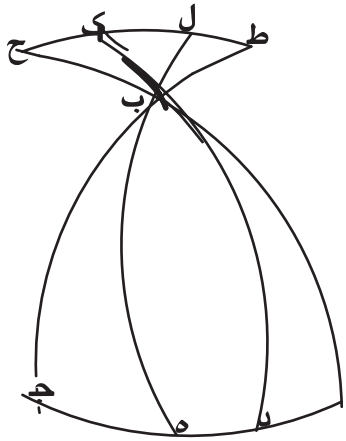
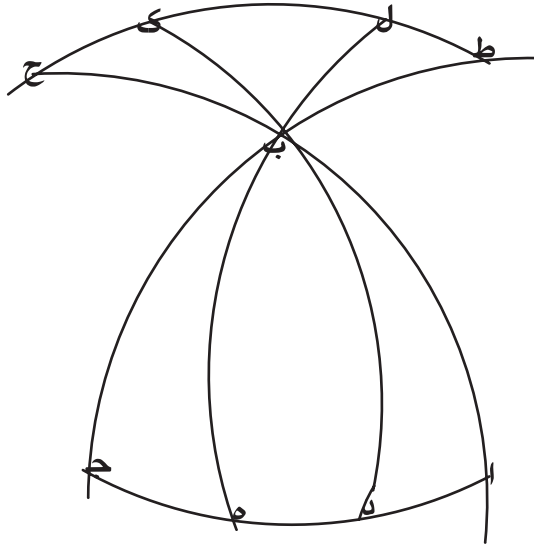


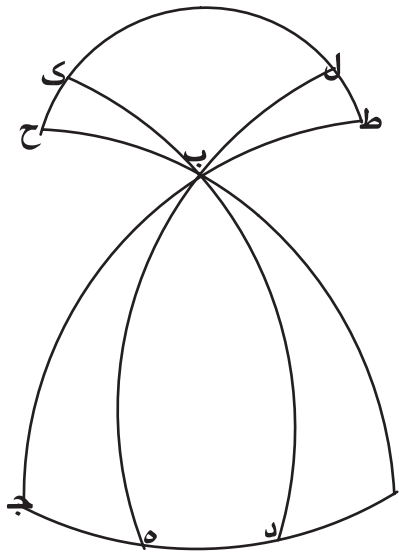
Figure 79: Book II, Prop. 15



(a) L

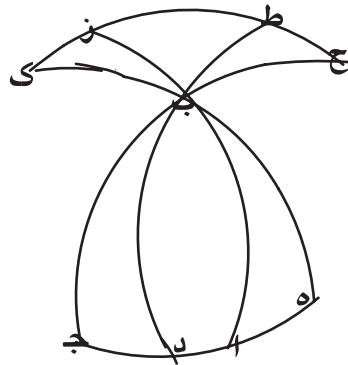


(b) A

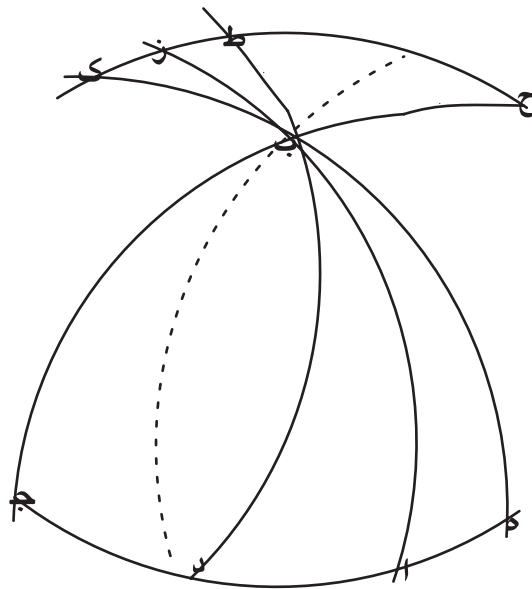


(c) K

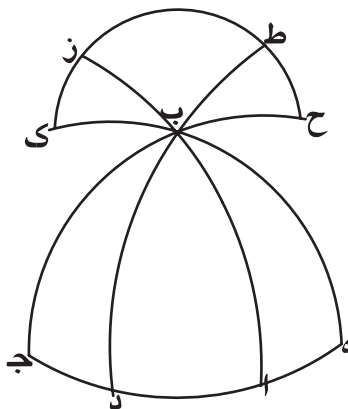
Figure 80: Book II, Prop. 14



(a) L



(b) A



(c) K

Figure 81: Book II, Prop. 16

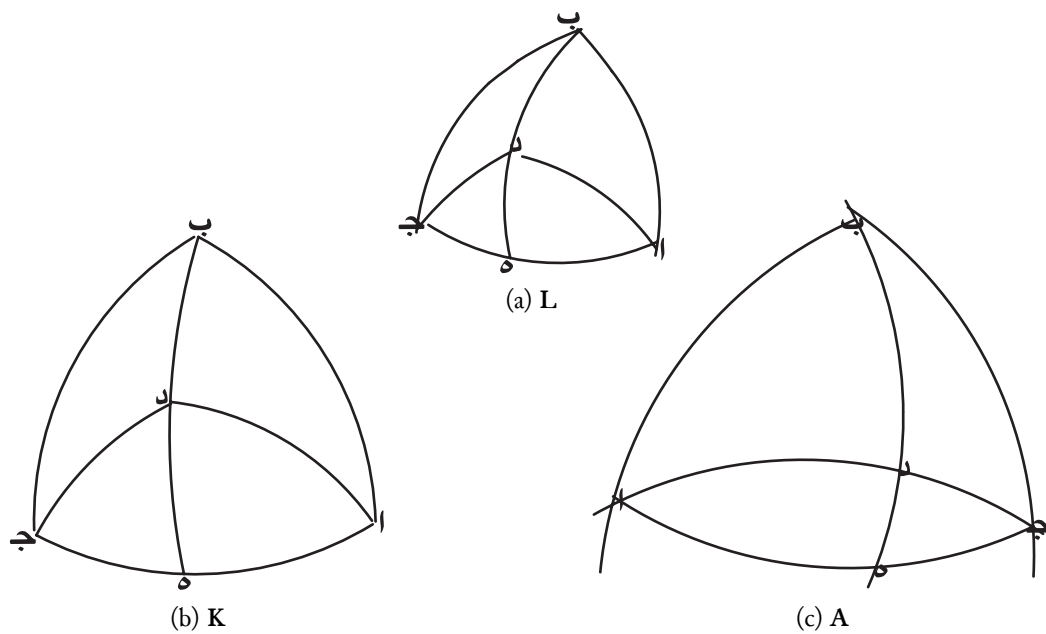


Figure 82: Book II, Prop. 17

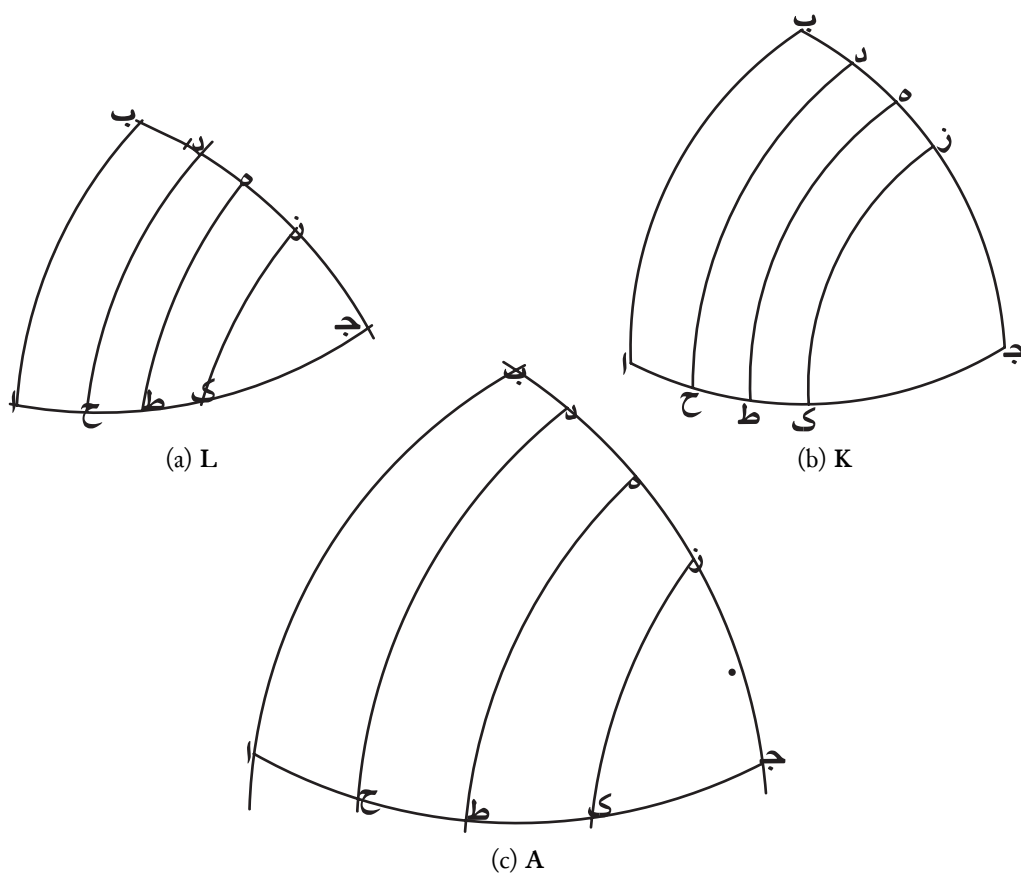
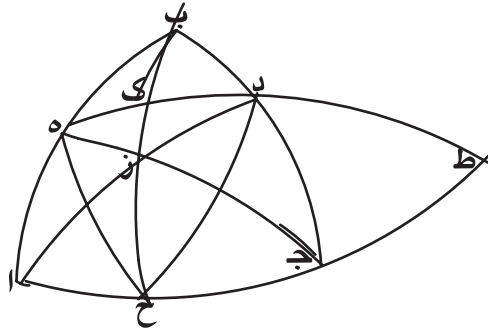
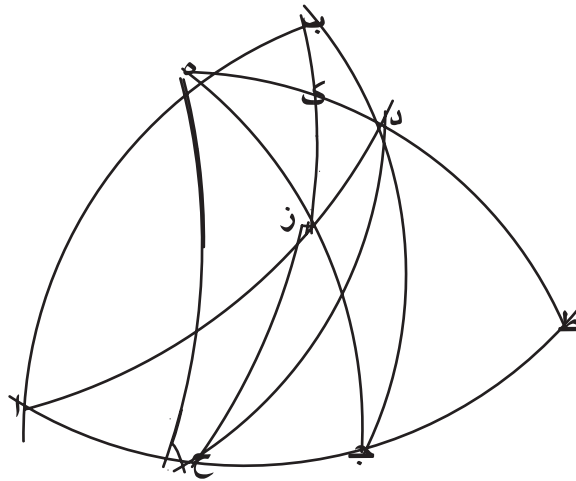


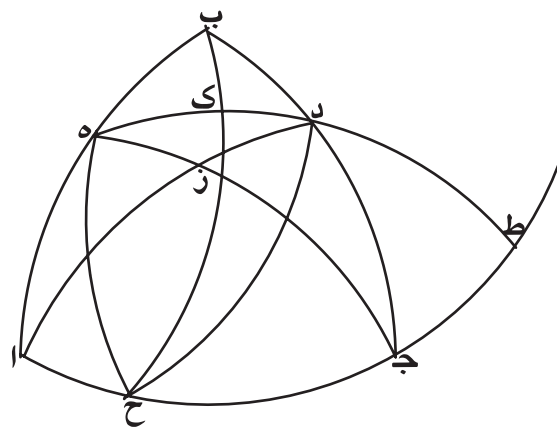
Figure 83: Book III, Prop. 1



(a) L



(b) A



(c) K

Figure 84: Book II, Prop. 18

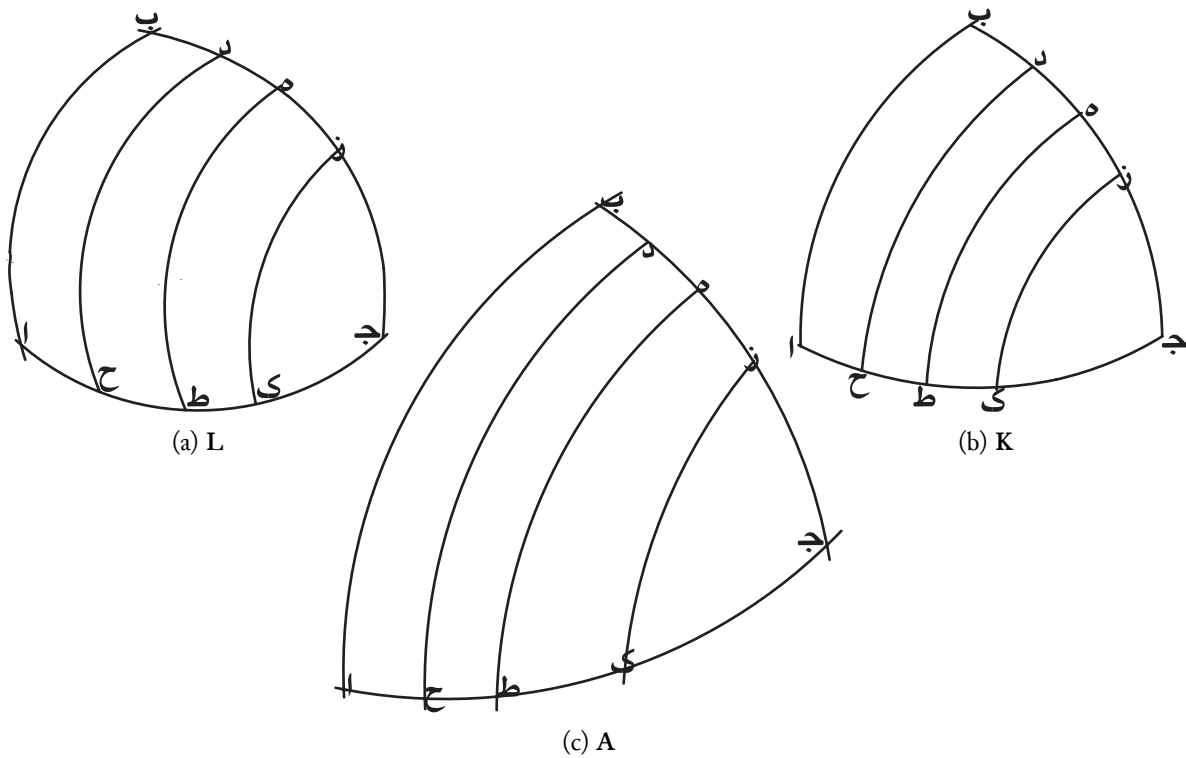


Figure 85: Book III, Prop. 2, First proof

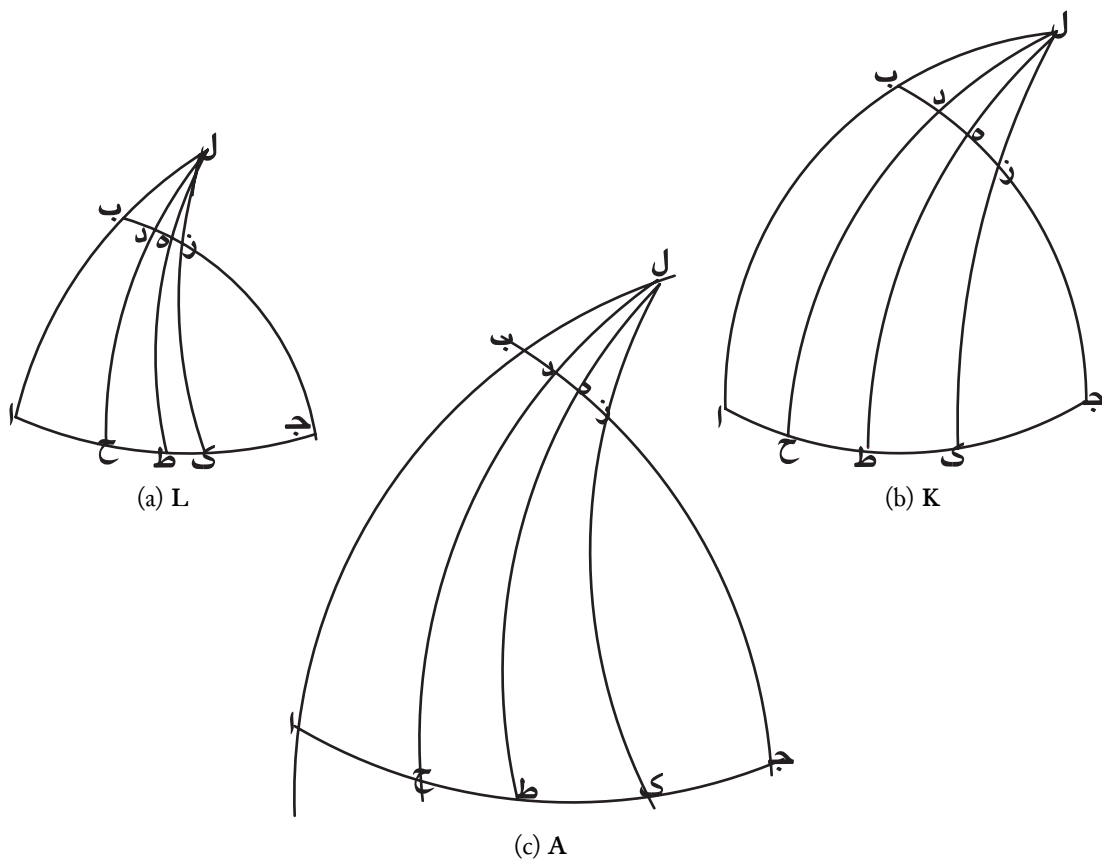
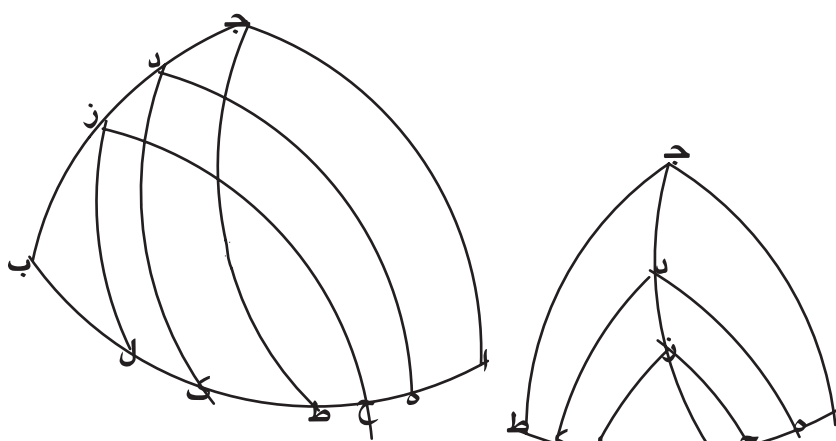
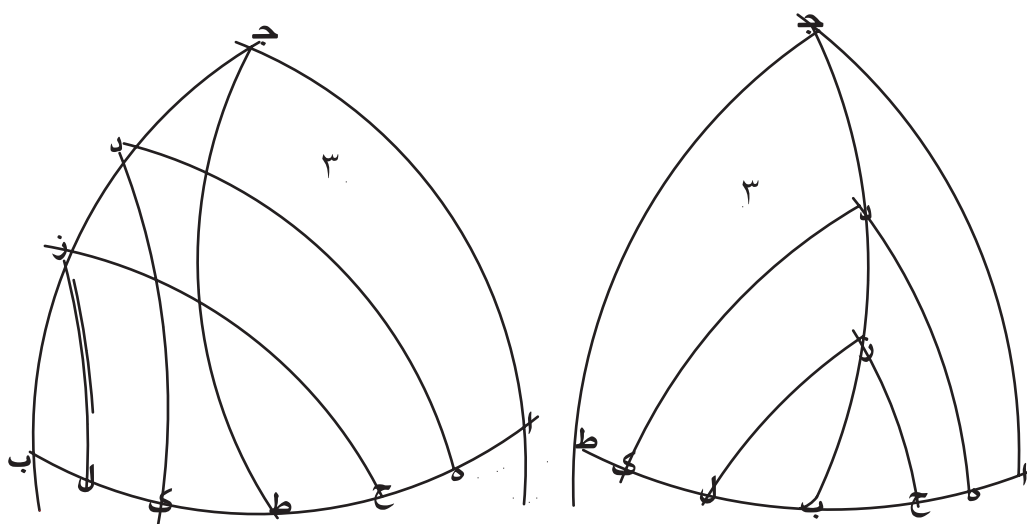


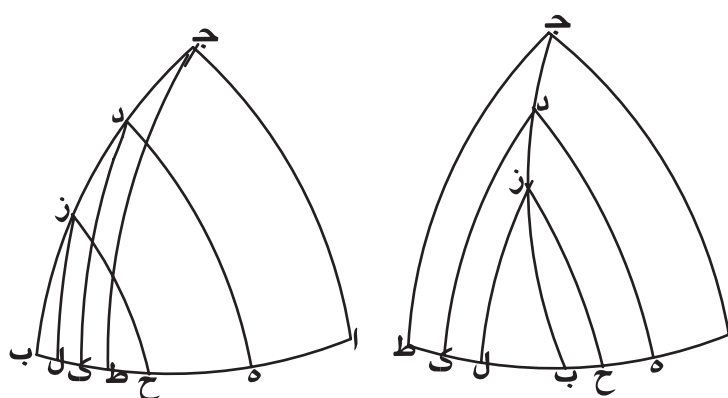
Figure 86: Book III, Prop. 2, Second proof



(a) L



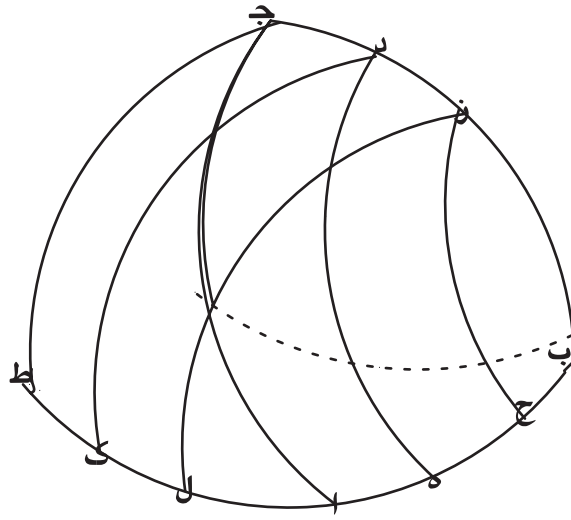
(b) A



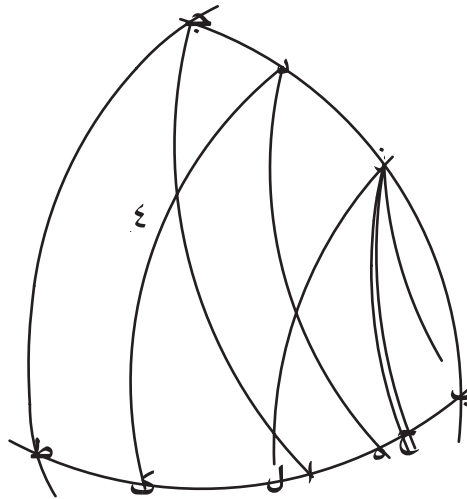
(c) K

Figure 87: Book III, Prop. 3

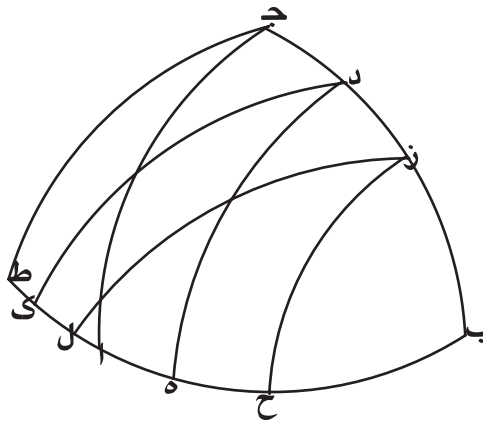
In A, the diagrams for III 3 and III 4 are in the same box and the left diagram for III 3 has been interchanged with the diagram for III 4. This was noted by a scribe with the numerals, written inside the figures in red ink. In the manuscript, the diagram for III 4, which follows, is found where we have placed the left figure for III 3. The diagram for III 3 is directly below these. 63



(a) L



(b) A



(c) K

Figure 88: Book II, Prop. 4

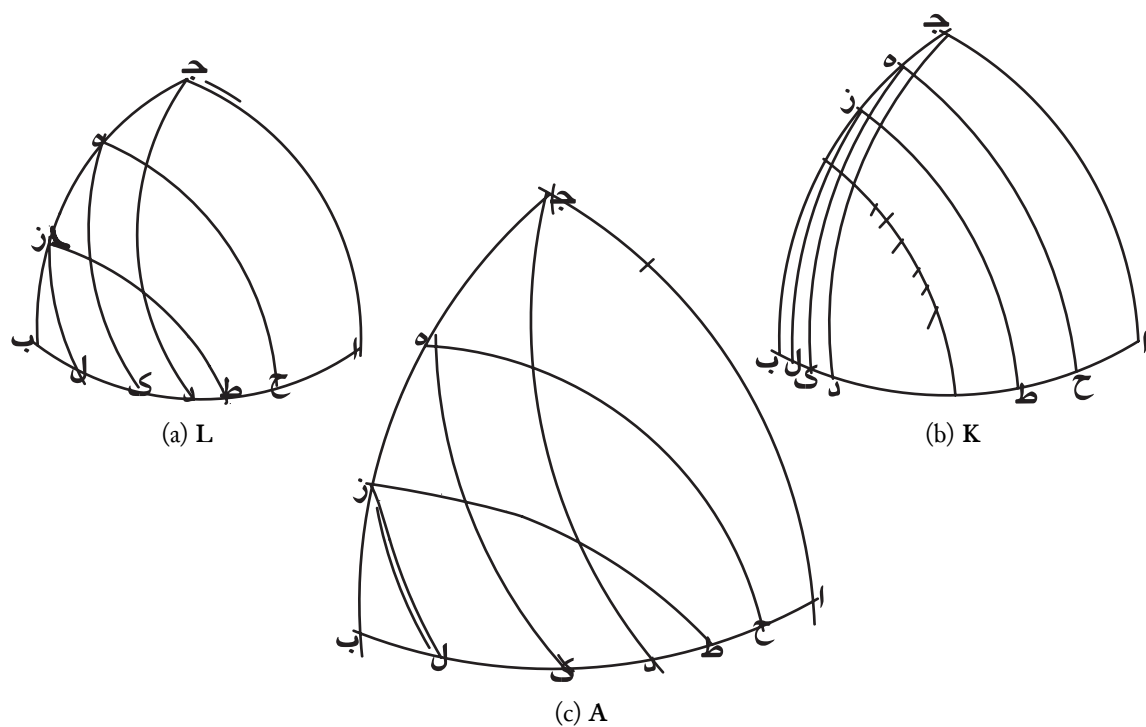


Figure 89: Book III, Prop. 5

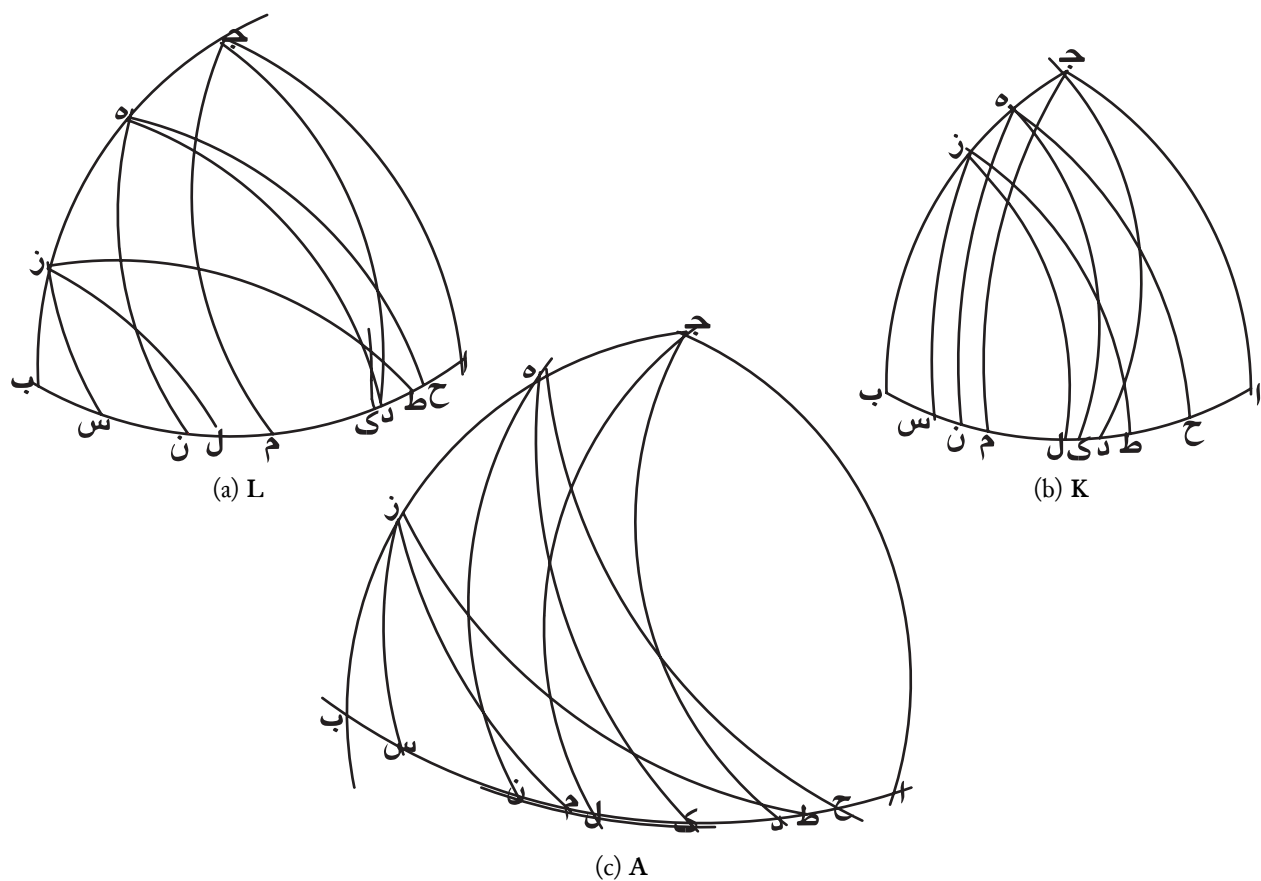
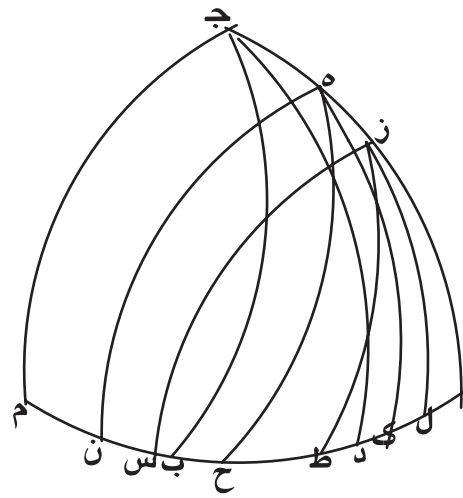
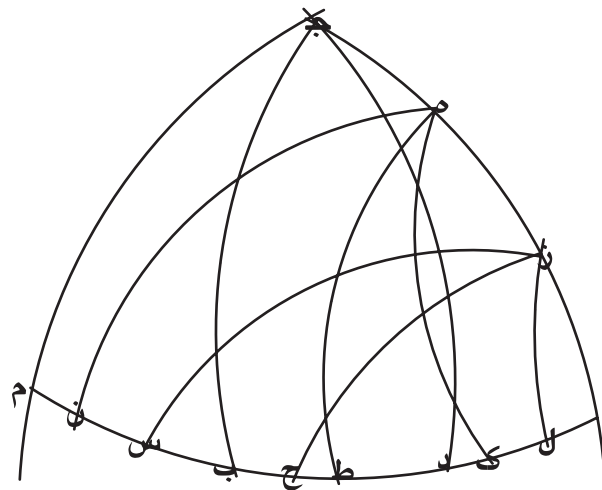


Figure 90: Book III, Prop. 6

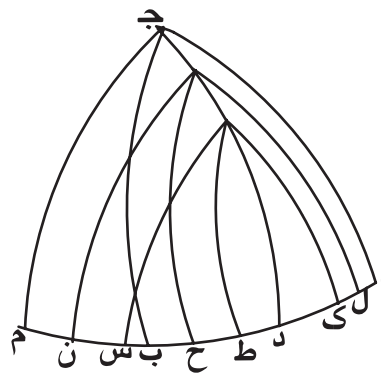
In L, the point labeled with $\bar{\rho}$ was first labeled $\bar{\rho}$ and then later corrected.



(a) L

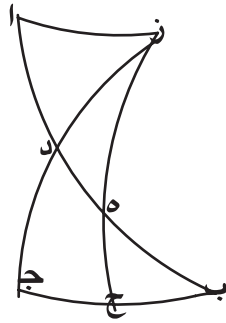


(b) A

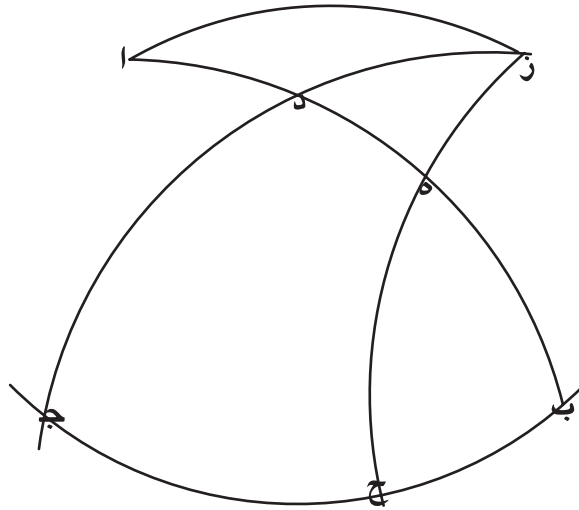


(c) K

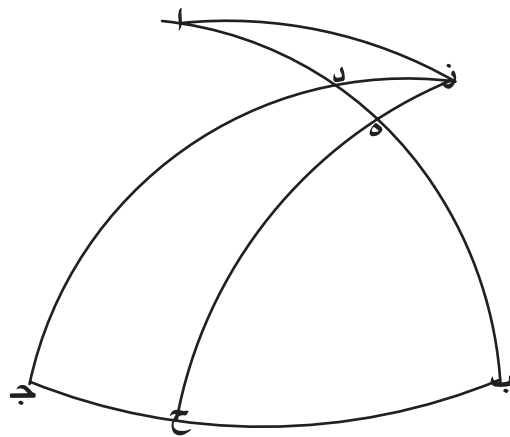
Figure 91: Book III, Prop. 7



(a) L

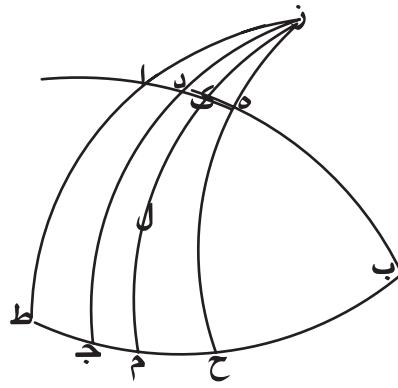


(b) A

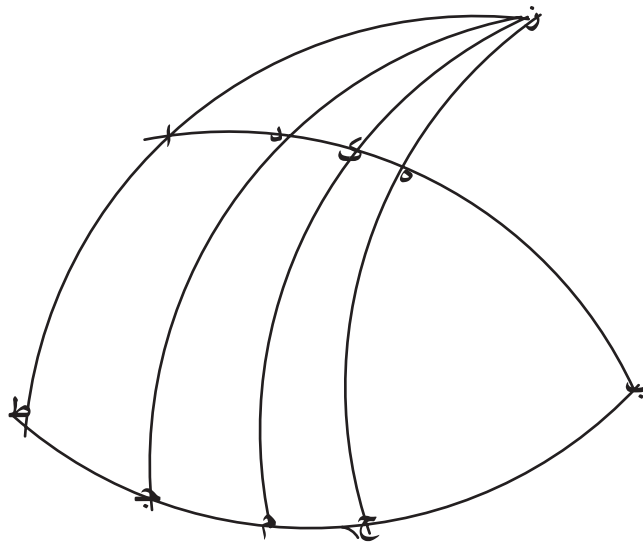


(c) K

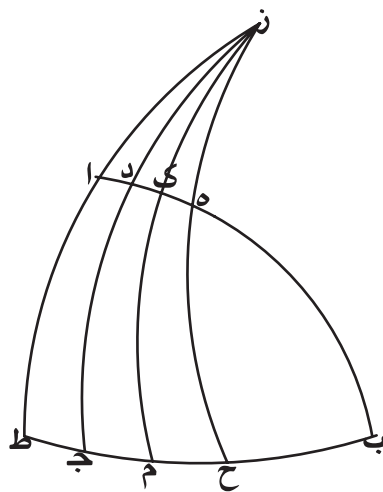
Figure 92: Book III, Prop. 8



(a) L

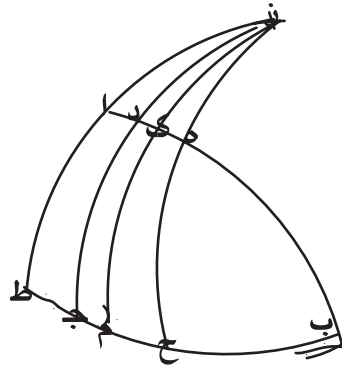


(b) A

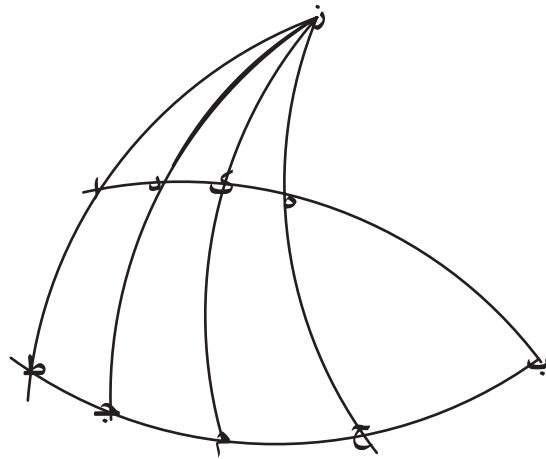


(c) K

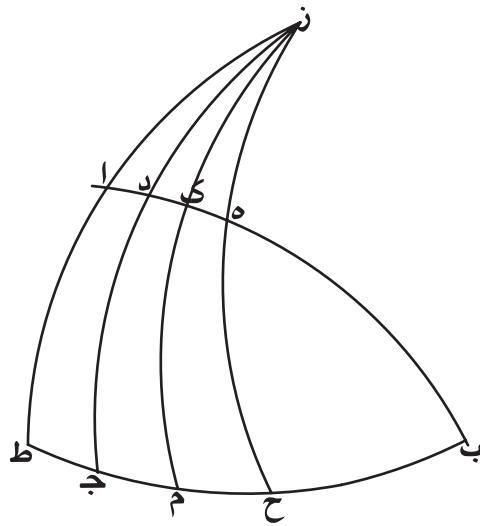
Figure 93: Book III, Prop. 9



(a) L



(b) A



(c) K

Figure 94: Book III, Prop. 10