BOOK REVIEW*

By Reviel Netz. *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*. Cambridge: Cambridge University Press, 1999, xvii + 327 pp., ISBN 0-521-62279-4 US$65 hardcover.

This book will be important and exciting for anyone interested in Greek mathematics and anyone interested in deduction in general. At present the Greek mathematical authors are read, if they are read, out of a purely historical interest. Certainly, in the classroom, very few instructors would think to teach young people about mathematics or deduction by taking them through one of the works of the ancient Greeks. Nevertheless, the Greeks did develop the first methods of deduction and, in one way or another, the rest of us formed our notions of what constitutes a proof in reaction to the texts that they produced. The habits of thought, the practices of language and the mathematical methods that inform these earliest deductive texts will have much to tell us about what we mean by proof, and how we go about proving. I make these claims because most of the pages of Netz's book are written for the specialist. The details of his arguments are directed at a reader well versed in the works of the Hellenistic geometers and sometimes even alert to the contingencies of the Ancient Greek language. Netz has, however, kept the non-specialist in mind and the introduction and summary of each chapter are intended to let one know where he is going and by what means he travels. He even lets it be known when a particularly thorny bit is coming up that one may want to skim over.

This is not a work in the history of Greek mathematics. Nor is it a study of Greek mathematical practices as a whole. It is a cognitive study of Greek geometrical texts that focuses on the work of the three great Hellenistic geometers: Euclid, Archimedes and Apollonius. Although the argument occasionally makes use of another author, such as Aristarchus (p. 40) or Autolycus (pp. 152–153), the reader feels a number of conspicuous vacancies. There is almost no mention of the arguably deductive, albeit

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idiosyncratic, work of Diophantus. The writings of Ptolemy and Hero, which are often deductive, are mentioned only in passing or in footnotes. Also, probably for practical reasons and because much of the study is linguistic, none of those Greek authors which survive to us only in Arabic translations, such as Diocles or Menelaus, are included. Wherever Netz speaks of Greek mathematics, we should read Hellenistic geometry. That said, the core texts of Hellenistic geometry are the place where we would begin to look if we were interested in how Greek mathematicians went about proving things, and Netz's reading of these texts is both new and informative.

Netz attempts to illuminate Greek deductive practices through a close study of the use of lettered diagrams and the use of mathematical language; two chapters are devoted to each study. Two critical chapters follow which aim to show that those deductive practices which the Greeks developed are able to successfully produce both necessity and generality. A final chapter situates the Greek mathematician in an historical and social context. I will make mention of some of Netz's more interesting claims and findings.

Netz's first investigation is into the use of the lettered diagram. Because, as Netz points out, "the diagrams of antiquity are not extant" (p. 12), he attempts to get around this difficulty by beginning his study with the text and asking whether we can construct the diagrams based on the text. What he finds, which may be surprising, is that in many cases we cannot. Many points, lines, and figures are, as far as concerns the text alone, what Netz calls "underspecified" or "completely unspecified" (pp. 19–26). The only way to get the information we need about these objects, in order to follow the proof, is to look at the diagram. In fact, information that we read directly off the diagram is often used in the course of a derivation (pp. 26–32) or as a starting point for reasoning (pp. 187–189). As a result of this, the text and the diagram are interrelated in a complex way. A linguistic and semiotic analysis of the way in which the text refers to the diagram leads Netz to the conclusion that the letters in the text, as in the phrase "the [line] AB", are used as Peircean indices and refer directly to the points, lines and objects in the concrete diagram (pp. 43–56). The implication of this claim is that the content of any individual proof is about the objects in the diagram to which it refers; and whatever it proves, it proves about these particular objects. While this assertion seems true, it opens the question of how the Greek geometrical propositions attained the generality that they claim. This question is one that Netz takes up in a later chapter devoted to the production of generality.

Netz advances our understanding of the relationship between the texts and the diagrams that accompany them by a study of the use of letters for names and naming. He finds that certain patterns emerge, such as the fact that objects are named alphabetically in the text or the fact that the names of rectangles admit of certain permutations but not of others (pp. 71–77). He notes, however, that these patterns are not strictly followed and where we find divergence it is often meaningful. Netz characterizes this state of textual and diagrammatic regularity as arising naturally from the discourse of a small group of readers and writers who are steeped in a corpus of texts unified by a common subject matter. He calls this state a "self-regulating conventionality" (pp. 71–79). It is only against the background of self-regulating conventionality that a text can deviate from what we would expect in meaningful ways (pp. 79–83). This material is an interesting study of the interrelationship of text and diagram as found in the Medieval manuscript tradition.

Unfortunately, Netz begs credulity by using the forgoing discussions to try to work out the temporal ordering of distinct processes such as the drawing of the diagram, the formulation of the proof, the lettering of the diagram, and the writing of the final text (pp. 83–86). Netz's conclusions would have us believe that the working habits of the Hellenistic geometers were all more or less the same, and that the texts and diagrams that we have, underwent no polishing or stylistic changes under the editorial and scribal hands through which we know they passed.

Netz's second investigation is into the use of mathematical language. He finds, perhaps surprisingly, that the terms defined in the definitions play little role in the bulk of the text, where the geometer is actually proving things. For example, a straight line in a mathematical text is not usually referred to by the words *hê eutheia grammê,* "the straight line", but simply by *hê eutheia AB*, "the straight AB", or very often simply the idiom *hê AB,* "the AB." The Greek geometer can make this ellipsis because he reserves the feminine article *hê*, in conjunction with two letters, for the idiom designating the line. The definitions, on the other hand, are a sort of second-order discourse: they tell us things we will want to know about the objects in question, and they are occasionally used as axioms to introduce necessity into a proof (pp. 91–103). The bulk of the text is written in a limited lexicon made up of such words as the lettered names, articles, prepositions and a small group of verbs and nouns; a small subset of these words makes up the great majority of the text (pp. 104–108). These texts avoid synonyms (pp. 108–113). Netz claims that this restriction of the mathematical lexicon is another example of a self-regulating convention. He goes on to look at how this narrow lexicon is employed in a set of idioms that differentiate the geometrical texts from other Greek texts with regard to the form of the language.

Netz calls this mathematical idiom a *formula*; a term he takes, and modifies, from Homeric scholarship. These are not mathematical formulae. Netz's formulae are words or phrases that are either "semantically marked" or "very markedly repetitive." By "very markedly repetitive," he means that the frequency of use is much higher in mathematical texts than in other Greek texts. By "semantically marked," he designates an idiom which is much more likely to express a particular idea than any equivalent expression (pp. 127–133). Netz collects the core group of these formulae and studies their behavior. What Netz finds is that these formulae have a hierarchic tree structure in which each formula is a node and the branches lead down to further formula which in turn may serve as further nodes. In this way, Netz shows that every proposition can be represented as a tree graph with the proposition itself at the top as the original trunk. The next level of nodes is occupied by the basic divisions of the proposition: the enunciation, the exposition, the construction, the proof and so forth. The tree branches out from these into formulae of greater complexity which contain formulae of lesser complexity until finally we reach the bottom level occupied by those formulae that express simple objects such as points and lines (pp. 133–148).

Formulae, like lexical restrictions, are not slavishly followed. They are transformed both through ellipsis and extension. In fact, they are flexible enough to be extended to new results and entirely new domains of research. The form and function of linguistic formulae give access and deductive transparency to the Greek geometric texts (pp. 148–158). They are accessible because they are composed out of simple objects; they are transparent "because their form mirrors logical relations" (p. 158).

The use of the lettered diagram and the application of a lexically restricted formulaic language are the practices out of which the Greek geometers construct necessity. Necessity is carried through the proof by strings of short arguments that take a variety of different "starting points" as givens. Necessity is injected into each argument though the use of the following types of starting points: (1) explicit references to previous results (which are extremely rare and can often be explained away as scribal or editorial interpolations), (2) the use of a "tool-box" of known results that are invoked in a formulaic way, (3) references to the diagram, and (4) mathematical intuition (pp. 168–198). Netz produces a graphical representation of the logical structure of Greek proofs, compares this structure to the structure found through his analysis of linguistic formulae, and studies the overall structure of Greek geometric proofs (pp. 193–216). He ends his chapter on the shaping of necessity with a discussion of the tool-box. The tool-box is a term used by Ken Saito to designate a non-codified body of known results that are invoked by Greek geometers in the course of their proofs.[1] These are results that Greek geometers felt they could require their readers to know. Netz finds, as we might expect, that the fundamental tool-box is coextensive with certain books of Euclid's *Elements,* particularly books I, III, V, and VI.

One may well wonder why book II is not included in this list, but this is an accident of Netz's selection. Had his study used book II or III of Apollonius' *Conics,* instead of book I, he would have found extensive use of the results of *Elements*, II.

It was noted above that the fact that Greek proofs are about particular objects raises the question of how these proofs produce the generality which they assert. Netz's answer to this question, although not likely one that a Greek thinker would put forward, is both interesting and helpful. Taking Mueller's study as his starting point,[2] Netz shows that Greek proofs are general because they demonstrate that the proof itself can be readily repeated for any other objects that satisfy the conditions of the proof. The proposition as a whole states that if we have some particular objects that satisfy the conditions of the propositions, then a particular proof can be written for these objects; generality is produced by the further insight that this proof is constructed in such a way that it can be repeated for any other like objects. The virtue of this explanation is that it stresses an aspect of Greek geometry that is often neglected: the ability of the geometer to "do" or to "make"; for example, the ability to do a construction or to produce a proof.

The final chapter starts out strong with an attempt to numerate the ancient mathematicians and locate them in their social context. It discusses the very small number of active mathematicians, their narrow position in the leisured classes, and the limited role of mathematics in the ancient curriculum (pp. 271–292). Unfortunately, in the following sections, in an attempt to place Greek mathematics within the context of Greek intellectual culture, the discussion degenerates into statements of such sweeping generality as to be neither interesting nor palatable. One is repeatedly struck by the fact that Netz's study has really only been about a few Hellenistic geometers and yet he insists on applying its results to all Greek mathematicians (pp. 292–311). These final sections, along with the introduction, are probably the weakest part of the book.

The great majority of Netz's book is a very fine reading of the core texts of Hellenistic geometry which gives us insight into the production of both necessity and generality in deductive processes. His claim to have revealed the detailed practices of Greek geometers should, however, be read with some skepticism, and his attempt to generalize from a limited selection of texts and authors to all Greek mathematics is, I think, in principle, unsound.

NOTES

1. Information on Saito's tool-box project can be found at http://wwwhs.cias.osakafu-u.ac.jp/~ksaito/.

2. Mueller, I. (1981) *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. Cambridge, MA.

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