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The Role of Geometrical Construction in Theodosius's *Spherics*

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Abstract This paper is a contribution to our understanding of the constructive nature of Greek geometry. By studying the role of constructive processes in Theodoius's *Spherics*, we uncover a difference in the function of constructions and problems in the deductive framework of Greek mathematics. In particular, we show that geometric problems originated in the practical issues involved in actually making diagrams, whereas constructions are abstractions of these processes that are used to introduce objects not given at the outset, so that their properties can be used in the argument. We conclude by discussing, more generally, ancient Greek interests in the practical methods of producing diagrams.

1 Introduction

This paper is a contribution to our understanding of the role of constructions and problems in Greek geometry.[1] The discussion of these issues by scholars has been

[1] In this paper, we use the term *problem* in the technical sense of a proposition of Greek geometry that sets out to perform a specific construction and then demonstrates that this construction is valid (for example, *Elem.* I 1). A *theorem*, on the other hand, shows that given some initial set of objects and conditions, some statement is true of these things (for example, *Elem.* I 5). This distinction between problems and theorems was discussed by Proclus in his commentary to the first book of Euclid's *Elements* (Friedlein 1873, pp. 77–81; Morrow 1970, pp. 63–67). As distinguished from a problem, a *construction* is a component of a Greek geometrical proposition found in every problem and in nearly every theorem (for example, the use of *Elem.* I 1 in the problem *Elem.* I 2, or the use of the problem *Elem.* I 3 in the theorem *Elem.* I 5). Again, see Proclus's commentary (Friedlein 1873, pp. 203–210; Morrow 1970, pp. 159–164). The parts of a Greek proposition are also discussed, for example, by Mueller (1981, pp. 11–13) and Netz (1999b).

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dominated by the claim made by Zeuthen (1896) that constructions were meant by Greek mathematicians to serve as proofs of the existence of key mathematical objects. Although few historians of Greek mathematics have accepted this idea, most of the work on constructions and problems has still centered around this as the central debate. For example, Knorr (1983) shows that whereas some problems do seem to be existence proofs, others can clearly not be read in this way. In fact, he argues that problems perform a range of functions in Greek mathematical texts. In this paper, we explore the ways in which problems and constructions are used in Theodosius's *Spherics*.[2]

Because the discussion has generally focused on the *Elements*, scholars have tended to conflate constructions with problems.[3] In the *Elements*, every construction makes use of problems that have already been demonstrated or one of the three construction postulates, *Elem.* I posts. 1–3. Hence, as Mueller (1981, pp. 15–41) has shown, the Euclidean problems are a fundamental feature of the deductive structure of the early books. As Knorr (1983, pp. 129–130) has pointed out, however, in other mathematical texts, even in other books of the *Elements*, problems play a variety of other roles as well. In particular, problems, like theorems, can be both goals of mathematical research and important tools in the production of new mathematical knowledge. Indeed, a Greek proposition, whether a theorem or a problem, can state a more or less inherently interesting result or it can be more or less useful in the development of further propositions, and the most significant propositions are both.[4] Constructions, however, when they are found in a theorem, are used to mobilize geometric objects so that their properties can be used in the course of the proof.[5] For example, the con-

[2] We have used both the editions of Heiberg (1927) and Czinczenheim (2000) since they are rather different in places. The translation of Heiberg's edition by Ver Eecke (1959), with its accompanying notes, is from a mathematical standpoint generally more useful than Czinczenheim's.

[3] Zeuthen (1896), for example, based his entire argument on *Elem.* I and Harari (2003) based her refutation of Zuethen on some considerations of Aristotelean philosophy and a single theorem in book I, *Elem.* I 5.

[4] Although these distinctions are somewhat subjective, the most important theorems, such as *Elem.* I 47, the so-called Pythagorean theorem, state results that act as the culmination of a certain part of the theory, but in turn are frequently used to develop new results (Heiberg and Stamatis 1969, vol. 1, pp. 63–65). There are also examples of theorems, such as Archimedes' *Sphere and Cylinder* I 34 on the measuration relations between a sphere and a containing cylinder, that state interesting results but are not of further use in the work (Heiberg 1910–1915, pp. 124–132). Finally, there are theorems such as those on the application of areas, *Elem.* II 5 & 6, whose importance derives from their frequent application in other areas of research (Heiberg and Stamatis 1969, vol. 1, pp. 73–76). As we will argue in this paper, these distinctions also hold true for the problems in Greek geometric texts. For example, the construction which sets out the sides of the regular solids, *Elem.* XIII 18, is interesting but not of further use, while the early problems in a number of books, such as *Elem.* I and III, are essential for the development of the theory, but not terribly interesting (Heiberg and Stamatis 1969, vol. 4, pp. 180–186, vol. 1 pp. 7–10, 94–95). An example from the *Elements* of a problem that is both a goal and a tool is *Elem.* II 14, to construct a square equal to a given rectilinear area (Heiberg and Stamatis 1969, pp. 91–92). In this paper, we will explore ways in which the problems in the *Spherics* also perform these dual roles.

[5] Harari (2003, p. 21) characterizes this mobilization as the introduction of new content as a means of measurement (by introducing given objects) or as a means of explicating spatial relationships. In fact, however, constructions are also used as a straightforward means of mobilizing definitions and previous theorems. Consider, for example, the use of the construction of a parallel line in *Elem.* I 32 (Heiberg and Stamatis 1969, vol. 1, pp. 44–45). At the beginning of this theorem, the geometer simply has a given triangle. Certain theorems about parallel lines have already been shown and would be useful for saying something about the equality of certain angles, but there are no parallel lines given, so they cannot be applied. By constructing a

struction of a circle allows the geometer to use the equality of the radii as stated stated in the definition, while the construction of a parallel line allows the geometer to use the properties of parallel lines established in the course of the theorems. Indeed, since the properties of the objects set out in the enunciation are generally insufficient to carry out the demonstration, it is the construction that allows the geometer to introduce the properties of other objects as stated in the definitions or demonstrated in the theorems.

In the *Spherics*, the difference between constructions and problems is more apparent than in the *Elements*. The arguments in the *Spherics* rely on a number of different constructive techniques. They make use of a set of unstated postulates, which, as we argue below, are generalizations of the Euclidian postulates and were probably meant to be understood *as* the Euclidian postulates. They use solid constructions, such as passing a cutting plane, which are not justified by any problem in the *Elements*, but which are used in other texts on solid geometry. They use the problems demonstrated in the *Elements* as well as the problems previously demonstrated in the *Spherics*. Moreover, as we will see below, there are constructions in the *Spherics* that produce the same objects as produced in a problem, but which do so without relying on that problem.[6] From this, it is clear that in the *Spherics*, the use of constructions in writing proofs was distinct from the use of constructions in showing how to solve a given problem.[7]

In the *Spherics*, all but one of the problems are carried out with a limited set of techniques that can actually be performed on the surface of a globe using the tools of elementary geometry.[8] Even those constructions used to justify these problems, however, do not have this limitation. In this way, the problems serve a practical purpose, not necessarily found in the constructions. This use of problems to show how to carry out a particular construction and the use of constructions, in turn, to write proofs highlights a distinction between the practical and theoretical aspects of the constructive nature of Greek geometry. That is, the techniques used in solving a problem are

Footnote 5 continued
parallel line, as shown in *Elem.* I 31, the geometer makes these theorems available for use in the argument. They are now not only true, but also useful.

[6] Examples of these different types of constructions can be found below, in our discussion of the problems.

[7] Greek geometrical texts do not distinguish between a problem and its solution. For our purposes, however, it will be useful to speak of the construction provided in a problem as its solution and the proof that this construction is valid as its demonstration. Indeed, as will be seen below, it is even possible within a given problem to distinguish between constructions that serve the solution and constructions that serve the demonstration (see the discussions of *Spher.* 18 & 19, below).

[8] Although never explicitly discussed, the construction postulates of the *Elements* effectively act as the logical foundation for the use of an unmarked ruler and a collapsing compass as the basic tools of construction. As we will show below, Theodosius appears to limit the constructions used to solve the problems in the *Spherics* to tools very similar to these, such as an unmarked ruler and a non-collapsing compass or, perhaps, a pair of calipers. The practical aspect of the problems in the *Spherics* was pointed out by Schmidt (1943, pp. 13–14). Neugebauer (1975, pp. 755–756) and Berggren (1991, p. 247) have objected to some gaps in the exposition of these practical constructions, but, as we show below, these steps can be justified by an appeal to the problems in the *Elements*. An explicit discussion of the use of a compass to draw circles on a sphere is found in a tenth century text by Muḥammad ibn Muḥammad ibn Yaḥyā al-Būzjānī Abū al-Wafāʾ (Woepcke 1855, p. 353). Although this text was written much later than Theodosius's *Spherics*, it is unlikely that the technology of compasses and globes had changed much even in this long interval.

constrained by the possible operations of some assumed set of geometric tools, while constructions employed in proofs—indeed, even the constructions employed in the proof of a problem—are not subject to this constraint.

In this paper, we explore this distinction between the different roles of construction by examining the details of how the problems are both solved and demonstrated and how constructions are used, both in theorems and in problems, to mobilize geometric objects for the geometer's deductive goals. Before we look at the problems themselves, however, it will be useful to survey briefly the *Spherics* as a whole.

2 Overview of the *Spherics*

The Greek text, as we have it now, is clearly the work of many hands over the course of many centuries.[9] Already when Theodosius first composed the treatise, sometime between the third and the middle of the first century BCE,[10] he reworked, and reorganized, material that was well-known and perhaps scattered around in various texts.[11] The *Spherics* was intended for readers who could be expected to know no more than the most basic plane and solid geometry of the *Elements* and, as such, it appears to have successfully secured a place in the canon of elementary geometric texts. Hence, it was read, commented upon, and rewritten so many times that it is now impossible to say precisely which words were written by Theodosius and which by other authors, earlier or later.

Through this process, comments and additional proofs, which were originally written in the margins of the manuscripts, were worked into the body of the treatise.[12] In fact, however, because of his didactic goals, when Theodosius originally drafted the *Spherics*, he may have written out many arguments as explicitly as possible. Certainly, passages that add simplified explanations, citations of the *Elements* or previous theorems of the *Spherics*,[13] and reiterated arguments are so prevalent that it would be difficult to attribute all of this material to the activities of later editors.[14] Moreover, it is

[9] The medieval transmission of the Greek text is covered by Czinczenheim (2000, pp. 180–377). The transmission of the work in Latin, Arabic and Hebrew is also more briefly described by Lorch (1996).

[10] For Theodosius's date see Czinczenheim (2000, pp. 10–11).

[11] Neugebauer (1975, p. 750, n. 23), in the course of justly criticizing the search for the *Urschriften* of the various surviving works on spherical astronomy, gives a long list of scholars who have attempted to identify the contents of these earlier, lost treatises. Although it is probably pointless to try to make definitive claims about the content and structure of lost works, there can be little doubt that there were previous treatises on spherical geometry.

[12] Clear examples of this are the second case of *Spher.* II 15, which begins with the remark "If someone says" (see footnote 57, below), or the final two theorems of the first book, *Spher.* I 22 & 23, which are obvious interpolations and are not found in the Arabic versions.

[13] In a Greek mathematical text, if a previous theorem is directly cited this is done by reiterating the enunciation either in full or in summary. The few cases where previous theorems are cited by book and number are generally held to be late interpolations.

[14] Many of these kinds of passages can, indeed, be identified as late additions, as can be seen by examining the passages marked as additions by Heiberg (1927), or Czinczenheim (2000), in conjunction with the critical notes concerning these passages. Nevertheless, there are numerous similar passages for which there is no manuscript evidence that can be brought to bear one way or the other.

still possible to detect in the text itself the traces of Theodosius's project of reworking older material in a new organization.[15] The result is that the text often spells out the details of the argument at an almost distracting level of detail.[16]

What is clear is that the text was intended for, and was indeed used by, readers who were encountering for the first time the geometry of the sphere, particularly as it pertains to the celestial sphere. For this reason, the text addresses two different, although related, projects: (1) the establishment of a deductive spherical geometry on the basis of a limited set of definitions and permitted constructions and (2) the use of this geometry to develop theorems that have direct applications in the topics of ancient spherical astronomy.[17] Hence, we should understand the problems in the *Spherics* within this elementary context. Since students were almost certainly introduced to spherical geometry on a real globe, it was important that an elementary text introduce methods of construction that could be used to draw diagrams on such a globe.

In composing the *Spherics* for an elementary readership, Theodosius seems to have been principally guided by the deductive structure of the *Elements*. The early theorems concerning the properties of the sphere are developed by analogy with the properties of the circle demonstrated in *Elem.* III (Heath 1921, pp. 247–248).[18] Moreover, the presentation of the constructions also assumes that the reader has a foundation in the constructive methods of the *Elements*. Any construction that can be carried out using the postulates and problems in the *Elements* is generally assumed with little or no justification. A construction that can be carried out on the sphere, using only the constructions of the *Elements*, is indeed carried out in this way. The primary exception

15 We may take one example drawn from considerations of deductive structure and one from the conventions of ancient Greek mathematical style. For the first, *Spher.* II 10 shows that parallel circles cut the arcs of great circles that go through their poles into equal sections. Nevertheless, both *Spher.* II 15 & III 5 include individual, albeit short, arguments that this is the case (see footnote 60, below). Generally, in a Greek mathematical text, once a particular fact is demonstrated in a proposition, there need be no further justification of the matter. For the second example, we consider the ordering of the letter names in the text. In all but four theorems, the letter names of geometric objects are introduced alphabetically (93%), following what was apparently the standard style. The exceptions are *Spher.* II 9, 16, 22 and III 4. In *Spher.* II 22 and III 4, this is obviously done because the figure and its labeling are retained from the previous proposition and the same objects received the same names to facilitate the reader's understanding of the proposition. This is also the case with *Spher.* II 16, except that the figure for *Spher.* II 16 is meant to represent the same objects, with the same names, as *Spher.* II 10 & 13, with which it forms a group. This leaves *Spher.* II 9 as the only proposition with an irregular ordering that is not explicable by reference to the previous theorems. Hence, in all likelihood, it was directly adopted from another text in which the alphabetic ordering of the letter names was so explicable.

16 As a typical example we quote the following passage, taken from *Spher.* I 21. "Since point *C* is the pole of circle *FAG*, therefore circle *ABC* cuts circle *FAG* through the poles. Since, now, in a sphere a great circle, *ABC*, cuts some circle in the sphere, *FAG*, through its poles, it cuts it in half and orthogonally. Therefore, circle *ABG* is orthogonal to circle *FAG*. Therefore, circle *FAG* is orthogonal to circle *ABG*" (Czinczenheim 2000, pp. 79–80; Heiberg 1927, p. 38). The arguments often seem needlessly repetitive, the transitivity of similarity and the reflexivity of perpendicularity are usually explicitly stated, and the reader is constantly reminded that the objects under discussion are in a sphere.

17 For an accessible overview of ancient spherical astronomy, see Evans (1998, pp. 75–161). Theodosius's treatment of these topics has been discussed by Schmidt (1943) and Berggren (1991).

18 See Fried and Unguru (2001, pp. 332–357) for a discussion of the way that Apollonius used analogy with *Elem.* III in composing his *Conics*.

to this is the operation of cutting a sphere with a plane, which is used in a number of constructions, particularly those of *Spher.* I 2 & 19. Indeed, there is no justification of this construction on the basis of the postulates or problems of elementary geometry. Hence, from the outset, the treatise contains a distinction between constructions that can be carried out with the postulates, and thus tools, of elementary geometry, and those that are more conceptual, or abstract, but are nevertheless required by the need to write proofs about solid objects.

We will treat this distinction in some detail in the following section, in our discussion of the problems. In order to understand the function of the problems within the treatise as a whole, it will be useful give an overview of the three books. In particular, it should be noted that each of the three books has a different range of topics and perhaps a somewhat different goal.

The first book develops the basic geometry of the sphere on analogy with the geometry of the circle developed in *Elem.* III. In particular, the properties of lesser circles and great circles in a sphere are demonstrated analogously to the properties of chords and diameters in a circle. The first book culminates in a series of problems, the final two of which are essential for the rest of the treatise.[19] Hence, the aim of the first book is to establish the basic properties of circles in the sphere and to develop some constructions that will be useful for producing a spherical geometry.

In the second book, the analogy with the plane objects of *Elem.* III changes, so that now the properties of lesser circles and great circles in a sphere are demonstrated analogously to the properties circles and lines in the plane. The book begins by developing a theory of the tangency of two circles in the sphere and uses this to treat the relationships that obtain between great circles and sets of parallel lesser circles. In particular, *Spher.* II explores the conditions under which great circles and sets of parallel circles cut each other in equal or similar arcs. The final two problems are introduced in this book where they are needed to draw great circles tangent to lesser circles, again on the analogy with lines and circles in the plane. Towards the end of this book, the subject matter becomes almost purely astronomical. *Spher.* II 19 makes explicit mention of "the visible pole," while the final two theorems, *Spher.* II 22 & 23, although expressed in purely geometric language, clearly deal with the changing inclinations of the ecliptic on a given horizon between the terrestrial equator and the arctic regions.[20]

The third book, although still purely geometric in presentation, is a series of propositions whose purpose can only be understood when interpreted in the context of spherical astronomy.[21] The first part of the book deals with what we would call the

[19] The two final theorems of this book in the Greek text (*Spher.* I 22 & 23) are interpolated (see footnote 12, above).

[20] Although Theodosius's treatment is quite brief, the instantaneous positions of the ecliptic in terms of local coordinates was apparently an abiding topic of ancient spherical astronomy. Ptolemy, writing around the middle of the second century, treats the problem in full detail in the *Almagest*, culminating in one of the most sophisticated tables in the work, *Alm.* II 13 (Toomer 1984, pp. 105–129; Heiberg 1898–1903, p. 1, 174–187).

[21] The purely astronomical subject matter of *Spher.* III has lead some scholars to assert that the aim of the entire treatise is astronomical (Neugebauer 1975, pp. 755–756; Lorch 1996, pp. 159–160). As an elementary treatise, however, we need not read all three books as having a single goal. Probably, the books were organized to address different basic topics in spherical geometry and its application to the celestial sphere.

transformation of coordinates, that is the projection of arcs of the principal great circles onto each other using great circles or sets of parallel circles.[22] The book culminates in some theorems giving a partial treatment of the rising times of arcs of the ecliptic, a traditional topic already discussed by Euclid in his *Phenomena* (Berggren and Thomas 1996; Menge 1916).

3 The problems in the *Spherics*

In our discussion of the seven problems, we explain the constructions in detail, with a justification of each step.[23] We pay particular attention to practical operations that would be used in the process of carrying out these constructions on real globes using the basic tools of Greek geometry, a straight edge and compass.[24] As our discussion will show, many features of Theodoius's approach to the problems are best understood by an appeal to such practical considerations.

As well as the constructions in the *Elements*, Theodosius appears to rely on two unstated postulates.[25]

Polar Circle Construction: In *Spher.* I 19 and following, Theodosius assumes the ability to draw a circle with a given point as pole and a given line as distance, where the distance is less than the diameter of the sphere.[26] Because this distance is a key mathematical object in a number of arguments, we will call it the *pole-distance*.[27]

[22] Discussions of Theodosius's approach to these topics can be found in Schmidt (1943), Neugebauer (1975, p. 766) and Berggren (1991).

[23] We justify each step of the constructions and many of the steps of the proofs by references to propositions in the *Elements* or the *Spherics*. There are virtually no references by book and number in the Greek mathematical corpus. A Greek author generally cited a previous theorem by briefly restating the enunciation, or a summary of it. There are many such citations in the *Spherics*. Nevertheless, in the case of theorems, it has long been the practice of modern scholars to supply references by book and number. In the case of constructions, this practice has been less systematically applied. In this paper, we reference all constructions in order to make explicit the fact that constructive steps, like logical steps, were situated in a context of assumed knowledge.

[24] It is possible that for drawing on larger globes an articulated compass or pair of calipers may have been used.

[25] These postulates, among others, are set out by Naṣīr al-Dīn al-Ṭūsī in his redaction of Theodosius's *Spherics*. Moreover, Ṭūsī notes that they "treat this in the same way with respect to what results in the course of the problems" (al-Ṭūsī 1940, p. 3). This, presumably, means that his additional postulates function in the same way for the problems as the original postulates do for the theorems. Ṭūsī also includes the following construction postulate, "we produce any arc that there is until it completes its circle" (al-Ṭūsī 1940, p. 3). In fact, however, although circles are sometimes completed in this way, this construction can almost always be immediately reduced to one of the following postulates or a demonstrated construction. The only exception is in *Spher.* III 4 (Czinczenheim 2000, p. 145; Heiberg 1927, p. 126).

[26] Ṭūsī states this postulate as, "we make any point that happens to be on the surface of the sphere a pole, and we draw about it and with any distance, less than the diameter of the sphere, a circle on that surface" (al-Ṭūsī 1940, p. 3).

[27] When this distance is found in the figure, having already been constructed, it is called "the [line] from the pole" (ἡ ἐκ τοῦ πόλου), on analogy with the Euclidean expression for a given radius, "the [line] from the center" (ἡ ἐκ τοῦ κέντρου) (Czinczenheim 2000, 75 ff; Heiberg and Stamatis 1969, vol. 1, 177 ff; Heiberg 1927, 34 ff).

Equal-Arc Sectioning: In *Spher.* I 21 and following, Theodosius assumes the ability to cut off, from a given point on the circumference of a circle, an arc equal to a given arc.[28] In fact, on the sphere, equal-arc sectioning is a direct consequence of polar circle construction.

We need not, however, follow Ṭūsī in asserting that these postulates are lacking in Theodosius's presentation. Indeed, Theodosius takes for granted any construction that can be carried out by Euclidean means, and he probably considered the forgoing constructions covered by the geometry of the *Elements*.

The Euclidian postulate for circle construction, *Elem.* I post. 3, is "to draw (γράφεσθαι) a circle with any center and distance (διαστήματι)" (Heiberg and Stamatis 1969, vol. 1, p. 5). In the *Elements*, the distance is always in a single plane and the generating point is the center of the circle (Fowler and Taisbak 1999; Sidoli 2004). If we consider an abstraction of this postulate, however, the generating point and distance need not always lie in the same plane, so that, on the surface of a sphere, the generating point is a pole of the circle.[29] In the spherical geometric works of Autolycus and Theodosius, the generated circle is in the surface of a sphere, however, other solid figures are possible. For example, Diodorus in his lost *Analemma*, apparently used this postulate to produce the arc of a circle on the surface of a cone.[30] Hence, we may understand all uses of these postulates as the application of a generalized version of *Elem.* I post. 3, which, in the interest of clarity, we will call *Elem.* I post. 3*. In fact, however, Greek geometers probably simply thought of the Euclidean postulate as applying to the solid cases as well.

A functional difference between the use of *Elem.* I post. 3* in plane and solid geometry should be noted. In general, when actually carrying out the construction of a circle, the geometer has both a given generating point and a given length. In the simplest case, the generating point is an endpoint of the generating length, so that a circle is simply drawn with some tool, such as a compass or pair of calipers. In many cases, however, the length will be given elsewhere in the figure so that it must somehow be transformed such that one of its endpoints coincides with the generating point. *Elem.* I 2 solves the problem of this transformation in the plane, so that the tool of construction need not be used to transfer the length.[31] *Elem.* I 2, however, can only be used if the given length lies in the same plane as the circle to be produced. Hence, whatever they

[28] Ṭūsī states this postulate as, "we cut off what is equal to a given arc from an arc greater than it, when they belong to equal circles" (al-Ṭūsī 1940, p. 3).

[29] Already in the beginning of the Hellenistic period, in *On the Moving Sphere* 6, Autolycus used this method of constructing a circle on the sphere (Mogenet 1950, p. 203; Aujac 1979, p. 53).

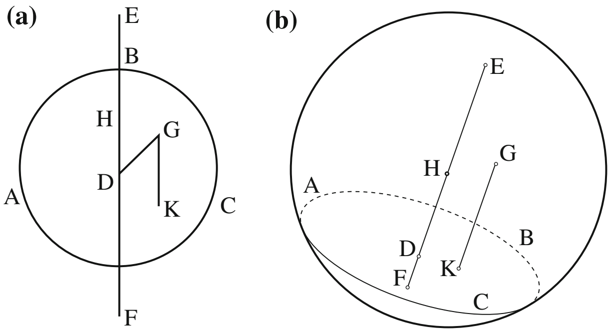
[30] Abū Saʿīd al-Ḍarīr al-Jurjānī transmitted a version of this construction, along with his own proof, in a short treatise on finding the meridian line given three shadows cast by the same gnomon throughout the course of a day (Schoy 1922; Hogendijk 2001). At the beginning of the construction he says, "we describe arc *EHT* around center *B* and with distance *BE* (وببعد)," where in fact *B* is the vertex of a cone and arc *EHT* is part of its base (Hogendijk 2001, p. 61, 68, n. 64). This indicates that Greek and Arabic geometers were willing to consider both the point and the distance used in constructing a circle as independent of its plane.

[31] Indeed, DeMorgan insisted that the tool must never be used for this transformation, stating that we should, "suppose the compasses to close of themselves the moment they cease to touch the paper," which perhaps raises the question of why they stay fixed when they are on the paper (Heath 1926, p. 246).

thought of *Elem.* I 2, in solid geometry most ancient mathematicians were probably willing to simply regard the tool as suitably fixed, and use it to transfer lengths.

***Spher.* I 2**: "To find (εὑρεῖν) the center of a given sphere."[32]

Construction: Let the sphere be cut (τετμήσθω) by some plane, producing circle *ABC* (*Spher.* I 1). The center of the circle, *D*, is taken (*Elem.* III 1). A perpendicular is erected at *D* and produced to meet the sphere at the two points *E* and *F* (*Elem.* XI 2, I post. 2). The line *EF* is then bisected at *H* (*Elem.* I 10).



***Spher.* I 2: a** MS Diagram,[33] **b** Perspective Reconstruction

Proof: A different center, *G*, is assumed and shown to be impossible.

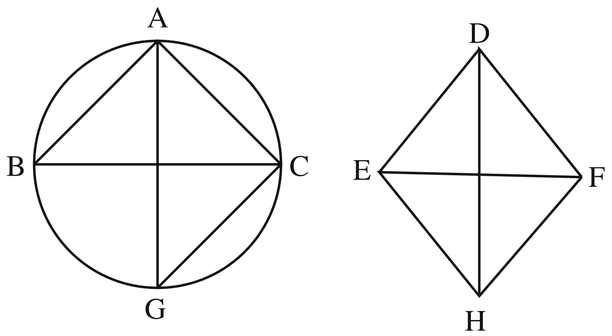
Comments: The object is to locate the point that is at the center of a given sphere. This proposition should be compared with *Elem.* III 1, which locates the center of a given circle. *Elem.* III, however, begins with a problem, whereas *Spher.* I 1 is a theorem. This structural difference is due to the fact that the construction in *Spher.* I 2 requires *Spher.* I 1, which shows that passing a plane through a sphere produces a circle. As discussed above, the construction involves passing a cutting plane through the sphere. There is no postulate for this, and it clearly cannot be carried out with the tools of elementary geometry.

***Spher.* I 18**: "To set out (ἐκθέσθαι) the diameter of a given circle in a sphere."[34]

[32] Czinczenheim (2000, p. 54) or Heiberg (1927, p. 4).

[33] For the MS diagrams we loosely follow those in *Vat. gr.* 204 and make no claims to exact reproduction. Czinczenheim (2000, p. 680) also claims to follow *Vat. gr.* 204, but there are a number of peculiar features to her diagrams, such as the regular use of curved lines that are not arcs of circles. For the diagram of *Spher.* I 2, there are a number of variants in the Greek and Arabic MSS (for the Greek diagrams see Czinczenheim (2000, p. 682)).

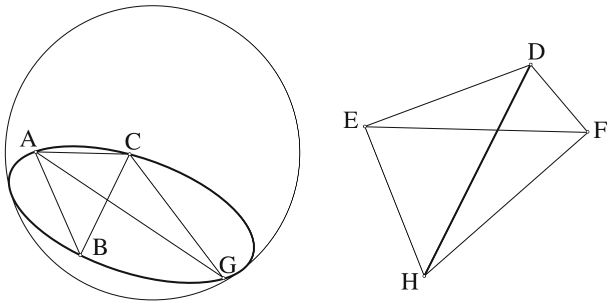
[34] Czinczenheim (2000, p. 74) or Heiberg (1927, p. 32).



***Spher.* I 18:** MS Diagram

Construction: Let three arbitrary points, A, B and C, be taken on the circumference of the circle. Triangle DEF is constructed such that $AB = DE$, $BC = EF$ and $CA = FD$ (*Elem.* I 22).[35] Two perpendiculars are erected such that $EH \perp ED$ and $FH \perp DF$, and extended to meet at H (*Elem.* I 11, I post. 2). For the sake of the proof, the diameter of the given circle, AG, is drawn and lines AB, AC, BC and CG are joined (*Elem.* III 1, I posts. 1 & 2).[36]

Proof: The congruency of figures $BACG$ and $EDFH$ is established, so that $DH = AG$.



***Spher.* I 18:** Perspective Reconstruction

[35] There is no indication in the text as to how this construction should be carried out. *Elem.* I 22 gives the construction of a triangle from three given lines, but it relies on *Elem.* I 3, and thence on *Elem.* I 2, and so can only be carried out on the plane. Probably, Theodosius expected that the circle-drawing tool would be used transfer the lengths by setting the endpoints of the tool on the two given points and then carrying this span to the plane diagram.

[36] The expression used for drawing the internal diameter, AG, is "let AG have been passed (ἤχθω)," which uses the perfect imperative form of ἄγω, one of the verbs most commonly used for the construction of lines (Mugler 1958, pp. 39–40).

Comments: The goal is to produce a line outside of the sphere that is equal to the diameter of a given circle in the sphere. A diameter of the circle is not given at the outset, but is constructed along the way. This internal diameter is constructed using standard terminology for the construction of a diameter in a plane and the implication is that it is to be done using the constructive methods of the *Elements*, that is, finding the center of the circle and joining a line through this center and *A* (*Elem.* III 1, I posts. 1 & 2). Although this internal diameter is constructed using standard techniques, it does not, in fact solve the problem. The internal diameter is, clearly, only constructed for the sake of the proof. It is constructed in order to introduce a new set of starting points from which Theodosius will argue that the external line is equal to the internal diameter. In this problem, it becomes clear that Theodosius intends a functional difference between the use of construction for the sake of proof and its use as a practical technique for drawing figures. The internal diameter, which is constructed for the sake of the argument, does not solve the problem. The problem is only solved when another line has been drawn outside the sphere, which is equal to this internal diameter. The difference must, in some sense, be practical. If the geometer is working with a real spherical body, the lengths *AC*, *AB* and *BC* can be transfered with the circle-drawing tool, so that triangle *DEF* can be drawn on a flat surface. The internal diameter, on the other hand, is not immediately accessible to the practical tools of the geometer.

***Spher.* I 19**: "To set out (ἐκθέσθαι) the diameter of a given sphere."[37]

Construction: Let a sphere be imagined (νενοήσθω), and two arbitrary points, *A* and *B*, taken on its surface. With *A* as a pole and *AB* as a distance (διαστήματι), circle *BCD* is drawn (*Elem.* I post. 3*). Then it is possible to set out the diameter of circle *BCD* (*Spher.* 18).[38] Triangle *EFH* is constructed such that $EF = EH$ are equal to the pole-distance of circle *BCD*, and the base, *FH*, is equal to its diameter (*Elem.* I 22).[39] Two perpendiculars are erected such that $FG \perp EF$ and $HG \perp EH$, and extended to meet at *G* (*Elem.* I 11, I post. 2).

Proof: Let a diameter, *AK*, and a plane passing through it be imagined (νενοήσθω). The figure *BADK* is then shown to be congruent with the figure *FEHG*. Hence, $EG = AK$.

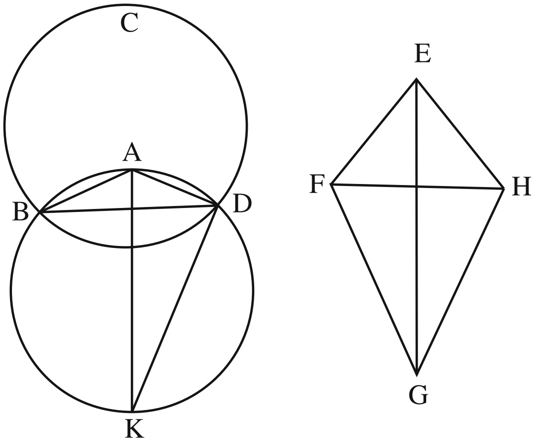
Comments: The goal is to produce a line outside of the given sphere that is equal to its diameter. Again, the diameter to be produced is not given at the outset but is constructed along the way for the sake of the proof. In this case, the internal diameter is said to have been produced by having been imagined.[40] Since this way of producing objects is common in Greek mathematical texts where the objects in question are

37 Czinczenheim (2000, p. 75) or Heiberg (1927, p. 34).

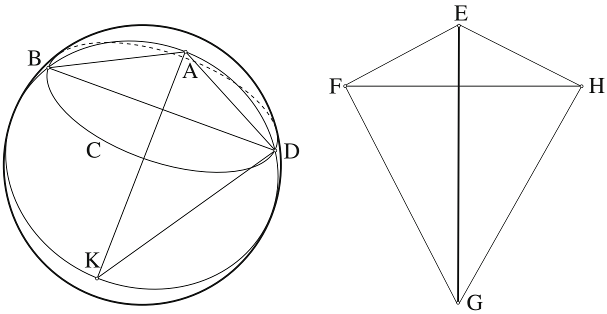
38 This is the only place in the text where *Spher.* 18 is used. Hence, it is interesting to note that Theodosius does not use the imperative, as usually done to cite a previous problem in the construction stage of a proof, but the infinitive, which is usually reserved for the enunciation of a problem.

39 Again, this was probably actually done by carrying the length *AB* with the circle-drawing tool.

40 In *Spher.* I 18, it was not necessary to introduce the idea of imagining the sphere or its internal objects because a circle in the sphere was given from the outset. Hence, the constructions in *Spher.* I 18 involved two plane objects, a circle whose plane happened to be in a sphere and triangle in a plane somehow external to the sphere.



***Spher.* I 19:** MS Diagram



***Spher.* I 19:** Perspective Reconstruction

three dimensional, it should be noted that *Spher.* I 19 is the only theorem in which the operation is used in this text, which is entirely on three dimensional geometry.[41] Hence, it is being used to highlight a distinction between the construction of the sphere itself and its internal diameter and the constructions which actually solve the problem. The difference, again, seems to be practical; whereas, the construction of the external diameter can actually be carried out using standard tools of geometry, that of the sphere and of the internal diameter cannot.

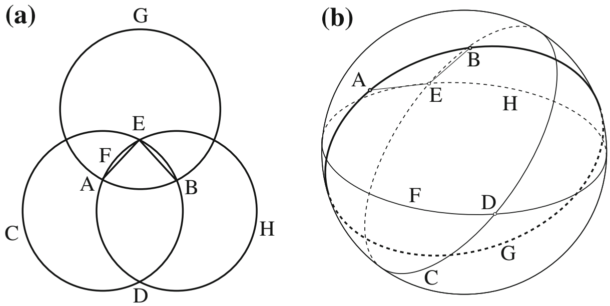
41 Netz (1999a, pp. 52–56) provides a discussion of the use of this concept in Greek mathematics. The idea of *imagination* is found with a range of meanings to do with the actual three dimensionality of objects not fully conveyed by the diagram (*Elem.* XI 12, Heiberg and Stamatis 1969, vol. 4, p. 18; Ptolemy's *Analemma* 6 Heiberg 1907, 196 ff.), the motion of all or parts of the diagram (Ptolemy's *Almagest*, Heiberg 1898–1903, 217 ff; *Planisphere* 10, Sidoli and Berggren 2007, p. 67), or simply auxiliary constructions not depicted in the diagram (*Elem.* IV 12, Heiberg and Stamatis 1969, vol. 1, p. 169).

***Spher.* I 20**: "To draw (γράψαι) a great circle[42] through two given points that are on the surface of a sphere."[43]

Construction: [Case 1] If the given points, *A* and *B*, are on a diameter of a sphere, it is possible to draw an infinite number of great circles through them. [Case 2] If they are not on a diameter of a great circle, great circle *CDE* is drawn with *A* as a pole and the side of the great square as a distance, and great circle *FEH* is drawn with *B* as a pole and the side of the great square as a distance (*Elem.* I post. 3*, *Spher.* I 19, *Spher.* I 17).[44] Lines *AE* and *EB* are joined, so that by construction they are equal to the side of the great square. Finally, *ABG* is drawn with *E* as a pole and *EB* as a distance.

Proof: In this proposition, there is no clear, structural division between the construction and proof, because the construction itself contains a number of deductive steps and the proof is a direct consequence of the construction. Nevertheless, a somewhat redundant proof follows, arguing that circle *ABG* is great, since its pole-distance is the side of a square described in a great circle (*Spher.* I 17).

Comments: This problem is addressed in two cases. The first case is dismissed out of hand with neither construction nor proof. There are a number of possible reasons



***Spher.* I 20: a** MS Diagram, **b** Perspective Reconstruction

[42] Literally, "greatest circle," a standard technical expression (Czinczenheim 2000, p. 77; Heiberg 1927, p. 36).

[43] Czinczenheim (2000, p. 77) or Heiberg (1927, p. 36).

[44] We use the expression "side of the great square" as an abbreviation for the standard expression "the side of the square inscribed in the great circle." The side of the great square is established as the characteristic pole-distance of a great circle in *Spher.* I 16 & 17. Although the text provides no practical construction for producing the side of the great square, it easily follows once the diameter of the sphere has been set out (*Spher.* I 19). For example, such a construction is found in *Elem.* IV 6. Hence, a great circle can be drawn about a given pole by setting out the diameter of the sphere (*Spher.* I 19), constructing a square with this length as its diagonal, and using the side of this square as the pole-distance of the circle.

for this. One is that a construction is trivial.[45] A second, and probably more important, reason is that a problem in Greek mathematics is about constructing specific objects and then demonstrating that these objects satisfy the requirements of the problem. When an infinite number of objects satisfy these requirements, the project becomes meaningless. Such a construction would be similar to that of drawing a line through a point in the plane, or a circle through two points on either the plane or the sphere. We have no evidence that Greek geometers saw such vague constructions as problems that required geometric solutions.[46] A third possibility is that this passage was added after Theodosius drafted the *Spherics* by a mathematically inclined reader who saw that another configuration was possible and could easily be solved. If Theodosius had, in fact, not mentioned this first case it would be in keeping with the common practice of simply solving the most dificult case and leaving the simpler cases to the reader.[47]

The second part of the problem establishes that a great circle can be drawn through the two points but says nothing about the fact that this great circle is unique. In fact, Theodosius seems to have only a vague notion of the uniqueness of this great circle. In the course of demonstrating *Spher.* II 5, Theodosius shows that assuming two great circles can pass through any two points not on the same diameter of the sphere leads to an absurdity.[48] He does not, however, state this as a proof of uniqueness. Moreover, the proof of *Spher.* II 5 would be simplified if Theodosius believed he could appeal to the uniqueness of this great circle. It would be easy to establish the uniqueness of a great circle through two non-diametrically opposite points, if the circle were constructed by passing a plane through the center of the sphere and the two given points and then pointing out that this plane intersects the sphere in a great circle. This construction, however, would give the geometer no means of drawing the great circle, which is, in fact, the stated aim of the proposition. That is, the goal of the problem is not *to find* the great circle through two given points, but *to draw* it.

***Spher.* I 21**: "To find (εὑρεῖν) the pole of a given circle in a sphere."[49]

Construction: A point, *D*, is taken, at random, on the circumference of the given circle *ABC*, and arc *AE* is cut off equal to arc *AD* (*Elem.* I post. 3*). Arc *DE* is bisected

[45] For example, a great circle is drawn with *A*, or *B*, as a pole and side of the great square as a distance (see footnote 44, above). A point is taken at random on this great circle and another great circle is drawn with this point as a pole, passing through *A* and *B*.

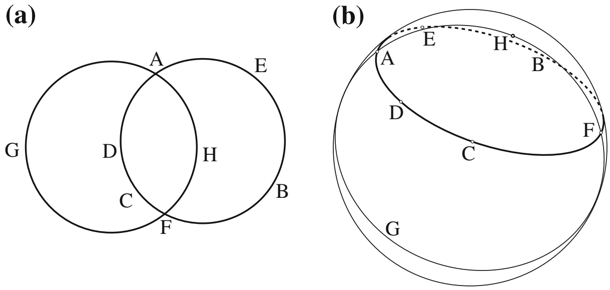
[46] It is useful to compare this case of the problem to similar problems in *Elem.* IV in which a given figure is constructed in, or around, a given circle (Heiberg and Stamatis 1969, vol. 1, pp. 152–179). In *Elem.* IV, however, while an infinite number of positions of the different figures solve the problem, the figures are constructed to satisfy certain conditions of magnitude and their position is considered to be irrelevant. In *Spher.* I 20, on the other hand, the magnitude of the great circle is assumed so that it is only its position that is relevant to the solution. Hence, in the first case, where a infinite number of positions satisfy the requirements, the problem becomes too open-ended.

[47] Indeed, we find a similar situation in *Spher.* II 15 (see footnote 57, below).

[48] See below for further discussion of this theorem.

[49] Czinczenheim (2000, p. 78) or Heiberg (1927, p. 36).

at F.[50] [Case 1] If *ABC* is not a great circle, the great circle *AFG*, passing through *A* and *F*, is drawn (*Spher.* I 20), and arc *FHA* is bisected at *H* (*Elem.* III 30). [Case 2] If *ABC* is a great circle, arc *AF* is bisected at *C* (*Elem.* I post. 3*),[51] and great circle *AFG* is drawn with *C* as a pole and *CF* as a distance (*Elem.* I post. 3*, *Spher.* I 17). Again, arc *FHA* is bisected at H.[52]



***Spher.* I 21: a** MS Diagram, **b** Perspective Reconstruction

Proof: A proof is given following the construction of each case, and in both there is no clear separation between the proof and the construction. Both proofs rely on an appeal to *Spher.* def. 5, which states that, "A pole of a circle in a sphere is a point on the surface of the sphere, from which point straight lines extended to the circumference of the circle are equal to one another."[53] In the first case, since great circle *AFG* bisects circle *ABC*, it will be perpendicular to it (*Spher.* I 14), so that *H* must be a pole of *AFC* (*Spher.* def. 5). In the second case, great circle *AFG* is drawn with *C* as a pole, such that it is perpendicular to great circle *ABC* and it passes through its poles (*Spher.* I 14). Hence, *H* must, again, be a pole of circle *ABC* (*Spher.* def. 5).

Comments: It would be possible to make a simpler construction, along the lines of *Spher.* I 2, by taking the center of the given circle (*Elem.* III 1), erecting a perpendicular, and extending it to meet the sphere at the two poles (*Elem.* XI 2, I post. 2). Moreover, this single construction would serve for both cases. Indeed, in the course of *Spher.* I

50 As Berggren (1991, pp. 246–247) has pointed out, if this construction is to be used on the sphere, the procedure is not made fully explicit in the text. As usual, much of the practical procedure must be supplied by the reader. Using *Spher.* I 18, we set out the diameter of circle *ABC*, bisect it (*Elem.* I 10), and draw a circle around it. We transfer arc *DE* to the outside circle as $D'E'$. We bisect arc $D'E'$ at F' (*Elem.* III 30), and cut off arc *DF* equal to $D'F'$ (*Elem.* I post. 3*). The process of transferring arc lengths between the sphere and a plane is similar to the solutions of various problems in spherical astronomy using analemma techniques. An example of the use of the analemma in a similar vein can be found in Heron's *Dioptra* 35 (Sidoli 2005).

51 Since *ABC* is a great circle, arc *AF*, which is a great semicircle, will be bisected by laying off the arc subtending the side of the great square (see footnote 44).

52 For this construction, see footnote 50, above.

53 Heiberg (1927, p. 2). Czinczenheim (2000, p. 52), prefers the primary verb λέγεται, found in *Par. gr.* 2448 and 2342, but we do not follow her in this.

8, such an internal construction is used to produce the poles of the circle, which are then used in the proof. This simplified construction, however, although adequate for the purpose of using the poles in the course of a proof, gives the geometer no practical means of carrying out the construction on the outside of the sphere. Hence, Theodosius is willing to forego a simple construction with a single proof in favor of two constructions that require separate proofs, because these constructions can be carried out on a globe using standard tools of ancient geometry.

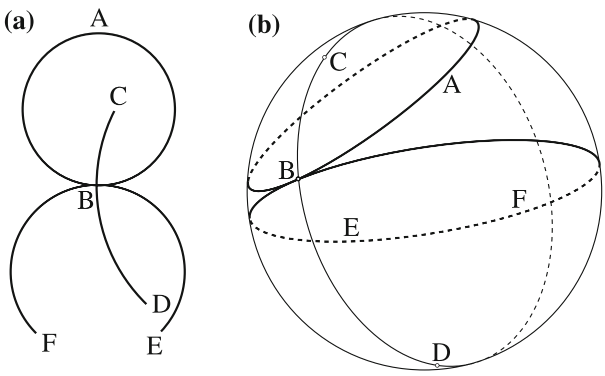
***Spher.* II 14**: "If a lesser circle[54] in a sphere and some point on its circumference be given, to draw (γράψαι) through the point a great circle tangent to the given [circle]."[55]

Construction: Where *AB* is the given circle and *B* a given point on it, the pole of *AB* is taken as *C* and a great circle is drawn through *C* and *B* (*Spher.* I 21 & 20). Arc *BD* is cut off equal to the arc subtending the side of the great square (*Elem.* I post. 3*, *Spher.* I 19). With *D* as a pole and *BD* as a distance, the great circle *BEF* is drawn (*Elem.* I post. 3*, *Spher.* I 17).

Proof: Since the great circle *CBD* intersects both circles *AB* and *BEF* at a single point, *B*, while their poles lie on it, circles *AB* and *BEF* are tangent (*Spher.* II 3).

Comments: There is no parallel for this problem in the *Elements*, probably because it would be too simple. As *Elem.* III 16 cor. shows, the tangent to a given circle at a point on it would simply be drawn as the perpendicular to the diameter at the given point.

***Spher.* II 15**: "If a lesser circle in a sphere and some point on the surface of the sphere, which is between it and the [circle] equal and parallel to it, be



***Spher.* II 14: a** MS Diagram, **b** Perspective Reconstruction

[54] Literally, "a circle less than the greatest" (Czinczenheim 2000, p. 101; Heiberg 1927, p. 68).

[55] Czinczenheim (2000, p. 101) or Heiberg (1927, p. 68).

given, to draw (γράψαι) through the point a great circle tangent to the given circle."[56]

Construction: Where *AB* is the given circle and *C* the given point between it and the circle equal and parallel to it, the pole of *AB* is taken as *D* (*Spher.* I 21). With *D* as a pole and line *DC* as a distance, the circle *CF* is drawn (*Elem.* I post. 3*), and the great circle *DCG* is drawn through points *D* and *C* (*Spher.* I 20). [Case 1][57] Where arc *BC* is less than a quadrant, arc *BG* is cut off equal to the arc subtending the side of the great square (*Elem.* I post. 3*, *Spher.* I 19), and with *G* as a pole and line *BG* as a distance, great circle *EBH* is drawn (*Elem.* I post. 3*, *Spher.* I 17), cutting circle *CF* at *E* and *H*. Hence, *EBH* is tangent to *AB* (*Spher.* II 3). Great circles *DEK* and *DHL* are drawn (*Spher.* I 20), and arcs *EK* and *HL* are cut off equal to arc *CG*. With *K* and *L* as poles and lines *KM* and *LN* as distances, circles *OMC* and *CNX* are drawn (*Elem.* I post. 3*). [Case 2][58] Where arc *BC* is a quadrant, essentially the same construction will serve. [Case 3][59] Where arc *BC* is greater than a quadrant, the arc of great circle *DBCG* between *C* and the circle equal and parallel to it will be less than a quadrant. Hence, the same construction can be effected on the opposite hemisphere.

Proof: [Case 1] Since arcs *EK* and *HL* are constructed equal to arc *CG* while arcs *ME*, *BC* and *NH* are equal,[60] arcs *KM* and *LN* are equal to *BG* and hence circles *OMC* and *CNX* are also great. Lines *KC*, *GE*, *GH* and *LC* are drawn and elementary geometry, along with *Spher.* II 12, is used to show that *LC* and *KC* are equal to $LN = KM$. Hence, since the poles of circles *CNX* and *AB* lie on the same great circle, which they both

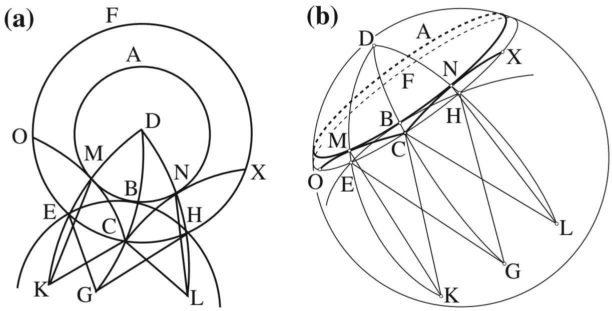
56 Czinczenheim (2000, p. 102) or Heiberg (1927, p. 70).

57 Heiberg (1927, pp. 70–76) followed a single, late MS (*Par. gr.* 2448) and produced a text with three cases for this proposition. Czinczenheim (2000, pp. 102–105), however, correctly placed the passage introducing the three cases and the third case in the scholia, where they are found in all but one of the Greek MSS. The Arabic tradition supports this decision. Although Ṭūsī's version has three cases and the exposition is clearer than that in Heiberg's edition (al-Ṭūsī 1940, pp. 22–23), the older Arabic versions have the same structure as Czinczenheim's edition. This includes a second case, which Czinczenheim marked as a later addition, probably correctly, since in both texts it begins with the phrase "if someone says" (εἰ δέ τις λέγοι, فإن قال قائل) (Czinczenheim 2000, p. 105; Kraus MS, 47r; Leiden Or. 1031, 45r). This is the only place in the work where such a phrase is used. Nevertheless, the fact that it is in both the early Greek and early Arabic versions means that, although this passage was probably not written by Theodosius, it must have been included in the Greek text prior to the eighth or ninth century. Hence, the different versions of the text show that Theodosius originally wrote this problem solving only the most difficult case. A second case was added to the Greek text sometime before the original Arabic translation was made, and then the third case was added independently in the Arabic tradition by al-Tūsī and in the Greek tradition by the anonymous copyist of *Par. gr.* 2448.

58 Although this case was probably not written by Theodosius, it is found in all versions of the text currently extant (see footnote 57, above).

59 This case was almost certainly not in Theodosius's text, since it is not found in the older Greek or Arabic traditions of the text. Nevertheless, any moderately competent mathematical reader will have realized that this case exists and can also be solved (see footnote 57, above).

60 The equality of these arcs was demonstrated in *Spher.* II 10, but here Theodosius argues that they are equal because arc $DE =$ arc $DC =$ arc DH and arc $DM =$ arc $DB =$ arc DN (Czinczenheim 2000, p. 103; Heiberg 1927, p. 72). The same argument is again repeated for the same purpose in *Spher.* III 5 (Czinczenheim 2000, p. 148; Heiberg 1927, p. 130).



Spher. **II 15: a** MS Diagram, **b** Perspective Reconstruction

intersect at N, while the poles of circles OMC and AB lie on the same great circle, which they both intersect at M, the two great circles OMC and CNX are tangent to AB and pass through C (*Spher.* II 3). Hence, the problem has two solutions.[61] [Case 2] Where arc BG = arc BC, a simplified version of the previous proof is provided. [Case 3] Where arc $BG <$ arc BC and the construction is carried out in the opposite hemisphere for the circle equal and parallel to circle AB, the foregoing proof also holds.

Comments: This problem is somewhat involved and its solution was probably derived through geometric analysis. For example, with lesser circle AB and point C assumed as given, let two great circles, say CNX and CMO, be assumed passing through point C and tangent to circle AB at two points, say N and M. Then, Theodosius's theory of tangency, as set out in *Spher.* II 3–5,[62] insures that the poles of the great circles CNX and CMO will fall on great circles that pass through points N, D and M, D respectively. Likewise, if a great circle is drawn through D and C, intersecting circle AB at B and a great circle is drawn tangent to circle AB at point B, then the polls of this tangential great circle will fall on great circle DB. If FC is drawn parallel to circle AB passing through point C, considerations of similarity show that just as great circles CNX and CMO pass through C of the parallel circle FC, so great circle EBH will intersect circle FC at H and E, the intersections of circle FC with great circles DN and DM, respectively. Hence, constructing points H and E is the key to Theodosius's solution.

This proposition should be compared to *Elem.* III 17, which solves the problem of drawing a line through a given point tangent to a given circle. Indeed, *Spher.* II 15 can be read as an application of the solution given in *Elem.* III 17 to the spherical situation. In the *Elements*, there is no mention of the obvious fact that the plane problem also has two solutions.

61 In the Greek text, this is expressed by stating that the problem itself is done in two ways, "the problem will be produced in two ways" (γίνεται διχῶς τὸ πρόβλημα) (Czinczenheim 2000, p. 105; Heiberg 1927, pp. 74–75). Czinczenheim has marked this as a later addition, probably correctly. Ṭūsī makes no mention of this obvious fact, and the Kraus MS sets out this double solution from the beginning, stating "we want to draw two great circles" [Kraus MS, 46v].

62 See Sect. 5, below.

4 Characteristic features of the problems

Having given a summary of the seven constructions individually, we now examine them as a group to better understand the special features that these propositions have that distinguish them as problems.

According to the wording of the enunciations, there are three different kinds of problems in the *Spherics*. There are two problems that demonstrate how *to find* (εὑρεῖν, *Spher.* I 2 & 21), three problems that demonstrate how *to draw* (γράψαι, *Spher.* I 20, II 14 & 15), and two problems that demonstrate how *to set out* (ἐκθέσθαι, *Spher.* I 18 & 19). In this text, points are found, circles are drawn, and diameters of circles in the sphere are set out. This usage follows that of the *Elements*, but Theodosius also uses these different verbs to mark off conceptually different processes.

In the problems in the *Elements*, the mathematical objects that are found are points (*Elem.* III 1), numbers (*Elem.* VII 2–3, 33–44, 36, 39; VIII 2, 4; X lemm. 29), magnitudes (*Elem.* X 3–4), and lines of a determinate length (*Elem.* X 27–28). In a sense, all of these objects can be said to necessarily exist from the outset, so that it really is a matter of *finding* them. For example, *Elem.* III 1 finds the center of a circle, the existence of which is guaranteed by *Elem.* I defs. 15 & 16, while the number-theoretical problems in *Elem.* VII & XIII find numbers satisfying certain requirements. The existence of the magnitudes and lines found in *Elem.* X can be said to be guaranteed by a vague concept of continuous magnitude.[63]

The points found in the *Spherics* are certainly known to exist prior to the propositions that show how they are found. In this text, both the sphere and a circle in it are defined in such a way that the existence of the center of the sphere and of the pole of a circle is guaranteed by the definitions.[64] Hence, these constructions are provided so that defined properties of the center of the sphere or the pole of a circle can be used in the course of a proof, even when these objects are not given at the outset. Moreover, *Spher.* I 21 has the additional feature of providing a practical solution to the problem of finding the pole of a circle given on a solid globe.

In the *Elements*, the only objects that are produced by the expression *let it have been drawn*, γεγράφθω, are circles and semicircles (*Elem.* I 1–3, 12, 22; II 14; III 17, 33; IV 1, 10, 13, 15; VI 13; X 13, 29–30, 33–35; XI 23; XIII 13–16, 18). When the vertices of a rectilinear figure fall on the circumference of a circle, we find both the circle and the rectilinear figure produced by the expression *let it have been drawn inside*, ἐγγεγράφθω, or *let it have been drawn around*, περιγεγράφθω (*Elem.* IV 4–5,

63 In the *Elements*, a related construction, *to find as well* (προσευρεῖν), is used to find third and fourth proportionals satisfying certain conditions (*Elem.* VI 11-13; IX 18–19; X 10). Since these are also continuous magnitudes or lines, it appears that this is a technical term for the kind of construction used to find proportionals.

64 The sphere is defined as follows. "A sphere is a solid figure contained under a single surface, from which the straight lines falling on one point inside the figure are equal to one another" (Czinczenheim 2000, p. 52; Heiberg 1927, p. 2). The definition of the pole of a circle in a sphere is given above, in the summary of the proof of *Spher.* I 21.

9–11, 14, 16; XI 23; XII 2, 10–12, 17–18; XIII 8, 10–13, 16, 18).[65] The circles drawn in the *Elements* are produced by means of the circle construction postulate, *Elem.* I post. 3, which allows the geometer *to draw* a circle using a fixed point and a given distance.[66] Theodosius's usage in the *Spherics* follows this model.

Circles in the *Spherics* that are constructed with a pole and a distance are also produced by the expression *let it have been drawn*, γεγράφθω.[67] As *Spher.* I 1 shows, however, in this text a circle may also be produced by passing a plane through the sphere. For Theodosius, this is apparently a different kind of circle construction and it is performed with a different verb. Circles constructed with a cutting plane are produced with the expression *let it have been made*, ποιείτω (*Spher.* I 1–4, 7, 19). Hence, we may understand *drawing* a circle as a technical expression for the construction of a circle with a fixed point and a given length, that is, as a geometrical abstraction of the operation of a circle-drawing tool.

Although letting an object be *set out* (ἐκκείσθω) is a common operation in Greek constructions, there is only one problem in the *Elements* that has as its goal a demonstration of how *to set out* (ἐκθέσθαι) certain lengths. In the last proposition of the original books of the *Elements*, *Elem.* XIII 18, Euclid shows how to set out and compare the sides of the five regular solids. This is the culmination of the book and follows the problems that construct the solids in a sphere—the tetrahedron in *Elem.* XIII 13, the octahedron in *Elem.* XIII 14, the cube in *Elem.* XIII 15, the icosahedron in *Elem.* XIII 16, and the dodecahedron in *Elem.* XIII 17. In the construction for *Elem.* XIII 18, Euclid begins by letting the diameter of the given sphere be set out, although in the *Elements* there is no problem demonstrating how this construction is carried out, such as *Spher.* I 19. The construction in *Elem.* XIII 18 then proceeds to show how to set out, in the plane, the sides of the solids that would be constructed in a sphere of the given diameter. In this usage, the verb is taken in the special sense of *to set outside*.

This problem was clearly the model for Theodosius's use of the expression *set out* in *Spher.* I 18 & 19. These two problem are quite interesting and, as Schmidt (1943, pp. 13–14) has pointed out, they form a strong argument that the problems in the *Spherics* were meant to be useful for making drawings on real globes. As usual, the choice of terminology is deliberate. As in *Elem.* XIII 18, the line segments are produced outside the sphere, in a plane space over which the geometer has full control. In the case of *Elem.* XIII 18, this is done so that the sides of the solids can be compared to each other. In *Spher.* I 18 & 19, it is done so that the diameters of circles in a solid sphere can be available for use in further problems. As noted in the commentary to these problems, the diameters in question are actually constructed in the sphere for the sake of the proof, but these internal diameters do not solve the problem. Hence, *Spher.* I 18 & 19 make quite clear the dual role of problems in the *Spherics*.

Thinking of a problem as theoretical when it it is used by the geometer as a starting point in geometric argumentation and as practical when it gives a technique for using a

[65] Two other, related expressions are also used for rectilinear figures, *let it have been drawn on*, ἀναγεᾳ γράφθω, or *let it have been drawn in*, καταγεγράφθω (*Elem.* I 47; II 2–8; VI 22, 25, 27–30; X 19–21, 24–25, 92–96; XI 37, XIII 1–5, 13).

[66] See Sect. 3, above.

[67] See Sect. 5, below, for a discussion of the uses of the constructions involving drawing circles.

real tool to draw a geometric object, helps us understand the presentation of the problems in the *Spherics*. This division, however, is not a dichotomy. For example, a circle may be constructed on the sphere either by passing a plane through the sphere, or by using a given point as a pole and a length as distance. A circle produced by a cutting plane is available to the geometer, as a conceptual object with known properties, but cannot actually be drawn until its poles are located. A circle which is drawn with a pole and a distance is no less useful in writing proofs, but it has the added advantage of actually being drawn.

In the *Spherics*, the only problem that is strictly theoretical is *Spher.* I 2 and its use is confined to the early part of *Spher.* I. In fact, however, there are other constructions that are provided in the course of demonstrations that have a purely theoretical function. For example, in *Spher.* I 8, the poles of a circle are constructed for the sake of the demonstration, and in *Spher.* I 19, a great circle is produced in a sphere by passing a plane through its center. When the related problems are later established, however, they show how to produce these same objects in such a way that they may actually be produced on a real globe. In this way, a problem may be more theoretical, as *Spher.* I 2, or more practical, as *Spher.* I 18, or, perhaps ideally, it may be both theoretical and practical, as *Spher.* I 21. The fact that the constructions provided in *Spher.* I 21 can be used on a real globe in no way diminishes its value as a theoretically useful proposition. Indeed, it is one of the most frequently used propositions in the second two books of the treatise.

In the propositions leading up to *Spher.* I 18–21, almost all the constructions are purely theoretical, and are introduced in order to demonstrate certain basic properties of circles in the sphere, such as the relationship between a circle and the diameter of the sphere through its center, or the relationship between a great circle and a perpendicular lesser circle. With these properties established, Theodosius can introduce the more practical constructions of *Spher.* I 18–21. Once the problems are solved, however, the fact that they employ practical constructions recedes into the background. For example, when *Spher.* I 21 is used, Theodosius simply says something such as, "let the pole have been taken." The method by which it is taken is immaterial. The pole is taken so that its mathematical properties as a pole, derived from the definitions and established in earlier theorems, can be used in the proof.

Thus, the relationship between the practical and theoretical aspects of the problems changes throughout the course of the work. In order to get a better idea of how these changes operate, we must examine the way that problems and constructions are used in the logical development of the text.

5 The application of the problems in the *Spherics*

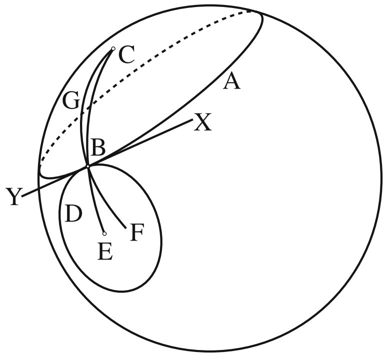
In the logical development of the treatise, constructions, and particularly the seven problems, play a vital role by introducing new mathematical objects whose properties can be used in the course of a proof. If every theorem in the work could be demonstrated solely on the basis of the objects set out in the enunciation, there would be no theoretical need for constructions. In many cases, however, the original objects are not sufficient for the proof.

One example will suffice to make this situation clear. In Spher. II 3–5, Theodosius establishes that two circles in the sphere are tangent at a point if and only if their poles lie on a single great circle and they cut that great circle at the same point.[68] The theorems are as follows.[69]

Spher. II 3: "If, in a sphere, two circles cut some great circle at the same point, having their poles on the same [great circle], they will be tangent to one another."

Spher. II 4: "If, in a sphere, two circles are tangent to one another, the great circle drawn through their poles will also pass through their point of contact."

Spher. II 5: "If, in a sphere, two circles are tangent to one another, the great circle drawn through the poles of one of them and the point of contact will also be through the poles of the other."



Spher. II 3–5: Perspective Reconstruction for *Spher.* II 3–5 (Theory of Tangency)

The proof of *Spher.* II 3 is established directly on the basis of the definition of the tangency of two circles in a sphere. At the beginning of *Spher.* II, Theodosius states that, "Circles in a sphere are said to be tangent when the common section of both of their planes is tangent to the circles."[70] Where circles *AB* and *BD* are two circles whose poles, *C* and *E*, lie on the great circle *CBE*, it is a simple matter to show that the common section of their planes, line *XY*, is perpendicular to their respective diameters at *B*. Hence, the theorem is established using objects either stated in the enunciation or directly implied by these.[71]

In *Spher.* II 5, however, an auxiliary construction is required for the proof. The argument is indirect and Theodosius begins by stating that the great circle, *CBE*, passing through the pole of one of the circles, *C*, and the point of contact, *B*, will

68 Berggren (1991, p. 242) calls this group of theorems Theodosius's "Fundamental Theorem of Tangency."

69 Czinczenheim (2000, pp. 83–85) or Heiberg (1927, pp. 44–46).

70 Czinczenheim (2000, p. 82) or Heiberg (1927, p. 42).

71 The only objects set out in the construction are the common sections of the various planes, including line *XY*.

be extended to the pole of the other circle, *E*. For if not, let it be extended as *CBF*. Then he constructs a great circle through the poles *C* and *E* (*Spher.* I 20), which must therefore pass through the point of tangency, point *B* (*Spher.* II 4). Then, since the two great circles *CBE* and *CGBF* intersect at the two points *C* and *B*, the line joining the intersections must be a diameter of the sphere (*Spher.* I 11). The line joining *C* and *B*, however, is also the pole-distance of circle *AB*, which is impossible. In this theorem, the construction mobilizes a fundamental property of great circles, established in *Spher.* I 11, so that it can be used to sink the false hypothesis. It should be noted that in this proposition the construction is purely theoretical. Despite the fact that the construction of a great circle through two points was carried out in a practical manner in *Spher.* I 20, in this case, the construction must in some sense be purely imaginary. As the argument shows, it is not actually possible to carry out the real construction of a second great circle through the two given points.

In such ways, constructions are used to mobilize geometric properties already established in order to introduce new starting points for the argument that are not provided by the objects discussed in the enunciation. From the perspective of the logical development of the treatise, problems serve to eliminate the need to repeat the same construction in numerous propositions. Once a problem has been solved, the objects it constructs can simply be assumed as having been constructed in later propositions.

Only objects that are not discussed in the enunciation need to be explicitly constructed. For example, in *Spher.* I 7, the center of the sphere and the center of a circle in the sphere are principal actors in the theorem, hence they need not be constructed. Furthermore, objects which could be constructed by relying on a problem are also constructed in other ways when the construction employed adds further elements required by the proof. For example, in *Spher.* I 16, the center of the sphere is constructed by drawing perpendicular diameters in a great circle. Since this theorem states that the pole-distance of a great circle is equal to the side of the great square, a pair of perpendicular diameters is also necessary for the proof.

Hence, we understand the explicit use of a problem to be a statement that the geometric object has been constructed, a statement using the passive imperative perfect.[72] This is the standard locution for expressing a geometrical operation in ancient Greek texts. Using this as our criteria, we may summarize the uses of the problems in the *Spherics* as follows.

Spher. I 2, finding the center of the sphere, is applied five times (*Spher.* I 3, 4, 6, 8 & 11). In fact, however, the center of the sphere is also constructed in *Spher.* I 12–14 & 16, without invoking *Spher.* I 2, as part of somewhat more involved constructions. It should be noted that the use of *Spher.* I 2, the only problem that relies on an internal construction, is confined to the earlier part of the treatise. Indeed, the general tendency of the treatise is to first use internal geometry to demonstrate some fundamental prop-

[72] In general, the same verb is used for the construction as for the statement of the problem, however, sometimes a more abstract verb can be used for the construction. For example, in *Spher.* II 14, the pole of a circle is found by simply letting a point be (ἔστω) the pole (Czinczenheim 2000, p. 101; Heiberg 1927, p. 68). We have already noted one exception to the use of the passive imperative perfect in the application of *Spher.* I 18 in *Spher.* I 19. See footnote 38, above.

erties of objects on the surface of the sphere and then use these properties to further investigate the surface geometry.

The pair of problems *Spher.* I 18 & 19, setting out the diameter of a circle in the sphere, functions as one of the constructive aims of the first book. *Spher.* I 18 is required by *Spher.* I 19, but, strangely, Theodosius merely asserts the possibility of setting out the internal diameter, without the standard assertion that the construction must have already been carried out.[73] Since setting out the diameter of sphere is never used in the text, from a theoretical perspective *Spher.* I 19 may appear to be unnecessary. Nevertheless, this construction is practically required every time a great circle is actually drawn, and specifically in the remaining problems, which all rely on the possibility of drawing great circles.[74] Hence, *Spher.* I 18 is explicitly used once (*Spher.* I 19), and *Spher.* I 19 is implicitly use four times (*Spher.* I 20, 21, II 14 & 15).

The constructive goal of the first book is completed with *Spher.* I 20 & 21, drawing a great circle through a pair of given points and locating the poles of a given circle. Before these problems are solved, constructions of the poles of a circle are also carried out in the first book using internal lines (for example, *Spher.* I 8 & 16). *Spher.* I 20 & 21 are the two most utilized problems in the text. *Spher.* I 20 is applied 23 times (*Spher.* I 21; II 5–6, 8–10, 13–17, 19–23; III 5–6, 8–10, 12–13), and *Spher.* I 21 fourteen times (*Spher.* II 1, 6, 8, 13-16, 17, 22–23; III 7–8, 12). This is hardly surprising, since Theodosius's approach to spherical geometry and astronomy is founded on the relationships between great circles and systems of parallel circles.[75]

The construction of a great circle tangent to a lesser circle is, on the other hand, relatively less frequent. *Spher.* II 14, which constructs the great circle through a given point of tangency, is used only twice (*Spher.* II 16, 22). The construction of a tangent great circle through a point not on the lesser circle, *Spher.* II 15, is used four times (*Spher.* II 16; III 7, 13, 14). These problems are clearly introduced where they are in book II because they are necessary for the important theorem *Spher.* II 16, which shows that there are only two possible configurations in which a pair of great circles cuts similar arcs from a set of parallel circles. Moreover, the rather involved *Spher.* II 15 requires a number of the theorems of book II as well as the two final problems of book I, *Spher.* I 20 & 21.

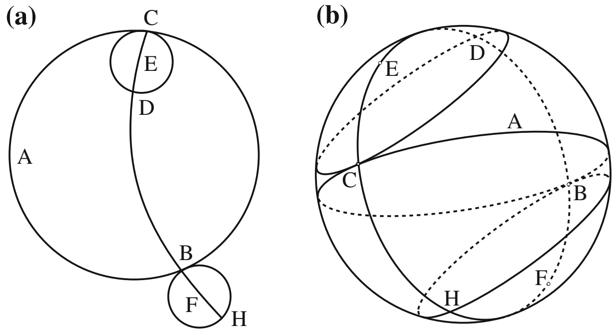
We have already noted that a number of theorems do not use any of the seven problems but, nevertheless, employ construction as an essential part of the deductive strategy. In fact, this is fairly frequent in the text. One case, however, warrants further comment. In *Spher.* II 6, a circle equal and parallel to a given circle is constructed, and in *Spher.* II 8 this construction is again implicitly employed.

In this text, *Spher.* II 6 is the only straightforward case of the use of construction in an existence proof. This takes place, however, in a theorem, not a problem. *Spher.* II 6 demonstrates that if, in a sphere, a great circle is tangent to a lesser circle, then it is also tangent to another equal and parallel circle. The theorem proceeds by constructing the second tangent circle and showing that it is both equal and parallel.

73 See footnote 38, above.

74 See footnote 44, above.

75 Theodosius' approach can be compared, for example, with that of Menelaus, who primarily works with arcs of great circles and spherical triangles (Nadal et al. 2004; Krause 1936).



Spher. II 6: a MS Diagram, **b** Perspective Reconstruction (Proof of Existence)

Where great circle *ABC* is tangent to lesser circle *CD*, the pole of *CD* is found at *E* (*Spher.* I 21), and great circle *CEB* is drawn through *E* and *C* (*Spher.* I 20), intersecting great circle *ABC* again at *B*. Then arc *BF* is cut off equal to arc *EC* (*Elem.* I post. 3*), and circle *BH* is drawn with pole *F* and distance *FB* (*Elem.* I post. 3*). It is then shown that circle *BH* is equal and parallel to circle *DC*. In this way, in order to show that there exists a circle with certain properties, the circle is first constructed and the requisite properties are demonstrated as following from the construction. Indeed, the properties so proven are a direct result of the construction.

In this section, we have seen a number of ways in which constructions are used as sources of necessity. Indeed, constructions are Theodosius's most important means of mobilizing the resources of the definitions and the properties established in previous theorems to bear on the deductive project of the proposition at hand. This similarity of deductive approach is another aspect of the way in which the *Spherics* has been modeled on geometrical books of the *Elements*.[76]

6 Conclusion

In this paper, we have shown how Theodosius's *Spherics* provides internal evidence for the practical origin of constructions in Greek geometry. Within a deductive treatise such as the *Spherics*, however, this practical origin appears to be essentially irrelevant.

76 Mueller (1981, pp. 15–41) discusses the important role of construction in Euclid's deductive approach, especially with respect to *Elem.* I. Netz (1999a, pp. 174–175) attempts to give a more restricted role to constructions in Greek mathematics, arguing that they are only used as a way of laying down new hypotheses which cannot be derived from the original objects. This may be true of the construction used in *Spher.* II 5, but it is an inadequate way of understanding the role of construction in *Spher.* II 6. Moreover, Netz (1999a, pp. 175–182, 187–188) claims that the diagram itself is a source of necessity. In fact, however, the diagram is drawn as a result of the construction. For example, Netz [176] argues that, in *Elem.* I 5, a certain angle between two lines that are differently named by points lying variously on the lines can be understood to be the same angle directly from the text "only with the greatest difficulty." The lines in question, however, were constructed two sentences prior. It is hard to imagine how anyone who has actually drawn these lines, said what they are, and given them labels could have forgotten all that two sentences later. No doubt, actually doing constructions helped ancient geometers to understand what a construction can and cannot imply.

The practical techniques of drawing diagrams are abstracted and reduced to some limited set of permitted constructions that allow the geometer to call directly on the necessary properties of geometric objects inherent in their definitions or established in the course of the theorems.

Deductively, constructions were a vital tool for producing proofs. Before a proof could be developed, however, the geometer would need to investigate the mathematical properties of the objects in question. This was almost certainly done by drawing figures and then making arguments about them. Although these arguments would often include operations on ratios, they would return again and again to the diagram, and to the construction which had produced it, as a vital source of necessity. Hence, metrical accuracy, although not necessary, would greatly facilitate this process.

In an elementary work, such as the *Elements* or the *Spherics*, the practical aspects of ancient geometry are obscured by the didactic goals of the treatise. In other works, which treated more practical areas of the exact sciences, however, the relationship between a given set of geometric tools and the geometric constructions used to solve problems is often made more explicit. For example, ancient gnomonics appears to have been based on constructions made with a compass and a set-square, as becomes clear from Vitruvius's *Architecture* IX 7 and Ptolemy's *Analemma* 11–14 (Granger 1934, pp. 248–254; Heiberg 1907, pp. 210–223), while Pappus, in *Collection* VIII, gives a series of geometric constructions that are carried out with an unmarked ruler and a compass set at some fixed radius (Jackson 1980). From these texts, it is clear that ancient mathematicians were interested in developing mathematical methods that directly modeled the possible operations of actual instruments.

Although the diagrams that have been preserved in the manuscript tradition are generally purely schematic, our investigation has shown that the problems in the *Spherics* were written in such a way that they could be carried out on an actual globe and, hence, must have derived from an interest in producing accurate diagrams. Indeed, there is evidence in other mathematical texts that Greek geometers were interested in working with instruments so as to produce metrically accurate diagrams. For example, Diocles, in *On Burning Mirrors*, describes the use of a bone ruler to drawn a parabola through a set of points (Toomer 1976, pp. 63–67), and Nicomedes is reported to have constructed a mechanical device for drawing conchoid lines, which could also be used for neusis constructions that could not be effected by Euclidean means (Heiberg 1910–1915, vol. 3, pp. 98–106; Netz 2004, pp. 298–301).

Moreover, the *Spherics* was written for students of spherical astronomy who would have been interested in representing the principal circles of the celestial sphere on a globe. Indeed, a globe inscribed with these lines could well have been produced using the kinds of constructive techniques set fourth in Theodosius's seven problems. There are a number of references to such inscribed globes in the ancient technical literature. For example, in his *Introduction to the Phenomena*, Geminus makes reference to inscribed globes in the course of his description of the celestial sphere (Aujac 1975, pp. 31–31; Evans and Berggren 2006, p. 159), and Ptolemy, in the final section of his *Planisphere*, makes reference to the systems of circles that were drawn on inscribed globes (Sidoli and Berggren 2007, p. 81). These globes would have been fairly simple and they appear to have been well known in antiquity. Three examples of ornamental globes inscribed with images of the celestial sphere survive from the Greco-Roman period.[77] Furthermore, there is textual evidence for the kinds of celestial globes that may have been used for teaching and research in spherical astronomy. Ptolemy, in *Alm.* VIII 3, provides instructions for the construction of a detailed star globe (Toomer 1984, pp. 404–407; Heiberg 1898–1903, pt. 2, pp. 179–185), and Leontios, in the *Construction of the Sphere of Aratus*, describes a demonstration globe (Maass 1898, pp. 561–567).

In this regard, we should also consider the evidence of Heron's *Dioptra* 35 (Schöne 1903, pp. 302–306). This chapter of the *Dioptra* describes the physical construction of a concave hemisphere that is used to find the great-arc distance between two locations given simultaneous observations of a lunar eclipse. The solution is effected by making constructions in the hemisphere and on a plane analemma figure. The spherical constructions used by Heron in *Dioptra* 35 are either those assumed by Theodosius as postulates or demonstrated in one of his seven problems, for example drawing a great circle through two points, drawing a parallel circle through a given point, transferring an arc-length, and so fourth.[78] The mathematical methods used in *Dioptra* 35 are somewhat advanced and this text should be read as evidence that the constructions and problems of the *Spherics* were of interest to those carrying out mathematical research as well as students.

It is within this context of material culture and practices that we should understand the constructive methods of the *Spherics*. Although the text is structured as a purely deductive treatise, it was written by and for individuals who used material objects to aid in their investigations of the mathematical aspects of their cosmos. As an elementary treatise, the *Spherics* not only develops the basics theorems necessary for understanding the geometry of the sphere, but also sets out a series of problems that would have been useful for anyone solving problems in spherical geometry by drawing diagrams on a real globe.

Although our discussion and claims have been restricted to the *Spherics*, we should make some remarks about what this implies for our understanding of Greek mathematics more generally. Our study of the *Spherics* has allowed us to make explicit, for the first time, the crucial distinction between constructions, which were used for deductive purposes, and problems, which added to the repertoire of techniques that could be used in the course of geometric research. Probably at a detailed level the research practices of every Greek mathematician were fairly different, especially as there do not appear to have been any schools teaching research methods in the Greco-Roman world. Nevertheless, in the case of geometry, it is reasonable to assume that Greek mathematicians generally proceeded by drawing diagrams and then making arguments about them. Thus, the basic toolbox of the geometer was the set of theorems used to draw inferences and the set of procedures used to draw diagrams. These procedures were set out in problems. From the perspective of deductive argument, however, the actual procedure used in a problem was largely irrelevant. Hence, while difficult problems, such as the duplication the cube, the trisection of the angle or the quadrature of the circle, provided a impetus for original research (Knorr 1986), the solutions to sim-

[77] Evans (1999, pp. 238–241) and Evans and Berggren (2006, pp. 27–31) provide discussions of the archaeological and textual evidence for inscribed globes.

[78] See Sidoli (2005, pp. 241–247) for a full treatment of the constructive procedures attested by Heron.

pler problems provided mathematicians with practical techniques that they could use in the course of their research. Through the process of developing the deductive structure in which the research results were eventually expressed, however, the practical motivation originally underlying the problems receded into the background.

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