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Historia Mathematica 39 (2012) 1–33

HISTORIA MATHEMATICA

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Comparative analysis in Greek geometry

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Available online 10 November 2011

Abstract

This article is a contribution to our knowledge of ancient Greek geometric analysis. We investigate a type of theoretic analysis, not previously recognized by scholars, in which the mathematician uses the techniques of ancient analysis to determine whether an assumed relation is *greater than*, *equal to*, or *less than*. In the course of this investigation, we argue that theoretic analysis has a different logical structure than problematic analysis, and hence should not be divided into Hankel's four-part structure. We then make clear how a comparative analysis is related to, and different from, a standard theoretic analysis. We conclude with some arguments that the theoretic analyses in our texts, both comparative and standard, should be regarded as evidence for a body of heuristic techniques.

Résumé

Cet article vise à contribuer à notre connaissance de l'analyse géométrique grecque. Nous explorons un type d'analyse théorique, jusqu'ici non reconnu d'un point de vue académique, dans lequel le mathématicien utilise les techniques de l'analyse ancienne afin de déterminer si une relation assumée est *supérieur* à, é*gal* à ou *inférieur* à. Au cours de cette exploration, nous défendons l'idée que l'analyse théorique a une structure logique différente de celle de l'analyse problématique, et qu'eelle ne devrait par conséquent pas être divisée en quatre parties selon la structure de Hankel. Nous clarifions ensuite la maniére dont une analyse comparative est liée à - et différente de – une analyse théorique standard. Nous concluons à l'aide d'argument qui indiquent que les analyses théoriques dans nos textes, tant comparatives que standard, doivent être considérées comme des preuve d'un ensemble de techniques heuristiques.

MSC: 01A20; 01A30; 01A35

Keywords: Ancient Greek analysis; Comparative analysis; Euclid; Apollonius; Pappus; Commandino

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doi:10.1016/j.hm.2011.09.002

1. Introduction

In most of the extant examples of analysis in Greek mathematical texts, the author begins by assuming that a certain object has been drawn, a number has been found, or a statement is true and then proceeds by investigating the mathematical consequences of this assumption. In a few cases, however, the author assumes that some relation holds between two objects, or ratios, and then uses the techniques of ancient analysis to determine *what this relation is*. We call this later type of argument *comparative analysis*, since what the mathematician is seeking is a comparison, or relation (σύγκρισις), between the two objects, or ratios.

This paper is an investigation of comparative analysis. We identify and read the ancient texts that preserve arguments of this type, and we situate these within the context of ancient analytical practices. Finally, we argue that comparative analysis was actually used as a heuristic technique, and although its presence in our texts represents some rhetorical choice on the part of the author, it was originally not a primarily rhetorical device. When we speak of *heuristic technique*, we do not mean an automatic method for *generating* proofs and constructions, but rather a set of techniques that were useful in *looking for* them.[1] Hence, we must understand the method of analysis, in the loose sense of Polya and Szego's method, as an idea that can be used more than once.[2]

Pappus, in the introduction to *Collection* VII, states that analyses are of two kinds, *problematic analysis*, which is "the supplying of what is proposed" (τὸποριρτικὸν τοῦ προταθέντος), and *theoretic analysis*, which is "the investigation of the truth" (τὸ ζητητικὸν τἀληθοῦς) [Jones, 1986, 83]. Problematic analysis, then, is primarily about producing something, or finding something, while theoretic analysis is primarily about showing something, or proving something. In some sense, comparative analysis might be said to fit into both of these categories. Nevertheless, since in geometrical texts — and all of the comparative analyses we will study are found in geometrical texts — a problematic analysis always involves the construction of a particular geometrical object, it makes better sense to think of comparative analysis as a type of theoretic analysis. Indeed, as we will see below, the components of a comparative analysis bear a closer conceptual and logical relation to those of a standard theoretic analysis than to those of a problematic analysis. Hence, we will describe comparative analyses in terms of theoretic analyses.

Modern scholars, following Hankel [1874, 137–150], have generally divided an analyzed proposition into two primary parts, which, in turn, each have two subparts: the *analysis* is divided into the *transformation* and the *resolution*, and the *synthesis* is divided into the *construction* and the *proof*.[3] In fact, however, most of the discussions of ancient texts in terms

[1] This follows the usage of Hintikka and Remes [1974, 1] and agrees with the general sense of heuristic found in Knorr [1986]. Netz [2000, 143–144], on the other hand, speaks of a heuristic method as a sort of mechanical procedure or algorithm that will automatically generate results and argues that Greek analysis fails to meet this restricted notion.

[2] Mahoney [1969, 319] points out that Greek analysis is not a method in the sense in which the word is usually understood, and speaks instead of "a body of techniques, which was suggestive rather than proscriptive."

[3] Note that the terminology is not always consistent. For example, Hintikka and Remes [1974, 24] call the transformation the "analysis proper" and Fournarakis and Christianidis [2006, 47] call it the "hypothetical part." Nevertheless, despite the difference in nomenclature, the divisions in the argument are maintained.

of this division have focused on problematic analyses.[4] In a theoretical analysis, however, a construction, although often present, is not an essential feature of the mathematical goal, which is the demonstration of some statement.[5] For this reason, in a theoretic analysis there is no resolution, and when there is a construction it is of a different logical status than a construction in a problematic analysis.[6] Hence, from a logical perspective a theoretical analysis simply consists of an analysis and a synthesis, both of which may, or may not, rely on an auxiliary construction.

Nevertheless, a number of the theoretic analyses preserved in our sources contain a section between the analysis and the synthesis. A theoretic analysis begins with the assumption of what is to be shown and then proceeds, by a series of deductive steps modeled on the argumentative practices of the *Elements*, sometimes combined with auxiliary constructions, to show that this implies some other statement that the geometer knows can be shown independently of the analytical assumption. In many cases this other statement is directly obvious, on the basis of the geometry of the figure, in which case the geometer will simply assert that it is so. In other cases, however, it may take some reasoning to show that this other statement is, in fact, true. We will call this assertion, or second stage of reasoning, the *verification.*[7]

The verification, then, shows that the final statement of the analysis is true, based on the geometry of the figure, or other considerations that are independent of the argument and assumptions of the analysis. The validity of the verification is sometimes so obvious that it can simply be asserted;[8] at other times, however, it requires an argument, which is again provided using the argumentative practices of the *Elements*. In this paper, we will encounter examples of both simple and extended verification.

It is important to note, however, that the length of the verification is, from a logical perspective, an arbitrary choice on the part of the geometer. As we will show below, even in cases where there is an extended verification in our sources, it is possible to continue the argument of the synthesis until the verification can be made directly on the basis of the geometry of the figure and the conditions of the theorem. Thus, the length of the verification represents, not an essential part of the logical structure, but a decision, on the part of the mathematician, about when it has become sufficiently obvious that the theorem can be written.

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[4] See, for example, Berggren and Van Brummelen [2000] and Saito and Sidoli [2010]. Hintikka and Remes [1974, 22–26] also use these four categories to discuss a problematic analysis, *Collection* IV prop. 4, but as we will see below, both the *resolution* and the *construction* are missing.

[5] A similar point is made by Mäenpää [1997, 221–226], using intuitionistic type theory.

[6] This different logical status becomes clear when we examine the structure of problems whose proofs themselves require auxiliary constructions, such as *Elements* III 1 or the analogous *Spherics* I 2. In these cases, we find that there are two phases of construction, the first of which solves the problem and the second of which provides geometric objects necessary to show that the solution is valid, but which were not provided by the initial construction. These secondary constructions, which are unnecessary in the majority of problems, have the same logical status as a construction in a theorem. See Sidoli and Saito [2009] for further discussion of these distinctions.

[7] The verification is sometimes called the "resolution," since it comes at the same position in the argument as the resolution in a problematic analysis; for example, Hintikka and Remes [1974, 25]. We restrict the term *resolution*, however, to an argument, using *Data* style arguments, that what is sought is given on the basis of the initial configuration and any auxiliary constructions. This follows the original formulation by Hankel [1874, 144], who states that the resolution is supported "durch ein Citat aus den 'Daten' Euklid's."

[8] Such is the case in the example of the theoretic analysis studied by Hintikka and Remes [1974, 24–26], *Collection* IV prop. 4.

The extant theoretic analyses do not include a separate construction at the beginning of the synthesis, and if an auxiliary construction is introduced in the analysis, it is simply assumed as already having been constructed for the purposes of the synthesis.[9] Hence, following the assertion or argument of the verification, the synthesis proceeds to show that the claim made at the end of the analysis implies the analytical assumption, which was what we set out to prove. The proof often proceeds along lines very similar to those of the analysis, but in the opposite direction. In some cases, as we will see below, it follows exactly the same steps in the reverse order, but in other cases, the order of the argument can diverge somewhat from that of the analysis.[10]

Of course, this schema is an idealization that was not always followed in ancient practice. To get a sense of how the parts of the schema function, and to see how an actual argument can diverge from the model, we will look at the example of *Collection* VII prop. 26.

In the course of his discussion of Apollonius's *Cutting off a Ratio*, Pappus provides a lemma that, like most of the lemmas in *Collection* VII, demonstrates a fairly uninteresting mathematical result that was meant to be read in conjunction with the ancient work to which it refers, but that is, nevertheless, of real value to the historian of ancient mathematics.

In *Collection* VII prop. 26, with respect to Fig. 1, Pappus shows that if a triangle ABG is cut by two lines, AD and AE, such that $\angle BAG + \angle DAE = 2R$, $R = 90°$, then $(BG \times GD) : (BE \times ED) = GA^2 : AE^2$.[11] The text reads as follows.[12]

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[9] Such is the case in the example studied by Hintikka and Remes [1974, 26] , in which we see that their *construction* is an empty place holder. Indeed, most of our examples of theoretic analysis have only a single diagram, whereas most of the extent examples of problematic analyses from Hellenistic authors include two diagrams. This stresses the fact that Greek geometric problems are about the construction of a particular figure.

[10] See, for example, the discussions by Saito [1986, 46–47] and Behboud [1994, 63–66].

[11] In this paper, we use the expression $(AB \times CD)$ for the *rectangle contained by lines AB and CD* and the expression AB^2 for the *square on line AB*. In the texts discussed in this paper, these expressions always refer to geometric objects and no reference to arithmetic operations is intended by our convention. Another feature of our convention is that we always inclose an individual object in parentheses. This produces some odd results in places. For example, "the line that is equal in square to four times TE by EZ" becomes $(\sqrt{4(TE \times EZ)})$ to indicate that it is a line that is not specifically named, but is rather defined by the terms and relations given in the symbolic expression. See Footnote 30, below.

[12] We have inserted various numbers and references in brackets into the text, here and in the following. We do this so that we can refer to specific parts of the text in our discussion and to refer the reader to theorems and problems that were available in the ancient literature that can be used as justifications of the mathematical argument.

The translated texts are divided into paragraphs, marked with numerals, as [1], and into individual statements marked with an alphanumeric key, as [A1]. The numbers indicate the individual mathematical steps of the argument, which are, of course, somewhat arbitrary. The letters are taken from the name of the part of the proposition, as discussed below. An asterisk indicates a reconstructed step.

Although we include references to ancient works in brackets, it is unlikely that ancient readers would have been expected to consult the written works. Rather they would simply have been assumed to know the mathematics in question. Nevertheless, we have included these specific references to help modern readers stay mindful of the fact that these arguments were being presented in a context of *assumed knowledge* — what Saito has called the toolbox — which is often somewhat different from the assumed knowledge of a modern reader of mathematics.

This lemma is also discussed by Acerbi [2007, 490–491].

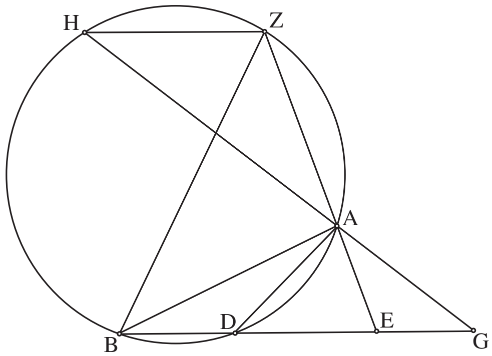

Fig. 1. *Collection* VII prop. 26.

> [1] If there is a triangle, *ABG*, and two lines, *AD* and *AE*, are passed through, such that angles *BAG* and *DAE* are equal to two right [angles], then it will be that as rectangle *BG*, *GD* to rectangle *BE*, *ED*, so is the square on *GA* to the square on *AE*.[13]
> [2, A1] For, if I circumscribe a circle around triangle *ABD* [Elements IV 5], and *EA* and *GA* are produced to *Z* and *H*, [A2] rectangle *BG*, *GD* transforms into (μεταβαίνει) rectangle *HG*, *GA* [Elements III 36], and rectangle *BE*, *ED* [transforms] into rectangle *ZE*,*EA* [Elements III 36], [A3] and it will be necessary to seek (δεήσει...ζητῆσαι), alternately, if as rectangle *HG*, *GA* to the square on *GA*, so is the rectangle *ZE*, *EA* to the square on *EA* [Elements V 16]. [A4] This, however, is the same as to seek if as *HG* to *GA* so is *ZE* to *EA* [Elements VI 1]. [A5] Therefore, if it is, *HZ* is parallel to *BG* [Elements I 28, VI 5, V 17].
> [3] But it is. [V1] For since the angles *BAG* and *DAE* are equal to two right [angles], angle *DAE* is equal to angle *BAH* [*Elements* I 13]. [V2] But, angle *DAE*, outside the quadrilateral, is equal to angle *ZBD* [*Elements* III 22 and I 13], [V3] while angle *BAH* is equal to angle *BZH* [*Elements* III 21]. [V4] Therefore, angle *ZBD* is equal to angle *BZH*. [V5] And they are alternate [angles], therefore, *HZ* is parallel to *BZ* [*Elements* I 27]. This, however, is what was sought; therefore it holds. [Jones, 1986, 147]

After the specification, in Section [1], the analysis begins with the construction of an auxiliary circle about triangle *ABD* and the secondary constructions of producing lines *EAZ* and *GAH* [A1]. As we see later, line *BZ* is also joined, but this is not mentioned by Pappus. The analysis, then, uses a series of deductive steps, [A2]–[A5], to show that the assumption that $(BG \times GD) : (BE \times ED) = GA^2 : AE^2$ — which Pappus never explicitly states *as* an assumption — leads to the claim that $HG : GA = ZE : EA$, which would be true if $HZ \parallel BG$. Notice, that although Pappus uses the expression "it is necessary to seek if" such-and-such is the case, the steps of this argument can all be justified deductively by appeal to propositions in the *Elements*. What Pappus is establishing is other things that will be true of this particular configuration of geometric objects if he assumes what he is trying to show.[14] In other words, he transforms the argument into one of showing that the, perhaps, more obscure fact that $(BG \times GD) : (BE \times ED) = GA^2 : AE^2$ can be shown as a result of the more geometrically apparent fact that $HZ \parallel BG$.

[13] We have not attempted to translate literally. Thus, we render τό ὑπὸ τῶν ΒΓΔ as "rectangle *BG*, *GD*" and τό ἀπὸ ΓΑ as "the square on *GA*," since a Greek reader would have understood this ellipsis as fully determinate.
[14] This point is also made by Hintikka and Remes [1974, 35].

The verification, [V1]–[V5], then shows that, indeed, $HZ \parallel BG$, independent of the analytical assumption. In this argument, Pappus relies on the auxiliary construction and basic propositions of the *Elements*, but makes no mention of what he is trying to prove.

As we pointed out above, however, including this extended verification is an arbitrary choice that Pappus has made. He could just as well have continued beyond [A5] as follows. Now, if $HZ \parallel BG$, by *Elements* I 29,

$$\angle ZBD = \angle BZH. \tag{A6*}$$

But, because they stand on the same arc, by *Elements* III 21,

$$\angle BZH = \angle BAH, \tag{A7*}$$

and since $\angle DAE$ is the external angle of cyclic quadrilateral $BZAD$, by *Elements* III 21 and I 13,

$$\angle ZBD = \angle DAE. \tag{A8*}$$

Substituting (A8*) and (A7*) into (A6*), it is necessary to see if

$$\angle DAE = \angle BAH. \tag{A9*}$$

But it is. Since $\angle BAH$ is the supplementary angle to $\angle BAG$, by *Elements* I 13, $\angle BAG + \angle BAH = 2R$, and by the original, nonanalytical assumption $\angle BAG + \angle DAE = 2R$; hence, $\angle DAE = \angle BAH$.

In this way, we see that the analysis could have been carried to the point where the verification follows directly from the original condition of the theorem to be shown, namely $\angle BAG + \angle DAE = 2R$. The fact that Pappus did not take this approach, but rather reduced the argument to one of showing that $HZ \parallel BG$ and then showed this separately, as a result of the condition, is indicative of a rhetorical, or expository, choice on his part. By dividing the argument in this way, he is drawing attention to the role that the fact that $HZ \parallel BG$ plays in the overall structure of the proof. He is indicating that the key to proving this theorem lays in first proving that $HZ \parallel BG$. That is, we can understand *why* $(BG \times GD) : (BE \times ED) = GA^2 : AE^2$ is the case, when we see that $HZ \parallel BG$.

In this proposition, Pappus presents no synthesis, presumably because he thought it would be obvious on the basis of what he has already said. Nevertheless, for the sake of completing our discussion of the structure of an analyzed proposition, we can reconstruct the synthesis as follows.

We will begin by imagining that we have no analysis and that we are constructing a full synthetic theorem. Considering Fig. 1, suppose we are given $\triangle ABG$ with internal lines AD and AE, such that $\angle BAG + \angle DAE = 2R$, but with none of the other objects drawn in the diagram.

If the argument were to be made fully synthetic, the construction would proceed by drawing a circle through the three points A, B, and D [*Elements* IV 5] and extending lines GA and EA until they meet the circle at points H and Z [*Elements* I post. 2]. It would then proceed by joining lines AH, BZ, and ZH [*Elements* I post. 1]. With the construction of these auxiliary objects, everything is in place to complete the argument. In fact, however, the synthesis of an analyzed proposition always simply assumes the construction of the analysis.[15]

[15] This should be contrasted with the synthesis of a problematic analysis, which begins with a new construction, and generally with a new figure; see Saito and Sidoli [2010, 586, 590, 599].

In this case, the proof can constructed following the steps of the analysis exactly. We need not go through the steps of the first part of the proof; nevertheless, some attention should be paid to the order in which we would proceed. Starting from the given objects, $\triangle ABG$ and lines AD and AE, along with the newly constructed objects, circle ADB and the lines inside it, we would show, following the same order as the verification, [V1]–[V5], that $HZ \parallel BG$. Indeed, this part of the full proof is simply a repetition of the verification and is never given in our sources. The second part of the proof, however, goes through the steps of the analysis, but in reverse order. Pappus' argument is fairly compact, but we may flesh it out in the synthesis. In the following, we give certain steps of our argument the numbers (S5)–(S1), to highlight the fact that each of these statements can be associated directly with the correspondingly numbered statement of the analysis. We begin with the claim, established in the first part of the proof, which is a repetition of the verification, that

$$HZ \parallel BG. \tag{S5}$$

From this it follows, by *Elements* I 29, that the angles of $\triangle HZA$ and $\triangle AEG$ are equal. Hence, by *Elements* VI 4, $AH : AG = ZA : AE$, and by *Elements* V 18, $(AH + AG) : AG = (ZA + AE) : AE$; that is,

$$HG : AG = ZE : AE. \tag{S4}$$

We can then use *Elements* VI 1 to point out that $HG : AG = (HG \times AG) : AG^2$ and $ZE : AE = (ZE \times AE) : AE^2$, so that, by substitution into (S4), we have

$$(HG \times AG) : AG^2 = (ZE \times AE) : AE^2. \tag{S3}$$

Then, by *Elements* V 16, we have.

$$(HG \times AG) : (ZE \times AE) = AG^2 : AE^2, \tag{S3.i}$$

which is not explicitly stated by Pappus, but is implied by his use of the expression "alternatively." Finally, by *Elements* III 36,

$$\begin{aligned}(HG \times AG) &= (BG \times GD), \text{and}\\ (ZE \times AE) &= (BE \times ED).\end{aligned} \tag{S2}$$

So, by substitution,

$$(BE \times ED) : (BG \times GD) = AG^2 : AE^2, \tag{S1.i}$$

which, again, is not explicitly stated in the analysis itself, but the assumption of which is implied by the very fact that Pappus proceeds by analysis.

In this way, we see that an argument exactly mirroring the analysis can be reconstructed for the synthesis. In this regard, some questions about the reversibility of steps of the analysis may arise. We first note that all steps involving assertions of equality and substitutions, such as (S4) → (S3) or (S2) → (S1.i), are simply reversible. Furthermore, *Elements* V 16, used at (S3) → (S3.i), is its own converse. Finally, the three propositions of the *Elements* that we used as justification for (S5) → (S4), namely *Elements* I 29, VI 4 and V 18, are all the converses of the related propositions that we referred to in the translation of the analysis, namely *Elements* I 28, VI 5, and V 17. Of course, Pappus does not refer to any propositions in his argument. Nevertheless, it seems clear that Pappus was well aware that the most obvious choices for propositions that could be used to justify his assertions all have direct converses. Hence, as Mahoney [1926, 326–327] has pointed out, Pappus probably treated the issue of reversibility, somewhat loosely, with the sensibility of a working

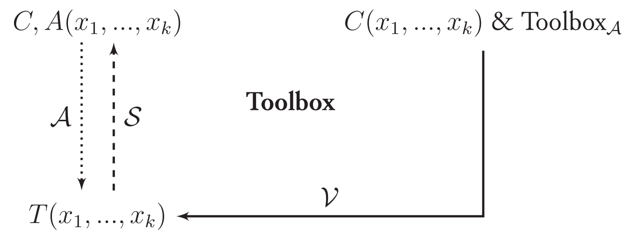

Fig. 2. Schematic of a standard theoretic analysis.

mathematician who has a good grasp on which deductive steps are strictly reversible and which are not. From this perspective, it was obvious to Pappus, having arrived at the end of the verification, that a synthesis could be constructed straightforwardly on the basis of what he has said. So he simply asserts that the theorem holds.

With this discussion as background, we may outline the general form of a theoretical analyzed proposition as follows.[16]

1. $\mathcal{A}$: Analysis
   (a) Construction (may be absent)
   (b) Deduction
2. $\mathcal{V}$: Verification
3. $\mathcal{S}$: Synthesis: deduction (inverse)

Moreover, as we saw above, the verification is sometimes so short as to be a simple declaration of the fact and the synthesis is sometimes omitted. Nevertheless, although the construction, verification, and synthesis are all sometimes absent in our sources, the former is left out because it is sometimes mathematically unnecessary, whereas the verification and the synthesis are only left out because the argument is presumed to be obvious.[17] Where the synthesis is given, we may understand the geometer as making explicit to the reader the fact that the deductive argument presented in the synthesis is, in fact, fully reversible. Where the geometer regarded this fact as obvious, or trivial, the proof could be omitted. Finally, the geometer could emphasize a key step in the argument by concluding the analysis at this point and then confirming this point in the verification.

In order to develop a clearer picture of how the standard theoretic analysis works, we present the schematic in Fig. 2.[18] The geometer begins with a conjunction of both the con-

[16] It is worth noting that the rhetorical structure of the standard theoretic analysis is sometimes found internally within a normal theorem. In these cases, the mathematician assumes what is to be shown and then uses ratio and arithmetic operations to reduce this to something obvious, or more readily shown. See, for example, *Conics* III 24–26 [Heiberg, 1891–1893, 368–376; Acerbi, 2007, 491–492].

[17] See, for example, how Hintikka and Remes [1974, 25–26] explain in detail the full argument of the verification for *Collection* IV prop. 4, which Pappus glosses over with the remark “but it is” (ἔστιν δέ).

[18] This schematic is not meant to describe the logical form of a theoretic analysis with any rigor, but rather to provide a visual representation of the argumentative structure we find in the texts. Nevertheless, these figures can be compared to those presented from a more rigorously logical perspective by Hintikka and Remes [1974, 36] and Mäenpää [1997, 217 & 224].

The only substantial difference between our schematic and that provided by Hintikka and Remes [1974, 36] is the relationship of the Toolbox and Toolbox$_{\mathcal{A}}$, which will be discussed below.

ditions of the theorem, C, and the analytical assumption, A, which are both statements about a given, and specific, set of objects, $x_1,\ldots,x_k$.[19] The analysis then proceeds by using the conditions of the theorem along with this assumption, $C,A(x_1,\ldots,x_k)$, in conjunction with a toolbox of known theorems, allowed operations and possible constructions, Toolbox,[20] to derive a transformed statement of something else that will consequently be true of these same objects, $T(x_1,\ldots,x_k)$. We represent the fact that the analysis is a series of steps with a broken line. Moreover, although the language of the mathematician often makes it clear that he or she is thinking of the analysis as a tentative argument about what will happen *if* $A(x_1,\ldots,x_k)$ is the case, the argumentative strategy, is nevertheless, deductive.

The overall synthetic argument, which is broken up into the verification and the synthesis, begins with the conditions of the theorem in question, $C(x_1,\ldots,x_k)$, now independent of the analytical assumption, and a toolbox of known results, possible operations, and auxiliary constructions now made specific by having actually been used in the analysis, Toolbox_A, and proceeds to show, first, that $C(x_1,\ldots,x_k)$ implies $T(x_1,\ldots,x_k)$ and then, through a series of deductive steps, that $T(x_1,\ldots,x_k)$ implies $A(x_1,\ldots,x_k)$. The verification is essentially a single step, represented by a solid line, while the synthesis then proceeds through a series of arguments drawing on Toolbox_A and more or less loosely modeled on the argument of the analysis, represented by a broken line. In many of the examples we will look at in this paper, the synthesis follows through the same series of steps as the analysis, but this is not at necessary.[21]

We have represented the general toolbox, **Toolbox**, as floating freely, unassociated with any particular set of starting points.[22] Of course, from a purely logical perspective, some

[19] That is, the objects with which a Greek analysis deals are always instantiated. This point is made quite clearly by Hintikka and Remes [1974, 35].

It may be appropriate to point out that $x_1,\ldots,x_k$ are not variables, but simply an unordered list of instantiated objects, such as *line AB* or *circle CD*. As Mäenpää [1997, 215] has made clear, it would be just as well to call these objects x. evertheless, we have used $x_1,\ldots,x_k$ simply to remind the reader that we are dealing with a set of instantiated objects that remains unchanged in the three parts of the analyzed proposition.

[20] For the time being, because we are focused on theoretic analysis, we ignore the role of auxiliary constructions, but see Hintikka and Remes [1974, 41–48] and Mäenpää [1997, 217–226].

[21] This point is made in detail by Saito [1986, 46–47] and Behboud [1994, 63–66].

[22] Hintikka and Remes [1974, 36] address this issue by asserting that both the analysis and the verification must have as premises an additional set of assumptions, which they call K, that are "a conjunction of axioms and suitable earlier theorems," and speak in terms of the conjunction of it with $C,A(x_1,\ldots,x_k)$, when carrying out the analysis, and with $C(x_1,\ldots,x_k)$, when carrying out the synthesis. As Behboud [1994, 61] points out, however, K is not given at the beginning of the analysis, but rather a set of useful theorems, operations and assumptions becomes established during the course of the analysis, which we call Toolbox_A. This is what we have tried to capture with our use of the ideas of the **Toolbox** and Toolbox_A.

The concept of the Toolbox introduced here is related, but not identical, to the concept of the toolbox as discussed by Netz [1999, 216–235], who refers to, and draws on, a research project by Saito. The toolbox, as it is usually understood, refers to a body of accepted results and procedures that the mathematician can assume to be understood by the reader. These are usually collected in works like the *Elements*, *Data* or *Spherics*, but are not necessarily so. In our discussion of analysis, the **Toolbox** is a somewhat broader concept, which consists of everything the mathematician knows and can bring to bear on any problem or proof. In fact, however, this is often a similar set of results, most of which can be found in the canonical texts. In both cases, the toolbox is an assumed body of knowledge and techniques.

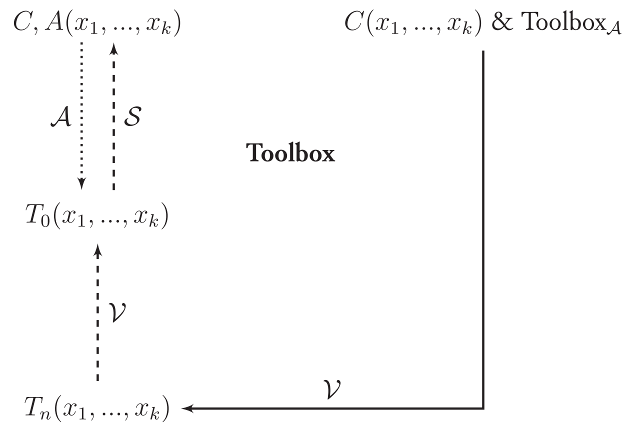

Fig. 3. Schematic of an extended theoretic analysis.

subset of this general toolbox forms the auxiliary assumptions that should be included at the beginning of both the analysis and the synthesis. From a mathematical perspective, however, the toolbox consists of everything that the mathematician assumes can be drawn into the discourse with no further justification — theorems that can be called upon, operations that can be performed, constructions that can be made, and so forth. The toolbox can be applied at any step of any part of the analyzed proposition. Hence, from the perspective of mathematical practice, the toolbox represents a sort of context of mathematical knowledge, in which the overall argument is assumed to take place.

One of the most important goals of the analysis is to decide what subset of the toolbox can be used in any particular theorem; that is, to determine of all the possible theorems, operations, and auxiliary constructions, which ones will be of service in producing the result we seek.[23] Although the synthesis may not mirror the analysis exactly, it will still draw on the same subset of the toolbox as the analysis and in roughly the same way. Hence, we may understand the analysis as both demonstrating that $T(x_1,\ldots,x_k)$ implies $A(x_1,\ldots,x_k)$ and producing a set of theorems, operations, and constructions that directly bear on the matter at hand, Toolbox$_{\mathcal{A}}$. This, then, becomes an explicit set of auxiliary assumptions for the synthetic argument, which we can be expressed by taking Toolbox$_{\mathcal{A}}$ in conjunction with the conditions of the theorem, $C(x_1,\ldots,x_k)$. From a practical perspective, Toolbox$_{\mathcal{A}}$ furnishes the mathematician with everything that will be necessary for the synthetic proof.[24]

A logician might be concerned about the general validity of this method, since it is possible that not every step of the analysis will have a direct converse, but a working mathematician would be less fazed by this. On the one hand, one can always think though what theorems and operations, have converses as one goes, and on the other, the analysis is an analysis of a specific configuration of geometric objects, showing that $A(x_1,\ldots,x_k)$ implies $T(x_1,\ldots,x_k)$. Hence, it should generally be fairly easy to show that for this specific set of objects it is also that case that $T(x_1,\ldots,x_k)$ implies $A(x_1,\ldots,x_k)$, even if the argument one uses is not a direct converse of the analysis. In both cases, what one is concerned with is

[23] Behboud [1994, 61] also makes this point, stating that finding this subset of the toolbox is "one of the heuristic tasks of an analysis."

[24] An examination of the discussions by Saito [1986, 44–47] and Behboud [1994, 63–64] shows that even in cases where the synthesis does not mirror the analysis, the Toolbox$_{\mathcal{A}}$, determined in the analysis and applied again in the synthesis, is the same.

not a chain of logical inferences that can be derived from a certain statement, but rather the implications of that statement for a specific configuration of geometric objects.

As we saw above, not every theoretic analyzed proposition has such a simple verification. Sometimes the verification is extended into a deductive chain of reasoning. We can represent such extended verifications with the schematic in Fig. 3.

In the case of an extended verification, the geometer argues from the conditions and the analytical assumption, C, $A(x_1,\ldots,x_k)$, to some transformed statement, $T_0(x_1,\ldots,x_k)$, which is, in fact, not a direct consequence of the conditions of the theorem, $C(x_1,\ldots,x_k)$. The verification then proves, through a series of steps, that $T_0(x_1,\ldots,x_k)$ can be derived from $C(x_1,\ldots,x_k)$, starting with some immediate first step $T_n(x_1,\ldots,x_k)$.

When we visualize this in a schematic, it becomes clear that a theoretic analysis with an extended verification has the same global structure as a standard theoretic analysis, and that $T_0(x_1,\ldots,x_k)$ can be understood as simply one of the claims that is made along the way in arguing that $A(x_1,\ldots,x_k)$ implies $T_n(x_1,\ldots,x_k)$. Hence, as we argued above, this form was probably not used for any compelling logical reason, but rather because the mathematician wanted to highlight the statement $T_0(x_1,\ldots,x_k)$ as somehow significant in understanding *how* the proof works, or *why* the theorem is true.

With this as a characterization of a theoretic analyzed proposition, we now have the background to begin an investigation of the type of theoretic analysis that we call comparative analysis. In a standard theoretic analysis, the mathematician assumes what is to be shown. In the case of our example, Pappus assumed that there is a relation of equality between two objects, in this case ratios. In the case of comparative theoretic analysis, as we will see below, the geometer assumes that there exists *some* relation between two objects, which is either *greater than*, *equal to*, or *less than*. In this case, the verification must determine which one of these relations holds, which is naturally also a proof that there is some relation.

Before discussing the structure of a comparative theoretic analysis and the relation between standard theoretic analysis and comparative theoretic analysis, we turn to some examples of this type of analysis that survive in our medieval sources for the ancient texts.

2. Examples of comparative analysis

So far, we have only identified two ancient texts that have clear cases of comparative analysis. These are Apollonius' *Cutting off a Ratio* and Book VI of Pappus' *Collection*. We will begin with an examination of the use of comparative analysis in *Cutting off a Ratio*.

2.1. Comparative analysis in Cutting off a Ratio

The earliest Greek author whose comparative analyses have survived is Apollonius. These analyses are found in his *Cutting off a Ratio*, a text presented entirely in the analytical mode. Although there are no extant copies of the Greek text, an Arabic translation, made by an unknown scholar, has survived [Rashed and Bellosta, 2010].[25]

Cutting off a Ratio is an extended analysis that, over the course of some 55 manuscript pages, exhaustively solves the problem of drawing a line through a given point such that it

[25] Since we will sometimes refer to the Arabic manuscripts of this text, we reference them as *CR* A for Aya Sofya 4830 and *CR* B for Bodleian Arch. Seld. A 32. These correspond to ط and ب, respectively, in Rashed and Bellosta [2010].

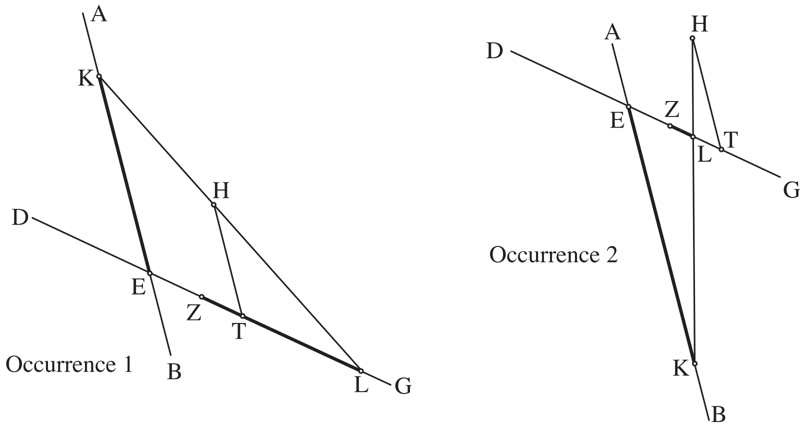

Fig. 4. *Cutting off a Ratio* I 6, Occurrences 1 and 2.

falls on two given lines and cuts from them lengths, determined from the given points, which have to one another a given ratio. That is, given two lines ℓ_1 and ℓ_2, and some point E on ℓ_1, another point Z on ℓ_2, and another point H on neither ℓ_1 nor ℓ_2, and some given ratio r, to draw a line, HKL, such that

$$EK : ZL = r.$$

This problem is then solved for all geometrically different configurations of the given initial lines and points and for all possible solutions for any given initial configuration. The text is divided into *dispositions* (τόπος, وضع), for the different configurations of the initially given objects, and *occurrences* (πτῶσις, وقوع), for the different possible positions of the cutting line.[26]

In order to see the full range of terminology dealing with comparative analysis in *Cutting off a Ratio*, we will look at the second and the fourth occurrence of the sixth disposition.

Cutting off a Ratio I 6 treats the following configuration. In Figs. 4 and 5, let ℓ_1 and ℓ_2 be AB and CD, and let the given point not on either of these lines be point H. The sixth disposition is specified by setting the given point E at the intersection of AB and CD, and setting the given point Z on DG, such that it falls between point E and point T, which is determined by drawing HT parallel to AB.

There are four occurrences, depending on how the line passing through H falls on the two given lines AB and GD.[27]

[26] The Greek terminology comes from Pappus's description of *Cutting off a Ratio* in his *Collection* VII [Jones, 1986, 87]. The Arabic terminology is used throughout the text [Rashed and Bellosta, 2010]. See Saito and Sidoli [2010, 596–597], for a more complete discussion of these terms and our translation decisions.

[27] In this summary, we call the intersection of HL and AB point K and the intersection of HL and GD point L. This conforms with our discussion of the four occurrences and should facilitate seeing the relationships between the various occurrences. In the text, however, following standard Greek practice, the objects were simply renamed, with no attempt to make them consistent from one occurrence to the next. Here we distinguish between points L, K, and M, introduced in the general analysis, and points L_0, K_0, and M_0, which are special cases of these points introduced in the diorism. In the text, there is no such distinction.

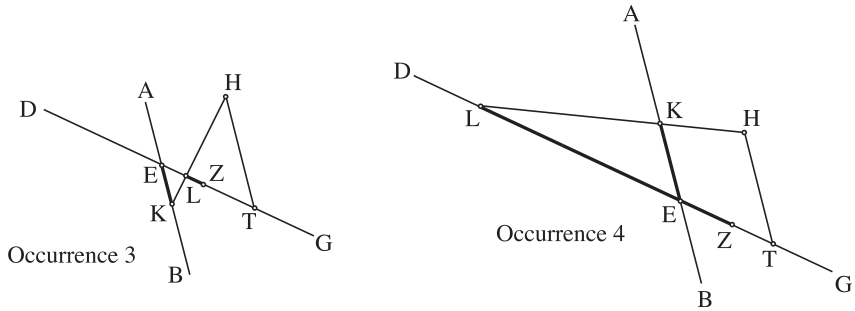

Fig. 5. *Cutting off a Ratio* I 6, Occurrences 3 and 4.

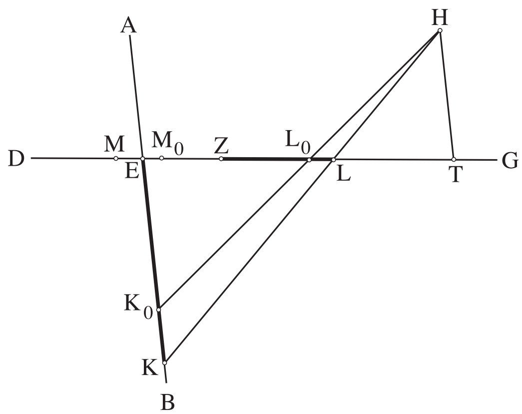

Fig. 6. *Cutting off a Ratio* I 6.2, summary.

Occurrence 1. In Fig. 4, line *KHL* cuts off segment *EK* from ray *EA* and segment *ZL* from ray *ZG*.

Occurrence 2. In Fig. 4, line *HLK* cuts off segment *EK* from ray *EB* and segment *ZL* from ray *ZG*.

Occurrence 3. In Fig. 5, line *HLK* cuts off segment *EK* from ray *EB* and segment *ZL* from ray *ZD*.

Occurrence 4. In Fig. 5, line *HKL* cuts off segment *EK* from ray *EA* and segment *ZL* from ray *ZD*.

The examples of comparative analysis that we will examine are taken from *Cutting off a Ratio* I 6.2 and 6.4. These two occurrences are fairly long and the comparative analysis makes up just a small component of the overall argument. In order to situate these proofs in their contexts, and in order to understand certain steps of the argument, it will be necessary to give a summary of the overall argument of *Cutting off a Ratio* I 6.2. In the summary, we do not provide the full arguments or their justifications. In this paper, the only arguments that we treat in full are those that are handled using comparative analysis, which in this summary are designated as Diorism (b) and (c) (see Fig. 6).

Analysis. The general analysis shows that if we draw HT such that $HT \parallel AB$, and if point M is determined by relation $TH : ZM = EK : ZL = r$, where TH and r are given, then it can be shown that $(ZM \times TE) = (TL \times LM)$, so that by applying the given rectangle $(ZM \times TE)$ to the given line TM and deficient by a square, LM^2, [*Elements* VI 28], point L will be given. Hence, line HL is given.

Diorism. (a) Since it is not always possible to apply the rectangle $(ZM \times TE)$ to TM deficient by a square — the condition is that $(ZM \times TE) \leqslant \frac{1}{4}TM^2$ — a special case, namely in which $TL_0 = L_0M_0$, $(TL_0 \times L_0M_0) = (ZM_0 \times TE)$ and $TH : ZM_0 = EK_0 : ZL_0 = r_0$, is solved using an analyzed proposition. That is, for the purposes of the diorism, we consider a new ratio, r_0, that is not given, but which is found satisfying the stated conditions.[28] It is shown that point M_0 is determined when L_0E is a mean proportional between TE and EZ, that is, $TE : L_0E = L_0E : EZ$.

(b) Since $TL_0 = L_0M_0$ when $(TL_0 \times L_0M_0)$ is applied to TM_0, the special case is a limiting case [*Elements* II 5]. It is shown through a comparative analyzed proposition that HL_0 cuts off the least ratio of any line drawn from H and falling on rays EB and ZG.

(c) It is shown through another comparative analyzed proposition that lines closer to HL_0 always cut of greater ratios from rays EB and ZG than lines farther from HL_0.

Synthesis. Using the special case of Diorism (a), line HL_0 is constructed so that there are three cases depending on the relation $r \gtreqless EK_0 : EL_0$. The solutions and the limits of solubility are handled separately.

Determination of the Limit. The limit[29] of the ratio is shown to be given as $r_0 = EK_0 : EL_0 = TH : ((TE + EZ) - (\sqrt{4(TE \times EZ)}))$.[30]

There are two comparative-analyzed propositions in this occurrence. We will examine that denoted as Diorism (b). We should note that, although this occurs within the diorism of *Cutting off a Ratio* I 6.2, it is a complete analyzed proposition, so it has, itself, both an analysis and a synthesis. For this proposition, we begin with the three conditions of the special case, Diorism (a), which in terms of the letter names for this proposition, Fig. 7, are as follows: $TE : EK = EK : EZ$, $KM = TK$, and $(TE \times MZ) = (MK \times KT)$. We then draw another line through H and falling on rays EB and ZG, say HNS, and seek a relation between ratio $EL : ZK$ and ratio $ES : ZN$. The text of *Cutting off a Ratio* I 6.2 Diorism (b) reads as follows:

> [1] First, we seek (نطلب) if line KL cuts a ratio, which is the ratio LE to ZK, greater or less than all of the [other] lines that extend from point H and cut the two lines EB and EG.

[28] See Saito and Sidoli [2010, 6–1–608] for further discussion of this type of diorism.

[29] The term used here is , whose plural often translates the plural of ὅρος and, in that case, means mathematical *definitions* [Rashed and Bellosta, 2010, 193]. Here, however, what is being expressed is the limit of the given ratio for which solutions are possible. Since the word literally means *boundary* or *extremity*, we have translated with *limit*, but we should point out that this does not mean a rigorously defined limit in anything like the modern sense. Rashed and Bellosta [2010, 192–193] have translated with “la détermination du rapport,” which is also possible, but which might give the reader the incorrect impression that the final section of the occurrence gives an evaluation of the general ratio, r, as opposed to an evaluation of the special case of the limiting ratio, r_0.

[30] This final, subtracted term is literally described as “the line that is equal in square to four times TE by EZ” [Rashed and Bellosta, 2010, 193]. Pappus, in *Collection* VII prop. 17, shows how to construct a line that is equal to the final difference of Apollonius’ proportion, $((TE + EZ) - (\sqrt{4(TE \times EZ)}))$ [Jones, 1986, 137]. This passage also gives a sense of the type of Greek that was used to express these objects, although Apollonius almost certainly did not express the line that was the side of the square by name, as is indicated in the Arabic.

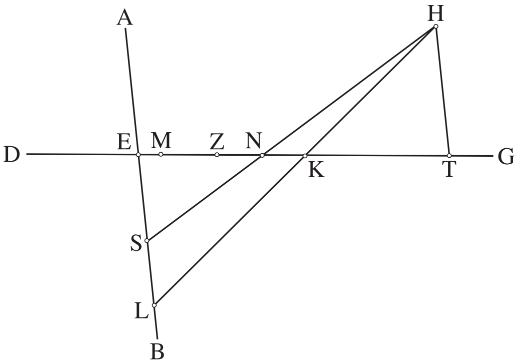

Fig. 7. *Cutting off a Ratio* 6.2, diorism.

We find this problem (ونجد هذه المسألة) as follows.[31] When the things maintain their configuration with respect to the parallel line, and we take a line as a mean between *TE* and *EZ*, with respect to ratio,[32] line *EK*. And we join line *HK* and we extend it rectilinearly. It is necessary that we seek if line *HL* cuts the ratio *LE* to *ZK* greatest or least of those which the [other] lines cut, which extend from point *H* and cutting the two lines *EB* and *ZG*.

[2] We cut off a line, line *KM*, equal to line *TK*.[33] So, rectangle *TE* by *MZ*[34] is equal to rectangle *MK* by *KT*,[35] and the ratio *LE* to *KZ* is equal to the ratio *TH* to *ZM*.[36] We produce another line on it, *HS*.[37]

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[31] In this place, and elsewhere, Rashed and Bellosta [2010, 185 l.22] read نحدّ in place of نجد, although the Aya Sofya manuscript clearly reads نجد [*CR*, A, 10v]. The word in the Bodleian manucript is completely undotted here, ىحد, but this is common for this manuscript [*CR*, B, 21v]. (See also Rashed and Bellosta [2010, 203 l.9, 223 l.4, 239 l.5].) The Greek mathematicians, however, often use *to find* (εὑρίσκω) fairly synonymously with *to solve* (λύω), as the verb for what one does with a problem. (See, for example, Pappus's discussion of different categories of ancient problems [Sefrin-Weis, 2010, 63–65].) Although Rashed and Bellosta [2010, 471–472] associate the usage in these places with those of the determination of the limit, in which the root حدّ is certainly being used, the two are conceptually unrelated. The former has to do with the verb which is used with the noun مسألة to express the idea of solving a problem, whereas the latter concerns the expression of a limiting case of the given ratio, what we have called r_0, in terms of given segments. Since the mathematical argument that follows is a theoretic analyzed proposition, one may legitimately ask what Apollonius means by the statement "we find the problem." Indeed, the problem in question is the general problem of determining whether *HL* cuts off the greatest or least ratio of all the lines falling from point *H* on the two the specified rays. Apollonius solves this problem in two steps; he shows, first, that *HL* cuts off the least (or greatest) ratio of any of these lines and, second, that the ratios cut off by such lines always increase (or decrease) as the lines are taken farther away from *HL*. We discuss this issue further at the end of this section.

[32] The expression "a mean between *AB* and *GD*, with respect to ratio" وسط بين ا ب ج د على نسبة is the idiom used to denote a mean proportional.

[33] This is the condition that determines point *M*, set out in Diorism (a).

[34] The expression can be read literally as "the surface *AB* by *CD*" (سطح ا ب في ج د).

[35] The equality of these two rectangles is demonstrated in the general analysis and follows as a resultof the construction of point *M* through the relation $TH : ZM = EL : ZK$.

[36] This is the principal construction of the general analysis, which determines the point *M*.

[37] The manuscripts disagree about whether this is *HS* or *HN* [*CR* A, 10v; *CR* B, 21v]. We have adopted the reading of Aya Sofya 4830.

[3, A1] So, it is necessary that we relate (ينبغي ان نقيس)[38] the ratio *SE* to *ZN* to the ratio *LE* to *ZK*. [A2] But, the ratio *LE* to *ZK* is as the ratio *TH* to *ZM*. [A3] So, it is necessary that we relate the ratio *SE* to *ZN* to the ratio *TH* to *ZM*. [A4] When we alternate, it is necessary that we relate the ratio *SE* to *TH* to the ratio *ZN* to *ZM*. [A5] But the ratio *SE* to *TH* is as the ratio *EN* to *NT* [*Elements* VI 4]. [A6] So is it necessary that we relate the ratio *EN* to *NT* to the ratio *NZ* to *ZM*. [A7] When we compose, it is necessary that we relate the ratio *ET* to *TN* to the ratio *MN* to *MZ*. [A8] So it is necessary that we relate the rectangle *TE* by *MZ* to the rectangle *MN* by *NT*.

[4, V1] We find a relation of that (قياس ذلك). Rectangle *MK* by *KT* is greater than rectangle *MN* by *NT*, because point *K* bisects line *TM*.[39]

[5, S8] So, because rectangle *MK* by *KT* is greater than rectangle *MN* by *NT*, but rectangle *ET* by *MZ* is equal to rectangle *MK* by *TK*, then rectangle *ET* by *MZ* is greater than rectangle *MN* to *NT*. [S7] So, the ratio *ET* to *NT* is greater than the ratio *NM* to *MZ* [*Elements* VI 14]. [S6] So, when we separate, the ratio *EN* to *NT* is greater than the ratio *NZ* to *ZM*. [S5] But the ratio *EN* to *NT* is as the ratio *SE* to *TH* [*Elements* VI 4], [S4] so the ratio *ES* to *TH* is greater than the ratio *NZ* to *ZM*. [S3] So, when we alternate, the ratio *SE* to *NZ* is greater than the ratio *TH* to *MZ*. [S2] But, the ratio *TH* to *ZM* is as the ratio *LE* to *ZK*,[40] [S1] so the ratio *SE* to *NZ* is greater than the ratio *LE* to *ZK*.

[6] Therefore, line *HL* cuts a ratio less than the ratio that *HS* cuts. Likewise, we prove that it cuts a ratio less than the ratios that all the [other] lines cut that are produced from point *H* and cut the two lines *ZG* and *EB*. So, line *HL* cuts a ratio less than the ratio that is cut by all the [other] lines that are produced from point *H* and cut the two lines *ZG* and *EB*. [Rashed and Bellosta, 2010, 185–189]

The steps of this comparative-analyzed proposition may be sketched as follows. Section [1] provides an exposition and description of the problem. We draw line *HL* according to the condition determined in the special case of Diorism (a), namely by joining point *H* with a point *K* taken such that $TE : EK = EK : EZ$ [*Elements* VI 13]. It is then necessary to see if *HL* cuts the greatest or the least ratio of all lines falling from point *H* to the rays *EB* and *ZG*. Section [2] sets out the other condition of the special case, $KM = TK$, and reminds the reader that $(TE \times MZ) = (MK \times KT)$ and $EL : ZK = TH : ZM$, as was shown in Diorism (a). A new line, *HS*, is drawn and the specification of the problem is given. The steps of the analysis, in section [3], are labeled [A1]–[A8] and the verification, section [4] has a single step, [V1]. Since this is a key argument, we give it in full. First, we have to relate the two ratios

$$ES : ZN \ (?) \ EL : ZK.^{41} \tag{A1}$$

But, in the general analysis, point *M* was determined by the proportion

$$EL : ZK = TH : ZM, \tag{A2}$$

[38] See Rashed and Bellosta [2010, 473–474] for a discussion of this and related expressions. A argument for reading the verb نقيس somewhat abstractly as "we relate" can be based on the introduction of Thābit ibn Qurra's *On the Composition of Ratios*, where he tells us that Euclid defined a ratio as "some relation belonging to homogeneous quantities" [Lorch, 2001, 168]. The word that Thābit uses for relation, or comparison, is قياس, based on the same root as قاس.

[39] This is a direct consequence of *Elements* II 5.

[40] This is how point *M* was originally determined.

[41] The use of (?) indicates a relation that is either greater than, equals, or less than and upon which various operations can be performed. See the commentary following this proposition.

so, by substitution into (A1),

$$ES : ZN \ (?) \ TH : ZM, \tag{A3}$$

and by alternation,[42]

$$ES : TH \ (?) \ ZN : ZM. \tag{A4}$$

But, since $\triangle SEN$ is similar to $\triangle HTN$, by *Elements* VI 4,

$$ES : TH = EN : NT, \tag{A5}$$

So, by substitution of (A5) into (A4),

$$EN : NT \ (?) \ ZN : ZM; \tag{A6}$$

and when we compose,[43]

$$EN + NT : NT \ (?) \ ZN + ZM : ZM;$$

that is,

$$ET : NT \ (?) \ MN : ZM. \tag{A7}$$

Hence, we must find the following relation:[44]

$$(ET \times ZM) \ (?) \ (MN \times NT). \tag{A8}$$

Apollonius then briefly points out, in Section [4], that this relation is, in fact, independently known. This forms the entire verification of the validity of the last step of the analysis. That is, since point K bisects line TM, by *Elements* II 5,

$$(MK \times KT) > (MN \times NT). \tag{VI}$$

The synthesis, Section [5], then begins by combining this with $(ET \times ZM) = (MK \times KT)$, which was shown in the general analysis, and states [S8] as an answer to [A8], namely that $(ET \times ZM) > (MN \times NT)$. The rest of the synthesis follows through the steps [S7]–[S1] in exactly the inverse of the order in which the corresponding steps were used in the analysis. Finally, Section [6] reiterates the overall goal of the analyzed proposition and points out that this serves as a proof that line HL cuts off the least ratio of any line falling from point H onto the rays EB and ZG.

What distinguishes this proposition as a comparative analysis is the argument of the analysis itself, which relies on the claim that there has to be *some* relationship between certain pairs of ratios, or rectangles, and which proceed by performing operations on these unknown relations in such a way that we can see that the unknown relation functions as a conceptual unit. This is what we have tried to capture with the, perhaps, strange symbolism $A(?)B$. The analysis consists of a series of operations carried out on this as yet unknown relation, which Apollonius knows must exist because it is simply one of the relations *greater than*, *equal to*, or *lesser than*. The verification, or determination, of the relation is then a statement about what this relation is, which can be determined on the basis of the geometry

[42] This is a generalization of the ratio operation *alternately*, defined in *Elements* V def. 12, and justified for proportions in *Elements* V 16. See below for further discussion.
[43] This is a generalization of the ratio operation *by composition*, defined in *Elements* V def. 14, and justified for proportions in *Elements* V 18. See below for further discussion.
[44] This is, presumably, a generalization of the fact that the sides of equal rectangles are proportional to one another, which is demonstrated in *Elements* VI 14. We are not aware of any ancient proof of this generalization. See below for further discussion.

of the figure and the assumptions of the theorem. In this case, this section is a single statement, but in other cases, as we will see below, the determination of this ratio may require a bit more work.

These sorts of ratio manipulations are justified in the *Elements* for proportions and for the transformation between a proportion and an equality made up of the sides of a rectangle, but readers of Aristarchus, Archimedes, and Apollonius will have long noticed that these mathematicians often use such operations on inequalities and ratio-inequalities as well.[45] Indeed, in this text, Apollonius assumes that these operations can be performed on generalized relations that may be either proportions, ratio-inequalities, equalities, or inequalities.[46]

Some five centuries later, Pappus, in his *Collection* VII, provided proofs that extended to ratio inequalities the applicability of all the ratio operations commonly in use, including a few that were not included in the *Elements* [Jones, 1986, 128–130]. This presumably means that in Pappus's time there was no treatise available to him that provided justification for these operations. Of course, it is possible that such a treatise had existed but was subsequently lost. It seems more likely, however, that no such treatise ever existed and that the Hellenistic geometers used these operations because they are simple, and obvious, extensions of a set of practices that were well established, assured in the conviction that if pressed they could supply a proof to justify their practice. This means that, in this regard, ancient mathematicians probably read the *Elements* V as a work that sought to give certain logical foundations to some of their practices, not as a work that told them what they could and could not do.[47]

Another characteristic feature of a comparative analysis is that it is clearly a heuristic tool. In most of the surviving standard theoretic analyzed propositions, and indeed in Pappus's description of theoretic analysis, the analysis begins with the assumption of what is to be shown and then proceeds, through an examination of the consequences of this assumption, to show something that is obviously true or that the geometer knows can be shown independently of the analytical assumption. The synthesis is then usually an inversion of the argument in the analysis, often in exactly the same steps, and is, hence, frequently omitted in the theoretic analyzed propositions that survive in our sources. Indeed, the argument in *Cutting off a Ratio* I 6.4 Diorism (b) can easily be turned into a standard analyzed proposition. To do this, we simply have to assume, in step (A1), that $ES : ZN > EL : ZK$. We can then work through the same series of steps to show that this implies that $(ET \times ZM) > (MN \times NT)$, which can be confirmed independently by means of *Elements* II 5. In this case, for all practical purposes, the synthesis is unnecessary.

In this way, comparative analysis provided techniques for investigating relations about which the geometer may initially have known nothing. Indeed, since Apollonius could

[45] See Berggren and Sidoli [2007, 225–227] for a recent discussion of these issues.

[46] Historians of Greek mathematics have come to distinguish between *equalities* of numbers or magnitudes and *proportions*, which are generally asserted by Greek mathematicians as *an identity* of two or more ratios. This distinction follows the linguistic practice of the ancient mathematicians. Indeed, Greek mathematicians often say that two ratios are the "same" when a proportion holds, but ratios are never said to be "equal" (see *Elements* V def. 3, where ratio is defined a sort of relation; thus two ratios or relations can be the same, but not equal). Nevertheless, as this paragraph makes clear, the ancient mathematicians clearly recognized the formal similarity between these different types of relations and subjected them to the same operations without comment.

[47] See Saito [2003, 336–341] for an argument that *Elements* V is not meant to be an exhaustive and general "theory of ratio" but was, rather, meant to provide a justification of various tools that were used in geometrical investigations.

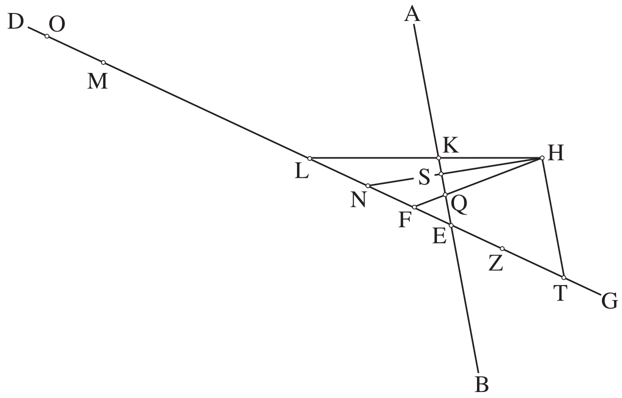

Fig. 8. *Cutting off a Ratio* 6.4, diorism.

easily have framed the comparative-analyzed propositions of *Cutting off a Ratio* in the form of a standard theoretic analyzed proposition, which assumes what it sets out to prove, he may well have written them up in the form that he did in order to help readers develop an idea of useful tactics that can be employed when investigating unknown relations.

Indeed, the comparative theoretic analyses provided by Apollonius illustrate a generally applicable approach. They show that, simply by assuming that there is *some* relation, we can use the techniques of analysis to investigate the nature of this relation. Once we have assumed some relation between any two objects that we want to relate, we can proceed, through a series of deductive steps, to look for an unknown relation that we can then establish by independent means. In the case of *Cutting off a Ratio* I 6.2 Diorism (b), we come to the relation $(ET \times ZM)(?)(MN \times NT)$, which Apollonius must have seen at once could be decided on the basis of *Elements* II 5, since he has already shown, in the general analysis, that $(ET \times ZM) = (MK \times TK)$.

In the case of comparative analysis, since the geometer has not assumed what is to be shown, it is both logically and heuristically important that the synthesis be given in full, since it will rarely be obvious, after a series of operations, what the established relation will imply for the unknown relation that was assumed at the beginning of the analysis. From a logical perspective, we must still verify that all of the steps of the analysis are convertible and from a heuristic perspective, we must follow through this chain of steps to see what the determined relation implies for the assumed relation.[48]

In *Cutting off a Ratio* I 6.2 Diorism (c), Apollonius goes on to use another comparative analyzed proposition to show that any line beyond *HNS* passing through point *H* and falling on lines *EB* and *ZG* will cut off a ratio still less than that cut off by *HNS*. We will omit this argument and in its place examine the analogous argument given in *Cutting off a Ratio* I 6.4.

This occurrence handles the situation represented in Fig. 5 (d). In terms of the diagram given in the text, Fig. 8, *Cutting off a Ratio* I 6.4 treats the case where line *HKL* falls on rays *EA* and *ZD*. In this occurrence, the ratio $EK : ZL = r_0$, which is cut off by line *HL*, is a maximum, not a minimum. Hence, in this regard the argument below is opposite to that found in *Cutting off a Ratio* I 6.2 Diorism (c), but in all other ways it is analogous.

[48] In the case where the comparative analysis should lead to an equality, a synthesis would generally not be necessary, so long as all of the operations used in the analysis were known to be reversible.

In *Cutting off a Ratio* I 6.4 Diorism (c), in order to show that the ratios continuously increase as the line approaches *HL*, Apollonius draws another line, *HF*, cutting *AB* and *GD* at *Q* and *F*, respectively, and shows that $ES : ZN > EQ : ZF$. For this purpose, he sets out a point *O*, analogous to point *M*, such that $ES : ZN = TH : ZO$, and points out that, by the same argument as in the general analysis, $(ZO \times TE) = (TN \times NO)$.

With this as preliminary, the text of *Cutting off a Ratio* I 6.4 Diorism (c) proceeds as follows.

> [1] . . . So, we produce another line on it, as *HF*.
>
> [2, A1] So, it is necessary that we link (ينبغي ان نقرن) ratio *ES* to *ZN* and ratio *EQ* to *ZF*. [A2] But ratio *ES* to *ZN* is as ratio *TH* to *ZO* [by construction], [A3] so it is necessary that we link ratio *TH* to *ZO* and ratio *EQ* to *ZF*.[49] [A4] By alternation, it is necessary that we link *TH* to *EQ*, that is *TF* to *FE* [*Elements* VI 4], and *OZ* to *ZF*,[50] [A5] and when we convert, it is necessary that we link ratio *TF* to *TE* and ratio *ZO* to *OF*.[51] [A6] So, it is necessary to link rectangle *ZO* by *TE* and rectangle *TF* by *FO*.[52] [A7] Rectangle *ZO* by *TE* is equal to rectangle *TN* by *NO*,[53] [A8] so it is necessary that we link rectangle *TN* by *NO* and rectangle *TF* by *FO*. [A9] Similarly, it is necessary to link rectangle *TL* by *LO* and rectangle *TN* by *NO*.[54] Rectangle *TN* by *NO* is equal to *ZO* by *TE*,[55] [A10] so it is necessary that we link rectangle *TL* by *LO* and rectangle *ZO* by *TE*.
>
> [3] We find the relation of that (قياس ذلك): [V1] As what we demonstrated, rectangle *TL* by *LM* is equal to rectangle *TE* by *ZM* [general analysis], [V2] and we take away rectangle *TL* by *LM* from rectangle *TL* by *LO*, and rectangle *TE* by *ZM* from rectangle *TE* by *ZO*, [V3] and we link the remainder to the remainder, so it is necessary that we link rectangle *TL* by *MO* and rectangle *TE* by *MO*.[56]
>
> [4, V4] Its relation (قياسه) is that it is greater, namely rectangle *TL* by *MO* than rectangle *TE* by *MO*, because *LT* is greater than *TE*. [V5, S10] Because rectangle *TE* by *MO* is less than rectangle *TL* by *MO*, and rectangle *TE* by *ZM* is equal to rectangle *TL* by *LM*, so the whole of rectangle *TE* by *ZO* is less than the whole of rectangle *TL* by *LO*.[57]
>
> [5, S9] Rectangle *TE* by *ZO* is the equal of rectangle *TN* by *NO*,[58] [S8] so rectangle *ON* by *NT* is less than rectangle *TL* by *LO*, [S7] therefore the rectangle *TF* by *FO* is less than

[49] Rashed and Bellosta [2010, 205 l.14] follow the manuscripts and read وينبغي where we read فينبغي. As their notes point out a few lines below, however, the manuscripts often read و where ف would make better sense [Rashed and Bellosta, 2010, 205 l.18].

[50] See Footnote 42, above.

[51] This is a generalization of the ratio operation *by conversion*, defined in *Elements* V def. 16. Although this operation was not justified in the *Elements*, it was widely used in practice for both proportions and ratio inequalities.

[52] See Footnote 44, above. Here again, Rashed and Bellosta [2010, 205 l.18], following the manuscripts, read وينبغي where we read فينبغي. See Footnote 49, above.

[53] This is a reminder of a statement shown in the preliminary to the analyzed proposition.

[54] This follows from a repetition of the preceding argument for the other pair of lines.

[55] This is a reminder of step [A7].

[56] That is, since, by the general analysis, $(TL \times LM) = (TE \times ZM)$, while

$$(TL \times LO) - (TL \times LM) = (TL \times MO),$$
$$(TE \times ZO) - (TE \times ZM) = (TE \times MO),$$

in order to relate $(TL \times LO)(?)(TE \times ZO)$, we must relate $(TL \times MO)(?)(TE \times MO)$.

[57] This is the conclusion of the section that determines the relation and is also the starting point of the synthesis.

[58] This is a reminder of a statement proved in the preliminary to the analyzed proposition.

> rectangle *TN* by *NO*. [S6] But rectangle *TN* by *NO* is the equal of rectangle *TE* by *ZO*,[59] so rectangle *TF* by *FO* is less than rectangle *TE* by *ZO*. [S5] So, the ratio *FT* to *TE* is less than the ratio *ZO* to *OF* [*Elements* VI 14]. [S4] And when we convert, the ratio *TF* to *FE*, namely the ratio *TH* to *EQ* [*Elements* VI 4], is greater than the ratio *OZ* to *ZF*,[60] [S3] and by alteration, the ratio *TH* to *ZO* is greater than the ratio *EQ* to *ZF* [*Elements* V 16]. [S2] And the ratio *TH* to *ZO* is as the ratio *ES* to *ZN* [by construction], [S1] so the ratio *ES* to *ZN* is greater than the ratio *EQ* to *ZF*.

In *Cutting off a Ratio* I 6.4 Diorism (c), the verb that is used for what we are seeking to do with the two ratios has changed from "to relate" or "to compare" (root قيس) into "to link" or "to join" (root قرن). Nevertheless, what we are seeking is still referred to with the same noun (قياس). As we will argue below, this may have been due to the fact that there was no verb in the Greek text and the translator was working to try to convey an idea that was handled in the Greek by prepositional phrases in a more natural Arabic style. We will return to this question once we have examined a Greek text that contains a comparative-analytic argument.

In *Cutting off a Ratio* I 6.4 Diorism (c), we encounter a longer verification that is expressed in five sentences, [V1]–[V5]. Whereas in *Cutting off a Ratio* I 6.2 Diorism (b) the verification was handled with a single sentence, here it is developed into a section of the proof that, itself, seems to have an internal comparative-analytical argument.[61] In fact, however, the verification of this proposition is also established directly on the basis of two assumptions of the theorem and a pair equations that are contrived for this purpose. Although the verification is carried out in five statements, and hence might be thought to be analogous to the extended verification we encountered in the introduction, from a logical perspective it has the same structure as that in *Cutting off a Ratio* I 6.2 Diorism (b).

In *Cutting off a Ratio* I 6.4 Diorism (c), the analysis, Section [1], establishes through the series of steps [A1]–[A10] that we can state the relation $ES : EN(?)EQ : ZF$ once we know the relation $(TL \times LO)(?)(ZO \times TE)$, while the synthesis, Section [4], shows that once this relation is found to be $(ZO \times TE) < (TL \times LO)$ we can work our way through the exact same series of steps, [S9]–[S1], to show that $ES : EN > EQ : ZF$. The argument of the verification for *Cutting off a Ratio* I 6.4 Diorism (c) can be summarized as follows.

In order to determine the relation $(TL \times LO)(?)(ZO \times TE)$, we can consider what happens when we take away equals from these. By the argument in the general analysis, we have

$$(TL \times LM) = (TE \times ZM), \tag{V1}$$

so we can subtract these from the terms in the relation we seek, producing a pair of equations,

$$\begin{aligned}(TL \times LO) - (TL \times LM) &= (TL \times MO), \quad \text{and} \\ (TE \times ZO) - (TE \times ZM) &= (TE \times MO).\end{aligned} \tag{V2}$$

But since, by construction, $TL > TE$, so that[62]

[59] This is a reminder of step [S9].
[60] See Footnote 51, above.
[61] As we will see below, however, this internal comparative-analytical statement, [V3], is not necessary to the argument (see Footnote 62).
[62] The statement, in the text [V3], that we have to seek the relation $(TL \times MO)(?)(TE \times MO)$, is simply an indication of how to understand the role of this pair of equations. Hence, we have omitted it in this summary.

$$(TL \times MO) > (TE \times MO), \tag{V4}$$

therefore, in (V2), it must be the case that

$$(TL \times LO) < (TE \times ZO). \tag{V5}$$

This statement, which is the conclusion of the verification, is also the starting point of the synthesis [V5, S10]. As we can see, this argument is simply the assertion of two claims that derive from the geometry of the figure, (V1) and (V4), and the contrivance of a pair of equations, (V2), that will lead directly to the result we seek. Moreover, there is no natural way of reversing the argument of this verification and turning it into an extension of the analysis, in such a way as to make this extended verification unnecessary, as was possible in our example of a standard theoretic analyzed proposition.

All that this verification does is show directly on the basis of statements already shown and the geometry of the figure that there is, indeed, a relation between $(TL \times LO)$ and $(TE \times ZO)$ — which was never really in doubt — by showing what this relation is. Indeed, the synthesis begins with the final statement of the verification, not the final statement of the analysis proper, as is the case in a standard theoretic analyzed proposition.

Hence, we have highlighted another key difference between a comparative and a standard theoretic analysis. In a standard theoretic analysis, because we assume precisely what we are setting out to show, it is enough to reduce the proposition to a claim that can be established on the basis of the conditions of the theorem and the geometry of the figure. In the case of a comparative analysis, however, because what is assumed is simply the existence of some relation, once we have reduced the assumed relation to a more manageable relation, we still have to determine *what* this relation is. This determination may be immediate, or it may take a little work, but its result is the necessary starting point of the synthesis. For these reasons, the longer verifications that we find in some of Apollonius's comparative analyses are an essential part of the argument and cannot be subsumed into the analysis by working them backward.

Apollonius's comparative theoretic analyses, then, have the following structure. They begin with the assumption that some relation exists between two objects that the geometer is seeking to compare. The analysis proceeds by treating this unknown relation as a conceptual unit and subjecting it to a series of operations until it has been transformed into a relation that the geometer knows can be established by independent means. The verification then establishes the nature of this relation based on the geometry of the figure and any assumptions of the theorem. With the relation established, the synthesis then begins from the claim of the verification and works back through the same steps as the analysis to produce the sought relation. Since the relation was assumed as unknown, the synthesis is needed to show what the originally assumed relation is.

In the context of the extended diorisms of *Cutting off a Ratio*, these two theoretic analyzed propositions can be taken as a proof that, where L bisects TM, line HL cuts off the greatest, or least, ratio drawn from point H to lines EA and ZD and that all other lines cut off ratios that always approach the maximum, or minimum, ratio cut off by HL as the lines themselves approach HL. Hence, this pair of analyzed propositions are about maxima and minima, as can be seen by comparing the arguments in these propositions with the theorems that demonstrate the properties of maximum and minimum lines in circles, *Elements* III 7 and 8, or conic sections, *Conics* V.

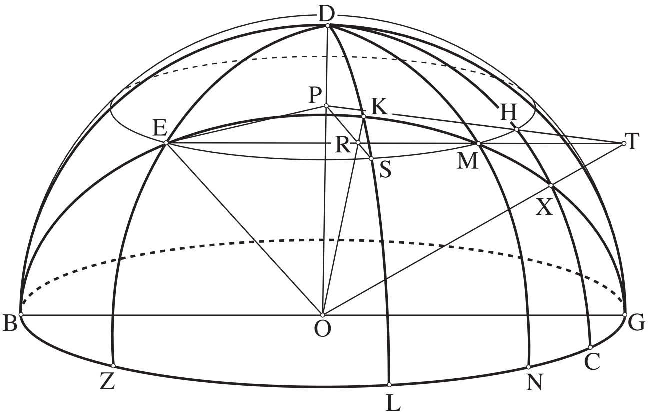

Fig. 9. *Collection* VI prop. 16.

In four of the five cases where this pair of comparative analyzed propositions is used, the text refers to the pair as handling a "problem" (مسألة).[63] In terms of the usual language of Greek geometry, this is a peculiar expression, because a *problem* is generally solved by the construction of a particular geometric object. In this case, we should probably think of the problem as determining whether or not the line falling from the given point onto the given rays cuts off a minimum or maximum ratio. In each case, the text states that we need to determine whether or not the line determined in Diorism (a) cuts off the greatest or the least ratio. This is then followed by the pair of comparative analyses that we have examined. Taken together, this pair of propositions shows that the ratio so cut off is a unique maximum, or minimum. Hence, we should understand the problem as being a determination of what kind of limiting ratio is cut off and whether or not it is unique.

2.2. *Comparative analysis in Pappus's Collection VI*

As part of his treatment of Theodosius's *Spherics* III 6, in *Collection* VI props. 16–20, Pappus uses comparative analysis four times [Hultsch, 1876, 494–512]. Ostensibly, these propositions all concern the size relations between certain arcs of one great circle and their orthogonal projections onto another great circle. In fact, however, they are a treatment of what we would call the right ascensions of arcs of the ecliptic, or what the ancients understood as the rising times of arcs of the ecliptic for observers at the terrestrial equator. The issue was not purely academic, however, since the rising times of arcs of the ecliptic as seen at latitudes where ancient observers actually did live, were determined on the basis of those at the terrestrial equator.[64]

As an example of Pappus's use of comparative analysis, we will look at *Collection* VI prop. 16. In Fig. 9 (see also Fig. 10), this theorem shows that where *BEKXG* is a great circle

[63] The statement "we find this problem" (نجد هذه المسألة) occurs at the beginning of the Diorism (b) section of *Cutting off a Ratio* I 6.2, 6.4, 7.2 and 7.4 [Rashed and Bellosta, 2010, 185, 203, 223, 239]. See Footnote 31 for our reading of these passages. The first time a similar pair of analyses occurs is at the beginning of *Cutting off a Ratio* I 5.3 Diorism (b). In this place, the text reads "We know this as follows" (ونعلم ذلك هكذا) [Rashed and Bellosta [2010, 167]].

[64] For a full discussion of Pappus' treatment of Theodosius's *Spherics*, see Malpangotto [2003].

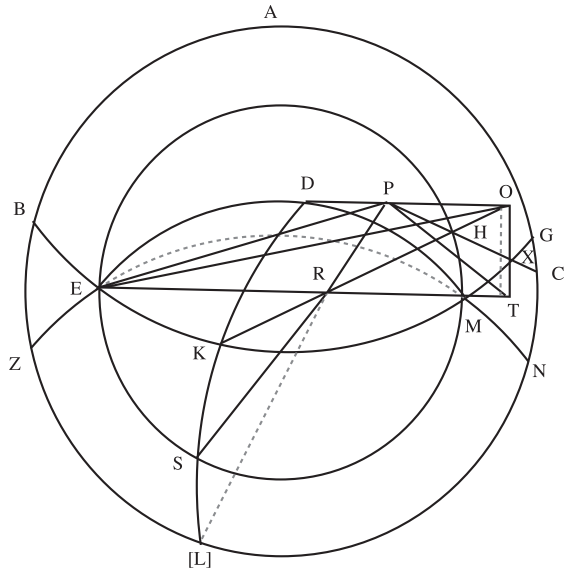

Fig. 10. Manuscript diagram for *Collection* VI prop. 16 [*Vat. gr.* 218, 92v]. The dotted, gray lines were drawn in and then erased. There is a problem with the placement of *H*, and the line *PHC* is neither necessary for the proof nor, indeed, should it be a straight line. The diagram that we have produced in Fig. 9 is essentially that of Commandino and all subsequent versions of the text.

inclined on another great circle, *BZLCG*, arc *BE* > arc *XG* and arc *EK* = arc *KX*, then the projections of arcs *EK* and *KX* on great circle *BZLCG*, determined by great circles drawn through the pole of circle *BZLCG*, are such that arc *ZL* > arc *LC*.

The argument proceeds as follows. Arc *GM* is set out equal to arc *BE* and great circle *DM* is drawn through to point *N*. Since arc *BE* = arc *GM*, a circle *ESM* can be drawn about pole *D*. The center of the sphere is taken at *O* [*Spherics* I 2], and line *DO* is joined perpendicular to circle *BZLCG* and passing through the center of circle *ESM* at point *P*. Line *EM* is joined and extended to point *T* and line *OX* is joined and extended to meet point *T*. Lines *EO*, *ORK*, *PR*, *RS*, *PH*, and *HT* are joined.

It is then shown that lines *PR* and *RS* form a straight line, *PRS*, by showing that the three points lie in the planes of the two circles *ESM* and *DKL*; and, likewise, that lines *PH* and *HT* form a straight line, *PHT*, by showing that the three points lie in the planes of the two circles *ESM* and *DXC*. Hence, in the plane of triangle *EOT*, since they subtend equal arcs, angle *EOK* = angle *ROT*, and *EO* : *OT* = *ER* : *RT* [*Elements* VI 3]. Pappus then uses a comparative-analytical argument, which we quote in full.

> [A1] Since, however, I seek what arc *ZL* is to *LC* (ζητῶ τίς ἡ ΖΛ περιφέρεια τῇ ΛΘ),
>
> [A2] that is *ES* to *SH* [*Spherics* I 10], [A3] therefore, I will seek (ζητήσω) what angle
>
> *EPR* is to angle *RPT*,[65] [A4] therefore, what ratio *EP* to *PT* is to ratio *ER* to *RT*.[66]

[65] As we saw in the introduction, Pappus uses a similar expression in *Collection* VII prop. 26 as well, where he says "it is necessary, alternately, to seek (ζητεῖν) . . . " [Jones, 1986, 147]. Although he uses this expression, however, he is actually assuming that a certain *proportion* holds. Hence, there is no mathematical difference between *Collection* VII prop. 26 and a standard theoretic analysis.

[66] This comparison comes from a consideration of *Elements* VI 3, which shows that if $\angle EPR = \angle RPT$ then $EP : PT = ER : RT$.

> [A5] Ratio ER to RT, however, is the same as ratio EO to OT [*Elements* VI 3], [A6] therefore I will seek what ratio EO to OT is to ratio EP to PT. [A7] Therefore, I will seek what the ratio of the square on EO to the square on OT is to the ratio of the square on EP to the square on PT [*Elements* VI 20], [A8] and alternately, what the ratio of the square on OE to the square on EP is to the ratio of the square on OT to the square on TP, [A9] and by separation, what the ratio of the square on OP to the square on PE is to the ratio of the square on OP to the square on TP,[67][A10] therefore what the square on TP is to the square on PE, [A11] therefore what TP is to PE.
>
> [V1] PE, however, is equal to PH. [V2] Clearly, a comparison obtains (ἔχει δὴ σύγκρισιν), for it is greater. [Hultsch, 1876, 496–498]

The synthesis then proceeds by starting with the relations $TP > PH = PE$, and working backward through the steps of the analysis to show that, in fact, arc $ZL >$ arc LC. Just as was the case with the comparative analyses of *Cutting off a Ratio*, since all of the operations used in this analyzed proposition are reversible, Pappus could have saved himself the trouble of writing out the synthesis if he had started the analysis with the assumption that arc $ZL >$ arc LC. Instead, however, he started with the more general assumption that there is *some* relation, arc ZL (?) arc LC, and used analysis to determine what this relation is. In this case, a synthesis is necessary, because it is not immediately obvious that $TP > PH$ implies that arc $ZL >$ arc LC.[68]

Hence, we see that the structure of these comparative analyses in *Collection* VI are the same as those in *Cutting off a Ratio*. The analysis starts with the relation that we wish to establish asserted as an unknown relation. It then performs a series of transformations on this relation, using ratio operations and considerations of the geometry of the figure, until it arrives at a relation that can be determined independently. The verification then determines this relation. In this case, the determination can be made in a single statement, based on the geometry of the figure. The synthesis, then, works back through the steps of the analysis. In both *Cutting off a Ratio* and *Collection* VI, the steps of the analysis are literally taken in reverse. Nevertheless, this reversal of the order of steps need not have been done so exactly.

Although the structure of the argument is the same in the diorisms of *Cutting off a Ratio* and the theoretic analyses of Pappus's comments on Theodosius's *Spherics* III 6, the language is somewhat different between the two texts. *Cutting off a Ratio* proceeds by stating that it is "necessary to relate (or link) two magnitudes (or ratios)," while, in the *Collection*, Pappus says "I will seek what some magnitude (or ratio) is to another." Whereas the Greek text expresses this using a pronoun and the dative case, the Arabic uses a verbal expression. It may well have been the case that Pappus's Greek expression was grammatically similar to what he read in Apollonius's Greek version of *Cutting off a Ratio* and that, when *Cutting off a Ratio* was translated into Arabic, this phrase was rendered into more natural Arabic with the verbal expressions that we saw. Both texts agree, however, that what is found is

[67] The operation of separation can be applied to this unknown relation by considering $\triangle POE$ and $\triangle POT$, so that by *Elements* I 47,

$$OE^2 = PO^2 + EP^2, \quad \text{and}$$
$$OT^2 = PO^2 + TP^2.$$

[68] It should be noted, however, that once Pappus gives the full synthesis for *Collection* VI prop. 16, he regards the production of the syntheses for *Collection* VI props. 17, 19, and 20, which are analogous, as too trivial to warrant full treatment.

some relation, or comparison (σύγκρισις, قياس). Since Pappus was well acquainted with the Greek text of *Cutting off a Ratio*, it clear that Pappus's procedure in *Collection* VI was influenced by the type of analysis used by Apollonius in the extended diorisms of *Cutting off a Ratio*.

Comparative analysis is thus the determination of the type of relation that obtains between two terms. Hence, although from a logical perspective the structure of this type of theoretic analysis is the same as that in which we assume a specific relation, from a practical perspective it is more open-ended than a standard analysis and it is indisputably a heuristic approach.

Whereas a standard theoretic analysis starts with the assumption that what one wants to prove holds, a comparative analysis begins with the more general claim that there is *some* relation between two terms. In other words, at the beginning of a standard theoretic analysis, the mathematician must have good reason to believe that theorem to be shown in fact holds, whereas a comparative analysis can begin with the trivial assumption that there is *a* relation and then proceed to use the propositions of the *Elements*, ratio manipulation, geometric constructions, and so on, to come to some understanding of the specifics of the relation in question.

Finally, we should add that Pappus's use of analysis in these propositions of *Collection* VI did not escape the careful attention of Commandino, in the 16th century. Although Commandino noted the use of analysis in these propositions and pointed out that the language is peculiar, he does not seem to have regarded the comparative analysis itself as in any way remarkable. In his note to the beginning of the comparative analysis quoted above, Commandino says, "In this place, Pappus somehow uses analysis, although he employs a novel and unusual way of speaking."[69]

3. Conclusions

We are now in position to give an overview of comparative analysis, which will allow us to relate this particular type of theoretic analysis to the standard form of theoretic analysis. For this purpose, we can use a schematic similar to that which we used to describe standard theoretic analysis.

Once again, we begin with the assumption of some relation, R_a, which we say obtains for some members of an assumed set of given objects, $x_1,\ldots,x_k$. We then perform a series of manipulations on this assumed relation, transforming it into a another relation, R_t, which holds between some other members of the same given set, and which can be immediately established on the basis of the conditions of the theorem, $C(x_1,\ldots,x_k)$.

The verification, then, confirms the general claim that there is *some* relation R_t, by showing a specific relation, R_{t_0}, that holds for the same objects as R_t, which are members of the original set of given objects, $x_1,\ldots,x_k$. In Fig. 11, we have represented this by depicting $R_{t_0}(x_1,\ldots,x_k)$ as a subset of $R_t(x_1,\ldots,x_k)$, so it is clear that establishing $R_{t_0}(x_1,\ldots,x_k)$ acts both as a confirmation of $R_t(x_1,\ldots,x_k)$ and provides the starting point for the synthesis, which will proceed by working back toward the original R_a using the same set of given theorems and operations, $\text{Toolbox}_{\mathcal{A}}$.

[69] *Utitur autem hoc loco Pappus resolutione quadam, quamquam nouum, & inusitatum loquendi modum usurpet* [Commandino, 1585, 193, n. O].

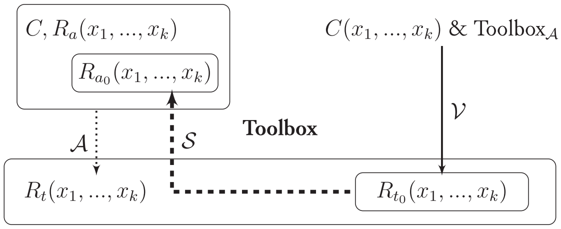

Fig. 11. Schematic of a comparative analysis.

In a comparative analysis, the synthesis can generally proceed by the same series of steps as the analysis, because these are manipulations and substitutions that all have straightforward converses. Nevertheless, because these steps can be drawn directly from the general toolbox as well as consequences of the assumption and Toolbox$_{\mathcal{A}}$, established in the analysis, as usual there is no need for the synthesis to mirror the temporal ordering of the analysis. In any case, the synthesis begins with $R_{t_0}(x_1, \ldots, x_k)$ and works back toward the relation which was assumed in the analysis until it arrives at $R_{a_0}(x_1, \ldots, x_k)$, which is a specific, instantiated relation that holds for the same set of objects as $R_a(x_1, \ldots, x_k)$. Hence in the schematic, we have depicted $R_{a_0}(x_1, \ldots, x_k)$ as a subset of $R_a(x_1, \ldots, x_k)$. Even in the case that all of the steps of the analysis are directly reversible, if the relation R_{t_0} is *greater* or *less than*, it will be necessary to go through with the synthesis to confirm which one of these applies to R_{a_0}.

It may be helpful to consider this schematic in terms of one of the examples we have seen. In *Collection* VI prop. 16, $R_a(x_1, \ldots, x_k)$ is the claim that there is some relation between arcs *ZL* and *LC*, [A1]. That is, one of the relations $ZL > LC$, $ZL = LC$, or $ZL < LC$. The analysis then transforms this into a new relation between lines *TP* and *PE*, [A11], namely one of the relations $TP > PE$, $TP = PE$, or $TP < PE$. Note that the objects related in the assumed and transformed statements, the arcs *ZL*, *LC* and the lines *TP* and *PE*, belong to the original set of instantiated objects, $x_1, \ldots, x_k$, but are different members. The verification then shows that $TP > PE$, [V2], which is represented by $R_{t_0}(x_1, \ldots, x_k)$, and is clearly a verification of the general claim of the transformation. The synthesis then works backwards to show that arc$ZL <$ arcLC, which is $R_{a_0}(x_1, \ldots, x_k)$.

We turn now to the important question of the extent to which comparative analysis, and theoretic analysis in general, were heuristic techniques for Greek mathematicians. It used to be taken more or less for granted that Greek geometric analysis was a heuristic technique.[70] More recently, however, the extent to which analysis can function as a set of heuristic techniques has been called in to question. Following a sustained account of the heuristic role of problematic analysis, Knorr [1986, 358–360] argued that theoretic analyzed propositions should not be understood as heuristic, and, indeed, were not proper analyses at all.[71] More recently still, Netz [2000] has argued that problematic analysis was also not heuristic, except

[70] This is expressed, for example, by Hintikka and Remes [1974, 1] in their opening sentence, with the statement that analysis is a method that was used "in looking for proofs ... and constructions...".

[71] In fact, Knorr [1986, 358–359] was apparently of the opinion that the idea of theoretic analysis was something that Pappus simply invented, but that had no basis in older mathematical practice. The fact that there are a number of theoretic analyses in Apollonius' *Cutting off a Ratio*, however, suffices to show that Pappus was discussing a category of text that he found in his sources, not just one that he used himself.

in the limited sense of revealing “the idea behind the solution.”[72] Since only Knorr’s argument is addressed to theoretic analysis, we will direct our attention to it in what follows.

In the first place, as many scholars have pointed out, there are two primary ways in which we can understand the claim the analysis was heuristic — that is, (1) that it could help mathematicians find a new solution to a problem or a new proof for a theorem, or (2) that it could lead them to previously unconsidered problems or to previously unknown theorems.

When we say that problematic analysis lead to new solutions, we mean that it provided a set of techniques that mathematicians could use to look for unknown solutions to extant problems. When we say that problematic analysis lead to new problems, we mean that the analytical investigation of some given configuration of objects, often involved a transformation to a new problem, whose solution was then sought. Mahoney [1969, 330–348] and Knorr [1986], among others, have argued at length that problematic analysis functioned in both of these ways for Greek mathematicians and that the problematic analyses that survive in our sources can be taken as evidence of this activity.[73]

When we say that theoretic analysis led to new proofs, we mean that it could be used to find a valid demonstration of some mathematical claim which the mathematician already believed to be true. When we say that theoretic analysis led to new theorems, we mean that it could be used to investigate a given configuration of geometric objects to reveal something that holds for them, but that was not known beforehand.

There can be little doubt that theoretic analysis could, in fact, be used to find a proof for something that the geometer believed to be true. Indeed, the standard theoretic analyses in Pappus’ *Collection* are of this type. After years of reading the Hellenistic geometers, it is unlikely that Pappus would have encountered a statement in Apollonius that was unclear to him and wondered if it were really true. Rather, he would, naturally, assume *that* it was true and wonder *why* it was true. The theoretic analysis that we encountered in the introduction is the natural answer to such a question. Looking back at Fig. 1, the original claim that $(BG \times GD) : (BE \times ED) = GA^2 : AE^2$ is true can be seen to follow as a result of the fact that $HZ \parallel BG$. Pappus’ theoretic analysis both shows how we could provide a proof of the theorem, and also highlights the key geometric fact that unlocks the demonstration. It is not only in reading and understanding the works of other mathematicians, however, that such methods would have been useful.[74] In the course of a mathematical investigation of new territory, the geometer will often have encountered results that seemed to be true, but that were unproven, or results that, if true, would lead to the development of new areas of the

[72] There are difficulties involved in interpreting Netz’s position exactly, because he seems to move effortlessly between the concept of analysis as a method, or a body of mathematical techniques, and analysis as a type of mathematical text, or an analyzed proposition. Although, his main focus is an investigation of the reasons that Greek mathematicians might have had for publishing analyzed propositions, when he comes to addressing the question “Is analysis ‘heuristic’?” he often appears to be discussing analysis as a body of techniques [Netz, 2000, 139–145]. Whatever his exact position, however, it should be clear that the question of whether or not a given analyzed proposition is an accurate depiction of a heuristic process that actually occurred has no direct bearing on the question of whether or not analytical techniques were generally used by mathematicians as heuristic tools.

[73] Notice that Netz [2000] appears to be arguing against this position, although he does not systematically refute the examples that others have advanced.

[74] We do not mean to underestimate the importance of reading and understanding the works of others as a mathematical activity in antiquity. The general uniformity of shared assumptions, applied techniques and even linguistic expressions make it clear that ancient mathematicians read each others work in detail, often mastering both the results and the methods.

theory. In such circumstances, the techniques of theoretical analysis would have been quite useful.[75]

The usefulness of theoretic analysis for the discovery of previously unknown mathematical theorems, however, is not so readily apparent. As Knorr [1986, 358] argued, the theoretic analysis that are found in the medieval manuscripts as alternative proofs for *Elements* XIII 1–5 seem to be rather contrived [Heiberg, 1969, 198–204].[76] Even if we read them as showing how to find a proof, given the theorem, they are somehow unsatisfactory because, for these theorems, it is not clear how the discovery of the theorem can be explained independently of the discovery of the proof.[77] Comparative analysis, however, as we saw above, is an approach that can be taken when we do not know what we want to show, but only what objects we want to relate. In order to see if these kinds of arguments would be useful in simultaneously discovering one of these theorems and motivating the demonstration, we will look at *Elements* XIII 1, which has been discussed by Mahoney [1969, 326–327] and Knorr [1986, 358].

If we take an analytical approach to *Elements* XIII 1, using the techniques of comparative analysis, as we have seen them developed in this paper, in Fig. 12, we can start with the conditions that the line AB has been divided such that $AB : AC = AC : CB$ and that $AD = \frac{1}{2}AB$. If we want a relation involving AD^2 and CD^2, then we take as an analytical assumption that such a relation exists. That is, we seek

$$AD^2 \ (?) \ CD^2. \tag{A1}$$

But since $AD = \frac{1}{2}AB$,

$$4AD^2 = AB^2; \tag{A2}$$

substituting (A2) into (A1), we can seek

$$AB^2 \ (?) \ CD^2. \tag{A3}$$

Then we complete the squares on AB and CD [*Elements* I 46], extend FC to meet KE at G and AK to H, and complete the square DH and the two parallelograms CH and HL [*Elements* I post. 2, I 46].

Now since, $AK = 2AH$, by *Elements* I 36, rectangle AG is twice rectangle CH, and since, by *Elements* I 43, rectangle CH is equal to rectangle HL,

$$\square HL + \square CH = \square AG. \tag{A4}$$

But one of the conditions of the theorem is that $AB : AC = AC : CB$, so by *Elements* VI 16,

$$AC^2 = AB \times CB; \text{ that is, } \square HF = \square CE. \tag{A5}$$

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[75] As Knorr [1986, 358] points out, the theoretic analyzed propositions in our sources arise in a context in which key elements of their proof will also be clear. Nevertheless, even in cases where one has strong reasons for believing that certain theorems will be key members of Toolbox$_{\mathcal{A}}$, confirming that this is actually the case and obtaining a full set of the members of Toolbox$_{\mathcal{A}}$ can done using a theoretical analysis.

[76] These analyzed propositions are generally held to be interpolations, but it is not known by whom they were written or precisely when they were included in the text.

[77] That is, here, as is often the case, the fact *that* that the theorem is true seems to arise in the same context in which one sees *why* the theorem is true.

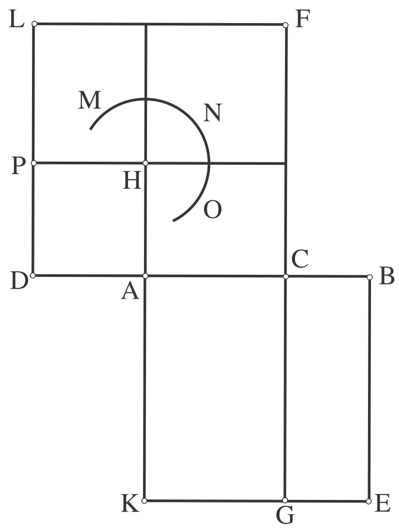

Fig. 12. *Elements* XIII 1, an analytical approach.

Therefore,

$$\text{gnomon } MNO = \square AE = AB^2. \tag{A6}$$

So, substituting (A6) into (A3), we can seek

$$\text{gnomon } MNO\,(?)\,CD^2. \tag{A7}$$

In this case, the verification is based immediately on the geometry of the figure. That is, this relation is known, since it is clear that

$$\text{gnomon } MNO + AD^2 = CD^2. \tag{V1}$$

With this as a starting point, we can easily see how to produce a synthesis. Using an argument similar to that given in (A4)–(A6), we can show that

$$AB^2 + AD^2 = CD^2. \tag{S1}$$

But, as we saw in (A2), $AB^2 = 4AD^2$, so that

$$5AD^2 = CD^2. \tag{S2}$$

In this example, we see that the synthesis is only loosely modeled on the analysis. Indeed, the analysis does not provide us with a chain of logical inferences that we can then simply work backwards through in the synthesis. Rather, the analysis provides us with a number of relations that are directly related to our assumed relation, and which may be more easily determined from the original conditions. Moreover, in the course of this investigation, we have settled on a subset of the total available toolbox of theorems, operations and constructions that are relevant to the theorem at hand, $\text{Toolbox}_{\mathcal{A}}$. It is clear that the structure of the argument that we have presented here is the same as that in the comparative analyses found in our sources and set out in Fig. 11.

It is not our intention to claim that any ancient mathematician actually made the argument we have given above. We merely wish to highlight the claim that comparative analysis could have been used as a heuristic tool, even in cases in which one did not know precisely what was to be shown. Of course, one must, first, have some idea of what objects one wants to compare in order to begin such an analysis, so it is not a completely unconstrained inquiry. This is not, however, an objection against a heuristic role for comparative analysis, since all meaningful mathematical investigation is directed toward some end. Another restriction is that comparative analysis can only investigate statements about relations, so that the theorems it produces would be statements of equalities or inequalities, proportions or ratio inequalities, and so forth. Nevertheless, this represents a fairly broad class of theorems.

In studying ancient analysis, we should try, so far as possible, to distinguish between *analyzed propositions*, a form of text that Greek mathematicians produced, a few of which are found in our sources, and the *field of analysis*, a loose body of techniques and practices that Greek mathematicians used in the process of actually doing mathematics — that is, reading and understanding mathematical texts and exploring and securing mathematical problems and results.

The analyzed propositions that survive belong, as Netz [2000, 145] has argued, to the "context of presentation," and should not be read as an accurate account of whatever heuristic processes lead to the discovery of the results they present. In this regard, we may take the analytical supplements to *Elements* XIII 1–5 as an example. These are almost certainly not meant to be read as an explanation of how the results of *Elements* XIII 1–5 were either discovered, or demonstrated, but rather as an exposition of a new style of mathematical argument, which was more formulaic and operational, in contrast to the older, strictly geometric style. They are very similar, in this regard, to the alternative, analytic reworking of *Elements* II, deriving from Heron's commentary to that book [Besthorn and Heiberg, 1897–1932, 8–78]. Probably, a full study should be carried out on these two groups of arguments; nevertheless, both seem to be using well-known examples to demonstrate a certain type of analytic approach. These propositions, then, should not be read as explanations of how certain theorems where shown, but rather as examples of how we can use a certain style of analytic approach. Apollonius' *Cutting off a Ratio* is also of this general class of text. It is not meant to answer any specific question we might have about how to cut two given lines in a given ratio, but rather to provide an exhaustive set of examples of how we can use analysis to approach problems.

In fact, probably all of the analyses that survive were written for some reason that can best be understood by considerations of the context of presentation. Nevertheless, this should not lead us to believe that the field of analysis as it was understood in antiquity, was primarily concerned with presentation, as opposed to discovery. The mere fact that entire treatises, such as the *Data* and the *Conics*, which were not themselves collections of analyzed propositions, were regarded by Pappus as essential to the field of analysis is evidence that Greek mathematicians wrote and organized their works to be of use, not just in understanding statements about mathematical objects, but also in the processes of actually doing mathematics. Indeed, the fact that Pappus himself regarded reading and teaching the classics as an important form of mathematical activity is an indication that he believed one could learn how to actually do mathematics through a careful reading of the Hellenistic mathematicians.[78] In order to understand the full extent of ancient Greek mathematical activity, from the limited sources available to us, we have to ask, not only what reasons these mathematicians would have had for structuring their specific arguments in the way they did, but also what general

[78] For discussions of Pappus's rhetorical and scholarly approach see Cuomo [2000] and Bernard [2003].

process of mathematical reasoning and techniques these sources reveal, and how these techniques might have been used in other applications, not directly found in our sources.

Acknowledgments

We thank Fabio Acerbi and the anonymous referees of this journal for helpful comments on earlier versions of this paper.

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